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Soft-*int* Almost Interior Ideals for Semigroups

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Abstract: Just as the concept of interior ideal of semigroups is a generalization of ideal in semigroups, the notion of soft intersection (soft-*int*) interior ideal is a generalization of soft-*int* ideal. In this paper, we propose the concepts of soft-*int* (weakly) almost interior ideal of a semigroup as a generalization of the nonnull soft-*int* interior ideals. We explore their algebraic properties in detail. We also show that an idempotent soft-*int* almost interior ideal is a soft-*int* almost subsemigroup. We additionally derive several intriguing relations related to semiprimeness, minimality, and (strongly) primeness between almost interior ideals and soft-*int* almost interior ideals.

Keywords: Soft Set; Interior Ideal; Soft Intersection (almost) Interior Ideal.

1. Introduction

Semigroups were first studied formally in the early twentieth century. Semigroups are significant in many mathematical areas because they give the abstract algebraic foundation for "memoryless" systems, which are time-dependent and restart with each iteration. Semigroups are essential mathematical models for linear time-invariant systems. In partial differential equations, any equation with time-independent spatial evolution has a semigroup associated with it. Finite semigroup theory has been particularly relevant in theoretical computer science.

Ideals are necessary to investigate algebraic structures and their applications. Dedekind initially proposed ideals to contribute to the study of algebraic numbers, and Noether developed them further to incorporate associative rings. In [1,2], bi-ideals and quasi-ideals were initially proposed for semigroups, respectively. Ideals are essential to encourage more study of mathematical structures. Some mathematicians offered novel developments of the concept of ideals displaying imperative consequences to describe the algebraic structures. While the bi-ideals are a generalization of quasi-ideals, the interior ideals are a generalization of left and right ideals.

Furthermore, the authors [3] presented the idea of almost left, right, and two-sided ideals of semigroups. In [4], the notion of almost bi-ideals in semigroups is a generalization of bi-ideals was presented. The introduction of the concept of almost quasi-ideals of semigroup was made in [5]. Using the notion of almost ideals and interior ideals of semigroups, the ideas of almost interior ideals and weakly almost interior ideals of semigroups were developed and their properties by investigated in [6]. Researchers have given considerable attention to the almost ideals of semigroups. The concept of almost subsemigroups, almost bi-quasi-interior ideals, almost bi-interior ideals, and almost bi-quasi ideals of semigroups was put forth by [7-10], respectively. Additionally, different kinds of almost fuzzy ideals of semigroups were studied [5, 7-12].

Molodtsov [13] presented the idea of a soft set to model uncertainty. Since then, soft sets have attracted the attention of researchers in several fields. The theory's cornerstone, soft set operations, was studied by [14-32]. The definition of a soft set and its operations were modified in [33]. The notion of soft-int groups was introduced in [34] leading to the analysis of several soft algebraic systems. In [35-36], the authors studied semigroups with soft-int left (right/sided) ideals, interior ideals, (generalized) bi-ideals, and quasi-ideals, and in [37], certain types of semigroups in terms of soft-int substructures of semigroups are characterized. Many soft algebraic structures were investigated in [38-50]. Recently, several new types of semigroup ideals were proposed in [51-55].

As a generalization of the soft-int ideal, soft-int interior ideal of semigroups was proposed in [33]. In this study, as a further generalization of the nonnull soft-int interior ideal, we present the concept of soft-int almost interior ideal, and its generalization, soft-int weakly almost interior ideals. Our results show that every soft-int weakly almost interior ideal of a semigroup is a soft-int almost interior ideal; however, the converse is not true for the counterexample. Furthermore, we demonstrate that an idempotent soft-int almost interior ideal is a soft-int almost subgroup. In addition, we demonstrate the relation between a semigroup's soft-int almost interior ideal and almost interior ideal in terms of (strongly) primeness, minimality, and semiprimeness.

2. Preliminaries

In this part, we go over some essential concepts related to soft sets and semigroups.

Definition 2.1. Let U be the universal set, E be the parameter set, P(U) be the power set of U, and $\mathbb{V} \subseteq$ E. A soft set $f_{\mathbb{V}}$ over U is a set-valued function such that $f_{\mathbb{V}}$: $E \to P(U)$ such that for all $x \notin \mathbb{V}$, $f_{\mathbb{V}}(x) = \emptyset$. A soft set over U can be represented by the set of ordered pairs

$$f_{\mathbb{V}} = \{(x, f_{\mathbb{V}}(x)) : x \in E, \mathrm{f}_{\mathbb{V}}(x) \in P(U)\}$$

[10, 33]. For all undefined basic concepts related to the soft set, we refer to [33].

Definition 2.2. The support of $f_{\mathbb{V}}$ is defined by

$$supp(f_{\mathbb{V}})=\{x \in \mathbb{V} : f_{\mathbb{V}}(x) \neq \emptyset\} \text{ [18].}$$

A soft set with an empty support is a null soft set, otherwise, it is nonnull.

Note 2.3. If $f_{\mathbb{V}} \tilde{\subseteq} f_{\mathbb{K}}$, then $supp(f_{\mathbb{V}}) \subseteq supp(f_{\mathbb{K}})$ [56].

In this paper, S stands for a semigroup. A nonempty subset $\mathbb{V}$ of S is called a subsemigroup of S if $\mathbb{V}\mathbb{V} \subseteq \mathbb{V}$; and is called an interior ideal of S if $S\mathbb{V}S \subseteq \mathbb{V}$. A nonempty subset $\mathbb{V}$ of S is called an almost interior ideal of S if $x\mathbb{V}y \cap \mathbb{V} \neq \mathbb{V}$, for all x, y $\in$ S.

Definition 2.4. Let f_S, $g_S \in S_S(U)$. Then, soft-int product $f_S \circ g_S$ is defined by [36]

$$(f_S \circ g_S)(x) = \begin{cases} \bigcup_{x=yz} \{f_S(y) \cap g_S(z)\}, & if\ \exists y, z \in S\ such\ that\ x = yz \\ \emptyset, & otherwise \end{cases}$$

Theorem 2.5. Let p_S, $\varkappa_S, \vartheta_S \in S_S(U)$. Then,

i. $(p_S \circ \varkappa_S) \circ \vartheta_S = p_S \circ (\varkappa_S \circ \vartheta_S)$.
ii. $p_S \circ \varkappa_S \neq p_S \circ \varkappa_S$
iii. $p_S \circ (\varkappa_S \tilde{\cup} \vartheta_S) = (p_S \circ \varkappa_S) \tilde{\cup} (p_S \circ \vartheta_S)$ and $(p_S \tilde{\cup} \varkappa_S) \circ \vartheta_S = (p_S \circ \vartheta_S) \tilde{\cup} (\varkappa_S \circ \vartheta_S)$.
iv. $p_S \circ (\varkappa_S \tilde{\cap} \vartheta_S) = (p_S \circ \varkappa_S) \tilde{\cap} (p_S \circ \vartheta_S)$ and $(p_S \tilde{\cap} \varkappa_S) \circ \vartheta_S = (p_S \circ \vartheta_S) \tilde{\cap} (\varkappa_S \circ \vartheta_S)$.
v. If $p_S \tilde{\subseteq} \varkappa$, then $p_S \circ t_S \tilde{\subseteq} \varkappa_S \circ t_S$ and $t_S \circ p_S \tilde{\subseteq} t_S \circ \varkappa_S$.
vi. If $\mathfrak{H}_S$, $y_S \in S_S(U)$ such that $\mathfrak{H}_S \tilde{\subseteq} p_S$ and $y_S \tilde{\subseteq} q_S$, then $\mathfrak{H}_S \circ y_S \tilde{\subseteq} p_S \circ q_S$ [36].

Definition 2.6. Let $\mathbb{V} \subseteq$ S. The soft characteristic function of $\mathbb{V}$, denoted by $S_{\mathbb{V}}$, is defined as [36]:

$$S_{\mathbb{V}}(x) = \begin{cases} U, & if\ x \in \mathbb{V} \\ \emptyset, & if\ x \in S \backslash \mathbb{V} \end{cases}$$

Corollary 2.7. $supp(S_{\mathbb{V}}) = \mathbb{V}$ [56].

Theorem 2.8. Let $\emptyset \neq$ Ꝟ, Ꝃ $\subseteq$ S. Then, [36,56]:

i) Ꝟ $\subseteq$ Ꝃ if and only if $S_{Ꝟ} \tilde{\subseteq} S_{Ꝃ}$

ii) $S_{Ꝟ} \tilde{\cap} S_{Ꝃ} = S_{Ꝟ \cap Ꝃ}$ and $S_{Ꝟ} \tilde{\cup} S_{Ꝃ} = S_{Ꝟ \cup Ꝃ}$

iii) $S_{Ꝟ} \circ S_{Ꝃ} = S_{ꝞꝂ}$

Definition 2.9. Let $x \in$ S. The soft characteristic function of x, denoted by S_x, is defined as [57]:

$$S_x(y) = \begin{cases} U, & if \ \ y = x \\ \emptyset, & if \ \ y \neq x \end{cases}$$

Definition 2.10. f_S is called a soft-int interior ideal of S over U if f_S(xyz)⊇ f_S(y), for all x,y,$z \in S$ [36].

If f_S(x) = U for all x∈ S, then f_S is a soft-int interior ideal, and it is denoted by 𝕊. Moreover, $\mathbb{S} = S_S$, that is, 𝕊(x) = U for all x∈ S [36].

Theorem 2.11. Let f_S be a soft set over U. Then, f_S is a soft-int interior ideal of S over U if and only $\mathbb{S} \circ f_S \circ \mathbb{S} \tilde{\subseteq} f_S$ [36].
$\mathcal{SI}$-I-ideal represents the soft-int interior-ideal from now on.
Definition 2.12. A soft set f_S is called a soft-int almost subsemigroup of S if $(f_S \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$ [56]. Referring to [58], one may discuss the potential consequences of graph applications and network analysis for soft sets, which are characterized by the divisibility of determinants, and we refer to [59] for soft int LA-semigroups.

3. Results on Soft-*int* Almost Interior Ideals of Semigroups

Definition 3.1. A soft set f_s is called a soft-int almost interior ideal of S if

$$(S_x \circ f_s \circ S_y) \tilde{\cap} f_s \neq \emptyset_s$$

For all x,y $\in$ S, and is called a soft-int weakly almost interior ideal of S if

$$(S_x \circ f_s \circ S_x) \tilde{\cap} f_s \neq \emptyset_s$$

For all $x, y \in$ S. Hereafter, soft-int almost interior-ideal of S and soft-int weakly almost interior ideal of S are denoted by $\mathcal{SI}$-almost I-ideal and $\mathcal{SI}$-weakly almost I-ideal, respectively.

Example 3.2. Consider the following semigroup S={ɒ, ɯ}:

Table 1. Cayley table of binary operation.

	ɒ	ɯ
ɒ	ɒ	ɯ
ɯ	ɯ	ɒ

Let $ƒ_s$, g_s, and $ꝗ_s$ be soft sets over U={$\bar{k}$ ∣ k∈ Z_{10}^{*}} as follows :

$$ƒ_s = \{(ɒ, \{\bar{1}, \bar{3}\}), (ɯ, \{\bar{1}, \bar{9}\})\}$$
$$g_s = \{(ɒ, \{\bar{7}, \bar{9}\}), (ɯ, \{\bar{3}, \bar{7}\})\}$$
$$ꝗ_s = \{(ɒ, \{\bar{1}, \bar{3}\}), (ɯ, \{\bar{9}, \bar{7}\})\}$$

Here, $ƒ_s$ ve g_s are both $\mathcal{SI}$-almost interior ideals. In fact, $ƒ_s$ is an $\mathcal{SI}$-almost I-ideal, that is, $(S_x \circ ƒ_s \circ S_y) \tilde{\cap} ƒ_s \neq \emptyset_s$, for all, $y \in S$.

Let's start with $S_{ɒ}, S_{ɒ}$:
$[(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} ƒ_{S}](ɒ) = (S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ})(ɒ) \cap ƒ_{S}(ɒ) = [(S_{ɒ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɒ}(ɒ)] \cup [(S_{ɒ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(a)$=$[[((S_{ɒ}(ɒ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɯ) \cap ƒ_{S}(b))] \cap S_{ɒ}(ɒ) \cup [((S_{ɒ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɒ) = f_{S}(ɒ) = \{\bar{1}, \bar{3}\}$

$[(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} ƒ_{S}](ɯ) = (S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} ɒ)(ɯ) \cap ƒ_{S}(ɯ) = [(S_{ɒ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɒ}(ɒ)] \cup [(S_{ɒ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɯ)$=$[[((S_{ɒ}(ɯ) \cap ƒ_{S}(a)) \cup ((S_{ɒ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɒ) \cup [((S_{ɒ}(ɒ) \cap ƒ(ɒ)) \cup ((S_{ɒ}(ɯ) \cap ƒ_{S}(b))] \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɯ) = ƒ_{S}(ɯ) = \{\bar{1}, \bar{9}\}$. Hence,

$$(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} ƒ_{S}=\{(ɒ,\{\bar{1}, \bar{3}\}), (ɯ, \{\bar{1}, \bar{9}\})\} \neq \emptyset_{S}$$

Let's continue with $S_{ɒ}, S_{ɯ}$:
$[(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S}](ɒ) = (S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ})(ɒ) \cap ƒ_{S}(ɒ) = [(S_{ɒ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɯ}(ɒ)] \cup [(S_{ɒ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɒ)$=$[[((S_{ɒ}(ɒ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɒ) \cup [((S_{ɒ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɒ) = ƒ_{S}(ɯ) \cap ƒ_{S}(ɒ) = \{\bar{1}\}$

$[(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S}](ɯ) = (S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ})(ɯ) \cap ƒ_{S}(ɯ) = [(S_{ɒ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɯ}(ɒ)] \cup [(S_{ɒ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɯ)$=$[[((S_{ɒ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɒ) \cup [((S_{ɒ}(ɒ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɒ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɯ) = ƒ_{S}(ɒ) \cap ƒ_{S}(ɯ) = \{\bar{1}\}$. Thus,

$$(S_{ɒ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S}=\{(ɒ,\{\bar{1}\}), (ɯ, \{\bar{1}\})\} \neq \emptyset_{S}$$

Let's continue with $S_{ɯ}, S_{ɯ}$:

$[(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S}](ɒ) = (S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ})(ɒ) \cap ƒ_{S}(ɒ) = [(S_{ɯ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɯ}(a)] \cup [(S_{ɯ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɒ)$=$[[((S_{ɯ}(ɒ) \cap ƒ_{S}(a)) \cup ((S_{ɯ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɒ) \cup [((S_{ɯ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɯ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɒ) = ƒ_{S}(ɒ) = \{\bar{1}, \bar{3}\}$

$[(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S}](ɯ) = (S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ})(ɯ) \cap ƒ_{S}(b) = [(S_{ɯ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{ɯ}(ɒ)] \cup [(S_{ɯ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɯ)$=$[[((S_{ɯ}(ɯ) \cap ƒ_{S}(a)) \cup ((S_{ɯ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(a) \cup [((S_{ɯ}(ɒ) \cap ƒ_{S}(a)) \cup ((S_{ɯ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɯ}(ɯ)] \cap ƒ_{S}(ɯ) = ƒ_{S}(ɯ) = \{\bar{1}, \bar{9}\}$. Therefore,

$$(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} ƒ_{S} = \{(ɒ,\{\bar{1}, \bar{3}\}), (ɯ, \{\bar{1}, \bar{9}\})\} \neq \emptyset_{S}$$

Let's continue with $S_{ɯ}, S_{ɒ}$:
$[(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{a}) \tilde{\cap} ƒ_{S}](ɒ) = (S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ})(ɒ) \cap ƒ_{S}(a)$=$[(S_{ɯ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɒ}(ɒ)] \cup [(S_{ɯ}{}^{\circ}ƒ_{S})(ɯ) \cap S_{a}(ɯ)] \cap ƒ_{S}(a)$=$[[((S_{ɯ}(ɒ) \cap ƒ_{S}(ɒ) \cup ((S_{ɯ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɒ) \cup [((S_{ɯ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɯ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɒ) = ƒ_{S}(ɯ) \cap ƒ_{S}(ɒ) = \{\bar{1}\}$

$[(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} ƒ_{S}](ɯ) = (S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ})(ɯ) \cap ƒ_{S}(ɯ)$= $[(S_{ɯ}{}^{\circ}ƒ_{S})(b) \cap S_{ɒ}(ɒ)] \cup [(S_{ɯ}{}^{\circ}ƒ_{S})(ɒ) \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɯ)$=$[[((S_{ɯ}(ɯ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɯ}(ɒ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɒ) \cup [((S_{ɯ}(ɒ) \cap ƒ_{S}(ɒ)) \cup ((S_{ɯ}(ɯ) \cap ƒ_{S}(ɯ))] \cap S_{ɒ}(ɯ)] \cap ƒ_{S}(ɯ) = ƒ_{S}(ɒ) \cap ƒ_{S}(ɯ) = \{\bar{1}\}$. Consequently,

$$(S_{ɯ}{}^{\circ}ƒ_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} ƒ_{S}=\{(ɒ,\{\bar{1}\}), (ɯ, \{\bar{1}\})\} \neq \emptyset$$

Therefore, $(S_{x}{}^{\circ}ƒ_{S}{}^{\circ} S_{y}) \tilde{\cap} ƒ_{S} \neq \emptyset_{S}$ for all $x, y \in$ S, so $ƒ_{S}$ is an $\mathcal{SI}$-almost I-ideal. Similarly, g_{S} is an $\mathcal{SI}$-almost I-ideal. In fact;

$$(S_{ɒ}{}^{\circ}g_{S}{}^{\circ} S_{ɒ}) \tilde{\cap} g_{S}=\{(ɒ,\{\bar{7}, \bar{9}\}), (ɯ, \{\bar{3}, \bar{7}\})\} \neq \emptyset_{S}$$
$$(S_{ɒ}{}^{\circ}g_{S}{}^{\circ} S_{ɯ}) \tilde{\cap} g_{S}=\{(ɒ,\{\bar{7}\}), (ɯ, \{\bar{7}\})\} \neq \emptyset_{S}$$

$$(S_{ɯ}{\circ}g_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ g_s=\{(ɒ,\{\bar{7},\bar{9}\}),(ɯ,\{\bar{3},\bar{7}\})\}\neq\ \emptyset_s$$
$$(S_{ɯ}{\circ}g_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ g_s=\{(ɒ,\{\bar{7}\}),(ɯ,\{\bar{7}\})\}\neq\ \emptyset_s$$

One can also show that $ꝗ_s$ is a weakly almost I-ideal; but not an $\mathcal{SI}$-almost I-ideal. In deed;

$[(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ h_s](ɒ)$=$(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɒ})(ɒ)\cap ꝗ_s(a)$= $ꝗ_s(ɒ)=\{\bar{1},\bar{3}\}$
$[(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ ꝗ_s](ɯ)$=$(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɒ})(ɯ)\cap ꝗ_s(ɯ)$= $ꝗ_s(ɯ)=\{\bar{9},\bar{7}\}$. Thus;

$$(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ ꝗ_s=\{(ɒ,\{\bar{1},\bar{3}\}),(ɯ,\{\bar{9},\bar{7}\})\}\neq\ \emptyset_s$$

And also

$[(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ ꝗ_s](ɒ)$=$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɯ})(ɒ)\cap ꝗ_s(ɒ)=\ ꝗ_s(ɒ)=\{\bar{1},\bar{3}\}$
$[(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ ꝗ_s](ɯ)$=$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɯ})(ɯ)\cap ꝗ_s(b)$= $ꝗ_s(ɯ)=\{\bar{9},\bar{7}\}$. Therefore,

$$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ ꝗ_s=\{(ɒ,\{\bar{1},\bar{3}\}),(ɯ,\{\bar{9},\bar{7}\})\}\neq\ \emptyset_s$$

Hence, $(S_x{\circ}ꝗ_s{\circ}\, S_x)\ \tilde{\cap}\ ꝗ_s\neq\emptyset_s$ for all $x\in S$, so $ꝗ_s$ is an $\mathcal{SI}$-weakly almost I-ideal. However,

$[(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ ꝗ_s](ɒ)$=$(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɯ})(ɒ)\cap ꝗ_s(ɒ)=\ ꝗ_s(ɯ){\cap}ꝗ_s(ɒ)=\emptyset$
$[(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}](ɯ)$=$(S_{ɒ}{\circ}\mathfrak{H}_s{\circ}\, S_{ɯ})(ɯ)\cap ꝗ(ɯ)=\ ꝗ_s(ɒ){\cap}ꝗ_s(ɯ)=\emptyset$. Thus,

$$(S_{ɒ}{\circ}ꝗ_s{\circ}\, S_{ɯ})\ \tilde{\cap}\ ꝗ_s=\{(ɒ,\emptyset),(ɯ,\emptyset)\}=\ \emptyset_s$$

$[(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ ꝗ_s](ɒ)$=$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɒ})(ɒ)\cap ꝗ_s(ɒ)=\ ꝗ_s(ɯ){\cap}ꝗ_s(ɒ)=\emptyset$
$[(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ ꝗ_s](ɯ)$=$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɒ})(ɯ)\cap ꝗ_s(ɯ)$= $ꝗ_s(ɒ){\cap}ꝗ_s(ɯ)=\emptyset$. Hence,

$$(S_{ɯ}{\circ}ꝗ_s{\circ}\, S_{ɒ})\ \tilde{\cap}\ ꝗ_s=\{(ɒ,\emptyset),(ɯ,\emptyset)\}=\ \emptyset_s$$

Consequently, $(S_x{\circ}ꝗ_s{\circ}\, S_y)\ \tilde{\cap}\ ꝗ_s=\emptyset_s$, for $\exists x,y\in$ S .Thus, $ꝗ_s$ is not an $\mathcal{SI}$-almost I-ideal.

Proposition 3.3. If $ʄ_s$ is an $\mathcal{SI}$-I-ideal such that $S_x{\circ}ʄ_s{\circ}\, S_y\neq\emptyset_s$ for all x, $y\in$ S, then $ʄ_s$ is an $\mathcal{SI}$-almost I-ideal.

Proof: Let $ʄ_s$ be an $\mathcal{SI}$-I-ideal, thus $\widetilde{\mathbb{S}}{\circ}ʄ_s{\circ}\widetilde{\mathbb{S}}\ \tilde{\subseteq}\ f_s$. We need to show that
$$(S_x{\circ}ʄ_s{\circ}\, S_y)\ \tilde{\cap}\ ʄ_s\neq\emptyset_s$$
for all x, $y\in$ S. Since $(S_x{\circ}ʄ_s{\circ}\, S_y)\ \tilde{\subseteq}\ \widetilde{\mathbb{S}}{\circ}ʄ_s{\circ}\widetilde{\mathbb{S}}\ \tilde{\subseteq}\ ʄ_s$, it follows that $S_x{\circ}ʄ_s{\circ}\, S_y\ \tilde{\subseteq}\ ʄ_s$. Thus,
$$(S_x{\circ}ʄ_s{\circ}\, S_y)\ \tilde{\cap}\ ʄ_s\ \tilde{\subseteq}\ S_x{\circ}ʄ_s{\circ}\, S_y\neq\emptyset_s$$
implying that $ʄ_s$ is an $\mathcal{SI}$-almost I-ideal.

Here, $S_x{\circ}ʄ_s{\circ}\, S_y\neq\emptyset_s$ implies that $ʄ_s\neq\emptyset_s$. Moreover, $\emptyset_s$ is an $\mathcal{SI}$-I-ideal as $(S_x{\circ}\emptyset_s{\circ}\, S_y)=\emptyset_s\ \tilde{\subseteq}\ \emptyset_s$; but $\emptyset_s$ is not an $\mathcal{SI}$-almost I-ideal since $(S_x{\circ}\emptyset_s{\circ}\, S_y)\ \tilde{\cap}\ \emptyset_s\ =\emptyset_s\ \tilde{\cap}\ \emptyset_s=\emptyset_s$.

If f_s is an $\mathcal{SI}$-almost I-ideal, then f_s needs not be an $\mathcal{SI}$-I-ideal as shown in the following example:

Example 3.4. In Example 3.2, it is shown $ʆ_s$ and g_s are $\mathcal{SI}$-almost I-ideal; but $ʆ_s$ and g_s are not $\mathcal{SI}$-I-ideal. In fact,

$\widetilde{\mathbb{S}}\circ ʆ_s\circ\widetilde{\mathbb{S}}(ɒ) = [(\widetilde{\mathbb{S}}\circ ʆ_s)(ɒ)\cap\widetilde{\mathbb{S}}(ɒ)]\cup[(\widetilde{\mathbb{S}}\circ ʆ_s)(ɰ)\cap\widetilde{\mathbb{S}}(ɰ)] = [\widetilde{\mathbb{S}}(ɒ)\cap ʆ_s(ɒ)]\cup[\widetilde{\mathbb{S}}(ɰ)\cap ʆ_s(ɰ)]\cup[\widetilde{\mathbb{S}}(ɰ)\cap ʆ_s(ɒ)]\cup[\widetilde{\mathbb{S}}(ɒ)\cap ʆ_s(ɰ)] = ʆ_s(ɒ)\cup ʆ_s(ɰ)\cup ʆ_s(ɒ)\cup ʆ_s(ɰ) = ʆ_s(ɒ)\cup ʆ_s(ɰ)\not\subseteq ʆ_s(ɒ)$

Thus, $ʆ_s$ is not an $\mathcal{SI}$-I-ideal. Similarly,

$\widetilde{\mathbb{S}}\circ g_s\circ\widetilde{\mathbb{S}}(ɒ) = [(\widetilde{\mathbb{S}}\circ g_s)(ɒ)\cap\widetilde{\mathbb{S}}(ɒ)]\cup[(\widetilde{\mathbb{S}}\circ g_s)(ɰ)\cap\widetilde{\mathbb{S}}(ɰ)] = [\widetilde{\mathbb{S}}(ɒ)\cap g_s(ɒ)]\cup[\widetilde{\mathbb{S}}(ɰ)\cap g_s(ɰ)]\cup[\widetilde{\mathbb{S}}(ɰ)\cap g_s(ɒ)]\cup[\widetilde{\mathbb{S}}(a)\cap g_s(ɰ)] = g_s(ɒ)\cup g_s(ɰ)\cup g_s(ɒ)\cup g_s(ɰ) = g_s(ɒ)\cup g_s(ɰ)\not\subseteq g_s(ɒ)$

Thus, g_s is not an $\mathcal{SI}$-I-ideal.

Proposition 3.5. Every $\mathcal{SI}$-almost I-ideal is an $\mathcal{SI}$-weakly almost I-ideal.

Proof: Let f_s be an $\mathcal{SI}$-almost I-ideal, then $(S_x\circ f_s\circ S_y)\,\widetilde{\cap}\, f_s \neq \emptyset_s$ for all $x, y \in S$. Hence, $(S_x\circ f_s\circ S_x)\,\widetilde{\cap}\, f_s \neq \emptyset_s$ for all all $x\in S$. Thereby, f_s is an $\mathcal{SI}$-weakly almost I-ideal. Since $\mathcal{SI}$-weakly almost I-ideal is a generalization of $\mathcal{SI}$-almost I-ideal, from now on all the theorems and proofs are given for $\mathcal{SI}$-almost I-ideal instead of $\mathcal{SI}$-weakly almost I-ideal.

The converse of Proposition 3.5. does not hold:

Example 3.6. In Example 3.2, $ꝗ_s$ is an $\mathcal{SI}$-weakly almost I-ideal; but $ꝗ_s$ is not $\mathcal{SI}$-almost I-ideal.

Proposition 3.7. Let $ʆ_S$ be an idempotent $\mathcal{SI}$-almost I-ideal. Then, $ʆ_S$ is an $\mathcal{SI}$-almost subsemigroup.

Proof: Assume that $ʆ_S$ is an idempotent $\mathcal{SI}$-almost I-ideal. Then, $ʆ_S\circ ʆ_S = ʆ_S$ and $[(S_x\circ ʆ_S\circ S_y)]\,\widetilde{\cap}\, ʆ_S \neq \emptyset_S$, for all $x, y\in S$. We need to show that

$$(ʆ_S\circ ʆ_S)\,\widetilde{\cap}\, ʆ_S \neq \emptyset_S$$

Since, $\emptyset_S \neq [(S_x\circ ʆ_S\circ S_y)]\,\widetilde{\cap}\, ʆ = [[(S_x\circ ʆ_S\circ S_y)]\,\widetilde{\cap}\, ʆ_S]\,\widetilde{\cap}\, ʆ_S$

$$= [[(S_x\circ ʆ_S\circ S_y)]\,\widetilde{\cap}\,(ʆ_S\circ ʆ_S)]\,\widetilde{\cap}\, ʆ_S$$

$$\widetilde{\subseteq}\,(ʆ_S\circ ʆ_S)\,\widetilde{\cap}\, ʆ_S$$

hence $(ʆ_S\circ ʆ_S)\,\widetilde{\cap}\, ʆ_S \neq \emptyset_S$, so $ʆ_S$ is an $\mathcal{SI}$-almost subsemigroup.

Theorem 3.8. Let $f_s\,\widetilde{\subseteq}\, ꝗ_s$. If f_s is an $\mathcal{SI}$-almost I-ideal, then $ꝗ_s$ is an $\mathcal{SI}$-almost I-ideal.

Proof: Let f_s is an $\mathcal{SI}$-almost I-ideal. Hence, $(S_x\circ f_s\circ S_y)\,\widetilde{\cap}\, f_s \neq \emptyset_s$, for all $x, y\in S$. We need to show that $(S_x\circ ꝗ_s\circ S_y)\,\widetilde{\cap}\, ꝗ_s \neq \emptyset_s$. In fact,

$$(S_x\circ f_s\circ S_y)\,\widetilde{\cap}\, f_s\,\widetilde{\subseteq}\,(S_x\circ ꝗ_s\circ S_y)\,\widetilde{\cap}\, ꝗ_s.$$

Since $(S_x\circ f_s\circ S_y)\,\widetilde{\cap}\, f_s \neq \emptyset_s$, $(S_x\circ ꝗ_s\circ S_y)\,\widetilde{\cap}\, ꝗ_s \neq \emptyset_s$, completing the proof.

Theorem 3.9. Let f_s and $ꝗ_s$ be $\mathcal{SI}$-almost I-ideals. Then, $f_s\,\widetilde{\cup}\, ꝗ_s$ is an $\mathcal{SI}$-almost I-ideal.

Proof: Since f_s is an $\mathcal{SI}$-almost I-ideal by assumption and $f_s\,\widetilde{\subseteq}\, f_s\,\widetilde{\cup}\, ꝗ_s$, $f_s\,\widetilde{\cup}\, ꝗ_s$ is an $\mathcal{SI}$-almost I-ideal by Theorem 3.6.

Corollary 3.10. The finite union of $\mathcal{SI}$-almost I-ideals is an $\mathcal{SI}$-almost I-ideals.

Corollary 3.11. Let f_s or $ꝗ_s$ be $\mathcal{SI}$-almost I-ideals. Then $f_s\,\widetilde{\cup}\, ꝗ_s$ is an $\mathcal{SI}$-almost I-ideals.

Note that if f_s and $ꝗ_s$ are $\mathcal{SI}$-almost I-ideals, then $f_s\,\widetilde{\cap}\, ꝗ_s$ needs not to be an $\mathcal{SI}$-almost I-ideals.

Example 3.12. Consider the $\mathcal{SI}$-almost I-ideals $ʆ_s$ and g_s in Example 3.2. Since,

$$ʆ_s\,\widetilde{\cap}\, g_s = \{(ɒ,\emptyset),(ɰ,\emptyset)\} = \emptyset_s$$

$ʆ_s\,\widetilde{\cap}\, g_s$ is not $\mathcal{SI}$-almost I-ideals.

Lemma 3.13. Let $x \in S$ and $\emptyset \neq Y \subseteq S$. Then $S_x \circ S_Y = S_{xY}$. If X is a nonempty subset of S and $y \in S$, then $S_X \circ S_y = S_{Xy}$ [57].

Theorem 3.14. Let $\emptyset \neq \mathbb{V} \subseteq S$. Then, $\mathbb{V}$ is an almost I-ideal if and only if $S_{\mathbb{V}}$, the soft characteristic function of $\mathbb{V}$, is an $\mathcal{SI}$-almost I-ideal.

Proof: Assume that $\emptyset \neq \mathbb{V}$ is an almost I-ideal. Then, $(x\mathbb{V}y) \cap \mathbb{V} \neq \emptyset$, for all all $x, y \in S$, and so there exist all $k \in S$ such that $k \in (x\mathbb{V}y) \cap \mathbb{V} \neq \emptyset$ Since,

$$((S_x \circ S_{\mathbb{V}} \circ S_y) \tilde{\cap} S_{\mathbb{V}})(k) = (S_{x\mathbb{V}y} \tilde{\cap} S_{\mathbb{V}})(k) = (S_{x\mathbb{V}y \cap \mathbb{V}})(k) = U \neq \emptyset$$

It follows that $(S_x \circ S_{\mathbb{V}} \circ S_y) \tilde{\cap} S_{\mathbb{V}} \neq \emptyset_S$. Thus, $S_{\mathbb{V}}$ is an $\mathcal{SI}$-almost I-ideal.

Conversely, let $S_{\mathbb{V}}$ be an $\mathcal{SI}$-almost I-ideal. Hence, we have $(S_x \circ S_{\mathbb{V}} \circ S_y) \tilde{\cap} S_{\mathbb{V}} \neq \emptyset_S$, for all $x, y \in S$. In order to show that $\mathbb{V}$ is an almost I-ideal, we should prove that $\mathbb{V} \neq \emptyset$ and $(x\mathbb{V}y) \cap \mathbb{V} \neq \emptyset$, for all $x, y \in S$. By assumption, $\mathbb{V} \neq \emptyset$ is obvious. Then,

$$\begin{aligned}\emptyset_S \neq (S_x \circ S_{\mathbb{V}} \circ S_y) \tilde{\cap} S_{\mathbb{V}} &\Rightarrow \exists k \in S; ((S_x \circ S_{\mathbb{V}} \circ S_y) \tilde{\cap} S_{\mathbb{V}})(k) \neq \emptyset \\ &\Rightarrow \exists k \in S; ((S_{x\mathbb{V}y} \tilde{\cap} S_{\mathbb{V}})(k) \neq \emptyset \\ &\Rightarrow \exists k \in S; ((S_{x\mathbb{V}y \cap \mathbb{V}})(k) \neq \emptyset \\ &\Rightarrow \exists k \in S; ((S_{x\mathbb{V}y \cap \mathbb{V}})(k) = U \\ &\Rightarrow k \in x\mathbb{V}y \cap \mathbb{V}\end{aligned}$$

Hence, $(x\mathbb{V}y) \cap \mathbb{V} \neq \emptyset$. Consequently, $\mathbb{V}$ is almost I-ideal.

Lemma 3.15. Let $\mathfrak{f}_S \in S_S(U)$. Then, $\mathfrak{f}_S \tilde{\subseteq} S_{supp(\mathfrak{f}_S)}$ [56].

Theorem 3.16. If $\mathfrak{f}_S$ is an $\mathcal{SI}$-almost I-ideal, then $supp(\mathfrak{f}_S)$ is an almost I-ideal.

Proof: Let $\mathfrak{f}_S$ be an $\mathcal{SI}$-almost I-ideal. Thus, $\mathfrak{f}_S \neq \emptyset_S$, thus $supp(\mathfrak{f}_S) \neq \emptyset$. Moreover, $(S_x \circ \mathfrak{f}_S \circ S_y) \tilde{\cap} f_S \neq \emptyset_S$, for all $x, y \in S$. To show that $supp(\mathfrak{f}_S)$ is an almost I-ideal, by Theorem 3.14, it is enough to show that $S_{supp(\mathfrak{f}_S)}$ is an $\mathcal{SI}$-almost I-ideal. By Lemma 3.15,

$$(S_x \circ \mathfrak{f}_S \circ S_y) \tilde{\cap} \mathfrak{f}_S \tilde{\subseteq} (S_x \circ S_{supp(\mathfrak{f}_S)} \circ S_y) \tilde{\cap} S_{supp(\mathfrak{f}_S)}$$

And since $(S_x \circ \mathfrak{f}_S \circ S_y) \tilde{\cap} \mathfrak{f}_S \neq \emptyset_S$, it implies that $(S_x \circ S_{supp(\mathfrak{f}_S)} \circ S_y) \tilde{\cap} S_{supp(\mathfrak{f}_S)} \neq \emptyset_S$, for all $x, y \in S$. Consequently, $S_{supp(\mathfrak{f}_S)}$ is an $\mathcal{SI}$-almost I-ideal and by Theorem 3.14, $supp(\mathfrak{f}_S)$ is an almost I-ideal.

The converse of Theorem 3.16 is not true in general, as shown in the following example.

Example 3.17. We know that $ꟻ_S$ is not an $\mathcal{SI}$--almost I-ideal in Example 3.2 and it is obvious that $supp(ꟻ_S)$={ɒ, ɯ}= S. Since,

$$[\{ɒ\}supp(ꟻ_S)\{ɒ\}] \cap supp(ꟻ_S) = [\{ɒ\}supp(ꟻ_S)\{ɯ\}] \cap supp(ꟻ_S) = [\{ɯ\}supp(ꟻ_S)\{ɯ\}] \cap supp(ꟻ_S) = [\{ɯ\}supp(ꟻ_S)\{ɒ\}] \cap supp(ꟻ_S) = \{ɒ, ɯ\} \neq \emptyset_S$$

It is seen that $[\{x\}supp(ꟻ_S)\}\ \{y\}] \cap supp(ꟻ_S) \neq \emptyset_S$, for all $x, y \in S$. That is to say, $supp(ꟻ_S)$ is an almost I-ideal; although $ꟻ_S$ is not an $\mathcal{SI}$-almost I-ideal.

Definition 3.18. An $\mathcal{SI}$-almost I-ideal f_S is called minimal if any $\mathcal{SI}$-almost I-ideal $ꟻ_S$ if whenever $\mathfrak{H}_S \tilde{\subseteq} f_S$, then $supp(ꟻ) = supp(f_S)$.

Theorem 3.19. Let $\emptyset \neq \mathbb{V} \subseteq S$. Then, $\mathbb{V}$ is a minimal almost I-ideal if and only if, is a minimal $\mathcal{SI}$-almost I-ideal.

Proof: Assume that $\mathbb{V}$ is a minimal almost I-ideal. Thus $\mathbb{V}$ is an almost I-ideal and $S_{\mathbb{V}}$ is an $\mathcal{SI}$-almost I-ideal by Theorem 3.14. Let f_S be an $\mathcal{SI}$-almost I-ideal such that $f_S \tilde{\subseteq} S_{\mathbb{V}}$. By Theorem 3.16, $supp(f_S)$ is an almost I-ideal and by Note 2.6, and Corollary 2.11,

$$supp(f_S) \subseteq supp(S_{\mathbb{V}}) = \mathbb{V}$$

Since $\mathbb{V}$ is a minimal almost I-ideal $supp(f_s)$=$supp(S_{\mathbb{V}})$= $\mathbb{V}$. Thus, $S_{\mathbb{V}}$ is a minimal $\mathcal{SI}$-almost interior by Definition 3.18.

Conversely, let $S_{\mathbb{V}}$ be a minimal $\mathcal{SI}$-almost I-ideal. Thus $S_{\mathbb{V}}$ is an $\mathcal{SI}$-almost I-ideal and $\mathbb{V}$ is an almost I-ideal by Theorem 3.14. Let B be an almost I-ideal such that $\mathbb{B} \subseteq \mathbb{V}$. By Theorem 3.14, S_B is an $\mathcal{SI}$-almost I-ideal and by Theorem 2.12 (i), $S_{\mathbb{B}} \tilde{\subseteq} S_{\mathbb{V}}$. Since $S_{\mathbb{V}}$ is a minimal $\mathcal{SI}$-almost I-ideal, by Corollary 2.11

$$B = \mathrm{supp}(S_{\mathbb{B}}) = \mathrm{supp}(S_{\mathbb{V}}) = \mathbb{V}$$

Thus, $\mathbb{V}$ is a minimal almost I-ideal.

Definition 3.20. Let f_s, g_s, and $\mathfrak{h}_s$ be any $\mathcal{SI}$-almost I-ideal. If $\mathfrak{h}_s \circ g_s \tilde{\subseteq} f_s$ implies that $\mathfrak{h}_s \tilde{\subseteq} f_s$ or $g_s \tilde{\subseteq} f_s$, then f_s is called an $\mathcal{SI}$-prime almost I-ideal.

Definition 3.21. Let f_s and $\mathfrak{h}_s$ be any $\mathcal{SI}$-almost I-ideal. If $\mathfrak{h}_s \circ \mathfrak{h}_s \tilde{\subseteq} f_s$implies that $\mathfrak{h}_s \tilde{\subseteq} f_s$, then f_s is called an $\mathcal{SI}$-semiprime almost I-ideal.

Definition 3.22. Let f_s, g_s and $\mathfrak{h}_s$ be any $\mathcal{SI}$-almost I-ideal. $(\mathfrak{h}_s \circ g_s) \tilde{\cap} (g_s \circ \mathfrak{h}_s) \tilde{\subseteq} f_s$ implies that $\mathfrak{h}_s \tilde{\subseteq} f_s$ or $g_s \tilde{\subseteq} f_s$, then f_s is called an $\mathcal{SI}$-strongly prime almost I-ideal.

Theorem 3.23. If $S_{\mathcal{P}}$, is an $\mathcal{SI}$-prime almost I-ideal, then $\mathcal{P}$ is a prime almost I-ideal, where $\emptyset \neq \mathcal{P} \subseteq S$.

Proof: Assume that $S_{\mathcal{P}}$ is an $\mathcal{SI}$-prime almost I-ideal. Thus $S_{\mathcal{P}}$ is an $\mathcal{SI}$-almost I-ideal and thus $\mathcal{P}$ is an almost I-ideal by Theorem 3.14. Let $\mathbb{V}$ and $\mathbb{B}$ be an almost I-ideal such that $\mathbb{V}\mathbb{B} \subseteq \mathcal{P}$. Thus, by Theorem 3.14, $S_{\mathbb{V}}$ and $S_{\mathbb{B}}$ are $\mathcal{SI}$-almost I-ideals, and by Theorem 2.12 (i) and (iii) $S_{\mathbb{V}} \circ S_{\mathbb{B}} = S_{\mathbb{V}\mathbb{B}} \tilde{\subseteq} S_{\mathcal{P}}$. Since $S_{\mathcal{P}}$ is an $\mathcal{SI}$-prime almost I-ideal and $S_{\mathbb{V}} \circ S_{\mathbb{B}} \tilde{\subseteq} S_{\mathcal{P}}$ it follows that $S_{\mathbb{V}} \tilde{\subseteq} S_{\mathcal{P}}$ or $S_{\mathbb{B}} \tilde{\subseteq} S_{\mathcal{P}}$. Thereby, $\mathbb{V} \subseteq \mathcal{P}$ or $\mathbb{B} \subseteq \mathcal{P}$. Consequently, $\mathcal{P}$ is a prime almost I-ideal.

Theorem 3.22. If $S_{\mathcal{P}}$ is an $\mathcal{SI}$-semiprime almost I-ideal then $\mathcal{P}$ is a semiprime almost I-ideal, where $\emptyset \neq \mathcal{P} \subseteq S$.

Proof: Assume that $S_{\mathcal{P}}$ is an $\mathcal{SI}$-semiprime almost I-ideal. Thus $S_{\mathcal{P}}$ is an $\mathcal{SI}$-almost I-ideal and thus $\mathcal{P}$ is an almost interior ideal by Theorem 3.14. Let $\mathbb{V}$ be an almost interior ideal such that $\mathbb{V}\mathbb{V} \subseteq P$. Thus, $S_{\mathbb{V}}$ is an $\mathcal{SI}$-almost I-ideals and $S_{\mathbb{V}} \circ S_{\mathbb{V}} = S_{\mathbb{V}\mathbb{V}} \tilde{\subseteq} S_{\mathcal{P}}$. Since $S_{\mathcal{P}}$ is an $\mathcal{SI}$-semiprime almost I-ideal and $S_{\mathbb{V}} \circ S_{\mathbb{V}} \tilde{\subseteq} S_{\mathcal{P}}$, it follows that $S_{\mathbb{V}} \tilde{\subseteq} S_{\mathcal{P}}$. Thereby, $\mathbb{V} \subseteq \mathcal{P}$. Consequently, $\mathcal{P}$ is a semiprime almost I-ideal.

Theorem 3.23. If $S_{\mathcal{P}}$ is an $\mathcal{SI}$-strongly prime almost I-ideal then $\mathcal{P}$ is a strongly prime almost I-ideal, where $\emptyset \neq \mathcal{P} \subseteq S$.

Proof: Assume that $S_{\mathcal{P}}$ is an $\mathcal{SI}$-strongly prime almost I-ideal. Thus $S_{\mathcal{P}}$ is an $\mathcal{SI}$-almost I-ideal and thus $\mathcal{P}$ is an almost I-ideal. Let $\mathbb{V}$ and $\mathbb{B}$ be an almost I-ideal such that $\mathbb{V}\mathbb{B} \cap \mathbb{B}\mathbb{V} \subseteq \mathcal{P}$. Thus, $S_{\mathbb{V}}$ and $S_{\mathbb{B}}$ are $\mathcal{SI}$-almost I-ideals, and

$$(S_{\mathbb{V}} \circ S_{\mathbb{B}}) \tilde{\cap} (S_{\mathbb{B}} \circ S_{\mathbb{V}}) = S_{\mathbb{V}\mathbb{B}} \tilde{\cap} S_{\mathbb{B}\mathbb{V}} \tilde{\subseteq} \mathcal{P}.$$

Since $S_{\mathcal{P}}$ is an $\mathcal{SI}$-strongly prime almost I-ideal and $(S_{\mathbb{V}} \circ S_{\mathbb{B}}) \tilde{\cap} (S_{\mathbb{B}} \circ S_{\mathbb{V}}) \tilde{\subseteq} S_P$ it follows that $S_{\mathbb{V}} \tilde{\subseteq} S_{\mathcal{P}}$ or $S_{\mathbb{B}} \tilde{\subseteq} S_{\mathcal{P}}$ Thus, $S_{\mathbb{V}}$ and $S_{\mathbb{B}}$ are $\mathcal{SI}$-almost I-ideals, and $\mathbb{V} \subseteq \mathcal{P}$ or $\mathbb{B} \subseteq \mathcal{P}$. Therein, $\mathcal{P}$ is a strongly prime almost I-ideal.

4. Conclusions

Soft-int interior ideal is a generalization of soft-int ideal [33]. In this study, as a further generalization of the nonnull soft-int interior ideal of semigroups, we introduced the concept of soft-int almost interior ideal and its generalization, soft-int weakly almost interior ideals, and studied their basic properties. We illustrate that every soft-int almost interior ideal of S is a soft intersection weakly almost interior ideal of S; nevertheless, the converse does not hold with the counterexample. Also, it was shown that idempotent soft-int almost interior ideal is also soft-int almost subsemigroup. We obtained the relation among soft-int almost interior ideal and almost interior ideal of a semigroup according to seemiprimeness, minimality, and (strongly) primeness. Many kinds of soft-int almost ideals of semigroups, including quasi-ideal, bi-ideal, bi-interior ideal, bi-quasi ideal, and bi-quasi interior ideal, may be studied in future studies. The relationships between these soft-int ideals and their generalized ideals are illustrated by the following Figure 1.

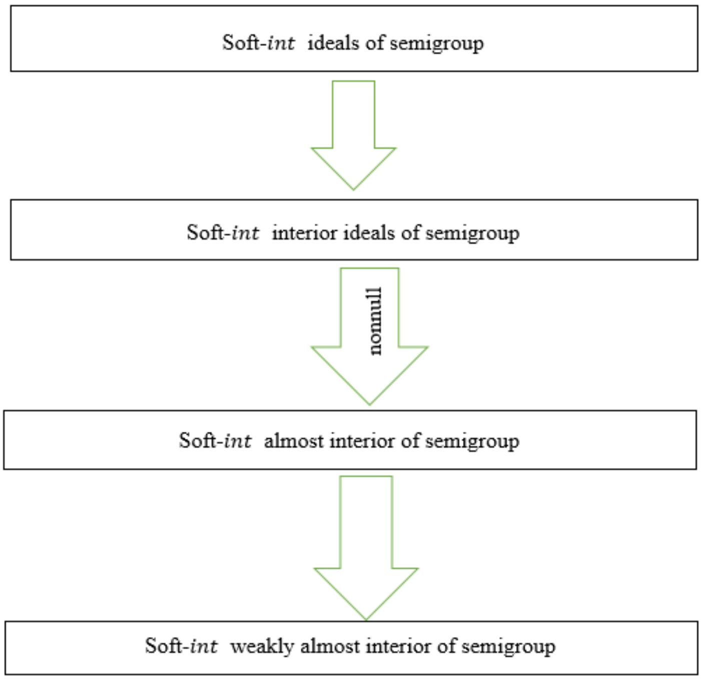

Figure 1. Relations of the certain soft intersection ideals.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

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