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Bonferroni Mean Operator of Interval Linguistic Neutrosophic Uncertain Linguistic Number and its Application in Multi-Attribute Group Decision-Making

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Abstract

This paper proposes the concept of interval linguistic neutrosophic uncertain linguistic number (IL-NULN). As a new and effective way of NS expression, ILNULN combines interval linguistic neutrosophic numbers and uncertain linguistic numbers, which can better handle uncertain and inconsistent information. In the ILNULN, the first part "uncertain linguistic number" reflects the attitude of the decision maker (DM) towards the evaluation object, and the second part "interval linguistic neutrosophic number" reflects the subjective linguistic judgment of the DM on the given uncertain linguistic number. In addition, considering the weighted arithmetic Bonferroni mean (WABM) operator integrates the correlation of aggregation parameters, this paper combines the ILNULN and WABM operator to propose the interval linguistic neutrosophic uncertain linguistic weighted arithmetic Bonferroni mean (ILNUL-WABM) operator. Finally, under the environment of interval linguistic neutrosophic uncertain linguistic number, this paper uses the ILNULWABM operator to make VIKOR decision based on the relative closeness, and gives a practical example to solve multi-attribute group decision making (MAGDM) problems.

Keywords: Multi-Attribute Group Decision-Making, Interval Linguistic Neutrosophic Uncertain Linguistic Number, Weighted Arithmetic Bonferroni Mean Operator, VIKOR.

1 | Introduction

Zadeh [1] put forward the concept of fuzzy set (FS). FS represents the uncertainty of decision information by the membership degree $T(x)$, which refers to the degree that which something belongs to a certain judgment. However, in the process of cognition, people tend to hesitate to different degrees, so Atanassov [2, 3] extended the FS and proposed the concept of an intuitionistic fuzzy set (IFS). IFS considers both membership and non-membership information, so it has a stronger performance in dealing with uncertain information. Atanassov and Gargov [4] extended the IFS to an interval-value intuitionistic fuzzy set (IVIFS). Smarandache [5] proposed the concept of a neutrosophic set (NS). NS includes the membership degree $T(x)$, uncertainty degree $I(x)$, and non-membership degree $F(x)$ of elements. NS can handle uncertain and

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inconsistent information. Wang and Zhang [6] further proposed the concept of an interval neutrosophic set (INS), where the representation of $T(x)$, $I(x)$ and $F(x)$ extend from a single value to an interval number respectively. Wang and Smarandache et al. [7] proposed the single-valued neutrosophic set (SVNS) theory. Ye and Fang [8] proposed the linguistic neutrosophic number (LNN), which was characterized independently by the truth, indeterminacy, and falsity of linguistic variables. Ye [9, 10]combined the uncertain linguistic set with INS to define the interval neutrosophic uncertain linguistic set (INULS). The first part of the interval neutrosophic uncertain linguistic variable represents the subjective evaluation value of the thing being evaluated, and the second part indicates membership degree, uncertainty degree, and non-membership degree. However, the interval neutrosophic part in INULS is still the real number rather than the linguistic number that easily expresses the linguistic information. To overcome this shortcoming, we introduce the concept of ILNULN, where the INULN is extended to an interval linguistic neutrosophic number.

Information integration is a common activity in our daily life. In decision-making problems, it is necessary to consider the relationship between attributes and eliminate the impact of awkward data. For this purpose, Bonferroni [11] proposed the Bonferroni mean (BM) operator. BM operator has a desirable characteristic that it can capture the interrelationship of input arguments. Yager [12] further extended the BM operator and proposed some more efficient integration operators. Since the arithmetic average only considers the group decision and ignores the individual decision, Zhou et al. [13] proposed the standardized weighted BM operator. Later the BM operator is extended to a neutrosophic environment. Wei et al. [14] developed an uncertain linguistic Bonferroni mean (ULBM) operator to aggregate the uncertain linguistic information. For the MAGDM problem with intuitionistic uncertain linguistic variables (IULVs) as attribute values, Liu et al. [15] developed a group decision-making method based on the Bonferroni mean (BM) aggregation operator. Liu and Wang [16] introduced a single-valued neutrosophic normalized weighted Bonferroni mean (SVNNWBM) operator. Wei et al. [17] proposed some single-valued neutrosophic Bonferroni power aggregation operators and single-valued neutrosophic geometric Bonferroni power aggregation operators. Wang et al. [18] developed a simplified neutrosophic linguistic Bonferroni mean (SNLBM) operator and a simplified neutrosophic linguistic normalized weighted Bonferroni mean (SNLNWBM) operator.

VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) [19] is a method of MADM based on the ideal point. This method gives the ranking index with the ideal closest to the ideal solution, which maximizes the group utility and minimizes individual regret when selecting a solution. At present, many scholars have studied the VIKOR method and its application. Lopez et al. [20] utilized fuzzy logic and the VIKOR method to analyze the linguistic terms collected from the DMs and to rank the best alternatives that prevent dengue fever. Chen et al. [21] combined social relation analysis with linguistic VIKOR to select a new project involving ambient intelligence products. Albahri et al. [22] combined GDP and AHP-VIKOR to evaluate and optimize decentralized telemedicine hospitals based on integrated techniques. Due to the traditional VIKOR method only considering the closeness among the alternatives and the positive ideal solution, Liu [23] proposed the VIKOR method based on the relative closeness coefficient. This method takes the closeness coefficient between alternatives and positive ideal solution as well as the closeness coefficient between alternatives and negative ideal solution into account.

The remainder of this paper is structured as follows. Section 2 briefly introduces some concepts of uncertain linguistic variables (ULVs), INS, INULS, related operators, and the VIKOR method. Section 3 introduces ILNULN and ILNULWABM operators. Section 4 introduces the VIKOR method based on the relative closeness coefficient under ILNULN and ILNULWABM operators. Section 5 gives a numerical example to illustrate the proposed MAGDM method. Section 6 makes a sensitivity analysis and related comparison. Section 7 is the conclusion.

2 | Preliminaries

Some basic concepts about ULVs, INS, INULS, and BM operators are reviewed to provide the mathematical support and theoretical guarantee for this paper.

2.1 | Uncertain Linguistic Variable

Let $S=\{s_0,s_1,...,s_{l-1}\}$ be a linguistic set, where s_i is a linguistic variable. In general, l is odd. For example, when $l=7$, a linguistic term set S can be expressed as [24, 25]:

$S=\{s_0,s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}=\{$extremely poor, very poor, poor, a little poor, medium, a little good, good, very good, excellent$\}$.

Definition 1. [26-28] Suppose $s=\left[s_\theta,s_\rho\right]$, $s_\theta,s_\rho\in S$ and $\theta\le\rho$. Then s is an ULV.

2.2 | Interval Neutrosophic Set

Definition 2. [5] Let X be a set of objects and x be the element in X. The NS A in X consists of the membership degree $T_A(x)$, uncertainty degree $I_A(x)$, and non-membership degree $F_A(x)$, and it is defined as $A=\left\{\left\langle x,T_A(x),I_A(x),F_A(x)\right\rangle|x\in X\right\}$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are non-standard subsets in $]0^-,1^+[$. Due to the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ is unlimited, so $0^-\le T_A(x)+I_A(x)+F_A(x)\le 3^+$.

Definition 3. [6] Let X be a set of objects and x be the element in X. The NS A on X consists of the membership degree $T_A(x)$, uncertainty degree $I_A(x)$ and non-membership degree $F_A(x)$. When $T_A(x)$, $I_A(x)$ and $F_A(x)$ are interval values in [0,1] respectively, then A is an INS which can be expressed as: $A=\left\{x,\left[T_A^L(x),T_A^U(x)\right],\left[I_A^L(x),I_A^U(x)\right],\left[F_A^L(x),F_A^U(x)\right]|x\in X\right\}$.

Similarly, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies $0\le T_A^U(x)+I_A^U(x)+F_A^U(x)\le 3$.

2.3 | Interval Neutrosophic Uncertain Linguistic Set

Definition 4. Let X be a set of objects and x be the element in X. An INULS A on X can be defined as $A=\left\{x,\left\langle\left[s_{\theta(x)},s_{\rho(x)}\right],\left(\left[T_A^L(x),T_A^U(x)\right],\left[I_A^L(x),I_A^U(x)\right],\left[F_A^L(x),F_A^U(x)\right]\right)\right\rangle|x\in X\right\}$, where $s_{\theta(x)}$ and $s_{\rho(x)}$ belong to linguistic set S, $\left[T_A^L(x),T_A^U(x)\right]\subseteq[0,1]$, $\left[I_A^L(x),I_A^U(x)\right]\subseteq[0,1]$ and $\left[F_A^L(x),F_A^U(x)\right]\subseteq[0,1]$ with the condition $0\le T_A^U(x)+I_A^U(x)+F_A^U(x)\le 3$ for any $x\in X$. The function $T_A(x)$, $I_A(x)$ and $F_A(x)$ represents the membership degree, uncertainty degree, and non-membership degree respectively with interval values of the element x in X to the uncertain linguistic variable $\left[s_{\theta(x)},s_{\rho(x)}\right]$.

Definition 5. For any two interval neutrosophic uncertain linguistic variables (INULVs):

$a_1=\left\langle\left[s_{\theta(a_1)},s_{\rho(a_1)}\right],\left(\left[T_A^L(a_1),T_A^U(a_1)\right],\left[I_A^L(a_1),I_A^U(a_1)\right],\left[F_A^L(a_1),F_A^U(a_1)\right]\right)\right\rangle$,

$a_2=\left\langle\left[s_{\theta(a_2)},s_{\rho(a_2)}\right],\left(\left[T_A^L(a_2),T_A^U(a_2)\right],\left[I_A^L(a_2),I_A^U(a_2)\right],\left[F_A^L(a_2),F_A^U(a_2)\right]\right)\right\rangle$, then the operational laws for INULVs are as follows:

(1) $a_1\oplus a_2=\left\langle\left[s_{\theta(a_1)+\theta(a_2)},s_{\rho(a_1)+\rho(a_2)}\right],\left(\left[T_A^L(a_1)+T_A^L(a_2)-T_A^L(a_1)T_A^L(a_2),T_A^U(a_1)+T_A^U(a_2)-T_A^U(a_1)T_A^U(a_2)\right],\left[I_A^L(a_1)I_A^L(a_2),I_A^U(a_1)I_A^U(a_2)\right],\left[F_A^L(a_1)F_A^L(a_2),F_A^U(a_1)F_A^U(a_2)\right]\right)\right\rangle$

(2) $a_1\otimes a_2=\left\langle\left[s_{\theta(a_1)\theta(a_2)},s_{\rho(a_1)\rho(a_2)}\right],\left(\left[T_A^L(a_1)T_A^L(a_2),T_A^U(a_1)T_A^U(a_2)\right],\left[I_A^L(a_1)+I_A^L(a_2)-I_A^L(a_1)I_A^L(a_2),\ I_A^U(a_1)+I_A^U(a_2)-I_A^U(a_1)I_A^U(a_2)\right],\left[F_A^L(a_1)+F_A^L(a_2)-F_A^L(a_1)F_A^L(a_2),F_A^U(a_1)+F_A^U(a_2)\text{-}F_A^U(a_1)F_A^U(a_2)\right]\right)\right\rangle$.

(3) $\lambda\otimes a_1=\left\langle\left[s_{\lambda\theta(a_1)},s_{\lambda\rho(a_1)}\right],\left(\left[1-\left(1-T_A^L(a_1)\right)^\lambda,1-\left(1-T_A^U(a_1)\right)^\lambda\right],\left[\left(I_A^L(a_1)\right)^\lambda,\left(I_A^U(a_1)\right)^\lambda\right],\left[\left(F_A^L(a_1)\right)^\lambda,\left(F_A^U(a_1)\right)^\lambda\right]\right)\right\rangle,(\lambda\ge 0)$

(4) $a_1^\lambda=\left\langle\left[s_{\theta^\lambda(a_1)},s_{\rho^\lambda(a_1)}\right],\left(\left[\left(T_A^L(a_1)\right)^\lambda,\left(T_A^U(a_1)\right)^\lambda\right],\left[1-\left(1-I_A^L(a_1)\right)^\lambda,1-\left(1-I_A^U(a_1)\right)^\lambda\right],\left[1-\left(1-F_A^L(a_1)\right)^\lambda,1-\left(1-F_A^U(a_1)\right)^\lambda\right]\right)\right\rangle,(\lambda\ge 0)$

Definition 6. Broumi et al. [9] For any two INULVs $a_1=\left\langle\left[s_{\theta(a_1)},s_{\rho(a_1)}\right],\left(\left[T_A^L(a_1),T_A^U(a_1)\right],\left[I_A^L(a_1),I_A^U(a_1)\right],\left(\left[F_A^L(a_1),F_A^U(a_1)\right]\right)\right\rangle$ $a_2=\left\langle\left[s_{\theta(a_2)},s_{\rho(a_2)}\right],\left(\left[T_A^L(a_2),T_A^U(a_2)\right],\left[I_A^L(a_2),I_A^U(a_2)\right],\left[F_A^L(a_2),F_A^U(a_2)\right]\right)\right\rangle$, the Hamming distance between a_1 and a_2 is defined as:

$$d(a_1,a_2)=\frac{1}{12(t-1)}\left(\left|\theta(a_1)T_A^L(a_1)-\theta(a_2)T_A^L(a_2)\right|+\left|\theta(a_1)T_A^U(a_1)-\theta(a_2)T_A^U(a_2)\right|+\left|\theta(a_1)I_A^L(a_1)-\theta(a_2)I_A^L(a_2)\right|+\left|\theta(a_1)I_A^U(a_1)-\theta(a_2)I_A^U(a_2)\right|+\left|\theta(a_1)F_A^L(a_1)-\theta(a_2)F_A^L(a_2)\right|+\left|\theta(a_1)F_A^U(a_1)-\theta(a_2)F_A^U(a_2)\right|+\left|\rho(a_1)T_A^L(a_1)-\rho(a_2)T_A^L(a_2)\right|+\left|\rho(a_1)T_A^U(a_1)-\rho(a_2)T_A^U(a_2)\right|+\left|\rho(a_1)I_A^L(a_1)-\rho(a_2)I_A^L(a_2)\right|+\left|\rho(a_1)I_A^U(a_1)-\rho(a_2)I_A^U(a_2)\right|+\left|\rho(a_1)F_A^L(a_1)-\rho(a_2)F_A^L(a_2)\right|+\left|\rho(a_1)F_A^U(a_1)-\rho(a_2)F_A^U(a_2)\right|\right)$$

Definition 7. [10] For an INULV $a=\left\langle[s_{\theta(a)},s_{\rho(a)}],\ \left(\left[T_A^L(a),T_A^U(a)\right],\left[I_A^L(a),I_A^U(a)\right],\left[F_A^L(a),F_A^U(a)\right]\right)\right\rangle$, then the score function of a can be expressed as:

$$S(a)=\frac{1}{12}(\theta(a)+\rho(a))\left(4+T_A^L(a)-I_A^L(a)-F_A^L(a)+T_A^U(a)-I_A^U(a)-F_A^U(a)\right).$$

2.4 | Related Operators

Definition 8. [11] Let $p,q\geq 0$, and $a_i\,(i=1,2,\ldots n)$ be a collection of nonnegative real numbers. If $BM^{p,q}(a_1,a_2,\ldots a_n)=\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}a_i^p a_j^q\right)^{\frac{1}{p+q}}$, then $BM^{p,q}$ is called the Bonferroni mean (BM) operator.

2.5 | VIKOR Method

Definition 9. [19] VIKOR is a method of MADM based on the ideal point. It is regarded as a pragmatic approach to search for a compromise solution appearing in a set that includes conflicting criteria. The multi-criterion measurement of compromise order is developed from the L_P measure and it is an aggregate function of distance functions. L_1 is the sum of all individual regrets, and L_∞ is the maximum of individual regrets. The assembly function of the VIKOR method is as follows:

$$L_{p,j}=\left\{\sum_{j=1}^{n}\left[\frac{\omega_j\left(f_{ij}^{+}-f_{ij}\right)}{\left(f_{ij}^{+}-f_{ij}^{-}\right)}\right]^p\right\}^{\frac{1}{p}},1\leq p\leq+\infty,\ j=1,2,\ldots n$$

Where $\omega_j\,(j=1,2,\ldots n)$ is the relevant weight of the criteria, L_{Pj} represents the distance of each alternative from the positive ideal solution, $f_{ij}^{+}=\max\limits_j f_{ij}$ represents the positive ideal solution, and $f_{ij}^{-}=\min\limits_j f_{ij}$ represents the negative ideal solution. The main advantage of this method is that it produces a solution by maximizing group utility and minimizing the opponent's individual regret.

2.5.1 | Calculation Steps of VIKOR Method

$A=\{A_1,A_2,\ldots A_m\}$ is a set of alternatives; $C=\{C_1,C_2,\ldots C_n\}$ represents n criteria; and $\omega=(\omega_1,\omega_2,\ldots\omega_n)^T$ denotes a weight vector of criteria with $\omega_j\geq 0\ (j=1,2,\cdots,n)$ and $\sum_{j=1}^{n}\omega_j=1$. The decision matrix is $Y=\left(y_{ij}\right)_{m\times n}$.

Step 1: Normalize the decision matrix $Y=\left(y_{ij}\right)_{m\times n}$

Step 2: Calculate the positive ideal alternative y_j^+ and the negative ideal alternative y_j^- by score function $y_j^+=\left\{\max y_{ij}\right\}=\left\{\max S\left(y_{ij}\right)\right\}$, $y_j^-=\left\{\min y_{ij}\right\}=\left\{\min S\left(y_{ij}\right)\right\}$.

Step 3: Compute the group utility values S_i and the individual regret values $R_i\ (i=1,2,\cdots,m)$. Liu [23] thought that the traditional VIKOR method was not reasonable to consider only the closeness of the alternative to the positive ideal solution. So she proposed the VIKOR method based on the relative closeness

coefficient. This method takes the closeness coefficient between alternatives and positive ideal solution as well as the closeness coefficient between alternatives and negative ideal solution into account and aims to obtain a relative optimal compromise solution through relative group utility and relative individual regret.

The utility value S_i and the regret value R_i $(i = 1, 2, \cdots, m)$ by following formulations:

$$\Delta S_i = \sum_{j=1}^{n} \omega_j \frac{d\left(y_j^- - y_{ij}\right) - d\left(y_j^+ - y_{ij}\right)}{d\left(y_j^- - y_j^+\right)}, \quad \Delta R_i = \max_j \omega_j \frac{d\left(y_j^- - y_{ij}\right) - d\left(y_j^+ - y_{ij}\right)}{d\left(y_j^- - y_j^+\right)}$$

Step 4: Calculate the values Q_i

$$Q_i = \varepsilon \frac{S_i - S^-}{S^+ - S^-} + (1-\varepsilon)\frac{R_i - R^-}{R^+ - R^-}$$

Where $S^+ = \max_i S_i$, $S^- = \min_i S_i$, $R^+ = \max_i R_i$, $R^- = \min_i R_i$ and ε represents the weight of the strategy of 'the majority of criteria'. In the comprehensive evaluation, the value of ε is determined according to the subjective tendency of the DM. If the DM pays more attention to group benefits, then $\varepsilon > 0.5$; if the DM is focused more on individual regret minimization, then $\varepsilon < 0.5$; otherwise if the DM pursues both the group benefit and the individual regret value minimum, then $\varepsilon = 0.5$.

Step 5: Sort the Q_i in ascending order.

Step 6: Test the compromise solution.

Condition 1: Acceptable advantage/

$Q\left(A^2\right)\text{-}Q\left(A^1\right) \geq 1/(m\text{-}1)$ where A^2 ranks second in the ordered list by Q;

Condition 2: Acceptable stability in the process of decision-making

A^1 must be the best sorted by S or/and R. This compromise solution holds steady during the whole decision-making process.

A set of compromise solutions is obtained if it does not satisfy one of the following conditions:

A^1 and A^2 are compromise solutions if only condition 2 is not satisfied; or $A^1, A^2, \ldots A^M$ are compromise solutions if condition 1 is not satisfied; and A^M is decided by the constraint: $Q\left(A^M\right)\text{-}Q\left(A^1\right) \leq 1/(m-1)$ for maximum M.

3 | ILNULN and ILNULWABM Operator

Definition 10. Let X be a set of objects and x be the element in X. An ILNULN A in X can be defined as $A = \left\langle \left[s_{\theta(x)}, s_{\rho(x)}\right], \left\langle \left[s_{T_A^L(x)}, s_{T_A^U(x)}\right], \left[s_{I_A^L(x)}, s_{I_A^U(x)}\right], \left[s_{F_A^L(x)}, s_{F_A^U(x)}\right] \right\rangle \right\rangle$, where $s \in S$. The function $\left[s_{T_A^L(x)}, s_{T_A^U(x)}\right]$, $\left[s_{I_A^L(x)}, s_{I_A^U(x)}\right]$ and $\left[s_{F_A^L(x)}, s_{F_A^U(x)}\right]$ represents the membership degree, uncertainty degree, and non-membership degree respectively with interval values of the element x in X to the uncertain linguistic number $\left[s_{\theta(x)}, s_{\rho(x)}\right]$.

Definition 11. For any two ILNULNs, $a_1 = \left\langle \left[s_{\theta(a_1)}, s_{\rho(a_1)}\right], \left\langle \left[s_{T_A^L(a_1)}, s_{T_A^U(a_1)}\right], \left[s_{I_A^L(a_1)}, s_{I_A^U(a_1)}\right], \left[s_{F_A^L(a_1)}, s_{F_A^U(a_1)}\right] \right\rangle \right\rangle$, $a_2 = \left\langle \left[s_{\theta(a_2)}, s_{\rho(a_2)}\right], \left\langle \left[s_{T_A^L(a_2)}, s_{T_A^U(a_2)}\right], \left[s_{I_A^L(a_2)}, s_{I_A^U(a_2)}\right], \left[s_{F_A^L(a_2)}, s_{F_A^U(a_2)}\right] \right\rangle \right\rangle$, then the operational laws for ILNULNs are as follows:

$$a_1 \oplus a_2 = \left\langle \left[s_{\theta(a_1)+\theta(a_2)}, s_{\rho(a_1)+\rho(a_2)} \right], \left(\left[s_{T_A^L(a_1)+T_A^L(a_2)-T_A^L(a_1)T_A^L(a_2)}, s_{T_A^U(a_1)+T_A^U(a_2)-T_A^U(a_1)T_A^U(a_2)} \right], \left[s_{I_A^L(a_1)I_A^L(a_2)}, s_{I_A^U(a_1)I_A^U(a_2)} \right], \right.\right.$$

(1)
$$\left.\left. \left[s_{F_A^L(a_1)F_A^L(a_2)}, s_{F_A^U(a_1)F_A^U(a_2)} \right] \right) \right\rangle$$

$$a_1 \otimes a_2 = \left\langle \left[s_{\theta(a_1)\theta(a_2)}, s_{\rho(a_1)\rho(a_2)} \right], \left(\left[s_{T_A^L(a_1)T_A^L(a_2)}, s_{T_A^U(a_1)T_A^U(a_2)} \right], \left[s_{I_A^L(a_1)+I_A^L(a_2)-I_A^L(a_1)I_A^L(a_2)}, s_{I_A^U(a_1)+I_A^U(a_2)-I_A^U(a_1)I_A^U(a_2)} \right] \right.\right.$$

(2)
$$\left.\left. \left[s_{F_A^L(a_1)+F_A^L(a_2)-F_A^L(a_1)F_A^L(a_2)}, s_{F_A^U(a_1)+F_A^U(a_2)-F_A^U(a_1)F_A^U(a_2)} \right] \right) \right\rangle$$

(3) $\lambda \otimes a_1 = \left\langle \left[s_{\lambda\theta(a_1)}, s_{\lambda\rho(a_1)} \right], \left(\left[s_{1-(1-T_A^L(a_1))^\lambda}, s_{1-(1-T_A^U(a_1))^\lambda} \right], \left[s_{(I_A^L(a_1))^\lambda}, s_{(I_A^U(a_1))^\lambda} \right], \left[s_{(F_A^L(a_1))^\lambda}, s_{(F_A^U(a_1))^\lambda} \right] \right) \right\rangle (\lambda \geq 0)$

(4) $a_1^\lambda = \left\langle \left[s_{\theta^\lambda(a_1)}, s_{\rho^\lambda(a_1)} \right], \left(\left[s_{(T_A^L(a_1))^\lambda}, s_{(T_A^U(a_1))^\lambda} \right], \left[s_{1-(1-I_A^L(a_1))^\lambda}, s_{1-(1-I_A^U(a_1))^\lambda} \right] \left[s_{1-(1-F_A^L(a_1))^\lambda}, s_{1-(1-F_A^U(a_1))^\lambda} \right] \right) \right\rangle (\lambda \geq 0)$

Definition 12. For any two ILNULNs $a_1 = \left\langle \left[s_{\theta(a_1)}, s_{\rho(a_1)} \right], \left(\left[s_{T_A^L(a_1)}, s_{T_A^U(a_1)} \right], \left[s_{I_A^L(a_1)}, s_{I_A^U(a_1)} \right], \left[s_{F_A^L(a_1)}, s_{F_A^U(a_1)} \right] \right) \right\rangle$, $a_2 = \left\langle \left[s_{\theta(a_2)}, s_{\rho(a_2)} \right], \left(\left[s_{T_A^L(a_2)}, s_{T_A^U(a_2)} \right], \left[s_{I_A^L(a_2)}, s_{I_A^U(a_2)} \right], \left[s_{F_A^L(a_2)}, s_{F_A^U(a_2)} \right] \right) \right\rangle$, then the operational properties for ILNULNs are as follows:

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$
(2) $a_1 \otimes a_2 = a_2 \otimes a_1$
(3) $\lambda(a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2, (\lambda \geq 0)$
(4) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2)\, a_1, (\lambda \geq 0)$
(5) $a_1^\lambda \otimes a_2^\lambda = (a_1 \otimes a_2)^\lambda, (\lambda \geq 0)$
(6) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{(\lambda_1+\lambda_2)}, (\lambda \geq 0)$

Definition 13. For any two ILNULNs $a_1 = \left\langle \left[s_{\theta(a_1)}, s_{\rho(a_1)} \right], \left(\left[s_{T_A^L(a_1)}, s_{T_A^U(a_1)} \right], \left[s_{I_A^L(a_1)}, s_{I_A^U(a_1)} \right], \left[s_{F_A^L(a_1)}, s_{F_A^U(a_1)} \right] \right) \right\rangle$, $a_2 = \left\langle \left[s_{\theta(a_2)}, s_{\rho(a_2)} \right], \left(\left[s_{T_A^L(a_2)}, s_{T_A^U(a_2)} \right], \left[s_{I_A^L(a_2)}, s_{I_A^U(a_2)} \right], \left[s_{F_A^L(a_2)}, s_{F_A^U(a_2)} \right] \right) \right\rangle$, the Hamming distance between a_1 and a_2 is defined as:

$$\begin{aligned} d(a_1,a_2) = \frac{1}{12} \Big(& \left| \theta(a_1) T_A^L(a_1) - \theta(a_2) T_A^L(a_2) \right| + \left| \theta(a_1) T_A^U(a_1) - \theta(a_2) T_A^U(a_2) \right| + \\ & \left| \theta(a_1) I_A^L(a_1) - \theta(a_2) I_A^L(a_2) \right| + \left| \theta(a_1) I_A^U(a_1) - \theta(a_2) I_A^U(a_2) \right| + \\ & \left| \theta(a_1) F_A^L(a_1) - \theta(a_2) F_A^L(a_2) \right| + \left| \theta(a_1) F_A^U(a_1) - \theta(a_2) F_A^U(a_2) \right| + \\ & \left| \rho(a_1) T_A^L(a_1) - \rho(a_2) T_A^L(a_2) \right| + \left| \rho(a_1) T_A^U(a_1) - \rho(a_2) T_A^U(a_2) \right| + \\ & \left| \rho(a_1) I_A^L(a_1) - \rho(a_2) I_A^L(a_2) \right| + \left| \rho(a_1) I_A^U(a_1) - \rho(a_2) I_A^U(a_2) \right| + \\ & \left| \rho(a_1) F_A^L(a_1) - \rho(a_2) F_A^L(a_2) \right| + \left| \rho(a_1) F_A^U(a_1) - \rho(a_2) F_A^U(a_2) \right| \Big) \end{aligned}$$

Definition 14. For an ILNULN $a = \left\langle \left[s_{\theta(a)}, s_{\rho(a)} \right], \left(\left[s_{T_A^L(a)}, s_{T_A^U(a)} \right], \left[s_{I_A^L(a)}, s_{I_A^U(a)} \right], \left[s_{F_A^L(a)}, s_{F_A^U(a)} \right] \right) \right\rangle$, the score function of a is expressed as: $S(a) = \frac{1}{12} (\theta(a) + \rho(a)) \left(4 + T_A^L(a) - I_A^L(a) - F_A^L(a) + T_A^U(a) - I_A^U(a) - F_A^U(a) \right)$.

Definition 15. Let $p, q \geq 0$, $a_i = \left\langle \left[s_{\theta(a_i)}, s_{\rho(a_i)} \right], \left(\left[s_{T_A^L(a_i)}, s_{T_A^U(a_i)} \right], \left[s_{I_A^L(a_i)}, s_{I_A^U(a_i)} \right], \left[s_{F_A^L(a_i)}, s_{F_A^U(a_i)} \right] \right) \right\rangle$ be a set of ILNULNs, and $\omega = (\omega_1, \omega_2, \ldots \omega_n)^T$ be the weight vector of a_i, $\omega_i \geq 0$ ($j = 1, 2, \cdots, n$) and $\sum_{j=1}^{n} \omega_i = 1$. Then the aggregated result by ILNULWABM operator is expressed as:

$$ILNULWABM_{s_\omega}^{p,q}\left(a_1,a_2,...a_n\right)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i a_i\right)^p\left(\omega_j a_j\right)^q\right)^{\frac{1}{p+q}}=\left\langle\left[s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\theta_i)^p(\omega_j\theta_j)^q\right)^{\frac{1}{p+q}}},s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\rho_i)^p(\omega_j\rho_j)^q\right)^{\frac{1}{p+q}}}\right],\right.$$

$$s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}},$$

$$\left.\left[s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\right],\left[s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}},\right.\right.$$

$$s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\Big],$$

$$\left.\left[s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}},s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\right]\right\rangle \quad (1)$$

Proof: Firstly, we need to prove that

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i a_i\right)^p\left(\omega_j a_j\right)^q=\left\langle\left[s_{\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\theta_i)^p(\omega_j\theta_j)^q},s_{\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\rho_i)^p(\omega_j\rho_j)^q}\right],\right.$$

$$\left(\left[s_{1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)},\right.\right.$$

$$\left.s_{1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)}\right],\left[s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)},\right.$$

$$\left.s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)}\right],\left[s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)},\right. \quad (2)$$

$$\left.\left.s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)}\right]\right)\right\rangle$$

By the operations of ILNULN defined in definition 16, we have

$$\omega_i a_i=\left\langle\left[s_{\omega_i\theta_i},s_{\omega_i\rho_i}\right],\left(\left[s_{1-\left(1-T_A^L(a_i)\right)^{\omega_i}},s_{(a_i)1-\left(1-T_A^U(a_i)\right)^{\omega_i}}\right],\left[s_{I_A^L(a_i)^{\omega_i}},s_{I_A^U(a_i)^{\omega_i}}\right],\left[s_{F_A^L(a_i)^{\omega_i}},s_{F_A^U(a_i)^{\omega_i}}\right]\right)\right\rangle,$$

$$\omega_j a_j=\left\langle\left[s_{\omega_j\theta_j},s_{\omega_j\rho_j}\right],\left(\left[s_{1-\left(1-T_A^L(a_j)\right)^{\omega_j}},s_{1-\left(1-T_A^U(a_j)\right)^{\omega_j}}\right]\right.\right.$$

$$\left.\left.\left[s_{I_A^L(a_j)^{\omega_j}},s_{I_A^U(a_j)^{\omega_j}}\right],\left[s_{F_A^L(a_j)^{\omega_j}},s_{F_A^U(a_j)^{\omega_j}}\right]\right)\right\rangle,$$

and

$$\left(\omega_i a_i\right)^p = \left\langle \left[s_{(\omega_i\theta_i)^p}, s_{(\omega_i\rho_i)^p} \right], \left(\left[s_{\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p}, s_{\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p} \right], \left[s_{1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p}, s_{1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p} \right], \left[s_{1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p}, s_{1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p} \right] \right) \right\rangle$$

$$\omega_j a_j \Big)^q = \left\langle \left[s_{(\omega_j\theta_j)^q}, s_{(\omega_j\rho_j)^q} \right], \left(\left[s_{\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q}, s_{\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q} \right], \left[s_{1-\left(1-I_A^L(a_j)^{\omega_j}\right)^q}, s_{1-\left(1-I_A^U(a_j)^{\omega_j}\right)^q} \right], \left[s_{1-\left(1-F_A^L(a_j)^{\omega_j}\right)^q}, s_{1-\left(1-F_A^U(a_j)^{\omega_j}\right)^q} \right] \right) \right\rangle.$$

Then

$$\left(\omega_i a_i\right)^p \times \left(\omega_j a_j\right)^q$$

$$= \left\langle \left[s_{(\omega_i\theta_i)^p(\omega_j\theta_j)^q}, s_{(\omega_i\rho_i)^p(\omega_j\rho_j)^q} \right], \left(\left[s_{\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q}, s_{1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q} \right], \left[s_{1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q}, \right.\right.$$

$$\left. \left[s_{\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q} \right], \left[s_{1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q}, \right. \right.$$

$$\left. \left. s_{1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q} \right] \right) \right\rangle.$$

(a) When n=2, we can get

$$\sum_{i=1}^{2}\sum_{\substack{j=1\\ i\neq j}}^{2}\left(\omega_i a_i\right)^p\left(\omega_j a_j\right)^q = \left(\omega_1 a_1\right)^p\left(\omega_2 a_2\right)^q + \left(\omega_2 a_2\right)^p\left(\omega_1 a_1\right)^q = \left\langle \left[s_{(\omega_1\theta_1)^p(\omega_2\theta_2)^q}, s_{(\omega_1\rho_1)^p(\omega_2\rho_2)^q} \right], \right.$$

$$\left(\left[s_{\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^q}, s_{\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^q} \right], \right.$$

$$\left[s_{1-\left(1-I_A^L(a_1)^{\omega_1}\right)^p\left(1-I_A^L(a_2)^{\omega_2}\right)^q}, s_{1-\left(1-I_A^U(a_1)^{\omega_1}\right)^p\left(1-I_A^U(a_2)^{\omega_2}\right)^q} \right], \right.$$

$$\left. \left. \left[s_{1-\left(1-F_A^L(a_1)^{\omega_1}\right)^p\left(1-F_A^L(a_2)^{\omega_2}\right)^q}, s_{1-\left(1-F_A^U(a_1)^{\omega_1}\right)^p\left(1-F_A^U(a_2)^{\omega_2}\right)^q} \right] \right) \right\rangle +$$

$$\left\langle \left[s_{(\omega_2\theta_2)^p(\omega_1\theta_1)^q}, s_{(\omega_2\rho_2)^p(\omega_1\rho_1)^q} \right], \left(\left[s_{\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^q}, \right.\right.$$

$$\left. s_{\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^q} \right], \left[s_{1-\left(1-I_A^L(a_2)^{\omega_2}\right)^p\left(1-I_A^L(a_1)^{\omega_1}\right)^q}, \right.$$

$$\left. \left. s_{1-\left(1-I_A^U(a_2)^{\omega_2}\right)^p\left(1-I_A^U(a_1)^{\omega_1}\right)^q} \right], \left[s_{1-\left(1-F_A^L(a_2)^{\omega_2}\right)^p\left(1-F_A^L(a_1)^{\omega_1}\right)^q}, s_{1-\left(1-F_A^U(a_2)^{\omega_2}\right)^p\left(1-F_A^U(a_1)^{\omega_1}\right)^q} \right] \right) \right\rangle$$

$$= \left\langle \left[s_{(\omega_1\theta_1)^p(\omega_2\theta_2)^q+(\omega_2\theta_2)^p(\omega_1\theta_1)^q}, s_{(\omega_1\rho_1)^p(\omega_2\rho_2)^q+(\omega_2\rho_2)^p(\omega_1\rho_1)^q} \right], \right.$$

$$\left(\left[s_{1-\left(1-\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^q\right)\times\left(1-\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^q\right)}, \right.\right.$$

$$s_{1-\left(1-\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^q\right)\times\left(1-\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^q\right)} \Big],$$

$$s_{\left(1-\left(1-I_A^L(a_1)^{\omega_1}\right)^p\left(1-I_A^L(a_2)^{\omega_2}\right)^q\right)\left(1-\left(1-I_A^L(a_2)^{\omega_2}\right)^p\left(1-I_A^L(a_1)^{\omega_1}\right)^q\right)}, s_{\left(1-\left(1-I_A^U(a_1)^{\omega_1}\right)^p\left(1-I_A^U(a_2)^{\omega_2}\right)^q\right)\left(1-\left(1-I_A^U(a_2)^{\omega_2}\right)^p\left(1-I_A^U(a_1)^{\omega_1}\right)^q\right)} \Big],$$

$$\left. \left. s_{\left(1-\left(1-F_A^L(a_1)^{\omega_1}\right)^p\left(1-F_A^L(a_2)^{\omega_2}\right)^q\right)\left(1-\left(1-F_A^L(a_2)^{\omega_2}\right)^p\left(1-F_A^L(a_1)^{\omega_1}\right)^q\right)}, s_{\left(1-\left(1-F_A^U(a_1)^{\omega_1}\right)^p\left(1-F_A^U(a_2)^{\omega_2}\right)^q\right)\left(1-\left(1-F_A^U(a_2)^{\omega_2}\right)^p\left(1-F_A^U(a_1)^{\omega_1}\right)^q\right)} \right] \right) \right\rangle$$

That is, when n=2, Eq. (2) is right.

(b) Suppose that when n=k, Eq. (2) is right; that is, $\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}(\omega_i a_i)^p(\omega_j a_j)^q$

$$=\left\langle\left[s_{\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}(\omega_i\theta_i)^p(\omega_j\theta_j)^q},s_{\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}(\omega_i\rho_i)^p(\omega_j\rho_j)^q}\right],\left(\left[s_{1-\prod_{i=1}^{k}\prod_{\substack{j=1\\ i\neq j}}^{k}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)},\right.\right.$$

$$s_{1-\prod_{i=1}^{k}\prod_{\substack{j=1\\ i\neq j}}^{k}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)}\Bigg],\left[s_{\prod_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)},s_{\prod_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)}\right]$$

$$\left.\left.\left[s_{\prod_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)},s_{\prod_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)}\right]\right)\right\rangle.$$

Then, when n=k+1, we have

$$\sum_{i=1}^{k+1}\sum_{\substack{j=1\\ i\neq j}}^{k+1}(\omega_i a_i)^p(\omega_j a_j)^q=\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}(\omega_i a_i)^p(\omega_j a_j)^q+\sum_{i=1}^{k}(\omega_i a_i)^p(\omega_{k+1}a_{k+1})^q+\sum_{j=1}^{k}(\omega_{k+1}a_{k+1})^p(\omega_j a_j)^q \tag{3}$$

Firstly, we prove that

$$\sum_{i=1}^{k}(\omega_i a_i)^p(\omega_{k+1}a_{k+1})^q=\left\langle\left[s_{\sum_{i=1}^{k}(\omega_i\theta_i)^p(\omega_{k+1}\theta_{k+1})^q},s_{\sum_{i=1}^{k}(\omega_i\rho_i)^p(\omega_{k+1}\rho_{k+1})^q}\right],\right.$$

$$\left[s_{1-\prod_{i=1}^{k}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_{k+1})\right)^{\omega_{k+1}}\right)^q\right)},s_{1-\prod_{i=1}^{k}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_{k+1})\right)^{\omega_{k+1}}\right)^q\right)}\right],\left[s_{\prod_{i=1}^{k}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_{k+1})^{\omega_{k+1}}\right)^q\right)},s_{\prod_{i=1}^{k}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_{k+1})^{\omega_{k+1}}\right)^q\right)}\right] \tag{4}$$

$$\left.s_{\prod_{i=1}^{k}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_{k+1})^{\omega_{k+1}}\right)^q\right)},s_{\prod_{i=1}^{k}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_{k+1})^{\omega_{k+1}}\right)^q\right)}\right]\Bigg)\Bigg\rangle$$

We also use the mathematical induction on k as follows:

(i) When k=2, we have

$$(\omega_i a_i)^p(\omega_3 a_3)^q=\left\langle\left[s_{(\omega_i\theta_i)^p(\omega_3\theta_3)^q},s_{(\omega_i\rho_i)^p(\omega_3\rho_3)^q}\right],\left(\left[s_{\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_3)\right)^{\omega_3}\right)^q},\right.\right.$$

$$\left[s_{\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_3)\right)^{\omega_3}\right)^q}\right],\left[s_{1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_3)^{\omega_3}\right)^q},\right.$$

$$\left.\left.s_{1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_3)^{\omega_3}\right)^q}\right],\left[s_{1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_3)^{\omega_3}\right)^q},\right.$$

$$s_{1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_3)^{\omega_3}\right)^q}\Bigg]\Bigg]\Bigg\rangle,$$

and

$$\begin{aligned}
\sum_{i=1}^{2}\left(\omega_i a_i\right)^p\left(\omega_3 a_3\right)^q &= \left(\omega_1 a_1\right)^p\left(\omega_3 a_3\right)^q + \left(\omega_2 a_2\right)^p\left(\omega_3 a_3\right)^q \\
&= \Bigg\langle\Bigg[s_{(\omega_1\theta_1)^p(\omega_3\theta_3)^q}, s_{(\omega_1\rho_1)^p(\omega_3\rho_3)^q}\Bigg], \Bigg(\Bigg[s_{\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^L(a_3)\right)^{\omega_3}\right)^q}, \\
&\quad s_{\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^U(a_3)\right)^{\omega_3}\right)^q}\Bigg], \Bigg[s_{1-\left(1-I_A^L(a_1)^{\omega_1}\right)^p\left(1-I_A^L(a_3)^{\omega_3}\right)^q}, \\
&\quad s_{1-\left(1-I_A^U(a_1)^{\omega_1}\right)^p\left(1-I_A^U(a_3)^{\omega_3}\right)^q}\Bigg], \Bigg[s_{1-\left(1-F_A^L(a_1)^{\omega_1}\right)^p\left(1-F_A^L(a_3)^{\omega_3}\right)^q}, \\
&\quad s_{1-\left(1-F_A^U(a_1)^{\omega_1}\right)^p\left(1-F_A^U(a_3)^{\omega_3}\right)^q}\Bigg]\Bigg)\Bigg\rangle, \\
&\quad + \Bigg\langle\Bigg[s_{(\omega_2\theta_2)^p(\omega_3\theta_3)^q}, s_{(\omega_2\rho_2)^p(\omega_3\rho_3)^q}\Bigg], \Bigg(\Bigg[s_{\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^L(a_3)\right)^{\omega_3}\right)^q}, \\
&\quad s_{\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^U(a_3)\right)^{\omega_3}\right)^q}\Bigg], \Bigg[s_{1-\left(1-I_A^L(a_2)^{\omega_2}\right)^p\left(1-I_A^L(a_3)^{\omega_3}\right)^q}, \\
&\quad s_{1-\left(1-I_A^U(a_2)^{\omega_2}\right)^p\left(1-I_A^U(a_3)^{\omega_3}\right)^q}\Bigg], \Bigg[s_{1-\left(1-F_A^L(a_2)^{\omega_2}\right)^p\left(1-F_A^L(a_3)^{\omega_3}\right)^q}, \\
&\quad s_{1-\left(1-F_A^U(a_2)^{\omega_2}\right)^p\left(1-F_A^U(a_3)^{\omega_3}\right)^q}\Bigg]\Bigg)\Bigg\rangle \\
&= \Bigg\langle\Bigg[s_{(\omega_1\theta_1)^p(\omega_3\theta_3)^q+(\omega_2\theta_2)^p(\omega_3\theta_3)^q}, s_{(\omega_1\rho_1)^p(\omega_3\rho_3)^q+(\omega_2\rho_2)^p(\omega_3\rho_3)^q}\Bigg], \\
&\quad \Bigg(\Bigg[s_{1-\left(1-\left(1-\left(1-T_A^L(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^L(a_3)\right)^{\omega_3}\right)^q\right)\times\left(1-\left(1-\left(1-T_A^L(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^L(a_3)\right)^{\omega_3}\right)^q\right)}, \\
&\quad s_{1-\left(1-\left(1-\left(1-T_A^U(a_1)\right)^{\omega_1}\right)^p\left(1-\left(1-T_A^U(a_3)\right)^{\omega_3}\right)^q\right)\times\left(1-\left(1-\left(1-T_A^U(a_2)\right)^{\omega_2}\right)^p\left(1-\left(1-T_A^U(a_3)\right)^{\omega_3}\right)^q\right)}\Bigg],
\end{aligned}$$

$$\left\langle \left[s_{\left(1-\left(1-I_A^L(a_1)^{\omega_1}\right)^p\left(1-I_A^L(a_3)^{\omega_3}\right)^q\right)\times\left(1-\left(1-I_A^L(a_2)^{\omega_2}\right)^p\left(1-I_A^L(a_3)^{\omega_3}\right)^q\right)}, s_{\left(1-\left(1-I_A^U(a_1)^{\omega_1}\right)^p\left(1-I_A^U(a_3)^{\omega_3}\right)^q\right)\times\left(1-\left(1-I_A^U(a_2)^{\omega_2}\right)^p\left(1-I_A^U(a_3)^{\omega_3}\right)^q\right)} \right], \left[s_{\left(1-\left(1-F_A^L(a_1)^{\omega_1}\right)^p\left(1-F_A^L(a_3)^{\omega_3}\right)^q\right)\times\left(1-\left(1-F_A^L(a_2)^{\omega_2}\right)^p\left(1-F_A^L(a_3)^{\omega_3}\right)^q\right)}, s_{\left(1-\left(1-F_A^U(a_1)^{\omega_1}\right)^p\left(1-F_A^U(a_3)^{\omega_3}\right)^q\right)\times\left(1-\left(1-F_A^U(a_2)^{\omega_2}\right)^p\left(1-F_A^U(a_3)^{\omega_3}\right)^q\right)} \right] \right\rangle$$

$$= \left\langle \left[s_{\sum_{i=1}^{2}(\omega_i\theta_i)^p(\omega_{k+1}\theta_{k+1})^q}, s_{\sum_{i=1}^{2}(\omega_i\rho_i)^p(\omega_{k+1}\rho_{k+1})^q} \right], \left(\left[s_{1-\prod_{i=1}^{2}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_{k+1})\right)^{\omega_{k+1}}\right)^q\right)}, s_{1-\prod_{i=1}^{2}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_{k+1})\right)^{\omega_{k+1}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{2}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_{k+1})^{\omega_{k+1}}\right)^q\right)}, s_{\prod_{i=1}^{2}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_{k+1})^{\omega_{k+1}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{2}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_{k+1})^{\omega_{k+1}}\right)^q\right)}, s_{\prod_{i=1}^{2}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_{k+1})^{\omega_{k+1}}\right)^q\right)} \right] \right) \right\rangle$$

(ii) Suppose that when $k=l$, the Eq.(4) is right; that is,

$$\sum_{i=1}^{l}(\omega_i a_i)^p(\omega_{l+1}a_{l+1})^q = \left\langle \left[s_{\sum_{i=1}^{l}(\omega_i\theta_i)^p(\omega_{l+1}\theta_{l+1})^q}, s_{\sum_{i=1}^{l}(\omega_i\rho_i)^p(\omega_{l+1}\rho_{l+1})^q} \right], \left(\left[s_{1-\prod_{i=1}^{l}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_{l+1})\right)^{\omega_{l+1}}\right)^q\right)}, s_{1-\prod_{i=1}^{l}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_{l+1})\right)^{\omega_{l+1}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{l}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_{l+1})^{\omega_{l+1}}\right)^q\right)}, s_{\prod_{i=1}^{l}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_{l+1})^{\omega_{l+1}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{l}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_{l+1})^{\omega_{l+1}}\right)^q\right)}, s_{\prod_{i=1}^{l}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_{l+1})^{\omega_{l+1}}\right)^q\right)} \right] \right) \right\rangle$$

Then, when k=*l*+1, we have

$$\sum_{i=1}^{l+1}(\omega_i a_i)^p(\omega_{l+2}a_{l+2})^q = \sum_{i=1}^{l}(\omega_i a_i)^p(\omega_{l+2}a_{l+2})^q + (\omega_{l+1}a_{l+1})^p(\omega_{l+2}a_{l+2})^q$$

$$= \left\langle \left[s_{\sum_{i=1}^{l}(\omega_i\theta_i)^p(\omega_{l+2}\theta_{l+2})^q}, s_{\sum_{i=1}^{l}(\omega_i\rho_i)^p(\omega_{l+2}\rho_{l+2})^q} \right], \left(\left[s_{1-\prod_{i=1}^{l}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_{l+2})\right)^{\omega_{l+2}}\right)^q\right)} \right.\right.\right.$$

$$\left. s_{1-\prod_{i=1}^{l}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_{l+2})\right)^{\omega_{l+2}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{l}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_{l+2})^{\omega_{l+2}}\right)^q\right)}, \right.$$

$$\left. \left. s_{\prod_{i=1}^{l}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_{l+2})^{\omega_{l+2}}\right)^q\right)} \right], \left[s_{\prod_{i=1}^{l}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_{l+2})^{\omega_{l+2}}\right)^q\right)}, \right.$$

$$\left.\left.\left.\prod_{i=1}^{l}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_{l+2})^{\omega_{l+2}}\right)^q\right)\right]\right)\right\rangle+\left\langle\left[s_{(\omega_{l+1}\theta_{l+1})^p(\omega_{l+2}\theta_{l+2})^q},s_{(\omega_{l+1}\rho_{l+1})^p(\omega_{l+2}\rho_{l+2})^q}\right],\left[s_{\left(1-\left(1-T_A^L(a_{l+1})\right)^{\omega_{l+1}}\right)^p\left(1-\left(1-T_A^L(a_{l+2})\right)^{\omega_{l+2}}\right)^q},\right.\right.$$

$$\left.\left(1-\left(1-T_A^U(a_{l+1})\right)^{\omega_{l+1}}\right)^p\left(1-\left(1-T_A^U(a_{l+2})\right)^{\omega_{l+2}}\right)^q\right],\left[s_{1-\left(1-I_A^L(a_{l+1})^{\omega_{l+1}}\right)^p\left(1-I_A^L(a_{l+2})^{\omega_{l+2}}\right)^q},s_{1-\left(1-I_A^U(a_{l+1})^{\omega_{l+1}}\right)^p\left(1-I_A^U(a_{l+2})^{\omega_{l+2}}\right)^q}\right],\left[s_{1-\left(1-F_A^L(a_{l+1})^{\omega_{l+1}}\right)^p\left(1-F_A^L(a_{l+2})^{\omega_{l+2}}\right)^q}\right.$$

$$\left.\left.\left.1-\left(1-F_A^U(a_{l+1})^{\omega_{l+1}}\right)^p\left(1-F_A^U(a_{l+2})^{\omega_{l+2}}\right)^q\right]\right)\right\rangle=\left\langle\left[s_{\sum_{i=1}^{l+1}(\omega_i\theta_i)^p(\omega_{l+2}\theta_{l+2})^q},s_{\sum_{i=1}^{l+1}(\omega_i\rho_i)^p(\omega_{l+2}\rho_{l+2})^q}\right],\right.$$

$$\left(\left[s_{1-\prod_{i=1}^{l+1}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_{l+2})\right)^{\omega_{l+2}}\right)^q\right)},s_{1-\prod_{i=1}^{l+1}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_{l+2})\right)^{\omega_{l+2}}\right)^q\right)}\right],\right.$$

$$\left[s_{\prod_{i=1}^{l+1}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_{l+2})^{\omega_{l+2}}\right)^q\right)},s_{\prod_{i=1}^{l+1}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_{l+2})^{\omega_{l+2}}\right)^q\right)}\right],$$

$$\left.\left.\left[s_{\prod_{i=1}^{l+1}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_{l+2})^{\omega_{l+2}}\right)^q\right)},s_{\prod_{i=1}^{l+1}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_{l+2})^{\omega_{l+2}}\right)^q\right)}\right]\right)\right\rangle$$

That is, for k=*l*+1, Eq. (4) is also right.

(iii) So, for all k, Eq. (4) is right.

Similarly, we can prove that

$$\sum_{=1}^{k}\left(\omega_{k+1}a_{k+1}\right)^p\left(\omega_ja_j\right)^q=\left\langle\left[s_{\sum_{j=1}^{k}(\omega_{k+1}\theta_{k+1})^p(\omega_j\theta_j)^q},s_{\sum_{j=1}^{k}(\omega_{k+1}\rho_{k+1})^p(\omega_j\rho_j)^q}\right],\right.$$

$$\left(\left[s_{1-\prod_{j=1}^{k}\left(1-\left(1-\left(1-T_A^L(a_{k+1})\right)^{\omega_{k+1}}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)},s_{1-\prod_{j=1}^{k}\left(1-\left(1-\left(1-T_A^U(a_{k+1})\right)^{\omega_{k+1}}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)}\right],\right.$$

$$\left[s_{\prod_{j=1}^{k}\left(1-\left(1-I_A^L(a_{k+1})^{\omega_{k+1}}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)},s_{\prod_{i=1}^{k}\left(1-\left(1-I_A^U(a_{k+1})^{\omega_{k+1}}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)}\right],$$

$$\left.\left.\left[s_{\prod_{i=1}^{k}\left(1-\left(1-F_A^L(a_{k+1})^{\omega_{k+1}}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)},s_{\prod_{i=1}^{k}\left(1-\left(1-F_A^U(a_{k+1})^{\omega_{k+1}}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)}\right]\right)\right\rangle$$

So, the Eq (3) can be transformed as

$$\sum_{i=1}^{k+1}\sum_{\substack{j=1\\i\neq j}}^{k+1}(\omega_ia_i)^p(\omega_ja_j)^q=\sum_{i=1}^{k}\sum_{\substack{j=1\\i\neq j}}^{k}(\omega_ia_i)^p(\omega_ja_j)^q+\sum_{i=1}^{k}(\omega_ia_i)^p(\omega_{k+1}a_{k+1})^q$$

$$+\sum_{j=1}^{k}(\omega_{k+1}a_{k+1})^p(\omega_ja_j)^q$$

$$=\left\langle\left[s_{\sum_{i=1}^{k+1}\sum_{\substack{j=1\\i\neq j}}^{k+1}(\omega_i\theta_i)^p(\omega_j\theta_j)^q},s_{\sum_{i=1}^{k+1}\sum_{\substack{j=1\\i\neq j}}^{k+1}(\omega_i\rho_i)^p(\omega_j\rho_j)^q}\right],\right.$$

$$\left(\left[s_{1-\prod_{i=1}^{k+1}\prod_{\substack{j=1\\i\neq j}}^{k+1}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)},\right.\right.$$

$$\left. s_{1-\prod_{i=1}^{k+1}\prod_{\substack{j=1\\ i\neq j}}^{k+1}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)}\right],$$

$$\left[s_{\prod_{i=1}^{k+1}\prod_{\substack{j=1\\ i\neq j}}^{k+1}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)}, s_{\prod_{i=1}^{k+1}\prod_{\substack{j=1\\ i\neq j}}^{k+1}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)}\right],$$

$$\left.\left.\left.\left[s_{\prod_{i=1}^{k+1}\prod_{\substack{j=1\\ i\neq j}}^{k+1}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)}, s_{\prod_{i=1}^{k+1}\prod_{\substack{j=1\\ i\neq j}}^{k+1}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)}\right]\right)\right)\right\rangle$$

So, when n=k+1, Eq (2) is also right. Thus, Eq.(2) is right for all n.

(2) Then, we prove Eq (1) is right. By Eq (2), we can get

$$ILNULWABM_{\omega}^{p,q}(a_1,a_2,\ldots a_n)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}(\omega_i a_i)^p(\omega_j a_j)^q\right)^{\frac{1}{p+q}}=\left\langle\left[s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\theta_i)^p(\omega_j\theta_j)^q\right)^{\frac{1}{p+q}}}, s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\omega_i\rho_i)^p(\omega_j\rho_j)^q\right)^{\frac{1}{p+q}}}\right],\right.$$

$$\left(\left[s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\right],\right.$$

$$\left[s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)^p\left(1-I_A^U(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\right],$$

$$\left.\left.\left[s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}\right]\right)\right\rangle$$

Next, some special cases of the ILNULWABM operator concerning the parameters p and q will be demonstrated respectively.

(1) When p=1 and q=0, then

$$ILNULWABM_{\omega}^{p=1,q=0}\left(s_1,s_2,...s_n\right)$$
$$=\frac{1}{n\left(n\text{-}1\right)}\sum_{i=1}^{n}\omega_i a_i$$
$$=\left\langle\left[s_{\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\omega_i\theta_i},s_{\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\omega_i\rho_i}\right],\left(\left[s_{1-\prod_{i=1}^{n}\left(1-T_A^L(a_i)\right)^{\frac{\omega_i}{n(n-1)}}},s_{1-\prod_{i=1}^{n}\left(1-T_A^U(a_i)\right)^{\frac{\omega_i}{n(n-1)}}}\right],\right.$$
$$\left.\left[s_{\prod_{i=1}^{n}\left(I_A^L(a_i)\right)^{\frac{\omega_i}{n(n-1)}}},s_{\prod_{i=1}^{n}\left(I_A^U(a_i)\right)^{\frac{\omega_i}{n(n-1)}}}\right],\left[s_{\prod_{i=1}^{n}\left(F_A^L(a_i)\right)^{\frac{\omega_i}{n(n-1)}}},s_{\prod_{i=1}^{n}\left(F_A^U(a_i)\right)^{\frac{\omega_i}{n(n-1)}}}\right]\right)\right\rangle$$

(2) When p=1 and q=1, then

$$ILNULWABM_{s_\omega}^{p=q=1}\left(s_1,s_2,...s_n\right)=\left(\frac{1}{n\left(n\text{-}1\right)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\omega_i\omega_j a_i a_j\right)^{\frac{1}{2}}$$
$$=\left\langle\left[s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\omega_i\omega_j\theta_i\theta_j\right)^{\frac{1}{2}}},s_{\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\omega_i\omega_j\rho_i\rho_j\right)^{\frac{1}{2}}}\right],\right.$$
$$\left(\left[s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}}\right.\right.$$
$$\left.\left.s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}}\right],$$
$$\left[s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)\left(1-I_A^L(a_j)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}},\right.$$
$$\left.s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-I_A^U(a_i)^{\omega_i}\right)\left(1-I_A^U(a_j)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}}\right],$$
$$\left[s_{\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)\left(1-F_A^L(a_j)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}},\right.$$

$$s_{1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)\left(1-F_A^U(a_j)^{\omega_j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}}\Big]\Big)\Big\rangle$$

(3) When p=0.5 and q=0.5, then

$$ILNULWABM_{s_\omega}^{p=q=0.5}(s_1,s_2,\ldots s_n)=\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}(\omega_i a_i)^{0.5}(\omega_j a_j)^{0.5}$$

$$=\Big\langle\Big[s_{\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}(\omega_i\theta_i)^{0.5}(\omega_j\theta_j)^{0.5}},s_{\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}(\omega_i\rho_i)^{0.5}(\omega_j\rho_j)^{0.5}}\Big],\Big(\Big[s_{1-\prod_{i=1}^{n}\prod_{\substack{j=1\\i\neq j}}^{n}\left(1-\left(\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}},$$

$$s_{1-\prod_{i=1}^{n}\prod_{\substack{j=1\\i\neq j}}^{n}\left(1-\left(\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}}\Big],\Big[s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\left(1-I_A^L(a_i)^{\omega_i}\right)\left(1-I_A^L(a_j)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}},$$

$$s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\left(1-I_A^U(a_i)^{\omega_i}\right)\left(1-I_A^U(a_j)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}}\Big],\Big[s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\left(1-F_A^L(a_i)^{\omega_i}\right)\left(1-F_A^L(a_j)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}},$$

$$s_{\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\left(1-F_A^U(a_i)^{\omega_i}\right)\left(1-F_A^U(a_j)^{\omega_j}\right)\right)^{0.5}\right)^{\frac{1}{n(n-1)}}}\Big]\Big)\Big\rangle$$

Definition 16. The following section investigates some additional properties of the ILNULWABM operator.

Theorem 1. (commutativity). Let $\left(a_1^{'},a_2^{'},\ldots a_n^{'}\right)$ be any permutation of $(a_1,a_2,\ldots a_n)$; then $ILNULWABM_{s_\omega}^{p,q}\left(a_1^{'},a_2^{'},\ldots a_n^{'}\right)=ILNULWABM_{s_\omega}^{p,q}\left(a_1,a_2,\ldots a_n\right)$

Proof: Let

$$ILNULWABM_{\omega}^{p,q}(a_1,a_2,\ldots a_n)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}(\omega_i a_i)^p(\omega_j a_j)^q\right)^{\frac{1}{p+q}},\ ILNULWABM_{\omega}^{p,q}\left(a_1^{'},a_2^{'},\ldots a_n^{'}\right)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\left(\omega_i a_i^{'}\right)^p\left(\omega_j a_j^{'}\right)^q\right)^{\frac{1}{p+q}}.$$

Since $\left(a_1^{'},a_2^{'},\ldots a_n^{'}\right)$ is any permutation $(a_1,a_2,\ldots a_n)$, we have

$$\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}(\omega_i a_i)^p(\omega_j a_j)^q\right)^{\frac{1}{p+q}}=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\left(\omega_i a_i^{'}\right)^p\left(\omega_j a_j^{'}\right)^q\right)^{\frac{1}{p+q}}.$$

Such, $ILNULWABM_{\omega}^{p,q}\left(a_1^{'},a_2^{'},\ldots a_n^{'}\right)=ILNULWABM_{\omega}^{p,q}\left(a_1,a_2,\ldots a_n\right)$

Theorem 2. (monotonicity) Let a_i and b_i be two collections of ILNULNs. If $a_i\leq b_i$ for all i, i.e. $s_{\theta(a_i)}\leq s_{\theta(b_i)},s_{\rho(a_i)}\leq s_{\rho(b_i)},s_{T_A^L(a_i)}\leq s_{T_A^L(b_i)},s_{T_A^U(a_i)}\leq s_{T_A^U(b_i)},$ $s_{I_A^L(a_i)}\geq s_{I_A^L(b_i)},s_{I_A^U(a_i)}\geq s_{I_A^U(b_i)},s_{F_A^L(a_i)}\geq s_{F_A^L(b_i)},s_{F_A^U(a_i)}\geq s_{F_A^U(b_i)},$ then $ILNULWABM_{\omega}^{p,q}(a_1,a_2,\ldots a_n)\leq ILNULWABM_{\omega}^{p,q}(b_1,\mathrm{b}_2,\ldots \mathrm{b}_n)$.

Proof:

(1) Since $s_{\theta(a_i)} \le s_{\theta(b_i)}$ and $s_{\rho(a_i)} \le s_{\rho(b_i)}$ for all i, we have $\theta(a_i) \le \theta(b_i)$ and $\rho(a_i) \le \rho(b_i)$. So

$$\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i\theta(a_i)\right)^p\left(\omega_j\theta(a_j)\right)^q\right)^{\frac{1}{p+q}} \le \left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i\theta(b_i)\right)^p\left(\omega_j\theta(b_j)\right)^q\right)^{\frac{1}{p+q}} \text{ and}$$

$$\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i\rho(a_i)\right)^p\left(\omega_j\rho(a_j)\right)^q\right)^{\frac{1}{p+q}} \le \left(\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i\rho(b_i)\right)^p\left(\omega_j\rho(b_j)\right)^q\right)^{\frac{1}{p+q}}$$

(2) Since $s_{T_A^L(a_i)} \le s_{T_B^L(b_i)}$, we have $T_A^L(a_i) \le T_B^L(b_i)$ and $\left(1-T_A^L(a_i)\right)^{\omega_i} \ge \left(1-T_B^L(b_i)\right)^{\omega_i}, \left(1-T_A^L(a_j)\right)^{\omega_j} \ge \left(1-T_B^L(b_j)\right)^{\omega_j}$. Due to $p,q \ge 0$, so $\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q \le \left(1-\left(1-T_B^L(b_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_B^L(b_j)\right)^{\omega_j}\right)^q$

Further, we get

$$\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\Big)^{\frac{1}{n(n-1)}} \ge \left(1-\left(1-T_B^L(b_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_B^L(b_j)\right)^{\omega_j}\right)^q\Big)^{\frac{1}{n(n-1)}}.$$

Finally, we get

$$\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-T_A^L(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^L(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\Bigg)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-T_B^L(b_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_B^L(b_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\Bigg)^{\frac{1}{p+q}}.$$

Similarly,

$$\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-T_A^U(a_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_A^U(a_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\Bigg)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ i\neq j}}^{n}\left(1-\left(1-T_B^U(b_i)\right)^{\omega_i}\right)^p\left(1-\left(1-T_B^U(b_j)\right)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\Bigg)^{\frac{1}{p+q}}.$$

(3)Since $s_{I_A^L(a_i)} \ge s_{I_B^L(b_i)}$ and $s_{I_A^L(a_j)} \ge s_{I_B^L(b_j)}$, we have $I_A^L(a_i) \ge I_B^L(b_i), I_A^L(a_j) \ge I_B^L(b_j)$ and $1-I_A^L(a_i)^{\omega_i} \le 1-I_B^L(b_i)^{\omega_i}, 1-I_A^L(a_j)^{\omega_j} \le 1-I_B^L(b_j)^{\omega_j}$. Due to $p,q \ge 0$, so

$$\left(1-I_A^L(a_i)^{\omega_i}\right)^p \le \left(1-I_B^L(b_i)^{\omega_i}\right)^p, \left(1-I_A^L(a_j)^{\omega_j}\right)^q \le \left(1-I_B^L(b_j)^{\omega_j}\right)^q,$$
$$\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q \le \left(1-I_B^L(b_i)^{\omega_i}\right)^p\left(1-I_B^L(b_j)^{\omega_j}\right)^q$$

Further, we get

$$\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}} \ge \left(1-\left(1-I_B^L(b_i)^{\omega_i}\right)^p\left(1-I_B^L(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}$$

and $\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}} \ge \prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_B^L(b_i)^{\omega_i}\right)^p\left(1-I_B^L(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}$.

Finally we get

$$\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_B^L(b_i)^{\omega_i}\right)^p\left(1-I_B^L(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},$$

i.e.

$$1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_A^L(a_i)^{\omega_i}\right)^p\left(1-I_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \ge 1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-I_B^L(b_i)^{\omega_i}\right)^p\left(1-I_B^L(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}.$$

Similarly,

$$1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^L(a_i)^{\omega_i}\right)^p\left(1-F_A^L(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \geq 1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_B^L(b_i)^{\omega_i}\right)^p\left(1-F_B^L(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_A^U(a_i)^{\omega_i}\right)^p\left(1-F_A^U(a_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \geq 1-\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-F_B^U(b_i)^{\omega_i}\right)^p\left(1-F_B^U(b_j)^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}.$$

In summary, we can prove

$$ILNULWABM_{\omega}^{p,q}(a_1,a_2,...a_n) \leq ILNULWABM_{\omega}^{p,q}(b_1,b_2,...b_n).$$

Theorem 3. (boundedness) The ILNULWABM operator lies between the max and min operators:

$$\frac{\min(a_1,a_2,...a_n)}{n} \leq ILNULWABM_{s_\omega}^{p,q}(a_1,a_2,...a_n) \leq \frac{\max(a_1,a_2,...a_n)}{n}.$$

Proof: Let $a^- = \min(a_1,a_2,...a_n)$ and $a^+ = \max(a_1,a_2,...a_n)$.

Since $a^- \leq a_i, a_j \leq a^+$, according to the monotonicity in Theorem 2, we know that

$ILNULWABM_{\omega}^{p,q}(a^-,a^-,...a^-) \leq ILNULWABM_{\omega}^{p,q}(a_1,a_2,...a_n)$,
$ILNULWABM_{\omega}^{p,q}(a_1,a_2,...a_n) \leq ILNULWABM_{\omega}^{p,q}(a^+,a^+,...a^+)$. Due to

$$LNULWABM_{\omega}^{p,q}(a^-,a^-,...a^-)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i a^-\right)^p\left(\omega_j a^-\right)^q\right)^{\frac{1}{p+q}}=\frac{a^-}{n},$$

$$ILNULWABM_{\omega}^{p,q}(a^+,a^+,...a^+)=\left(\frac{1}{n(n\text{-}1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\omega_i a^+\right)^p\left(\omega_j a^+\right)^q\right)^{\frac{1}{p+q}}=\frac{a^+}{n},$$

So, $\frac{a^-}{n} \leq ILNULWABM_{\omega}^{p,q}(a_1,a_2,...a_n) \leq \frac{a^+}{n}$.

4 | The VIKOR Method Based on Relative Closeness Coefficient under ILNULN and ILNULWABM Operator

For a MAGDM problem, there are a discrete set of alternatives $A=\{A_1,A_2,...,A_m\}$ and attributes $C=\{C_1,C_2,...C_n\}$ with weight vector $\omega=(\omega_1,\omega_2,...\omega_n)^T$. There are λ DMs $D=\{D_1,D_2,...D_\lambda\}$ assess this problem and the relative importance vector is $W=(W_1,W_2,...W_\lambda)^T$. For the DM D^k, the evaluation value of A_i on attribute C_j is represented by the decision matrix $R^k=\left(r_{ij}^k\right)_{m\times n}$, where

$r_{ij}^k=\left\langle\left[s_{\theta(r_{ij}^k)},s_{\rho(r_{ij}^k)}\right],\left(\left[s_{T^L(r_{ij}^k)},s_{T^U(r_{ij}^k)}\right],\left[s_{I^L(r_{ij}^k)},s_{I^U(r_{ij}^k)}\right],\left[s_{F^L(r_{ij}^k)},s_{F^U(r_{ij}^k)}\right]\right)\right\rangle$. The steps of the VIKOR method based on the relative closeness coefficient under ILNULNs and ILNULWABM operators are shown as follows:

Step 1: Normalize the decision matrix $R^k=\left(r_{ij}^k\right)_{m\times n}$.

The normalized matrix F is calculated by:

$$F^k=\left(f_{ij}^k\right)_{m\times n}=\left(\frac{r_{ij}^k}{\sum_{i=1}^{m}\left(r_{ij}^k\right)^2}\right)_{m\times n}\quad(i=1,2,...,m;\ j=1,2,...,n).$$

Step 2: Aggregate information from each DM. To aggregate the evaluation values of DMs, we use the ILNULWABM operator to aggregate the evaluation information matrix F^k to obtain the integration matrix F : $F=\left(f_{ij}\right)_{m\times n}$, $f_{ij}=ILNULWABM\left(f_{ij}^1, f_{ij}^2,...f_{ij}^\lambda\right)$.

Step 3: Compute the positive ideal alternative f_{ij}^+ and the negative ideal alternative f_{ij}^-.

We can use the score function to obtain the positive ideal alternative and the negative ideal alternative: $f_j^+=\left\{\max f_{ij}\right\}=\left\{\max S\left(f_{ij}\right)\right\}$, $f_j^-=\left\{\min f_{ij}\right\}=\left\{\min S\left(f_{ij}\right)\right\}$

$$S(f_{ij})=\frac{1}{12}\left(\theta(f_{ij})+\rho(f_{ij})\right)\left(4+T_A^L(f_{ij})-I_A^L(f_{ij})-F_A^L(f_{ij})+T_A^U(f_{ij})-I_A^U(f_{ij})-F_A^U(f_{ij})\right.$$

Step 4: Compute the group utility values ΔS_i and individual regret values ΔR_i.

$$\Delta S_i=\sum_{j=1}^{n}\omega_j\frac{d\left(f_j^-f_{ij}\right)-d\left(f_j^+-f_{ij}\right)}{d\left(f_j^+-f_j^-\right)},\ \Delta R_i=\max_j\omega_j\frac{d\left(f_j^-f_{ij}\right)-d\left(f_j^+-f_{ij}\right)}{d\left(f_j^+-f_j^-\right)}$$

Step 5: Compute the values Q_i

$$\Delta Q_i=\varepsilon\frac{\Delta S_i-\Delta S^-}{\Delta S^+-\Delta S^-}+(1-\varepsilon)\frac{\Delta R_i-\Delta R^-}{\Delta R^+-\Delta R^-}$$

Where $\Delta S^+=\max_i\Delta S_i$, $\Delta S^-=\min_i\Delta S_i$, $\Delta R^+=\max_i\Delta R_i$, $\Delta R^-=\min_i\Delta R_i$ and ε represents the weight of the strategy of the "the majority of criteria".

Step 6: Sort the ΔQ_i in descending order.

Step 7: Test the compromise solution.

5 | A Numerical Example

This article proposes the concept of interval linguistic neutrosophic uncertain linguistic numbers. ILNULN consists of two parts: interval linguistic neutrosophic and uncertain linguistic number. The interval linguistic neutrosophic reflects the subjective linguistic judgment of the decision maker on the given uncertain linguistic number, and the uncertain linguistic number reflects the attitude of the decision maker towards the evaluation object. Now we consider a MAGDM problem. Suppose there are four alternatives labeled A_1, A_2, A_3, A_4 and three attributes labeled C_1, C_2, C_3 whose weight vector is $\omega=(0.35, 0.4, 0.25)^T$. Three DMs assess this problem and the relative importance vector is $W=(0.33, 0.17, 0.5)^T$. Here, we let $S=\{s_i|i=0,1,2,...8\}$ where S_i represents a possible value for a linguistic number, and $S=\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}=\{$extremely poor, very poor, poor, a little poor, medium, a little good，good, very good, excellent$\}$. The DMs assign values to the alternatives through ILNULNs to form three decision matrices, as shown in Tables 1-3.

Step 1: Normalize the decision matrix R^k.

Step 2: Aggregate information from each DM. We use the ILNULWABM operator to gather decision information from all DMs. Here we let p=1 and q=1. The group decision matrix is shown in Table 4.

Table 1. Decision matrix R1 of the DM D^1.

	C_1	C_2	C_3
A_1	<(S_3,S_4),([S_6,S_7][S_2,S_4][S_0,S_1])>	<(S_5,S_6),([S_5,S_6][S_2,S_3][S_1,S_2])>	<(S_4,S_5),([S_4,S_5][S_2,S_3][S_3,S_4])>
A_2	<(S_4,S_5),([S_5,S_6][S_4,S_5][S_2,S_3])>	<(S_4,S_5),([S_4,S_5][S_3,S_4][S_2,S_3])>	<(S_7,S_8),([S_5,S_6][S_1,S_2][S_2,S_3])>
A_3	<(S_7,S_8),([S_4,S_5][S_2,S_3][S_3,S_4])>	<(S_3,S_4),([S_6,S_7][S_2,S_4][S_1,S_2])>	<(S_3,S_4),([S_4,S_5][S_3,S_4][S_3,S_4])>
A_4	<(S_5,S_6),([S_5,S_6][S_2,S_3][S_2,S_3])>	<(S_6,S_7),([S_3,S_4][S_2,S_4][S_4,S_5])>	<(S_7,S_8),([S_7,S_8][S_2,S_3][S_0,S_1])>

Table 2. Decision matrix R2 of the DM D^2.

	C_1	C_2	C_3
A_1	<(S_3,S_4),([S_7,S_8][S_3,S_4][S_1,S_2])>	<(S_4,S_5),([S_5,S_6][S_2,S_3][S_2,S_3])>	<(S_2,S_3),([S_5,S_6][S_3,S_4][S_2,S_3])>
A_2	<(S_6,S_7),([S_4,S_5][S_3,S_4][S_2,S_3])>	<(S_7,S_8),([S_5,S_7][S_3,S_4][S_1,S_2])>	<(S_7,S_8),([S_4,S_5][S_2,S_3][S_3,S_4])>
A_3	<(S_2,S_3),([S_6,S_7][S_1,S_2][S_0,S_1])>	<(S_3,S_4),([S_4,S_5][S_2,S_3][S_3,S_4])>	<(S_5,S_6),([S_6,S_7][S_3,S_4][S_0,S_1])>
A_4	<(S_7,S_8),([S_5,S_6][S_2,S_3][S_2,S_3])>	<(S_5,S_6),([S_6,S_7][S_3,S_4][S_1,S_2])>	<(S_4,S_5),([S_5,S_6][S_2,S_3][S_2,S_3])>

Table 3. Decision matrix R3 of the DM D^3.

	C_1	C_2	C_3
A_1	<(S_4,S_5),([S_6,S_7][S_2,S_3][S_1,S_2])>	<(S_4,S_5),([S_5,S_6][S_2,S_3][S_1,S_2])>	<(S_2,S_3),([S_6,S_7][S_2,S_4][S_1,S_2])>
A_2	<(S_3,S_4),([S_5,S_6][S_1,S_2][S_2,S_3])>	<(S_6,S_7),([S_7,S_8][S_1,S_2][S_0,S_1])>	<(S_5,S_6),([S_5,S_6][S_2,S_3][S_3,S_4])>
A_3	<(S_2,S_3),([S_4,S_5][S_2,S_3][S_3,S_4])>	<(S_2,S_3),([S_6,S_7][S_2,S_3][S_1,S_2])>	<(S_3,S_4),([S_5,S_6][S_3,S_4][S_2,S_3])>
A_4	<(S_6,S_7),([S_7,S_8][S_2,S_4][S_0,S_1])>	<(S_5,S_6),([S_5,S_6][S_2,S_4][S_2,S_3])>	<(S_7,S_8),([S_4,S_5][S_2,S_3][S_3,S_4])>

Table 4. Group decision matrix F.

	C_1	C_2
A_1	<($S_{0.1099}$,$S_{0.1197}$),([$S_{0.2445}$,$S_{0.2372}$],[$S_{0.8138}$,$S_{0.8241}$],[$S_{0.5002}$,$S_{0.6911}$])>	<($S_{0.1465}$,$S_{0.1498}$),([$S_{0.1940}$,$S_{0.1907}$],[$S_{0.7770}$,$S_{0.7668}$],[$S_{0.7353}$,$S_{0.7676}$])>
A_2	<($S_{0.1503}$,$S_{0.1541}$),([$S_{0.1614}$,$S_{0.1671}$],[$S_{0.8399}$,$S_{0.8241}$],[$S_{0.8376}$,$S_{0.8347}$])>	<($S_{0.1880}$,$S_{0.1847}$),([$S_{0.1967}$,$S_{0.2086}$],[$S_{0.8159}$,$S_{0.7975}$],[$S_{0.4487}$,$S_{0.7226}$])>
A_3	<($S_{0.1118}$,$S_{0.1243}$),([$S_{0.1664}$,$S_{0.1716}$],[$S_{0.7220}$,$S_{0.7376}$],[$S_{0.7819}$,$S_{0.8162}$])>	<($S_{0.0928}$,$S_{0.1052}$),([$S_{0.2041}$,$S_{0.1996}$],[$S_{0.7770}$,$S_{0.7931}$],[$S_{0.7986}$,$S_{0.8068}$])>
A_4	<($S_{0.2036}$,$S_{0.2004}$),([$S_{0.2013}$,$S_{0.2011}$],[$S_{0.7699}$,$S_{0.7900}$],[$S_{0.6150}$,$S_{0.7802}$])>	<($S_{0.1803}$,$S_{0.1778}$),([$S_{0.1716}$,$S_{0.1726}$],[$S_{0.8163}$,$S_{0.8386}$],[$S_{0.8793}$,$S_{0.8562}$])>
	C_3	
A_1	<($S_{0.0824}$,$S_{0.0969}$),([$S_{0.1917}$,$S_{0.1896}$],[$S_{0.8077}$,$S_{0.8219}$],[$S_{0.7800}$,$S_{0.7971}$])>	
A_2	<($S_{0.2058}$,$S_{0.2002}$),([$S_{0.1765}$,$S_{0.1776}$],[$S_{0.7149}$,$S_{0.7424}$],[$S_{0.8514}$,$S_{0.8430}$])>	
A_3	<($S_{0.1183}$,$S_{0.1268}$),([$S_{0.1968}$,$S_{0.1936}$],[$S_{0.8723}$,$S_{0.8492}$],[$S_{0.6841}$,$S_{0.7536}$])>	
A_4	<($S_{0.1788}$,$S_{0.1778}$),([$S_{0.2219}$,$S_{0.2143}$],[$S_{0.7684}$,$S_{0.7762}$],[$S_{0.6211}$,$S_{0.7423}$])>	

Step 3: Calculate the positive ideal alternative f_{ij}^{+} and the negative ideal alternative f_{ij}^{-}. Use the score function to obtain the positive ideal alternative and negative ideal alternative. The score values are as follows:

$S(f_{11}) = 0.0316$, $S(f_{12}) = 0.033$, $S(f_{13}) = 0.0176$, $S(f_{21}) = 0.0251$,
$S(f_{22}) = 0.0503$, $S(f_{23}) = 0.0407$, $S(f_{31}) = 0.0252$, $S(f_{32}) = 0.0203$
$S(f_{33}) = 0.0251$, $S(f_{41}) = 0.0487$, $S(f_{42}) = 0.0285$, $S(f_{43}) = 0.0454$

Due to $f_j^{+} = \{\max f_{ij}\} = \{\max S(f_{ij})\}$ and $f_j^{-} = \{\min f_{ij}\} = \{\min S(f_{ij})\}$, apparently, the f_{ij}^{+} and f_{ij}^{-} are shown as follows: $f_1^{+} = f_{41}$, $f_1^{-} = f_{21}$, $f_2^{+} = f_{22}$, $f_2^{-} = f_{32}$, $f_3^{+} = f_{43}$, $f_3^{-} = f_{13}$.

Step 4: Compute the group utility values ΔS_i and individual regret values ΔR_i

$\Delta S_1 = -0.4896$, $\Delta S_2 = 0.6922$, $\Delta S_3 = -0.8550$, $\Delta S_4 = 0.7706$,
$\Delta R_1 = 0.032$, $\Delta R_2 = 0.4$, $\Delta R_3 = -0.105$, $\Delta R_4 = 0.35$.

Step 5: Compute the values ΔQ_i. Here we make $\varepsilon = 0.5$. The VIKOR values ΔQ_i for each alternative can be calculated as follows: $\Delta Q_1 = 0.2480$, $\Delta Q_2 = 0.9759$, $\Delta Q_3 = 0$, $\Delta Q_4 = 0.9505$

Step 6: Sort the ΔQ_i in descending order.

We can sort the alternatives according to the values of ΔS_i, ΔR_i and ΔQ_i. The larger the value, the better the alternative. Then, according to the ranking process, three ordered lists can be obtained as displayed in Table 5.

Table 5. Group utility value, individual regret value.

	A_1	A_2	A_3	A_4	Ranking results
ΔS_i	-0.4896	0.6922	-0.855	0.7706	$A_4 \succ A_2 \succ A_1 \succ A_3$
ΔR_i	0.032	0.4	-0.105	0.35	$A_2 \succ A_4 \succ A_1 \succ A_3$
ΔQ_i	0.2480	0.9759	0.0000	0.9505	$A_2 \succ A_4 \succ A_1 \succ A_3$

Step 7: Test the compromise solutions. The alternatives are ranked by ΔQ: $\Delta Q_2 > \Delta Q_4 > \Delta Q_1 > \Delta Q_3$. The best alternative is A_2 and the second alternative is A_4. Due to $\Delta Q_2 - \Delta Q_4 = 0.0254 \le \frac{1}{4-1} = 0.3333$, so it doesn't satisfy condition 1- acceptable advantage. So A_2 is the best alternative and A_4 could be the compromise solution.

6 | Sensitivity Analysis and Related Comparison

6.1 | Sensitivity Analysis

Due to the decision result being related to the parameters *p, q*, so it is necessary to analyze different *p* and *q*. The sorting result is shown in Table 6.

It can be seen that the optimal solution is always A_2 based on different *p* and *q*. But the overall order is a little different. So *p* and *q* have a limited impact on the ranking result.

Similarly, in the VIKOR method, the compromise evaluation value of each alternative is affected by the group utility weight ε. In order to consider the impact of different values of ε on the evaluation results, the analysis is performed by setting different ε to observe their impact. The impact of the sorting result is shown in Table 7. It can see that when $\varepsilon = \{0, 0.2, 0.4, 0.5, 0.6\}$, the best alternative is A_2; when $\varepsilon = \{0.8, 1\}$, the best alternative is A_4. ε has an effect to decision result.

Table 6. Ranking results under different p, q

	A_1	A_2	A_3	A_4	Ranking results	Best alternative
p=1,q=1	0.2480	0.9759	0	0.9505	$A_2 > A_4 > A_1 > A_3$	A_2
p=1,q=0	0.0588	1	0	0.8907	$A_2 > A_4 > A_1 > A_3$	A_2
p=0,q=1	0.0588	1	0	0.8907	$A_2 > A_4 > A_1 > A_3$	A_2
p=0.5,q=0.5	0	1	0.6824	0.9102	$A_2 > A_4 > A_3 > A_1$	A_2

Table 7. Ranking results under different ε (p=1,q=1).

		A_1	A_2	A_3	A_4	Ranking results	Best alternative
ΔQ	**ε=0**	0.2712	1.0000	0.0000	0.9010	$A_2 > A_4 > A_1 > A_3$	A_2
	ε=0.2	0.2642	1.0000	0.0000	0.9309	$A_2 > A_4 > A_1 > A_3$	A_2
	ε=0.4	0.2572	1.0000	0.0000	0.9608	$A_2 > A_4 > A_1 > A_3$	A_2
	ε=0.5	0.2480	0.9759	0.0000	0.9505	$A_2 > A_4 > A_1 > A_3$	A_2
	ε=0.6	0.2502	1.0000	0.0000	0.9908	$A_2 > A_4 > A_1 > A_3$	A_2
	ε=0.8	0.2432	1.0000	0.0000	1.0207	$A_4 > A_2 > A_1 > A_3$	A_4
	ε=1	0.0472	1.0000	0.0000	1.0506	$A_4 > A_2 > A_1 > A_3$	A_4

6.2 | Related Comparison

To illustrate the effectiveness and superiority, we compared the proposed MAGDM method with the WAA operator, TOPSIS, and the original VIKOR method, respectively. For convenient comparison, Table 8 lists all the MAGDM results.

As shown in the table, similar sorting results are obtained through the calculation of the same example, and the best alternative is always A_2. However, different from the WAA operators, the ILNULWABM operators

depend on input parameters and consider the interaction between different attributes. And the VIKOR method based on the relative closeness coefficient applies the TOPSIS method's closeness to the VIKOR method to make the decision result more reasonable.

Table 8. Comparison table based on the numerical example above.

Method	Ranking results	Best alternative
Method of this article	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
Method based on WAA operator	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
TOPSIS	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

7 | Conclusion

This article proposes the concept of ILNULN. ILNULN consists of two parts: interval linguistic neutrosophic and uncertain linguistic number. The interval linguistic neutrosophic reflects the subjective linguistic judgment of the DM on the given uncertain linguistic number, and the uncertain linguistic number reflects the attitude of the DM towards the evaluation object. Based on ILNULN, this paper studies its basic properties, algorithms, scores function, and Hamming distance. WABM operator integrates the correlation of aggregation parameters. So we combine the ILNULN and WABM operator to propose the ILNULWABM operator. In addition, this paper applies ILNULN and ILNULWABM operators to the VIKOR method based on the relative closeness coefficient and discusses the impact of different parameters p, q, and ε on the MAGDM.

This article discusses and studies the WABM operator with ILNULN, and it has achieved certain results. But this research still needs to be further improved:

This article only considers the MAGDM problem in which the attribute weights and DM weights are crisp numbers but doesn't consider the linguistic value. However, this situation is common in practical decision-making problems. Therefore, we can conduct further research in the future.

In future research, it will be necessary and meaningful to apply the proposed interval linguistic neutrosophic uncertain linguistic MAGDM method to solve some practical problems in other areas, such as personnel evaluation, medical artificial intelligence, and pattern recognition.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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