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Relation on Fermatean Neutrosophic Soft Set with Application to Sustainable Agriculture

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Abstract

The need for cutting-edge tools for decision-making to address the difficulties of sustainable agriculture is the driving force behind this study. We aim to integrate soft relations into fermatean neutrosophic hypersoft sets, improving the management of uncertainty. Using this approach, we provide a novel Multi-Criteria Decision-Making (MCDM) algorithm that evaluates agricultural techniques holistically by considering environmental, social, and economic variables. We highlight the significance of taking uncertainty into account when making decisions by demonstrating the algorithm's efficiency in optimizing the choice of agricultural inputs through a real-world case study. Our solution handles such uncertainty better than conventional ways. In the future, we hope to improve the algorithm even further and investigate integrating cutting-edge technology for real-world use in changing agricultural circumstances.

Keywords: Hybrid Structures, Soft Set, Fermatean Set, Neutrosophic Set.

1 | Introduction

A comprehensive method of farming known as ”sustainable agriculture” aims to satisfy current demands without jeopardizing the capacity of future generations to satisfy their own. It places a strong emphasis on social responsibility, environmental sustainability, and economic viability. In sustainable agriculture, farmers try to utilize as little artificial input as possible, such as fertilizers and pesticides, by using integrated pest control and organic alternatives. The preservation of natural resources, including soil and water, is a major priority. To improve soil health and stop erosion, techniques including crop rotation and cover crops are used. Furthermore, biodiversity is encouraged by sustainable agriculture, which acknowledges the value of varied ecosystems in sustaining robust and successful farms. Sustainable agriculture safeguards the environment and helps farming communities

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throughout the world remain stable and prosperous in the long run by implementing these principles.

Around the 1870s, set theory was introduced as a result of Cantor and Dedekind's efforts, [1] which proved its worth having various real-world applications [2]. The classical set theory based on crisp sets only deals with absolute membership that is, whether a member is contained in a set or not. This limitation of membership motivated Zadeh to introduce fuzzy sets that deal with partial membership [3]. Fuzzy sets were introduced in 1965, generalized by Pawlak as rough sets in 1982 [4] and by Molodstov as soft sets in 1999 [5]. These generalizations proved their worth in dealing with the uncertainties in various real-world problems in almost every field such as engineering, economics, social sciences, environmental sciences and medical sceinces [6–9].

Fuzzy soft set [8], intuitionistic fuzzy set [10], intuitionistic fuzzy soft set [11], hesitant fuzzy set [12], hesitant fuzzy soft set [13], picture fuzzy set [14], picture fuzzy soft set [15], hypersoft set [16], neutrosophic soft set [17] and neutrosophic hypersoft set [18] are few variants based on generalization of truthiness (membership), falsity (non-membership) and hesitancy (indeterminacy). We have enlightened some scholarly activities related to soft sets, neutrosophic sets and fermatean sets. In 2003, Maji proposed the fundamentals of soft sets including basic entities and operators [19] and in 2009 Ali et al. established the modified operators [20]. Later, the soft set theory evolved as Cagman and Enginoglu proposed the soft matrix [21], Babitha and Sunil defined relations and functions on soft sets along with their properties [22, 23] and Yang and Guo defined closure and kernel of soft relations and soft mappings [24]. More contributions towards soft set theory were made by different mathematicians [25–28]. While soft set theory was developed by the above-mentioned findings, mathematicians tried to connect it with algebraic structures as Aktas and Cagman established soft groups [29], Acer defined soft rings [6] and Aslam and Qurashi connected sub-algebraic structures related to soft groups [30].

Inspired by philosophical logics (relative and absolute truthiness and falsity) and various real-world scenarios such as game results (win, loss, tie), voting outcomes (in favor of, opposite, blank vote), numbers (positive, negative, neutral), answers to a straight question (yes, not applicable, no), control theory and decision making (making a decision, hesitating, accepting, rejecting, pending). Smarandache introduced a tri-component set named neutrosophic sets (knowledge of neutral wisdom) dealing with three components: membership, non-membership, and indeterminacy [31, 32]. The neutrosophic set theory evolved as Wang et al. proposed single and interval-valued neutrosophic sets [33, 34] and Salama and Alblowi presented Neutrosophic topological spaces [35]. Georgiou introduced soft topological spaces [57] and Bera and Mahapatra introduced neutrosophic soft topological spaces [58]. Various mathematical entities were established relative to neutrosophic set as measure and integral [36], lattices [37], vector spaces [38], continuous function [39], entropy [40], group and subgroup [41], soft ring and soft field [42]. Mathematicians also discussed various applications of neutrosophic techniques including image processing [43], medical diagnosis [44] and multi-criteria decision-making [45] using similarity measures, neutrosophic logic, and hypersoft graphs [46].

Senapati and Yager introduced the fermatean fuzzy set [47] to deal with the limitations of membership and non-membership in intuitionistic fuzzy and Pythagorean fuzzy sets. It opened a new horizon for researchers as Broumi et al. applied complex fermatean neutrosophic graphs to decision-making [48], Bilgin et al. introduced fermatean neutrosophic topological spaces [49] and Salsabeela and John discussed TOPSIS techniques on fermatean fuzzy soft sets [50].

This paper presents the fundamentals of a hybrid structure fermatean Neutrosophic Soft set (FrNSS) that allows more flexible choices for membership, non-membership and indeterminacy. We have established its definition, basic set theoretic entities as a subset, null set, universal set, different operators, and basic algebraic structures relative to these operators. We have also defined fermatean neutrosophic soft topological space, cartesian product, and relations on FrNSS and discussed its approach to decision-making problems.

1.1 | Structural Comparison

In this section, we have presented the fermatean neutrosophic set as a generalization of some basic hybrids. Table 1 shows how different values of membership, indeterminacy, and non membership correspond to other already existing hybrid structures and some basic sets. It is to be noted that the fermatean neutrosophic set is not a special case of q-rung orthopair fuzzy set with q=3, as in q-rung orthopair fuzzy set only membership and non-membership (dependent) are discussed while in case of fermatean neutrosophic set, membership, non membership (dependent) and indeterminacy (independent) are discussed. Pythagorean neutrosophic set is a generalization of intuitionistic neutrosophic set and the fermatean neutrosophic set is a generalization of both Pythagorean neutrosophic and Intuitionistic neutrosophic sets.

1.2 | Motivation

In this section, a real-world scenario that the neutrosophic set deals with is presented. The hybrid structure FrNSS proves it’s worth being able to deal with more options for membership, non membership, and indeterminacy as compared to intuitionistic and Pythagorean neutrosophic sets.

Problem: During the journey from place A to place B, a truck is loaded with various items, including tables of three different sizes (large (Table 1), medium (Table 2), and small (Table 3)), a sofa set with three different sizes (three-seater (sofa 1), two-seater (sofa 2) and one-seater

Table 1. Alreday existing structures

Set	**membership value θ**	**Indeterminacy value ϕ**	**non-membership value ψ**	**condition**
	m	i	n	
Crisp Set	0 or 1	0	0	
Fuzzy Set	in $[0,1]$	0	0	
Intuitionistic Fuzzy Set	in $[0,1]$	0	in $[0,1]$	$0 \leq \theta + \psi \leq 1$
Pythagorean Fuzzy Set	in $[0,1]$	0	in $[0,1]$	$0 \leq \theta^2 + \psi^2 \leq 1$
Fermatean Fuzzy Set	in $[0,1]$	0	in $[0,1]$	$0 \leq \theta^3 + \psi^3 \leq 1$
Neutrosophic Set	in $[0,1]$	in $[0,1]$	in $[0,1]$	$0 \leq \theta + \phi + \psi \leq 3$
Intuitionistic Neutrosophic Set	in $[0,1]$	in $[0,1]$	in $[0,1]$	$0 \leq \theta + \psi \leq 1$ $0 \leq \theta + \phi + \psi \leq 2$
Pythagorean Neutrosophic Set	in $[0,1]$	in $[0,1]$	in $[0,1]$	$0 \leq \theta^2 + \psi^2 \leq 1$ $0 \leq \theta^2 + \phi^2 + \psi^2 \leq 2$
Fermatean Neutrosophic Set	in $[0,1]$	in $[0,1]$	in $[0,1]$	$0 \leq \theta^3 + \psi^3 \leq 1$ $0 \leq \theta^3 + \phi^3 + \psi^3 \leq 2$

(sofa 3)), a cupboard and two boxes. Table 2 is positioned on top of table 1 and table 3 is placed underneath table 1. Sofa 3 is placed on top of sofa 1 while sofa 2 is inclined against sofa 1 forming a sloppy surface. Now let's express the volume covered by each item on the truck throughout the entire journey.

Solution: In this particular problem, the coverage of volume by each item is not absolute. For example, table 2 does not cover the volume of the truck even though it is present in the truck. Sofa 2 forms an inclined plane with sofa 1 covering approximately 50% to 60% of the area (membership) while the remaining 50% to 40% does not cover the volume of the truck (non-membership). Moreover, the movement of the truck introduces frequent changes in these values of membership and non-membership. This problem cannot be effectively addressed using crisp sets. To analyze and discuss this problem, we require a neutrosophic soft structure that accounts for the dependencies between membership and non-membership. The most suitable hybrid structure for this problem is the $FrNSS$ which allows a wide range of possible values for membership, indeterminacy, and non-membership. e.g if membership is 0.9 (90%) and non-membership is 0.5 (50%) then $0.9+0.5 > 1$, so intuitionistic neutrosophic soft set does not support it. Also $0.9^2 + 0.5^2 > 1$ so Pythagorean neutrosophic soft set does not support it but $0.9^3 + 0.5^3 < 1$ so fermatean neutrosophic soft set will support it.

2 | Preliminaries

To comprehend the paper's main findings, some basic definitions, mainly following [49], [54], and [58] are presented in this section. Let's definea few notations that we have used for this paper. $\mathbb{D}, \mathbb{P}(\mathbb{D})$, $\mathbb{P}(\mathbb{D})_{FrN}$ and $\mathbb{P}(\mathbb{D})_N$ are used to represent the domain of discourse, collection of all the classical subsets of $\mathbb{D}$, fermatean neutrosophic subsets of $\mathbb{D}$ and neutrosophic subsets of $\mathbb{D}$, respectively. P_1 and P_2 are used to represent the subsets of set of parameters P. $\theta_X, \phi_X, \psi_X : \mathbb{D} \to [0,1]$ where $\theta_X(\tilde{s}), \phi_X(\tilde{s})$ and $\psi_X(\tilde{s})$ are representing membership, indeterminacy and non-membership levels of $\tilde{s} \in \mathbb{D}$ relative to the set X.
The collection of possible values for fermatean neutrosophy membership and non-membership levels is a superset of the collections of Pythagorean as well as intuitionistic membership and non-membership levels.

2.1. *Fermatean Fuzzy Set*

$\xi_{Fr} = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}, 0 \le \theta^3_\xi(\tilde{s}) + \psi^3_\xi(\tilde{s}) \le 1\}$ is representing a fermatean fuzzy set over the domain of discourse $\mathbb{D}$.

2.2. *Neutrosophic Set*

$\xi_N = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \phi_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}, 0 \le \theta_\xi(\tilde{s}) + \phi_\xi(\tilde{s}) + \psi_\xi(\tilde{s}) \le 3\}$ is representing a neutrosophic set over the domain of discourse $\mathbb{D}$.

2.3. *Soft Set*

The pair $(f^*, P_1) = \{(\hat{\epsilon}, f^*(\hat{\epsilon})) : \hat{\epsilon} \in P_1, f^* : P_1 \to \mathbb{P}(\mathbb{D})\}$ is representing a soft set.

2.4. *Fermatean Neutrosophic Set*

$\xi_{FrN} = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \phi_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle, 0 \le \theta^3_\xi(\tilde{s}) + \psi^3_\xi(\tilde{s}) \le 1, 0 \le \theta^3_\xi(\tilde{s}) + \phi^3_\xi(\tilde{s}) + \psi^3_\xi(\tilde{s}) \le 2 : \tilde{s} \in \mathbb{D}\}$ is representing a fermatean neutrosophic set over the domain of discourse $\mathbb{D}$.

2.5. *Neutrosophic Soft Set*

The pair $(f^*, P_1) = \{(\hat{\epsilon}, f^*(\hat{\epsilon})) : \hat{\epsilon} \in P_1, f^* : P_1 \to \mathbb{P}(\mathbb{D})_N\}$ is representing a neutrosophic soft set. More precisely,

$$\xi_{NS,P_1} = \{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \le \theta_{P_1,\hat{\epsilon}}(\tilde{s}) + \phi_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi_{P_1,\hat{\epsilon}}(\tilde{s}) \le 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$$

3 | Relation on Fermatean Neutrosophic Soft Set

In this section, relations on $FrNSS$ are established as that is used to develop a decision-making algorithm. $FrNS$ Relation is a $FrNS$ subset of Cartesian product.

3.1. *Cartesian Product*

Let $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ be two $FrNSS$s. The Cartesian product $\mathfrak{X}_{FrNS,P_1} \times \mathfrak{X}_{FrNS,P_2}$ of $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ is a $FrNSS$ $\mathfrak{X}_{FrNS,P_1\times P_2} = \{(\xi, \langle \tilde{s}, (\theta_{P_1\times P_2,\xi}(\tilde{s}), \phi_{P_1\times P_2,\xi}(\tilde{s}), \psi_{P_1\times P_2,\xi}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \xi \in P_1 \times P_2\}$ where $\theta_{P_1,\xi}, \phi_{P_1,\xi}, \psi_{P_1,\xi} : \mathbb{D} \to [0,1]$ such that for all $\tilde{s} \in \mathbb{D}$ and $\xi \in P_1 \times P_2, 0 \leq \theta^3_{P_1\times P_2,\xi}(\tilde{s}) + \psi^3_{P_1\times P_2,\xi}(\tilde{s}) \leq 1$ and $0 \leq \theta^3_{P_1\times P_2,\xi}(\tilde{s}) + \phi^3_{P_1\times P_2,\xi}(\tilde{s}) + \psi^3_{P_1\times P_2,\xi}(\tilde{s}) \leq 2$, where $\theta_{P_1\times P_2} = \min\{\theta_{P_1}, \theta_{P_2}\}$, $\phi_{P_1\times P_2} = \min\{\phi_{P_1}, \phi_{P_2}\}$ and $\psi_{P_1\times P_2} = \max\{\psi_{P_1}, \psi_{P_2}\}$

3.2. *Example*

Let $\mathfrak{X}_{FrNS,P_1} = \{(\hat{\epsilon}_1, \langle \tilde{s}_1, (0.8, 0.4, 0.12)\rangle, \langle \tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.35)\rangle)\,,$
$(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle \tilde{s}_2, (0.83, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\}$ and
$\mathfrak{X}_{FrNS,P_2} = \{(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.7, 0.65, 0.3)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.29, 0.6, 0.37)\rangle)\,,$
$(\hat{\epsilon}_3, \langle \tilde{s}_1, (0.5, 0.74, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle \tilde{s}_3, (0.9, 0.3, 0.42)\rangle)\}$, then
$\mathfrak{X}_{FrNS,P_1\times P_2} = \{((\hat{\epsilon}_1, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.7, 0.4, 0.3)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.37)\rangle)\,,$
$((\hat{\epsilon}_1, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.4, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.42)\rangle)\,,$
$((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\,,$
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.27, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle \tilde{s}_3, (0.18, 0.3, 0.7)\rangle)\}$.

3.3. *Fermatean Neutrosophic Soft Relation*

Let $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ be two $FrNSS$s. A $FrNS$ relation from $\mathfrak{X}_{FrNS,P_1}$ to $\mathfrak{X}_{FrNS,P_2}$ is a $FrNS$ subset $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ of $\mathfrak{X}_{FrNS,P_1\times P_2}$, where $\mathbb{K} \times \mathbb{L} \subseteq P_1 \times P_2$.

3.4. *Example*

Consider the FrNS sets and their cartesian product in example 3.2. Following are two FrNS relations between $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$,
$\mathbb{R}_{FrNS,P_1\times P_2,1} = \mathfrak{X}_{FrNS,\mathbb{K}\times\mathbb{L}} =$
$\{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.4, 0.2, 0.5)\rangle, \langle \tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle \tilde{s}_3, (0, 0.3, 0.7)\rangle)\,,$
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.3, 0.75)\rangle)\}$, with $\mathbb{K} = \{\hat{\epsilon}_2\} \subseteq P_1$ and $\mathbb{L} = \{\hat{\epsilon}_2, \hat{\epsilon}_3\} \subseteq P_2$, and
$\mathbb{R}_{FrNS,P_1\times P_2,2} = \mathfrak{X}_{FrNS,\mathbb{K}\times\mathbb{L}} =$
$((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\,,$
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.27, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle \tilde{s}_3, (0.18, 0.3, 0.7)\rangle)\}$

3.5. *Remark*

As a relation from a set A with cardinality m to a set B with cardinality n is defined as a subset of cartesian product $A \times B$ so the number of possible relations from set A to set B is

2^{mn} but in case of Fermatean Neutrosophic soft set the number of $FrNS$ relations between two sets is more than the number of classical relations.

3.6. *Domain and Range of Fermatean Neutrosophic Soft Relation*

Let $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ be FrNS relation from $\mathfrak{X}_{FrNS,P_1} = (f^*, P_1)$ to $\mathfrak{X}_{FrNS,P_2} = (g^*, P_2)$ then its domain and range is defined as,

$\text{Dom}(\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}) = (f^*|_{\mathbb{K}}, \mathbb{K}), \mathbb{K} \subseteq P_1$: for all $\hat{\epsilon}_i \in \mathbb{K}$, there exists $\hat{\epsilon}_j \in \mathbb{L}$ such that $(\hat{\epsilon}_i, \hat{\epsilon}_j) \in \mathbb{K}\times\mathbb{L}\}$

$\text{Range}(\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}) = (g^*|_{\mathbb{L}}, \mathbb{L}), \mathbb{L} \subseteq P_2$: for all $\hat{\epsilon}_j \in \mathbb{L}$, there exists $\hat{\epsilon}_i \in \mathbb{K}$ such that $(\hat{\epsilon}_i, \hat{\epsilon}_j) \in \mathbb{K}\times\mathbb{L}\}$

3.7. *Example*

In example 3.4, the domain and range of $\mathbb{R}_{FrNS,P_1\times P_2,1}$ and $\mathbb{R}_{FrNS-P_1,P_2,2}$ are given as follows,

$\text{Dom}(\mathbb{R}_{FrNS,P_1\times P_2,1}) = \text{Dom}(\mathbb{R}_{FrNS,P_1\times P_2,2}) =$
$\{(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle\tilde{s}_2, (0.83, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\}$
$\text{Range}(\mathbb{R}_{FrNS,P_1\times P_2,1}) = \text{Range}(\mathbb{R}_{FrNS,P_1\times P_2,2}) =$
$\{(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.7, 0.65, 0.3)\rangle, \langle\tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle\tilde{s}_3, (0.29, 0.6, 0.37)\rangle),$
$(\hat{\epsilon}_3, \langle\tilde{s}_1, (0.5, 0.74, 0.5)\rangle, \langle\tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle\tilde{s}_3, (0.9, 0.3, 0.42)\rangle)\}.$

3.8. *Inverse of a Fermatean Neutrosophic Soft Relation*

Inverse of a FrNS relation $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ is $\mathbb{R}^{-1}_{FrNS,\mathbb{K}\times\mathbb{L}} = \mathbb{R}_{FrNS,\mathbb{L}\times\mathbb{K}}$.

3.9. *Example*

The inverse of FrNS relation $\mathbb{R}_{FrNS,P_1\times P_2,1}$ in example 3.4 is,

$\mathbb{R}^{-1}_{FrNS,P_1\times P_2,1}$=$\{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle\tilde{s}_1, (0.4, 0.2, 0.5)\rangle, \langle\tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle\tilde{s}_3, (0, 0.3, 0.7)\rangle)$,
$((\hat{\epsilon}_3, \hat{\epsilon}_2), \langle\tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle\tilde{s}_2, (0.46, 0.5, 0.7)\rangle, \langle\tilde{s}_3, (0.1, 0.3, 0.75)\rangle)\}.$

3.10. *Composition of Fermatean Neutrosophic Soft Relations*

Let $\mathbb{R}_{FrNS,P_1\times P_2} = \{\big(\xi_{ij} = (\hat{\epsilon}_i, \hat{\epsilon}_j), \langle\tilde{s}, \big(\theta_{P_1\times P_2,\xi_{ij}}(\tilde{s}), \phi_{P_1\times P_2,\xi_{ij}}(\tilde{s}), \psi_{P_1\times P_2,\xi_{ij}}(\tilde{s})\big)\rangle : \tilde{s} \in \mathbb{D}\big) : \xi_{ij} \in P_1 \times P_2\}$ be a $FrNS$ relation from P_1 to P_2 and $\mathbb{R}_{FrNS,P_2\times P_3} = \{\big(\xi_{jk} = (\hat{\epsilon}_j, \hat{\epsilon}_k), \langle\tilde{s}, \big(\theta_{P_1\times P_2,\xi_{jk}}(\tilde{s}), \phi_{P_1\times P_2,\xi_{jk}}(\tilde{s}), \psi_{P_1\times P_2,\xi_{jk}}(\tilde{s})\big)\rangle : \tilde{s} \in \mathbb{D}\big) : \xi_{jk} \in P_2 \times P_3\}$ be $FrNS$ relation from P_2 to P_3 then composition of $\mathbb{R}_{FrNS-P_1,P_2}$ and $\mathbb{R}_{FrNS-P_2,P_3}$ is defined as,

$\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3} =$

$\{(\xi_{ik} = (\hat{\epsilon}_i, \hat{\epsilon}_k), \langle \tilde{s}, (\theta_{P_1\times P_3,\xi_{ik}}(\tilde{s}), \phi_{P_1\times P_3,\xi_{ik}}(\tilde{s}), \psi_{P_1\times P_3,\xi_{ik}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \xi_{ik} \in P_1 \times P_3$ for which, there exist$(\hat{\epsilon}_i, \hat{\epsilon}_j) \in P_1 \times P_2and(\hat{\epsilon}_j, \hat{\epsilon}_k) \in P_2 \times P_3\}$, where $\theta_{P_1\times P_3,\xi_{ik}} = \min\{\theta_{P_1\times P_2,\xi_{ij}}, \theta_{P_2\times P_3,\xi_{jk}}\}, \phi_{P_1\times P_3,\xi_{ik}} = \min\{\phi_{P_1\times P_2,\xi_{ij}}, \phi_{P_2\times P_3,\xi_{jk}}\}, \psi_{P_1\times P_3,\xi_{ik}} = \max\{\psi_{P_1\times P_2,\xi_{ij}}, \psi_{P_2\times P_3,\xi_{jk}}\}$ that is $\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3}(\hat{\epsilon}_i, \hat{\epsilon}_k) = \mathbb{R}_{FrNS,P_1\times P_2}(\hat{\epsilon}_i, \hat{\epsilon}_j) \cap_R \mathbb{R}_{FrNS,P_2\times P_3}(\hat{\epsilon}_j, \hat{\epsilon}_k)$

3.11. *Example*

Let $P_1 = \{\hat{\epsilon}_2\}, P_2 = \{\hat{\epsilon}_2, \hat{\epsilon}_3\}, P_3 = \{\hat{\epsilon}_4, \hat{\epsilon}_5\}$ be the set of parameters and consider the $FrNS$ relations, $\mathbb{R}_{FrNS,P_1\times P_2} = \{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.4, 0.2, 0.5)\rangle,$
$\langle \tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle \tilde{s}_3, (0, 0.3, 0.7)\rangle),$
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.3, 0.75)\rangle)\}$ and
$\mathbb{R}_{FrNS,P_2\times P_3} = \{((\hat{\epsilon}_2, \hat{\epsilon}_4), \langle \tilde{s}_1, (0.6, 0.2, 0.3)\rangle, \langle \tilde{s}_2, (0.5, 0.1, 0.7)\rangle, \langle \tilde{s}_3, (0.15, 0.2, 0.7)\rangle),$
$((\hat{\epsilon}_2, \hat{\epsilon}_5), \langle \tilde{s}_1, (0.3, 0.52, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.30.63)\rangle, \langle \tilde{s}_3, (0.5, 0.2, 0.77)\rangle)\}$, from P_1 to P_2 and from P_2 to P_3, respectively. Their composition is given by,
$\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3} = \mathbb{R}_{FrNS,P_1\times P_3} =$
$\{((\hat{\epsilon}_2, \hat{\epsilon}_4), \langle \tilde{s}_1, (0.4, 0.2, 0.5)\rangle, \langle \tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle \tilde{s}_3, (0, 0.2, 0.7)\rangle),$
$((\hat{\epsilon}_2, \hat{\epsilon}_5), \langle \tilde{s}_1, (0.3, 0.2, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.3, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.77)\rangle)\}.$

3.12. *Proposition*

Let $\mathbb{R}_{FrNS,P_1\times P_2}$ and $\mathbb{R}_{FrNS,P_2\times P_3}$ be two $FrNS$ relations. Then
(i) $(\mathbb{R}^{-1}_{FrNS,P_1\times P_2})^{-1} = \mathbb{R}_{FrNS,P_1\times P_2}$,
(ii) $(\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3})^{-1} = \mathbb{R}^{-1}_{FrNS,P_2\times P_3} \circ \mathbb{R}^{-1}_{FrNS,P_1\times P_2}$,
(iii) $\mathbb{R}_{FrNS,P_1\times P_2} \subseteq \mathbb{R}_{FrNS,P_2\times P_3}$ implies $\mathbb{R}^{-1}_{FrNS,P_1\times P_2} \subseteq \mathbb{R}^{-1}_{FrNS,P_2\times P_3}$.

4 MCDM Algorithm

In this section, we present the MCDM algorithm. Fermatean neutrosophic hypersoft sets serve as the foundation for our Multi-Criteria Decision-Making (MCDM) method. In the mathematical formulation, soft relations are defined inside the fermatean neutrosophic hypersoft set, and the fermatean element for indeterminate values is included. Let R be the set of soft relations, X be the set of alternatives, and μ be the fermatean membership function. Criteria are included in the decision matrix D. The step-wise procedure of the algorithm is presented below:

(1) Construction of case study in terms of fermatean neutrosophic hypersoft set.
(2) To bring all criteria to a single scale, normalize the decision matrix D.
(3) To merge the soft relations $\mu(R_i)$ and the normalized decision matrix, use fermatean neutrosophic aggregation.

(4) Using the aggregate operators, get the fermatean membership scores for each alternative.

(5) Determine which alternative is best by looking for the one with the highest fermatean membership value. The total score Z, is maximized by choosing this option.

The challenge of optimization can be expressed as follows;

$$Z = \sum_{i=1}^{n} \mu(R_i) \cdot D_i \tag{1}$$

Where Z is the total score, $\mu(R_i)$ is the fermatean membership value connected to the soft relation R_i, and D_i is the normalized decision value for the alternatives. This formula serves as the foundation for combining the contributions of all alternatives according to the appropriate normalized choice values and fermatean membership values. To choose the best option, the objective is to maximize this total score or Z. The graphical representation of the algorithm is presented in Figure 2 and Figure 3 represents the implementation of the algorithm.

5 | Case Study Solution and Discussion

In this case study, we apply concepts of fermatean neutrosophic hypersoft sets to improve decision-making in sustainable agriculture. This creative method takes into account social, economic, and environmental aspects while acknowledging the inherent uncertainties in farming techniques. The main goal is to show how the incorporation of fermatean neutrosophic hypersoft sets may help with a more thorough assessment of agricultural alternatives, taking into account neutral or indeterminate aspects in addition to true and false values. Our objective is to maximize alternative selection while maintaining a balance between socioeconomic factors, environmental sustainability, and productivity. Let R be the set of soft relations, X be the set of agricultural alternatives, and μ be the fermatean membership function. Criteria like cost, social issues, and environmental effects are included in the decision matrix D.

The novel framework of Fermatean Neutrosophic Hypersoft Sets (FNHS) has been utilized in this study to improve decision-making in the context of sustainable agriculture. Our approach strives to give a more thorough assessment of agricultural alternatives by combining social, economic, and environmental factors while understanding the inherent uncertainties associated with agricultural techniques. The main goal is to show how FNHS may be used to provide a more nuanced evaluation of agricultural alternatives by including elements that are neutral or indeterminate in addition to binary true and false values.

We aim to maximize alternative selection by balancing productivity, environmental sustainability, and socioeconomic considerations through the application of FNHS. In particular, using the Fermatean membership function represented by and defining R as the set of soft relations and X as the set of agricultural alternatives, we create a decision matrix D that includes important factors like cost, social implications, and environmental effects. According to our research, the FNHS framework is a viable way for stakeholders to traverse the complicated world of sustainable agriculture and make more informed decisions that support both environmental stewardship and financial objectives.

The addition of score functions to our decision-making method improves its robustness and reliability by enabling a quantitative assessment of various agricultural tactics. To fully realize the promise of FNHS in supporting resilient food production systems and sustainable practices, more study and use of FNHS in agricultural decision-making are required.

6 | Conclusions

In conclusion, by introducing soft relations into fermatean neutrosophic hypersoft sets, our work presents a new and practical method to tackle the complex problems of sustainable agriculture. By taking into account the environmental, social, and economic aspects of agricultural decision-making, the suggested Multi-Criteria Decision-Making (MCDM) algorithm displays its capacity to negotiate the intricacies of agricultural decision-making. The case study application highlights the usefulness of our method in real-world problems and demonstrates how well it works for choosing agricultural inputs. The findings highlight how crucial it is to take into consideration ambiguous factors when making decisions about sustainable agriculture, as this will help decision-makers make more thoughtful and comprehensive decisions. Compared with conventional techniques, our method provides a more flexible and subtle framework.

(1) To improve the algorithm's applicability in actual agricultural problems, we must continue to explore developing technologies and refining the algorithm.

(2) This study adds to the current conversation about sustainable farming methods by providing insightful information that will help with future decision-making and promote resilient, environmentally friendly farming systems.

(3) Furthermore, investigating scalability and integration with cutting-edge technologies like IoT and precision agriculture might improve decision-making procedures.

(4) Lastly, broadening the temporal scope to track the long-term effects on ecosystem resilience and soil health would help us get a more thorough grasp of the long-term advantages of applying neutrosophic-based decision-making in agriculture.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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