HyperSoft Set Methods in Engineering

Journal Homepage: sciencesforce.com/hsse

HyperSoft Set Meth. Eng. Vol. 1 (2024) 84–94

Paper Type: Original Article

# A One-Step Hybrid Obrechkoff-Type Block Method for First-Order Initial Value Problems in Ordinary Differential Equations

**Lawrence Osa Adoghe** [1], **Ukpebor Luke** [1], **Benjamin Ononogbo** [1,*], **and Edward Airemen** [1]

[1] Department of Mathematics, Ambrose Alli University, Ekpoma, Edo State, Nigeria;
Emails: adolaw@aauekpoma.edu.ng; upkebor@aauekpoma.edu.ng; ononogbochibuike58@gmail.com; airemen26@gmail.com.

**Received:** 25 Oct 2023 **Revised:** 14 Feb 2024 **Accepted:** 09 Mar 2024 **Published:** 12 Mar 2024

## Abstract

In this paper, we propose a one-step hybrid block method of the Obrechkoff-type block method for solving initial value problems in ordinary differential equations using Taylor series expansion. The new method was implemented with some selected initial value problems to determine the efficiency and accuracy of the method. The convergence and stability properties were also examined. It was discovered that the new method compared favorably with other existing methods in the selected literature.

**Keywords:** One-Step, Hybrid Block Method, Obrechkoff-Type Block Method, Taylor's Series Expansion, Consistency, Convergence, Error Analysis.

## 1 | Introduction

Numerous challenges in science and engineering can be mathematically described through linear or nonlinear ordinary differential equations, often tied to specific initial or boundary conditions. Examples abound, from predicting the trajectory of a ballistic missile or charting the path of an artificial satellite in its orbit, to modeling theories related to electrical networks, beam bending, and aircraft stability [3].

Regrettably, most of these differential equations resist analytical solutions, necessitating the reliance on numerical treatment as a potent alternative to solve these equations, typically formulated as initial value problems [1]. Within the realm of numerical methods, the Obrechkoff method, with its efficiency and accuracy, has emerged prominently and has seen increased interest for problem-solving systems in recent years, as referenced by [10].

Scholar [11] proposed a one-step method for addressing first-order initial value problems using interpolation and collocation approaches. This method was subsequently implemented through Taylor series expansion. Concurrently, research by [12] introduced and applied a three-step backward differential formula specifically tailored for stiff initial value problems in ordinary differential equations, proving efficient and accurate for such scenarios.

Corresponding Author: ononogbochibuike58@gmail.com

https://doi.org/10.61356/j.hsse.2024.111850

Recent studies, such as those detailed in [4], have contrasted Obrechkoff against super-implicit methods for solving first and second-order initial value problems. Additionally, [10] suggested a numerical solution for first-order initial value problems utilizing a self-starting implicit two-step Obrechkoff-type block method, among other approaches.

Notably, the one-step hybrid block method, reminiscent of Obrechkoff's type, demonstrates superior accuracy compared to existing methods for solving initial value problems within ordinary differential equations. Further efforts have been directed towards simplifying the complexity involved in deriving this method.

With this in mind, the present paper endeavors to introduce a novel, efficient numerical algorithm designed specifically for solving initial value problems of the following type:

$$y^{(1)} = f\left(x, y\right), y\left(x_0\right) = y_0 \in \left[a, b\right] \tag{1.1}$$

Where gradient function f(x, y), may have points of discontinuities and the specific objective are;

- Derive a one-step hybrid block method of Obrechkoff type for solving initial value problems.
- Determine the order, consistency and stability of the method.
- Draw the absolute stability region of the method.
- Implement and compare the performance of the method with other existing methods using some tested initial value problems.

According to [2], the general form of the *k*-step Obrechkoff method with *l* derivatives of *y* is given as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=1}^{k} \beta_{ij} y_{n+j}^{(')} \; ; \; \alpha_k = +1 \tag{1.2}$$

With the implicit $k$ = 1; $l$ = 2. The error constant decreases more rapidly with increasing $l$ rather than the step k. It is difficult to satisfy the zero-stability for large k. The weak stability interval appears to be small [2]. In addition, the implicit one –step method of order 4 is given by:

$$y_{n+1} - y_n = \frac{h}{2}\left(y_{n+1}^{(1)} + y_n^{(1)}\right) - \frac{h^2}{2}\left(y_{n+1}^{(2)} + y_n^{(2)}\right) \tag{1.3}$$

According to [2], we define k-step hybrid formula to be:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=1}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v} \qquad (1.4$$

*where* $\alpha_k = +1, \alpha_0, \beta_0$ *are not both zero,* $v \in \{0, 1, ...k\}$. These methods are similar to linear multi-step methods in predictor-corrector mode, but with one essential modification, an additional predictor is introduced at off-step point. This means that the final (corrector) stage has an additional derivative approximation to work from. This greater generality allows the consequences of the Dahlquist barrier to be avoided and it is actually possible to obtain convergent at k-step methods with $2k+1$ *up to* $k = 7$. This paper has the following structure: Section 2 presents, derivation of the method. Section 3, presents the consistency, convergence, the stability and the region of absolute stability of the method. Section 4, presents the implementation of the method, also a display of solution tables will be provided by way of comparison with other existing methods. Finally, we gave the conclusion.

## 2 | Derivation of One-Step Hybrid Block Method

In this section, we proposed One-Step Hybrid Obrechkoff Type Block method. An approximation solution of the form (2.1) is considered.

$$y_{n+j}=\sum_{j=0}^{k-1}\alpha_j y_{n+j}+\sum_{i=1}^{l}h^i\sum_{j=1}^{k}\beta_{ij}y_{n+j}^{(i)},\quad k=1,l=2$$

$$y_{n+1}=\alpha_0 y_n+h\left[\beta_{10}y_n^{(1)}+\beta_{1\frac{1}{2}}y_{n+\frac{1}{2}}^{(1)}+\beta_{11}y_{n+1}^{(1)}\right]+h^2\left[\beta_{20}y_n^{(2)}+\beta_{2\frac{1}{2}}y_{n+\frac{1}{2}}^{(2)}+\beta_{21}y_{n+1}^{(2)}\right] \tag{2.1}$$

Expanding (2.1) by Taylor's series about $x_n$ we have:

$$y(x)+hy^{(1)}(x_n)+\frac{(h)^2}{2!}y^{(2)}(x_n)+\frac{(h)^3}{3!}y^{(3)}(x_n)+\frac{(h)^4}{4!}y^{(4)}(x_n)+\frac{(h)^5}{5!}y^{(5)}(x_n)+\frac{(h)^6}{6!}y^{(6)}(x_n)+\ldots=\alpha_0 y(x_n)$$

$$h\left[\begin{array}{l}\beta_{10}y^{(1)}(x_n)+\beta_{1\frac{1}{2}}\left(\begin{array}{l}y^{(1)}(x_n)+\frac{1}{2}hy^{(2)}(x_n)+\frac{\left(\frac{1}{2}h\right)^2}{2!}y^{(3)}(x_n)+\frac{\left(\frac{1}{2}h\right)^3}{3!}y^{(4)}(x_n)+\frac{\left(\frac{1}{2}h\right)^4}{4!}y^{(5)}(x_n)\\+\frac{\left(\frac{1}{2}h\right)^5}{5!}y^{(6)}(x_n)+\ldots\end{array}\right)+\\ \beta_{11}\left(y^{(1)}(x_n)+hy^{(2)}(x_n)+\frac{(h)^2}{2!}y^{(3)}(x_n)+\frac{(h)^3}{3!}y^{(4)}(x_n)+\frac{(h)^4}{4!}y^{(5)}(x_n)+\frac{(h)^5}{5!}y^{(6)}(x_n)+\ldots\right)\end{array}\right]$$

$$+h^2\left[\begin{array}{l}\beta_{20}y^{(2)}(x_n)+\beta_{2\frac{1}{2}}\left(\begin{array}{l}y^{(2)}(x_n)+\left(\frac{1}{2}\right)hy^{(3)}(x_n)+\frac{\left(\frac{1}{2}h\right)^2}{2!}y^{(4)}(x_n)+\frac{\left(\frac{1}{2}h\right)^3}{3!}y^{(5)}(x_n)+\frac{\left(\frac{1}{2}h\right)^4}{4!}y^{(6)}(x_n)\\+\frac{\left(\frac{1}{2}h\right)^5}{5!}y^{(7)}(x_n)+\ldots\end{array}\right)+\\ \beta_{21}\left(y^{(2)}(x_n)+hy^{(3)}(x_n)+\frac{(h)^2}{2!}y^{(4)}(x_n)+\frac{(h)^3}{3!}y^{(5)}(x_n)+\frac{(h)^4}{4!}y^{(6)}(x_n)+\frac{(h)^5}{5!}y^{(7)}(x_n)+\ldots\right)\end{array}\right]$$

Where $y_{n+a}=y(x_n+ah)=y(x_n)+ahy^{(1)}(x_n)+\frac{(ah)^2}{2!}y^{(2)}(x_n)+\frac{(ah)^3}{3!}y^{(3)}(x_n)+\ldots$

Rewriting this expression in matrix form, where the coefficients of $h^i y^{(i)}(x_n)$ are equated to give:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{1}{2!} & 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{1}{3!} & 0 & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{1}{2!} \\ 0 & 0 & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{1}{4!} & 0 & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{1}{3!} \\ 0 & 0 & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{1}{5!} & 0 & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{1}{4!} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_{10} \\ \beta_{1\frac{1}{2}} \\ \beta_{11} \\ \beta_{20} \\ \beta_{2\frac{1}{2}} \\ \beta_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{24} \\ \frac{1}{120} \\ \frac{1}{720} \end{bmatrix}$$

Solving the matrix we have:

$$\left(\alpha_0, \beta_{10}, \beta_{1\frac{1}{2}}, \beta_{11}, \beta_{20}, \beta_{2\frac{1}{2}}, \beta_{21}\right)^T = \left(1, \frac{7}{30}, \frac{8}{15}, \frac{7}{30}, \frac{1}{60}, 0, -\frac{1}{60}\right)^T$$

Substituting the value of $\alpha_0, \beta_{10}, \beta_{1\frac{1}{2}}, \beta_{11}, \beta_{20}, \beta_{2\frac{1}{2}}, and\ \beta_{21}$ into Eq. (2.1) we have

$$y_{n+1} = y_n + \frac{h}{30}\left[7y_n^{'} + 16y_{n+\frac{1}{2}}^{'} + 7y_{n+1}^{'}\right] + \frac{h^2}{60}\left[y_n^{''} - y_{n+1}^{''}\right] \tag{2.2}$$

To implement the method derived in Eq. (2.2), an additional method is needed. This method is obtained by considering the one-step method given as:

$$y_{n+\frac{1}{2}} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=1}^{k} \beta_{ij} y_{n+j}^{(')} \tag{2.3}$$

Following the same steps adopted, we have the additional method as:

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{480}\left[101y_n^{'} + 128y_{n+\frac{1}{2}}^{'} + 11y_{n+1}^{'}\right] + \frac{h^2}{960}\left[13y_n^{''} - 40y_{n+\frac{1}{2}}^{''} - 3y_{n+1}^{''}\right] \tag{2.4}$$

Hence, Eqs. (2.2) and (2.4) are the required One-Step Hybrid Obrechkoff-Type Block method for the solution of Eq. (1.1)

# 3 | Analysis of the Method

## 3.1 | Order of the Methods

We define a linear operator L by:

$$L\left[y(x):h\right]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h\beta_j y'(x_n+jh)\right] \tag{2.5}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. Expanding $y(x_n+jh)$ *and* $y'(x_n+jh)$ by Taylor's series about the point $x_n$ and collecting like terms in h and y give:

$$L\left[y(x):h\right]=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)+\ldots+C_ph^py^p(x) \quad [12] \tag{2.6}$$

**Definition 3.1:** According to [5], the differential Eq. (2.5) and the associated LMM are said to be of order p if

$$C_0=C_1=C_2=\ldots C_p=0,\, C_{p+1}\neq 0 \tag{2.7}$$

**Definition 3.2:** The term $C_{p+1}$ is called error constant and it implies that the local truncation error is given by: $E_{p+1}=C_{p+1}h^{p+1}y^{(p+1)}(x_n)+O\left(h^{p+2}\right)$[11] (2.8)

Following Definition 3.1 and 3.2, we obtained the order of the method to be;

$$C_0=\begin{pmatrix}1-1\\1-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_1=\begin{pmatrix}1-\left[\frac{7}{30}+\frac{8}{15}+\frac{7}{30}\right]\\ \frac{1}{2}-\left[\frac{101}{480}+\frac{4}{15}+\frac{11}{480}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_2=\begin{pmatrix}\frac{1}{2}-\left[\left(\frac{1}{2}\right)\frac{8}{15}+\frac{7}{30}+\frac{1}{60}+0-\frac{1}{60}\right]\\ \frac{\left(\frac{1}{2}\right)^2}{2!}-\left[\left(\frac{1}{2}\right)\frac{4}{15}+\frac{11}{480}+\frac{13}{960}-\frac{1}{24}-\frac{1}{320}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_3=\begin{pmatrix}\frac{1}{6}-\left[\frac{\left(\frac{1}{2}\right)^2}{2!}\frac{8}{15}+\left(\frac{1}{2!}\right)\frac{7}{30}+\left(\frac{1}{2}\right)0-\frac{1}{60}\right]\\ \frac{\left(\frac{1}{2}\right)^3}{3!}-\left[\frac{\left(\frac{1}{2}\right)^2}{2!}\frac{4}{15}+\left(\frac{1}{2!}\right)\frac{11}{480}-\left(\frac{1}{2}\right)\frac{1}{24}-\frac{1}{320}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_4=\begin{pmatrix}\frac{1}{24}-\left[\frac{\left(\frac{1}{2}\right)^3}{3!}\frac{8}{15}+\left(\frac{1}{3!}\right)\frac{7}{30}+\frac{\left(\frac{1}{2}\right)^2}{2!}0-\left(\frac{1}{2!}\right)\frac{1}{60}\right]\\ \frac{\left(\frac{1}{2}\right)^4}{4!}-\left[\frac{\left(\frac{1}{2}\right)^3}{3!}\frac{4}{15}+\left(\frac{1}{3!}\right)\frac{11}{480}-\frac{\left(\frac{1}{2}\right)^2}{2!}\frac{1}{24}-\left(\frac{1}{2!}\right)\frac{1}{320}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_5=\begin{pmatrix}\frac{1}{120}-\left[\frac{\left(\frac{1}{2}\right)^4}{4!}\frac{8}{15}+\left(\frac{1}{4!}\right)\frac{7}{30}+\frac{\left(\frac{1}{2}\right)^3}{3!}0-\left(\frac{1}{3!}\right)\frac{1}{60}\right]\\ \frac{\left(\frac{1}{2}\right)^5}{5!}-\left[\frac{\left(\frac{1}{2}\right)^4}{4!}\frac{4}{15}+\left(\frac{1}{4!}\right)\frac{11}{480}-\frac{\left(\frac{1}{2}\right)^3}{3!}\frac{1}{24}-\left(\frac{1}{3!}\right)\frac{1}{320}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$C_6=\begin{pmatrix}\frac{1}{720}-\left[\frac{\left(\frac{1}{2}\right)^5}{5!}\frac{8}{15}+\left(\frac{1}{5!}\right)\frac{7}{30}+\frac{\left(\frac{1}{2}\right)^4}{4!}0-\left(\frac{1}{4!}\right)\frac{1}{60}\right]\\ \frac{\left(\frac{1}{2}\right)^6}{6!}-\left[\frac{\left(\frac{1}{2}\right)^5}{5!}\frac{4}{15}+\left(\frac{1}{5!}\right)\frac{11}{480}-\frac{\left(\frac{1}{2}\right)^4}{4!}\frac{1}{24}-\left(\frac{1}{4!}\right)\frac{1}{320}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$C_0=C_1=C_2=C_3=C_4=C_5=C_6=0$ . Therefore, the new method is order six with error constant as; $\left(\frac{1}{604800},\frac{1}{1209600}\right)^T$

## 3.2 | Consistency and Zero Stability

The linear multi-step method is said to be consistent if it has order $p\geq 1$. Therefore, our method is consistent.

**Definition 3.3:** A hybrid Block method is said to be zero- stable if the roots Z of the characteristic polynomia $\bar{p}(Z)$ defined by:

$$\bar{p}(Z)=\det\left[ZA^0-A^{'}\right]$$

Satisfies $|Z|\leq 1$ and every root with $|Z_0|=1$ has multiplicity not exceeding two in the Limit as $h\rightarrow 0$

Putting Eqs. (2.2) and (2.4) in matrix form as a block we obtain:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\ y_{n+1}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{2}}\\ y_{n}\end{bmatrix}+h\left(\begin{bmatrix}0 & \frac{7}{30}\\ 0 & \frac{101}{480}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{2}}\\ f_{n}\end{bmatrix}+\begin{bmatrix}\frac{16}{30} & \frac{7}{30}\\ \frac{128}{480} & \frac{11}{480}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\end{bmatrix}\right)$$

$$+h^{2}\left(\begin{bmatrix}0 & \frac{1}{60}\\ 0 & \frac{13}{960}\end{bmatrix}\begin{bmatrix}g_{n-\frac{1}{2}}\\ g_{n}\end{bmatrix}+\begin{bmatrix}0 & -\frac{1}{60}\\ -\frac{40}{960} & \frac{-3}{960}\end{bmatrix}\begin{bmatrix}g_{n+\frac{1}{2}}\\ g_{n+1}\end{bmatrix}\right)$$

The following matrix difference equation will be in form of:

$$A^{0}y_{n+1}=A^{(')}y_{n-1}+h\left[B^{(')}f_{n-1}+B^{'}f_{n}\right]+h^{2}\left[C^{(')}g_{n-1}+C^{'}g_{n}\right]$$

The first characteristics polynomial of the above matrix is given by

$$P(Z)=\det\left[ZA^{0}-A^{'}\right]$$

$$A^{0}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\qquad A^{'}=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

$$P(Z)=\det\left[Z\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right]$$

$$=\det\left[\begin{bmatrix}Z & 0\\ 0 & Z\end{bmatrix}-\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right]=\det\begin{bmatrix}Z & -1\\ 0 & Z-1\end{bmatrix}$$

$$P(Z)=\left[Z(Z-1)\right]$$

Therefore, $Z=0 \; and \; Z=1$. The hybrid method is zero stable.

## 3.3 | Convergence

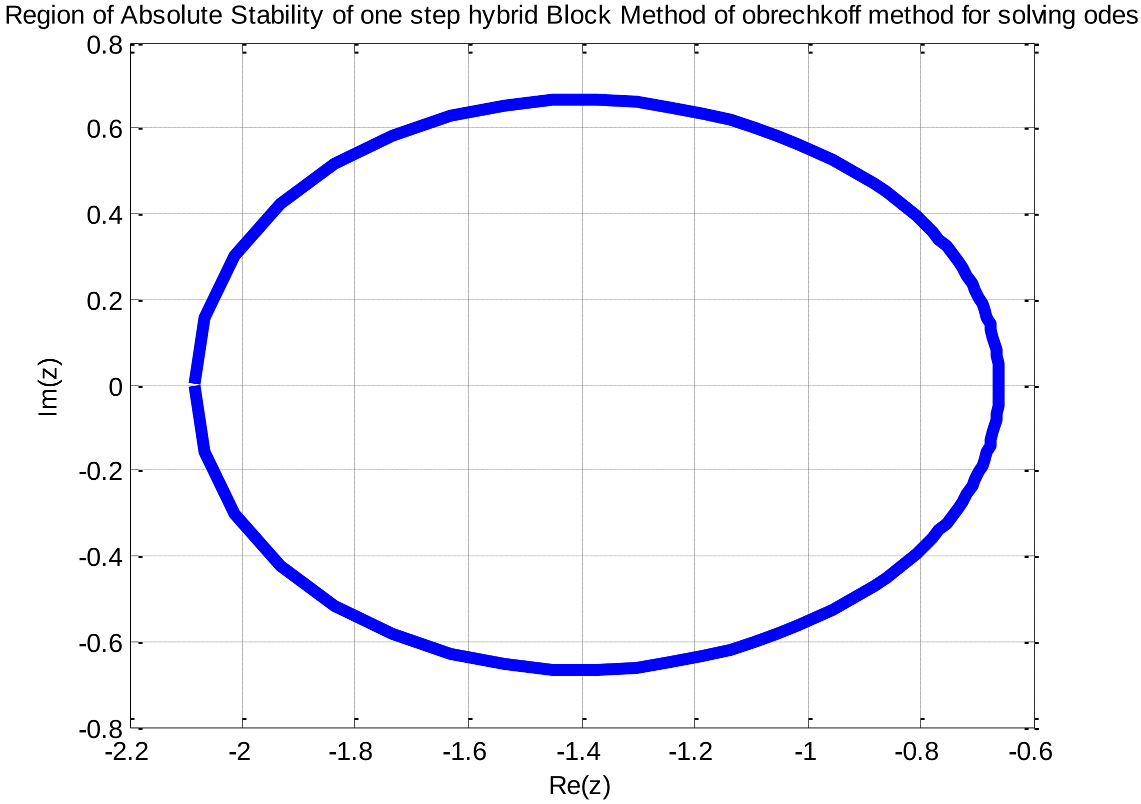
The primary goal of a numerical method revolves around generating solutions that closely resemble the theoretical solution consistently. Assessing the convergence of the One-Step Hybrid Obrechkoff Type Block method involves considering its alignment with fundamental properties discussed earlier, in correlation with Dahlquist's fundamental theorem [5] for linear multi-step methods. Without delving into the proof, we present Dahlquist's theorem as outlined in [5].

**Theorem 3.4:** The necessary and sufficient condition for a multi-step method to be convergent is for it to be consistent and zero stable.

Therefore, since the block method is consistent and zero-stable, it is likewise convergent. The region of absolute stability is determined by obtaining the stability polynomial obtained as:

$$\frac{-2z^{4}-36z^{3}-225z^{2}-162z+2880}{2z^{4}-36z^{3}+135z^{2}+1602z-2880},\; z=\lambda h$$

Plotting the roots of the stability polynomial with MATLAB, then we have the region of absolute stability as:




## 4 | Implementation of the New Method

In this section we present some numerical results to test the efficiency and usability of our new method with other existing methods. The notations are used in the tables of results:

- 1SHBM: The New One-Step Hybrid Block Method Of Obrechkoff Type.
- Error = Computed solution Minus Exact Solution.

**Problem 1:** [10]

$$y' = 0.5(1-y), y(0) = 0.5, h = 0.1 \ \ with\ exact\ \ solution\ y(x) = 1-0.5e^{-0.5x}$$

**Problem 2:** [1]

$$y' = -y,\ y(0) = 1,\ \ h = 0.1\ \ with\ exact\ \ solution\ y(x) = e^{-x}$$

**Problem 3:** [10] In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In the preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of $40 galmin^{-1}$the well-mixed solution is pumped out at a rate of $40 galmin^{-1}$. Using a numerical integrator, how much of the additive is in the tank 0.1, 0.5 and 1 min after the pumping process begins?. Let y be the amount (in pounds) of additive in the tank at time t. we know that y=100 when t=0. Thus, the initial value problem modeling the mixture process is;

$$y' = 80 - \frac{45}{(2000-5t)}, y(0) = 100, h = 0.1$$

With theoretical solution:

$$y(t) = 2(2000-5t) - \frac{3900}{(2000)^9}(2000-5t)^9$$

**Table 1a.** Comparison of computed results for problem 1.

| X | Exact solution | Computed solution (Zurni, et al.,2016) | Computed solution (Sunday *et al.*, 2013) | Computed solution ( 1 SHBM) |
|---|---|---|---|---|
| **0.1** | 0.52438528774964299546 | 0.52438528774960472804 | 0.5243852877552174 | 0.52438528774964238100 |
| **0.2** | 0.54758129098202021342 | 0.54758129098194536511 | 0.5475812909859664 | 0.54758129098201904445 |
| **0.3** | 0.56964601178747109638 | 0.56964601178736527269 | 0.5696460117956543 | 0.56964601178746942845 |
| **0.4** | 0.59063462346100907066 | 0.59063462346087361956 | 0.5906346234953703 | 0.59063462346100695522 |
| **0.5** | 0.61059960846429756588 | 0.61059960846413739010 | 0.6105996086572718 | 0.61059960846429505054 |
| **0.6** | 0.62959088965914106696 | 0.62959088965895722513 | 0.6295908898470451 | 0.62959088965913819578 |
| **0.7** | 0.64765595514064328282 | 0.64765595514044005788 | 0.6476559553183269 | 0.64765595514064009648 |
| **0.8** | 0.66483997698218034963 | 0.66483997698195855368 | 0.6648399771546479 | 0.66483997698217688570 |
| **0.9** | 0.68118592418911335343 | 0.68118592418887672320 | 0.6811859243738679 | 0.68118592418910964657 |
| **1.0** | 0.69673467014368328820 | 0.69673467014343242661 | 0.6967346704442603 | 0.69673467014367937035 |

**Table 1b.** Comparison of ERROR for problem 1.

| x | ERROR (Zurni et al.,2016) | ERROR (Sunday *et al.*, 2013) | ERROR ( 1 SHBM) |
|---|---|---|---|
| **0.1** | 3.826740E-14 | 5.574430E-12 | $6.1446\ 10^{-16}$ |
| **0.2** | 7.484830E-14 | 3.946177E-12 | $1.16897\ 10^{-15}$ |
| **0.3** | 1.058240E-13 | 8.183232E-12 | $1.66793\ 10^{-15}$ |
| **0.4** | 1.354510E-13 | 3.436118E-11 | $2.11544\ 10^{-15}$ |
| **0.5** | 1.601760E-13 | 1.929473E-10 | $2.51534\ 10^{-15}$ |
| **0.6** | 1.838420E-13 | 1.879040E-10 | $2.87118\ 10^{-15}$ |
| **0.7** | 2.032250E-13 | 1.776835E-10 | $3.18634\ 10^{-15}$ |
| **0.8** | 2.217960E-13 | 1.724676E-10 | $3.46393\ 10^{-15}$ |
| **0.9** | 2.366300E-13 | 1.847545E-10 | $3.70686\ 10^{-15}$ |
| **1.0** | 2.508620E-13 | 3.005770E-10 | $3.91785\ 10^{-15}$ |

**Table 2a**. Comparison of computed results for problem 2.

| X | Exact solution | Computed solution (Zurni et al.,2016) | Computed solution ( Badmus *et al.*, 2015) | Computed solution ( 1 SHBM) |
|---|---|---|---|---|
| **0.1** | 0.90483741803595957316 | 0.90483741804503260091 | 0.904837417881202 | 0.90483741803610926985 |
| **0.2** | 0.81873075307798185867 | 0.81873075309534995788 | 0.818730752939751 | 0.81873075307825276100 |
| **0.3** | 0.74081822068171786607 | 0.74081822070486153894 | 0.740818220548903 | 0.74081822068208554991 |
| **0.4** | 0.67032004603563930074 | 0.67032004606407889464 | 0.670320045918305 | 0.67032004603608289288 |
| **0.5** | 0.60653065971263342360 | 0.60653065974444846468 | 0.606530659599218 | 0.60653065971313514706 |
| **0.6** | 0.54881163609402643263 | 0.54881163612895298782 | 0.548811635994641 | 0.54881163609457120641 |
| **0.7** | 0.49658530379140951470 | 0.49658530382799175192 | 0.496585303694640 | 0.49658530379198460169 |
| **0.8** | 0.44932896411722159 14 | 0.44932896415534885121 | 0.449328964033219 | 0.44932896411781628883 |
| **0.9** | 0.40656965974059911188 | 0.40656965977917485733 | 0.406569659658082 | 0.40656965974120447940 |
| **1.0** | 0.36787944117144232160 | 0.36787944121046227174 | 0.367879441100594 | 0.36787944117205094291 |

**Table 2b**. Comparison of ERROR for problem 2.

| x | ERROR (Zurni et al.,2016) | ERROR (Badmus *et al.*, 2015) | ERROR ( 1 SHBM) |
|---|---|---|---|
| **0.1** | 9.0730E-12 | 1.5476E-10 | $1.4969669\ 10^{-13}$ |
| **0.2** | 1.1768E-11 | 1.3823E-10 | $2.7090233\ 10^{-13}$ |
| **0.3** | 2.3144E-11 | 1.3282E-10 | $3.6768384\ 10^{-13}$ |
| **0.4** | 2.8440E-11 | 1.1733E-10 | $4.4359214\ 10^{-13}$ |
| **0.5** | 3.1815E-11 | 1.1342E-10 | $5.0172346\ 10^{-13}$ |
| **0.6** | 3.4927E-11 | 9.9385E-11 | $5.4477378\ 10^{-13}$ |
| **0.7** | 3.6582E-11 | 9.6770E-11 | $5.7508699\ 10^{-13}$ |
| **0.8** | 3.8127E-11 | 8.4003E-11 | $5.9469740\ 10^{-13}$ |
| **0.9** | 3.8576E-11 | 8.2517E-11 | $6.0536752\ 10^{-13}$ |
| **1.0** | 3.9020E-11 | 7.0848E-11 | $6.0862131\ 10^{-13}$ |

**Table 3a.** Comparison of computed results and error for problem 3.

| x | Exact solution | Computed solution ((2SEM)Zurni et al.,2016) | Computed solution ( 1 SHBM) | ERROR ((2 SEM) Zurni et al.,2016) | ERROR ( 1SHBM) |
|---|---|---|---|---|---|
| **0.1** | 107.7662301168309486 | 107.76623267141251405 | 107.7662301168309486 | 2.554000E-06 | $7.11492085\ 10^{-9}$ |
| **0.2** | 115.5149409193028512 | 115.51494346840455900 | 115.51494094899193368 | 2.549000E-06 | $2.968908248\ 10^{-9}$ |
| **0.3** | 123.2461630508845221 | 123.24616814117862409 | 123.24616312018877335 | 5.090000E-06 | $6.930425125\ 10^{-8}$ |
| **0.4** | 130.9599271090910725 | 130.95993218819786255 | 130.95992722081203719 | 5.079000E-06 | $1.1172096469\ 10^{-7}$ |
| **0.5** | 138.6562636455413535 | 138.65627125250773431 | 138.65626380247096727 | 7.607000E-06 | $1.5692961377\ 10^{-7}$ |
| **0.6** | 146.3352031660153396 | 146.33521075612409816 | 146.33520337093594185 | 7.590000E-06 | $2.0492060225\ 10^{-7}$ |
| **0.7** | 153.9967761305114566 | 153.99678623520317743 | 153.99677638619580276 | 1.010000E-05 | $2.5568434616\ 10^{-7}$ |
| **0.8** | 161.6410129533038516 | 161.64102303550463010 | 161.64101326251512505 | 1.008000E-05 | $3.0921127345\ 10^{-7}$ |
| **0.9** | 169.2679440029996051 | 169.26795658656269977 | 169.26794436849142965 | 1.258000E-05 | $3.6549182455\ 10^{-7}$ |
| **1.0** | 176.8775996025958863 | 176.87761215807155490 | 176.87760002711233848 | 1.256000E-05 | $4.2451645218\ 10^{-7}$ |

## 5 | Conclusions

A recently introduced One-Step Hybrid Obrechkoff Type Block method has been created and utilized to address first-order initial value problems. This method is self-initiating and specifically crafted for solving a wide range of general first-order initial value problems within ordinary differential equations. Evaluations of the method's characteristics indicate its zero stability, consistency, convergence, and a sixth-order algebraic precision. Moreover, numerical assessments affirm its efficiency, demonstrating favorable comparisons against various existing methods detailed in Tables 1-3 within the literature.

## Acknowledgments

The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

## Author Contribution

All authors contributed equally to this work.

## Funding

This research has no funding source.

## Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

## Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

## Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

# References

[1] Omar, Z. and Ra'ft A.(2016), New uniform order single step hybrid block method for solving second order ordinary differential equations. International Journal of Applied Engineering Research 11(4), pp. 2402-2406. http://www.ripublication.com

[2] Lambert, J.D. (1973), Computational Methods in Ordinary Differential Equations. John Wiley and Sons, London New York. pp 114-149,

[3] Refiqul, I. and Jalal, U. (2015), Reformation on Modified Runge-Kutta third order method for solving Initial Value Problem. International Journal of Engineering Research and Technology.(IJERT). Vol 4,

[4] Neta, . B. and Fukushima, T. (2003). Obrechkoff versus-Implicit methods for the solution of first and second order initial value problems. International Journal of computer and mathematics with application Vol. 45 pg 383-390.

[5] Yakusak, N. S. and Adeniyi, R. B. (2015) A Four-Step Hybrid Block Method for First Order Initial Value Problems in Ordinary Differential Equations. AMSE JOURNALS –2015-Series: Advances Vol. 52(1), pp 17-30 .

[6] Shokri, A. and Shokri, A.A. (2013), The new class of implicit L-stable hybrid Obrechkoff method for the numerical solution of first order initial value problems. Comput. Phys. Commun., 184: 529-531.

[7] Adesanya, A.O., B. Abdulqadri and Ibrahim, Y.S. (2014), Hybrid one step block method for the solution of third order initial value problems of ordinary differential equations. Int. J. Applied Math. Comput., 6: 10-16. DOI: 10.12732/ijpam.v97i1.1

[8] Badmus, A.M., Yahaya Y.A. and Pam, Y.C. (2015), Adams type hybrid block methods associated with chebyshev polynomial for the solution of ordinary differential equations. British J. Math. Comput. Sci., 6: 464-474. DOI: 10.9734/BJMCS/2015/14945.

[9] Sunday, J., Odekunle, M.R. and Adesanya, A.O. (2013), Order six block integrator for the solution of first order ordinary differential equations. Int. J. Math. Soft Comput., 3: 87-96.

[10] Omar, Z. and Adeyeye O. (2016) , Numerical solution of first order initial value problems using a self-starting implicit two-step obrechkoff-type block method. Journal of mathematics and statistics 12(2): 27-134.

[11] Olanegan O.O, Ogunware, B.G, Omole, E. O, Oyinloye T.S, Enoch B.T. (2016), Some Classes of Continous Hybrid Linear Multistep Methods for the solution of First Order Ordinary Differential Equations. International Organisation of Science Research (IOSR) 11(5): 08-13 DIO: 10.9790/5728-11510813.

[12] Ukpebor L. A. and Omole E. O. (2020), Three-step Optimized Block Backward Differential Formulae (TOBBDF) for Solving Stiff Ordinary Differential Equations, African Journal of mathematics and Computer Science Research, 13(1), pp 51-57, DIO: 10.5897/AJMCSR2019.081.