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Interval graphs and proper interval graphs in Fuzzy and Neutrosophic Graphs

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Abstract: Interval graphs represent vertices as intervals on the real line, with edges denoting overlapping intervals, while proper interval graphs prevent one interval from being fully contained within another. This paper explores interval and proper interval graphs within the frameworks of fuzzy, neutrosophic, and Turiyam Neutrosophic graphs. We examine how these types of graphs can represent relationships involving uncertainty and imprecision, focusing on their properties and relationships.

Keywords: Neutrosophic Graph; Interval Graphs; Proper Interval Graphs; Fuzzy Graph; Intersection Graphs.

1 Introduction

1.1 *Interval graphs and proper interval graphs*

Graph theory is a fundamental area of mathematics that examines networks made up of nodes (vertices) and connections (edges), essential for analyzing paths, structures, and properties of these networks [28].

A notable example in graph theory is the intersection graph, which represents sets where vertices correspond to these sets, and edges exist between vertices if their corresponding sets intersect [82, 53, 100]. Variants such as intersection digraphs[23, 121], random intersection graphs[56, 122, 88] , Intersection hypergraphs [87, 72, 7], and geometric intersection graphs[31, 58] have also been studied. And several related graph classes have been extensively studied, including interval graphs [57, 46], proper interval graphs [60, 65, 16], Mixed interval hypergraphs[12], interval hypergraphs[94, 22, 85, 54], almost interval graphs [13], weighted interval graphs [9, 17, 118], semi-proper interval graphs [97], mixed interval graphs[61, 63, 62], Unit mixed interval graphs[99], Rigid interval graphs[77], Minimum proper interval graphs[60], circular arc graphs [52, 114, 59], and polygon-circle graphs [73].

In this paper, we focus on interval and proper interval graphs. Interval and proper interval graphs are well-known types of intersection graphs. In interval graphs, each vertex corresponds to an interval on the real line, with edges between vertices whose intervals overlap [57, 46]. Proper interval graphs, a subclass of interval graphs, ensure that no interval is fully contained within another, thereby avoiding nested intervals [60, 65, 16]. A graph is an interval graph if and only if it is chordal and AT-free [74]. Additionally, key graph parameters such as the interval number, pathwidth [70, 27], and boxicity [15] are associated with interval graphs. Interval graphs have various applications, including in food webs [21, 78, 18], scheduling problems [67, 51, 50], and DNA analysis [68, 84, 116, 83].

1.2 *Fuzzy Graphs and Neutrosophic Graphs*

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, representing the degree of uncertainty or imprecision [95, 39, 89]. Essentially, fuzzy graphs are a graphical representation of fuzzy sets [76, 120, 119]. They are widely used in fields like social networks, decision-making, and transportation systems, where relationships are uncertain or not clearly defined [95, 86]. Neutrosophic graphs [1, 33, 44], based on neutrosophic set theory [3, 101, 111], extend classical and fuzzy logic by incorporating three components: truth, indeterminacy, and falsity, providing greater flexibility in handling uncertainty.

Building on these ideas, Turiyam Neutrosophic graphs were introduced as an extension of neutrosophic and fuzzy graphs, where each vertex and edge is assigned four attributes: truth, indeterminacy, falsity, and a liberal state [47, 48]. Plithogenic graphs have also emerged as a more generalized form, and are actively being researched [112, 38, 43, 36, 103, 110].

While significant progress has been made in studying fuzzy and neutrosophic graphs, including their intersection variants (e.g., fuzzy intersection graphs [98, 81, 20, 55] and neutrosophic intersection graphs [11]), there has been limited exploration of interval graphs and proper interval graphs within the context of fuzzy, neutrosophic, and Turiyam Neutrosophic graphs.

1.3 Our Contribution

Based on the above, this paper defines interval graphs and proper interval graphs within the context of fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, and examines their properties as well as the relationships between these graph classes.

2 Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper.

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [28].

Definition 1 (Graph)**.** [28] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Degree)**.** [28] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^{-}(v)$ is the number of edges directed into v, and the *out-degree* $\deg^{+}(v)$ is the number of edges directed out of v.

Definition 3 (Subgraph)**.** [28] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 4 (Connected graph)**.** (cf.[115, 64]) A graph $G = (V, E)$ is said to be a **connected graph** if for any two distinct vertices $u, v \in V$, there exists a path in G that connects u and v. In other words, every pair of vertices in the graph is reachable from each other, meaning there is a sequence of edges that allows traversal between any two vertices.

Mathematically, for all $u, v \in V$, there exists a sequence of vertices $v_1 = u, v_2, \dots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $1 \le i < k$.

Definition 5 (Induced subgraph)**.** [75, 66] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Definition 6 (Complete Graph)**.** (cf.[29, 8]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

2.2 Intersection graph and Interval graphs

In this paper, we focus on Interval graphs, which are known as intersection graphs. Intersection graphs have been extensively studied[82, 53, 100]. The definition is provided below[82, 53, 100].

Definition 7 (Intersection graph)**.** [82, 53, 100] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \dots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

Next, we will consider interval graphs and proper interval graphs. The definitions are provided below[57, 46].

Definition 8. [57, 46] An *interval graph* is an undirected graph $G = (V, E)$ that can be represented by a family of intervals on the real line. For each vertex $v \in V$, there exists a corresponding interval I_v on the real line. Two vertices $u, v \in V$ are adjacent, i.e., $(u, v) \in E$, if and only if their corresponding intervals I_u and I_v overlap. Formally, the edge set E of the graph G is defined as:

$$E(G) = \{(u, v) \mid I_u \cap I_v \neq \emptyset\}.$$

Definition 9. [60, 65, 16] A *proper interval graph* is a special case of an interval graph where no interval is strictly contained within another. That is, for any two intervals I_u and I_v corresponding to vertices u and v, neither $I_u \subset I_v$ nor $I_v \subset I_u$. This restriction ensures that no interval is nested within another. A graph is a proper interval graph if and only if it has a proper interval representation.

The above graphs have been the subject of numerous published papers and studies [80, 96, 61]. An interval graph is both chordal (i.e., every induced cycle has length 3 [32]) and AT-free (i.e., it contains no asteroidal triple, a set of three vertices such that any two are connected by a path that avoids the neighborhood of the third vertex[19]). Below are examples of interval graphs and proper interval graphs.

Example 10 (Interval Graph)**.** Consider a set of intervals on the real line:

$$I_1 = [1, 5], \quad I_2 = [4, 8], \quad I_3 = [6, 9], \quad I_4 = [2, 3]$$

The vertices $V = \{v_1, v_2, v_3, v_4\}$ correspond to the intervals I_1, I_2, I_3, I_4, and the edges are drawn between vertices whose intervals overlap:

- $I_1 \cap I_2 \neq \emptyset$, so $(v_1, v_2) \in E$,
- $I_2 \cap I_3 \neq \emptyset$, so $(v_2, v_3) \in E$,
- $I_1 \cap I_4 \neq \emptyset$, so $(v_1, v_4) \in E$,

- $I_1 \cap I_3 = \emptyset$, so $(v_1, v_3) \notin E$,
- $I_3 \cap I_4 = \emptyset$, so $(v_3, v_4) \notin E$,
- $I_2 \cap I_4 = \emptyset$, so $(v_2, v_4) \notin E$.

The resulting interval graph $G = (V, E)$ is:

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3)\}$$

Example 11 (Proper Interval Graph)**.** Consider a set of intervals that do not nest:

$$I_1 = [1, 3], \quad I_2 = [4, 6], \quad I_3 = [7, 9], \quad I_4 = [2, 4]$$

None of these intervals are strictly contained within another. The vertices $V = \{v_1, v_2, v_3, v_4\}$ correspond to the intervals, and the edges are drawn between overlapping intervals:

- $I_1 \cap I_4 \neq \emptyset$, so $(v_1, v_4) \in E$,
- $I_2 \cap I_4 \neq \emptyset$, so $(v_2, v_4) \in E$,
- $I_3 \cap I_2 = \emptyset$, so $(v_2, v_3) \notin E$,
- No other intervals overlap.

The proper interval graph G is:

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{(v_1, v_4), (v_2, v_4)\}$$

Since no interval is strictly contained within another, this forms a proper interval graph.

Let p and q be integers. The generalized concepts of p-Proper Interval Graphs [91, 24] and q-Improper Interval Graphs[10] are well known. Like standard interval graphs, they have been the subject of various studies. The definitions of these graphs are provided below.

Definition 12 (p-Proper Interval Graph)**.** [91] A graph $G = (V, E)$ is called a *p-proper interval graph* if there exists an interval representation of G such that no interval in the representation is properly contained within more than p other intervals. Formally, let $\mathcal{I} = \{I_v \mid v \in V\}$ be a set of intervals corresponding to the vertices of G. The graph G is p-proper if for every interval $I_u \in \mathcal{I}$, the number of other intervals $I_v \in \mathcal{I}$ such that $I_v \subset I_u$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid I_v \subset I_u\}| \leq p.$$

Definition 13 (q-Improper Interval Graph)**.** [10] A graph $G = (V, E)$ is called a *q-improper interval graph* if there exists an interval representation of G such that no interval in the representation properly contains more than q other intervals. Formally, let $\mathcal{I} = \{I_v \mid v \in V\}$ be a set of intervals corresponding to the vertices of G. The graph G is q-improper if for every interval $I_u \in \mathcal{I}$, the number of other intervals $I_v \in \mathcal{I}$ such that $I_u \subset I_v$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid I_u \subset I_v\}| \leq q.$$

p-Proper Interval Graph generalizes the concept of proper interval graphs, where a 0-proper interval graph is a proper interval graph (i.e., no interval is properly contained within any other interval). q-Improper Interval Graph extends the notion of proper interval graphs, where a 0-improper interval graph is a proper interval graph (i.e., no interval properly contains any other interval).

2.3 Interval graph in Fuzzy Graphs

Now, we explore interval graphs within the context of fuzzy graphs. Fuzzy graphs extend classical graph theory by incorporating the principles of fuzzy sets [119, 14, 117, 25]. Extensive research has been conducted on fuzzy graphs [95]. The definition of a fuzzy graph is given below.

Definition 14. [95] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ is defined as follows:

- V is a set of vertices.

- $\sigma : V \to [0, 1]$ is a function that assigns a membership degree to each vertex $v \in V$, indicating the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \le \min\{\sigma(u), \sigma(v)\}$.

In this definition, the following properties hold:

- The fuzzy function μ is symmetric, meaning $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$.
- Additionally, $\mu(v, v) = 0$ for all $v \in V$, meaning that there is no self-loop in the fuzzy graph.

The fuzzy graph ψ allows for the representation of uncertainty in the presence or strength of connections between vertices, making it a valuable tool for modeling complex systems with ambiguous or imprecise relationships.

Next, we define the fuzzy interval graph, which combines the concepts of a fuzzy graph and an interval graph, as follows.

Definition 15 (Fuzzy Interval Graph)**.** Let V be a finite set of vertices, and let

$$\mathcal{F} = \{\mu_1, \mu_2, \dots, \mu_n\}$$

be a finite family of fuzzy intervals on the real line $\mathbb{R}$. Each fuzzy interval $\mu_i : \mathbb{R} \to [0, 1]$ is a normal, convex fuzzy subset of $\mathbb{R}$, meaning there exists a point $x_i \in \mathbb{R}$ such that $\mu_i(x_i) = 1$, and for any $y, z \in \mathbb{R}$ with $y \le z$, the following holds:

$$\mu_i(w) \ge \min\{\mu_i(y), \mu_i(z)\} \quad \text{for all } w \text{ between } y \text{ and } z.$$

The *fuzzy interval graph* $G = (V, \mu_V, \rho)$ is defined as follows:

- The vertex membership function $\mu_V : V \to [0, 1]$ is given by:

  $$\mu_V(v_i) = h(\mu_i) = \sup_{x \in \mathbb{R}} \mu_i(x) = 1,$$

  where $h(\mu_i)$ denotes the height of the fuzzy interval μ_i.
- The fuzzy adjacency relation $\rho : V \times V \to [0, 1]$ is defined by:

  $$\rho(v_i, v_j) = \begin{cases} h(\mu_i \cap \mu_j) = \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \ne j, \\ 0, & \text{if } i = j. \end{cases}$$

In this definition, the edge membership function $\rho(v_i, v_j)$ measures the degree of overlap between the fuzzy intervals μ_i and μ_j. The fuzzy interval graph captures the intersection properties of fuzzy intervals, extending the concept of interval graphs to the fuzzy context.

Definition 16 (Fuzzy Proper Interval Graph)**.** A *fuzzy proper interval graph* is a fuzzy interval graph $G = (V, \mu_V, \rho)$ where the family of fuzzy intervals $\mathcal{F} = \{\mu_1, \mu_2, \dots, \mu_n\}$ satisfies the additional condition that no fuzzy interval is strictly contained within another.

Formally, for any two distinct fuzzy intervals μ_i and μ_j in $\mathcal{F}$, neither of the following holds:

- $\mu_i(x) \le \mu_j(x)$ for all $x \in \mathbb{R}$ and there exists $x_0 \in \mathbb{R}$ such that $\mu_i(x_0) < \mu_j(x_0)$.
- $\mu_j(x) \le \mu_i(x)$ for all $x \in \mathbb{R}$ and there exists $x_0 \in \mathbb{R}$ such that $\mu_j(x_0) < \mu_i(x_0)$.

This restriction ensures that no fuzzy interval is nested within another. The vertex membership function μ_V and the fuzzy adjacency relation ρ are defined similarly to those in the fuzzy interval graph, but the non-nesting condition on fuzzy intervals ensures that no interval is entirely contained within another.

2.4 interval graph in Intuitionistic fuzzy Graphs

Next, we consider Interval graphs in Intuitionistic Fuzzy Graphs. Intuitionistic fuzzy graphs are an extended version of fuzzy graphs and have been the subject of extensive study for over 15 years [92, 71, 90, 49]. Intuitionistic fuzzy graphs are related to the concept of intuitionistic fuzzy sets [113, 30, 6, 5]. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy Interval graphs are provided below.

Definition 17 (Intuitionistic Fuzzy Graph (IFG))**.** [90] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

(1) (μ_A, v_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $v_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + v_A(x) \leq 1$$

The value $1 - \mu_A(x) - v_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

(2) (μ_B, v_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $v_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + v_B(x, y) \leq 1$$

Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$v_B(x, y) \leq v_A(x) \vee v_A(y)$$

In this definition:

- $\mu_A(x)$ and $v_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $v_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $v_A(x) = 0$ and $v_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

We define the Intuitionistic Fuzzy Interval Graph as follows. he Intuitionistic Fuzzy Interval Graph is a concept that combines the ideas of an Intuitionistic Fuzzy Graph and Interval graphs.

Definition 18 (Intuitionistic Fuzzy Interval Graph)**.** Let V be a finite set of vertices, and let

$$\mathcal{I} = \{(\mu_1, v_1), (\mu_2, v_2), \dots, (\mu_n, v_n)\}$$

be a finite family of intuitionistic fuzzy intervals on the real line $\mathbb{R}$. Each pair (μ_i, v_i) consists of a membership function $\mu_i : \mathbb{R} \to [0, 1]$ and a non-membership function $v_i : \mathbb{R} \to [0, 1]$, such that:

$$\mu_i(x) + v_i(x) \leq 1 \quad \text{for all } x \in \mathbb{R},$$

where μ_i is convex and normal, meaning there exists a point $x_i \in \mathbb{R}$ such that $\mu_i(x_i) = 1$, and for any $y, z \in \mathbb{R}$ with $y \leq z$, the following holds:

$$\mu_i(w) \geq \min\{\mu_i(y), \mu_i(z)\} \quad \text{for all } w \text{ between } y \text{ and } z.$$

The *Intuitionistic Fuzzy Interval Graph* $G = (V, \mu_V, v_V, \rho_\mu, \rho_v)$ is defined as follows:

- The vertex membership function $\mu_V : V \to [0, 1]$ and non-membership function $v_V : V \to [0, 1]$ are given by:

$$\mu_V(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x), \quad v_V(v_i) = \inf_{x \in \mathbb{R}} v_i(x).$$

- The fuzzy adjacency relations $\rho_\mu : V \times V \to [0,1]$ and $\rho_v : V \times V \to [0,1]$ are defined by:

$$\rho_\mu(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_v(v_i, v_j) = \begin{cases} \inf_{x\in\mathbb{R}} \max\{v_i(x), v_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

This definition extends the classical interval graph by incorporating intuitionistic fuzzy intervals, allowing for both membership and non-membership degrees for vertices and edges.

Definition 19 (Intuitionistic Fuzzy Proper Interval Graph)**.** An *Intuitionistic Fuzzy Proper Interval Graph* is an Intuitionistic Fuzzy Interval Graph $G = (V, \mu_V, v_V, \rho_\mu, \rho_v)$ where the family of intuitionistic fuzzy intervals $\mathcal{I} = \{(\mu_1, v_1), (\mu_2, v_2), \ldots, (\mu_n, v_n)\}$ satisfies the additional condition that no intuitionistic fuzzy interval is strictly contained within another.

Formally, for any two distinct intuitionistic fuzzy intervals (μ_i, v_i) and (μ_j, v_j) in $\mathcal{I}$, neither of the following holds:

- $\mu_i(x) \leq \mu_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $\mu_i(x_0) < \mu_j(x_0)$,
- $v_i(x) \geq v_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $v_i(x_0) > v_j(x_0)$,

ensuring that no interval is nested within another in terms of both membership and non-membership degrees. The vertex membership function μ_V, non-membership function v_V, and fuzzy adjacency relations ρ_μ and ρ_v are defined similarly to those in the Intuitionistic Fuzzy Interval Graph.

2.5 interval graph in Neutrosophic Graphs

First, the definition of a neutrosophic graph is provided. As mentioned in the introduction, neutrosophic graphs are an extension of fuzzy graphs and Intuitionistic Fuzzy Graphs. A Neutrosophic Graph assigns truth, indeterminacy, and falsity membership degrees to each vertex and edge, representing uncertainty. Similar to fuzzy graphs, neutrosophic graphs have been the subject of extensive research [69, 2, 108, 106]. Neutrosophic graphs are related to the concept of Neutrosophic sets [4, 26, 109, 79, 105]. The definition is provided below[108].

Definition 20. [108] A neutrosophic graph $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a graph where:

- $\sigma : V \to [0,1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.
- $\mu : E \to [0,1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.
- For every edge $e = v_i v_j \in E$, the following condition holds:

$$\mu_T(e) \leq \min(\sigma_T(v_i), \sigma_T(v_j)).$$

(1) σ is called the *neutrosophic vertex set*.

(2) μ is called the *neutrosophic edge set*.

(3) The number of vertices $|V|$ is the *order* of G, denoted by $O(G)$.

(4) The sum of the truth values over all vertices, $\sum_{v\in V} \sigma_T(v)$, is the *neutrosophic order* of G, denoted by $On(G)$.

(5) The number of edges $|E|$ is the *size* of G, denoted by $S(G)$.

(6) The sum of the truth values over all edges, $\sum_{e\in E} \mu_T(e)$, is the *neutrosophic size* of G, denoted by $Sn(G)$.

We define the Neutrosophic Interval Graph as follows. The Neutrosophic Interval Graphis a concept that combines the ideas of an Neutrosophic Graph and a Interval Graph.

Definition 21 (Neutrosophic Interval Graph)**.** Let V be a finite set of vertices, and let

$$\mathcal{N} = \{(\mu_1, \tau_1, \zeta_1), (\mu_2, \tau_2, \zeta_2), \dots, (\mu_n, \tau_n, \zeta_n)\}$$

be a finite family of neutrosophic intervals on the real line $\mathbb{R}$. Each triple (μ_i, τ_i, ζ_i) represents the truth-membership function $\mu_i : \mathbb{R} \to [0, 1]$, the indeterminacy-membership function $\tau_i : \mathbb{R} \to [0, 1]$, and the falsity-membership function $\zeta_i : \mathbb{R} \to [0, 1]$ such that:

$$\mu_i(x) + \tau_i(x) + \zeta_i(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

The *Neutrosophic Interval Graph*

$$G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$$

is defined as follows:

- The vertex membership functions $\mu_V : V \to [0, 1]$, $\tau_V : V \to [0, 1]$, and $\zeta_V : V \to [0, 1]$ are given by:

$$\mu_V(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x), \quad \tau_V(v_i) = \sup_{x \in \mathbb{R}} \tau_i(x), \quad \zeta_V(v_i) = \sup_{x \in \mathbb{R}} \zeta_i(x).$$

- The neutrosophic adjacency relations $\rho_\mu : V \times V \to [0, 1]$, $\rho_\tau : V \times V \to [0, 1]$, and $\rho_\zeta : V \times V \to [0, 1]$ are defined by:

$$\rho_\mu(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\tau(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{\tau_i(x), \tau_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\zeta(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{\zeta_i(x), \zeta_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

This definition extends classical interval graphs to the neutrosophic framework, accounting for truth, indeterminacy, and falsity degrees for both vertices and edges.

Definition 22 (Neutrosophic Proper Interval Graph)**.** A *Neutrosophic Proper Interval Graph* is a Neutrosophic Interval Graph

$$G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$$

where the family of neutrosophic intervals

$$\mathcal{N} = \{(\mu_1, \tau_1, \zeta_1), (\mu_2, \tau_2, \zeta_2), \dots, (\mu_n, \tau_n, \zeta_n)\}$$

satisfies the condition that no neutrosophic interval is strictly contained within another.

Formally, for any two distinct neutrosophic intervals (μ_i, τ_i, ζ_i) and (μ_j, τ_j, ζ_j) in $\mathcal{N}$, neither of the following holds:

- $\mu_i(x) \leq \mu_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $\mu_i(x_0) < \mu_j(x_0)$,
- $\tau_i(x) \leq \tau_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $\tau_i(x_0) < \tau_j(x_0)$,
- $\zeta_i(x) \leq \zeta_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $\zeta_i(x_0) < \zeta_j(x_0)$,

ensuring that no neutrosophic interval is nested within another in terms of truth, indeterminacy, and falsity degrees. The vertex and edge membership functions μ_V, τ_V, ζ_V and adjacency relations ρ_μ, ρ_τ, ρ_ζ are defined similarly to those in the Neutrosophic Interval Graph.

2.6 Interval Graph in Turiyam Neutrosophic Graph

This section explores interval graphs and proper interval graphs within the framework of Turiyam Neutrosophic Graphs. A Turiyam Neutrosophic Graph extends classical graph theory by assigning four distinct parameters—truth, indeterminacy, falsity, and liberal state—to each vertex and edge.

Recent research on Turiyam Neutrosophic Graphs, which build upon and extend Neutrosophic Graphs by introducing additional parameters, has gained significant attention [47, 45, 40, 34]. It is also known that Turiyam Neutrosophic Graphs can be generalized to Quadripartitioned Neutrosophic Graphs and related frameworks (cf. [107]).

The formal definition is presented below.

Definition 23 (Turiyam Neutrosophic Graph)**.** [47, 40, 34] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \dots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \dots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

(1) *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$

where:

- $t(v_i)$ is the truth value (tv) of the vertex v_i,
- $iv(v_i)$ is the indeterminacy value (iv) of v_i,
- $fv(v_i)$ is the falsity value (fv) of v_i,
- $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

for all $v_i \in V$, such that the following condition holds for each vertex:

$$0 \le t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \le 4.$$

(2) *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \le t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \le 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

We define the Turiyam Neutrosophic Interval Graph as follows. The Turiyam Neutrosophic Interval Graph is a concept that combines the ideas of an Turiyam Neutrosophic Graph and a Interval Graph.

Definition 24 (Turiyam Neutrosophic Interval Graph)**.** Let V be a finite set of vertices, and let

$$\mathcal{T} = \{(\mu_1, iv_1, fv_1, lv_1), (\mu_2, iv_2, fv_2, lv_2), \dots, (\mu_n, iv_n, fv_n, lv_n)\}$$

be a family of Turiyam Neutrosophic intervals on the real line $\mathbb{R}$. Each quadruple

$$(\mu_i, iv_i, fv_i, lv_i)$$

consists of four membership functions representing truth $\mu_i : \mathbb{R} \to [0,1]$, indeterminacy $iv_i : \mathbb{R} \to [0,1]$, falsity $fv_i : \mathbb{R} \to [0,1]$, and liberal $lv_i : \mathbb{R} \to [0,1]$. These membership functions satisfy:

$$0 \leq \mu_i(x) + iv_i(x) + fv_i(x) + lv_i(x) \leq 4 \quad \text{for all } x \in \mathbb{R}.$$

The *Turiyam Neutrosophic Interval Graph* $G^T = (V, \mu_V, iv_V, fv_V, lv_V, \rho_\mu, \rho_i v, \rho_f v, \rho_l v)$ is defined as follows:

- The vertex membership functions $\mu_V, iv_V, fv_V, lv_V : V \to [0,1]$ are defined as:

$$\mu_V(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x), \quad iv_V(v_i) = \sup_{x \in \mathbb{R}} iv_i(x),$$

$$fv_V(v_i) = \sup_{x \in \mathbb{R}} fv_i(x), \quad lv_V(v_i) = \sup_{x \in \mathbb{R}} lv_i(x).$$

- The Turiyam Neutrosophic adjacency relations $\rho_\mu, \rho_i v, \rho_f v, \rho_l v : V \times V \to [0,1]$ are defined as:

$$\rho_\mu(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_i v(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{iv_i(x), iv_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_f v(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{fv_i(x), fv_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_l v(v_i, v_j) = \begin{cases} \sup_{x \in \mathbb{R}} \min\{lv_i(x), lv_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

This definition extends the classical interval graph to the Turiyam Neutrosophic framework, allowing for truth, indeterminacy, falsity, and liberal state values for both vertices and edges.

Definition 25 (Turiyam Neutrosophic Proper Interval Graph)**.** A *Turiyam Neutrosophic Proper Interval Graph* is a Turiyam Neutrosophic Interval Graph

$$G^T = (V, \mu_V, iv_V, fv_V, lv_V, \rho_\mu, \rho_i v, \rho_f v, \rho_l v)$$

where the family of Turiyam Neutrosophic intervals

$$\mathcal{T} = \{(\mu_1, iv_1, fv_1, lv_1), (\mu_2, iv_2, fv_2, lv_2), \ldots, (\mu_n, iv_n, fv_n, lv_n)\}$$

satisfies the additional condition that no interval is strictly contained within another.

Formally, for any two distinct Turiyam Neutrosophic intervals $(\mu_i, iv_i, fv_i, lv_i)$ and $(\mu_j, iv_j, fv_j, lv_j)$ in $\mathcal{T}$, neither of the following holds:

- $\mu_i(x) \leq \mu_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $\mu_i(x_0) < \mu_j(x_0)$,
- $iv_i(x) \leq iv_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $iv_i(x_0) < iv_j(x_0)$,
- $fv_i(x) \leq fv_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $fv_i(x_0) < fv_j(x_0)$,
- $lv_i(x) \leq lv_j(x)$ for all $x \in \mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $lv_i(x_0) < lv_j(x_0)$,

ensuring that no Turiyam Neutrosophic interval is nested within another in terms of truth, indeterminacy, falsity, and liberal state values. The vertex membership functions μ_V, iv_V, fv_V, lv_V and the adjacency relations $\rho_\mu, \rho_i v, \rho_f v, \rho_l v$ are defined similarly to those in the Turiyam Neutrosophic Interval Graph.

3 Result in this paper

In this section, we present the results of this paper.

3.1 Property of neutrosophic interval raphs

We consider about neutrosophic interval graph. These properties also hold similarly for fuzzy graphs, intuitionistic fuzzy graphs and Turiyam Neutrosophic graphs.

Theorem 26. *a Neutrosophic Interval Graph can be transformed into a classic Interval Graph.*

Proof: To transform a Neutrosophic Interval Graph into a Classic Interval Graph, we can set all membership functions related to truth, indeterminacy, and falsity to specific values. Specifically, by setting the truth membership μ_V to 1 and both indeterminacy τ_V and falsity ζ_V to 0 for all vertices and edges, we eliminate the need for representing uncertainty or partial truth in the graph.

1. Set the membership values for vertices:

$$\mu_V(v) = 1, \quad \tau_V(v) = 0, \quad \zeta_V(v) = 0 \quad \forall v \in V.$$

This implies that every vertex fully belongs to the graph, with no indeterminacy or falsity.

2. Set the membership values for edges:

$$\rho_\mu(v_i, v_j) = 1, \quad \rho_\tau(v_i, v_j) = 0, \quad \rho_\zeta(v_i, v_j) = 0 \quad \forall (v_i, v_j) \in E.$$

This implies that if two intervals overlap, the edge between the corresponding vertices is fully included in the graph, with no indeterminacy or falsity.

After this transformation, all vertices and edges in the Neutrosophic Interval Graph behave exactly as in a classic Interval Graph. Specifically:

- The adjacency of vertices depends solely on the overlap of their intervals.
- There is no longer any notion of partial membership, uncertainty, or falsity affecting the structure of the graph.

Thus, the transformed graph is equivalent to a *classic Interval Graph*.

By setting the *truth membership* $\mu_V(v) = 1$ for all vertices and edges and setting the *indeterminacy* and *falsity memberships* $\tau_V(v) = 0$ and $\zeta_V(v) = 0$, the Neutrosophic Interval Graph is transformed into a classic Interval Graph. The adjacency structure based on interval overlap remains unchanged, and the resulting graph adheres to the traditional definition of an Interval Graph in graph theory. This transformation ensures that all uncertainty and fuzziness are eliminated, leaving a purely classical graph structure.

Therefore, the *Neutrosophic Interval Graph* can indeed be transformed into a *classic Interval Graph* by setting all membership values to specific constants, completing the proof. □

Theorem 27. *A Neutrosophic Proper Interval Graph can be transformed into a Classic Proper Interval Graph by assigning specific values to the truth, indeterminacy, and falsity membership functions.*

Proof: To transform a Neutrosophic Proper Interval Graph into a Classic Proper Interval Graph, we assign the following values to the truth, indeterminacy, and falsity membership functions:

- Set the truth membership $\mu_V(v) = 1$ for all vertices $v \in V$, and set the indeterminacy and falsity memberships to zero:

  $$\mu_V(v) = 1, \quad \tau_V(v) = 0, \quad \zeta_V(v) = 0, \quad \forall v \in V.$$

  This implies that each vertex is fully present in the graph without uncertainty or falsity.

- Similarly, set the truth membership $\rho_\mu(v_i, v_j) = 1$ for all edges $(v_i, v_j) \in E$, and set the indeterminacy and falsity memberships for edges to zero:

  $$\rho_\mu(v_i, v_j) = 1, \quad \rho_\tau(v_i, v_j) = 0, \quad \rho_\zeta(v_i, v_j) = 0, \quad \forall (v_i, v_j) \in E.$$

  This implies that all edges in the graph represent complete adjacency between vertices without uncertainty or falsity.

In both Neutrosophic and Classic Proper Interval Graphs, the property of ”properness” is essential. This property ensures that no interval is strictly contained within another. In the Neutrosophic context, this condition is enforced by the membership functions μ_V, τ_V, and ζ_V. By setting $\mu_V(v) = 1$ and $\tau_V(v) = \zeta_V(v) = 0$, we preserve this condition, as no vertex or edge will exhibit uncertainty or falsity, and the intervals remain distinct and non-nested.

By setting the truth membership $\mu_V = 1$ and $\rho_\mu = 1$ for all vertices and edges, and setting the indeterminacy and falsity memberships $\tau_V = 0$ and $\zeta_V = 0$ for all vertices and edges, the Neutrosophic Proper Interval Graph is transformed into a Classic Proper Interval Graph. The adjacency structure remains based solely on the overlap of intervals, and the non-nesting condition is preserved. Thus, the transformation is complete. □

Corollary 28. *A Fuzzy Interval Graph, Intuitionistic Fuzzy Interval Graph, or Turiyam Neutrosophic Interval Graph can be transformed into a classic Interval Graph.*

Proof: It can be proven in the same way as above. □

Corollary 29. *A Fuzzy proper Interval Graph, Intuitionistic Fuzzy proper Interval Graph, or Turiyam Neutrosophic proper Interval Graph can be transformed into a classic proper Interval Graph.*

Proof: It can be proven in the same way as above. □

Theorem 30. *Neutrosophic Graph can be represented as a Neutrosophic Interval Graph.*

Proof: We consider about Mapping vertices to intervals. Each vertex $v \in V$ in the Neutrosophic Graph is mapped to a neutrosophic interval on the real line, where:

$$\mu_V(v) = \sigma_T(v), \quad \tau_V(v) = \sigma_I(v), \quad \zeta_V(v) = \sigma_F(v).$$

This mapping ensures that the neutrosophic interval for each vertex reflects the truth, indeterminacy, and falsity degrees as defined in the Neutrosophic Graph.

We consider about mapping edges to intervals. For each edge $(v_i, v_j) \in E$, we define the neutrosophic adjacency relations ρ_μ, ρ_τ, and ρ_ζ in the Neutrosophic Interval Graph. These relations are determined based on the overlap of the neutrosophic intervals corresponding to v_i and v_j:

$$\rho_\mu(v_i, v_j) = \min(\mu_V(v_i), \mu_V(v_j)) = \min(\sigma_T(v_i), \sigma_T(v_j)),$$
$$\rho_\tau(v_i, v_j) = \min(\tau_V(v_i), \tau_V(v_j)) = \min(\sigma_I(v_i), \sigma_I(v_j)),$$
$$\rho_\zeta(v_i, v_j) = \min(\zeta_V(v_i), \zeta_V(v_j)) = \min(\sigma_F(v_i), \sigma_F(v_j)).$$

These relations mirror the conditions for the neutrosophic edge membership function μ in the original Neutrosophic Graph.

Since we have mapped both the vertices and edges of the Neutrosophic Graph to the corresponding neutrosophic intervals and relations in the Neutrosophic Interval Graph, we conclude that any Neutrosophic Graph can be represented as a Neutrosophic Interval Graph by this transformation. □

Corollary 31. *A Fuzzy Graph, Intuitionistic Fuzzy Graph, or Turiyam Neutrosophic Graph can be represented as a Fuzzy Interval Graph, Intuitionistic Fuzzy Interval Graph, or Turiyam Neutrosophic Interval Graph, respectively.*

Proof: It can be proven in the same way as above. □

Theorem 32. *Neutrosophic Proper Interval Graph is special case of Neutrosophic Interval Graph.*

Proof: Obviously holds. □

Theorem 33. *A Neutrosophic Interval Graph can be transformed into a Fuzzy Interval Graph, Intuitionistic Fuzzy Interval Graph, or Turiyam Neutrosophic Interval Graph.*

Proof: An interval graph is an undirected graph $G = (V, E)$ where each vertex corresponds to an interval on the real line, and two vertices are adjacent if and only if their intervals overlap. Formally, the edge set E of an interval graph is defined as:

$$E(G) = \{(u, v) \mid I_u \cap I_v \neq \emptyset\},$$

where I_u and I_v represent intervals on the real line associated with vertices u and v, respectively.

A *Neutrosophic Interval Graph* $G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$ extends the traditional interval graph by associating *truth* (μ), *indeterminacy* (τ), and *falsity* (ζ) membership degrees with each vertex and edge. Two vertices u and v are adjacent if and only if their intervals overlap:

$$\text{Adjacency Condition: } (u, v) \in E \quad \text{if and only if } I_u \cap I_v \neq \emptyset.$$

Additionally, the neutrosophic membership functions satisfy:

$$\mu_V(v) + \tau_V(v) + \zeta_V(v) = 1, \quad \forall v \in V,$$

and similarly for edges.

We need to show how a Neutrosophic Interval Graph can be transformed into a *Fuzzy Interval Graph*, *Intuitionistic Fuzzy Interval Graph*, or *Turiyam Neutrosophic Interval Graph*, while preserving the core interval graph structure (i.e., adjacency based on overlapping intervals).

Next, we consider about Transformation into Fuzzy Interval Graph. A *Fuzzy Interval Graph* is a graph where each vertex and edge has a single membership degree, and two vertices are adjacent if their intervals overlap:

$$E(G^{fuzzy}) = \{(u, v) \mid I_u \cap I_v \neq \emptyset\}.$$

To transform a Neutrosophic Interval Graph into a Fuzzy Interval Graph, we focus on the *truth-membership function* $\mu_V(v)$ for each vertex and $\mu_\rho(v_i, v_j)$ for each edge, ignoring the indeterminacy and falsity components.

Define the transformed fuzzy membership functions as:

$$\mu_V^{fuzzy}(v_i) = \mu_V(v_i), \quad \mu_\rho^{fuzzy}(v_i, v_j) = \mu_\rho(v_i, v_j).$$

The resulting fuzzy interval graph $G^{fuzzy} = (V, \mu_V^{fuzzy}, \mu_\rho^{fuzzy})$ retains the interval graph structure, where adjacency is determined by interval overlap, and the edge membership function $\mu_\rho^{fuzzy}(v_i, v_j)$ quantifies the fuzzy strength of the connection.

Thus, this transformation ensures that G^{fuzzy} is a valid *Fuzzy Interval Graph*.

Next, we consider about Transformation into Intuitionistic Fuzzy Interval Graph. An *Intuitionistic Fuzzy Interval Graph* uses both membership and non-membership degrees for vertices and edges, subject to the condition:

$$\mu_V(v_i) + v_V(v_i) \leq 1, \quad \forall v_i \in V.$$

In the Neutrosophic Interval Graph, the truth-membership $\mu_V(v)$ and falsity-membership $\zeta_V(v)$ can be mapped to the intuitionistic fuzzy membership and non-membership degrees, respectively.

Define the transformed intuitionistic fuzzy membership and non-membership functions as:

$$\mu_V^{intuitionistic}(v_i) = \mu_V(v_i), \quad v_V^{intuitionistic}(v_i) = \zeta_V(v_i),$$

$$\mu_\rho^{intuitionistic}(v_i, v_j) = \mu_\rho(v_i, v_j), \quad v_\rho^{intuitionistic}(v_i, v_j) = \zeta_\rho(v_i, v_j).$$

Thus, the resulting Intuitionistic Fuzzy Interval Graph

$$G^{intuitionistic} = (V, \mu_V^{intuitionistic}, v_V^{intuitionistic}, \mu_\rho^{intuitionistic}, v_\rho^{intuitionistic})$$

maintains the interval-based adjacency condition.

Thus, this transformation results in a valid *Intuitionistic Fuzzy Interval Graph*.

Next, we consider about Transformation into Turiyam Neutrosophic Interval Graph. A *Turiyam Neutrosophic Interval Graph* includes four membership functions: truth μ, indeterminacy iv, falsity fv, and liberal lv, satisfying:

$$0 \leq \mu(v) + iv(v) + fv(v) + lv(v) \leq 4, \quad \forall v \in V.$$

In a Neutrosophic Interval Graph, the truth-membership $\mu_V(v)$, indeterminacy $\tau_V(v)$, and falsity $\zeta_V(v)$ can be mapped to the corresponding Turiyam Neutrosophic membership degrees. Define the liberal state lv to be 0 or context-specific.

Define the transformed membership functions as:

$$\mu_V^{turiyam}(v_i) = \mu_V(v_i), \quad iv_V^{turiyam}(v_i) = \tau_V(v_i)$$

$$fv_V^{turiyam}(v_i) = \zeta_V(v_i), \quad lv_V^{turiyam}(v_i) = 0.$$

The edge functions are similarly defined:

$$\mu_\rho^{turiyam}(v_i, v_j) = \mu_\rho(v_i, v_j), \quad iv_\rho^{turiyam}(v_i, v_j) = \tau_\rho(v_i, v_j)$$

$$fv_\rho^{turiyam}(v_i, v_j) = \zeta_\rho(v_i, v_j), \quad lv_\rho^{turiyam}(v_i, v_j) = 0.$$

The resulting *Turiyam Neutrosophic Interval Graph*

$$G^{turiyam} = (V, \mu_V^{turiyam}, iv_V^{turiyam}, fv_V^{turiyam}, lv_V^{turiyam}, \mu_\rho^{turiyam}, iv_\rho^{turiyam}, fv_\rho^{turiyam}, lv_\rho^{turiyam})$$

preserves the interval adjacency condition.

Thus, this transformation results in a valid *Turiyam Neutrosophic Interval Graph.* □

Corollary 34. *A Neutrosophic Interval Graph can be transformed into a Fuzzy Interval Graph, Intuitionistic Fuzzy Interval Graph, or Turiyam Neutrosophic Interval Graph.*

Proof: Obviously holds. □

Theorem 35. *In a Neutrosophic Interval Graph, the neutrosophic adjacency relations ρ_μ, ρ_τ, and ρ_ζ are symmetric.*

Proof: By definition, the neutrosophic adjacency relations are given by:

$$\rho_\mu(v_i, v_j) = \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\},$$

$$\rho_\tau(v_i, v_j) = \sup_{x \in \mathbb{R}} \min\{\tau_i(x), \tau_j(x)\},$$

$$\rho_\zeta(v_i, v_j) = \sup_{x \in \mathbb{R}} \min\{\zeta_i(x), \zeta_j(x)\}.$$

For any $v_i, v_j \in V$, the minimum function is symmetric, i.e.,

$$\min\{\mu_i(x), \mu_j(x)\} = \min\{\mu_j(x), \mu_i(x)\}.$$

Similarly, the supremum over $x \in \mathbb{R}$ preserves this symmetry. Therefore,

$$\rho_\mu(v_i, v_j) = \rho_\mu(v_j, v_i).$$

The same argument applies to ρ_τ and ρ_ζ. Thus, all neutrosophic adjacency relations are symmetric. □

Corollary 36. *In a Neutrosophic Proper Interval Graph, the neutrosophic adjacency relations ρ_μ, ρ_τ, and ρ_ζ are symmetric.*

Proof: Obviously holds. □

Theorem 37. *In a Neutrosophic Proper Interval Graph, no neutrosophic interval is properly contained within another with respect to the truth-membership function μ_i.*

Proof: By definition, a Neutrosophic Proper Interval Graph is a Neutrosophic Interval Graph where, for any two distinct neutrosophic intervals (μ_i, τ_i, ζ_i) and (μ_j, τ_j, ζ_j), the following does not hold:

$$\mu_i(x) \leq \mu_j(x) \quad \forall x \in \mathbb{R}, \quad \text{and} \quad \exists x_0 \in \mathbb{R} \text{ such that } \mu_i(x_0) < \mu_j(x_0).$$

This condition explicitly states that no truth-membership function μ_i is entirely within another μ_j with a strict inequality at some point x_0. Therefore, no neutrosophic interval μ_i is properly contained within another μ_j in terms of the truth-membership functions. The same reasoning applies to the indeterminacy τ_i and falsity ζ_i functions. □

Theorem 38. *The class of Neutrosophic Interval Graphs is closed under taking induced subgraphs.*

Proof: Let $G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$ be a Neutrosophic Interval Graph, and let $V' \subseteq V$. Consider the induced subgraph $G' = (V', \mu_{V'}, \tau_{V'}, \zeta_{V'}, \rho'_\mu, \rho'_\tau, \rho'_\zeta)$ where:

$$\mu_{V'}(v_i) = \mu_V(v_i), \quad \tau_{V'}(v_i) = \tau_V(v_i), \quad \zeta_{V'}(v_i) = \zeta_V(v_i), \quad \forall v_i \in V',$$

$$\rho'_\mu(v_i, v_j) = \rho_\mu(v_i, v_j), \quad \rho'_\tau(v_i, v_j) = \rho_\tau(v_i, v_j), \quad \rho'_\zeta(v_i, v_j) = \rho_\zeta(v_i, v_j), \quad \forall v_i, v_j \in V'.$$

Since G is a Neutrosophic Interval Graph, there exists a family of neutrosophic intervals $\mathcal{N} = \{(\mu_i, \tau_i, \zeta_i) \mid v_i \in V\}$. The subgraph G' corresponds to the subset of intervals $\mathcal{N}' = \{(\mu_i, \tau_i, \zeta_i) \mid v_i \in V'\}$. The adjacency relations in G' are determined by the overlaps of intervals in $\mathcal{N}'$ using the same definitions as in G.

Therefore, G' is a Neutrosophic Interval Graph corresponding to $\mathcal{N}'$. Thus, the class of Neutrosophic Interval Graphs is closed under taking induced subgraphs. □

3.2 Neutrosophic p-proper interval graph

Let p and q be integers. The definitions of a *neutrosophic p-proper interval graph* and a *neutrosophic q-improper interval graph*, which extend the concepts of p-proper interval graphs and q-improper interval graphs, are provided below.

Definition 39. A *neutrosophic p-proper interval graph* is a neutrosophic graph

$$G = (V, E, \sigma_T, \sigma_I, \sigma_F)$$

where the truth-membership intervals I_v, corresponding to the vertices $v \in V$, satisfy the condition that no interval is properly contained within more than p others. Formally, for each vertex $u \in V$, let $\mathcal{I}_T = \{I_v \mid v \in V\}$ represent the truth-membership intervals. The graph G is p-proper if for every interval $I_u \in \mathcal{I}_T$, the number of intervals $I_v \subset I_u$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid I_v \subset I_u\}| \leq p.$$

This definition ensures that the intervals corresponding to the truth-membership values in the neutrosophic framework adhere to the p-proper constraint.

Definition 40. A *neutrosophic q-improper interval graph* is a neutrosophic graph

$$G = (V, E, \sigma_T, \sigma_I, \sigma_F)$$

where the truth-membership intervals I_v, corresponding to the vertices $v \in V$, satisfy the condition that no interval properly contains more than q others. Formally, for each vertex $u \in V$, let $\mathcal{I}_T = \{I_v \mid v \in V\}$ represent the truth-membership intervals. The graph G is q-improper if for every interval $I_u \in \mathcal{I}_T$, the number of intervals $I_u \subset I_v$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid I_u \subset I_v\}| \leq q.$$

This ensures that the truth-membership intervals in the neutrosophic graph follow the q-improper constraint.

Theorem 41. *Neutrosophic p-proper Interval Graph is a special type of neutrosophic proper Interval Graph.*

Proof: Obviously holds. □

Corollary 42. *Neutrosophic p-proper Interval Graph is a special type of neutrosophic Interval Graph.*

Proof: Obviously holds. □

Corollary 43. *Neutrosophic q-improper Interval Graph is a special type of neutrosophic Interval Graph.*

Proof: Obviously holds. □

Theorem 44. *A neutrosophic 0-proper interval graph is a neutrosophic proper interval graph.*

Proof: By definition, a neutrosophic p-proper interval graph is a neutrosophic interval graph where no interval is properly contained within more than p other intervals. When $p = 0$, no interval is properly contained within any other interval. This is precisely the condition for a neutrosophic proper interval graph, where no interval properly contains any other interval. Hence, a neutrosophic 0-proper interval graph is a neutrosophic proper interval graph. □

Theorem 45. *A neutrosophic 0-improper interval graph is a neutrosophic proper interval graph.*

Proof: A neutrosophic q-improper interval graph is defined as a neutrosophic interval graph where no interval properly contains more than q other intervals. For $q = 0$, no interval properly contains any other interval, which is exactly the condition for a neutrosophic proper interval graph. Therefore, a neutrosophic 0-improper interval graph is a neutrosophic proper interval graph. □

Theorem 46. *A Fuzzy p-proper Interval Graph, Intuitionistic Fuzzy p-proper Interval Graph, Neutrosophic p-proper Interval Graph or Turiyam Neutrosophic p-proper Interval Graph can be transformed into a classic p-proper Interval Graph.*

Proof: Obviously holds. □

Corollary 47. *A Fuzzy q-improper Interval Graph, Intuitionistic Fuzzy q-improper Interval Graph, Neutrosophic q-improper Interval Graph or Turiyam Neutrosophic q-improper Interval Graph can be transformed into a classic q-improper Interval Graph.*

Proof: Obviously holds. □

3.3 Fuzzy Intersection Graph and Fuzzy Interval Graph

We will examine the relationship between a Fuzzy Intersection Graph and a Fuzzy Interval Graph. The definition of a Fuzzy Intersection Graph is provided below [98, 81, 93, 20, 55] .

Definition 48 (Fuzzy Intersection Graph)**.** A *Fuzzy Intersection Graph* is a graph $G = (V, E, \sigma, \mu)$ where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $\sigma : V \to [0, 1]$ is a membership function that assigns a degree of membership to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the strength of the connection (degree of membership) between each pair of vertices $(u, v) \in V \times V$.

The edge set E of the fuzzy intersection graph is defined based on the membership functions of the vertices and the fuzzy relation. Specifically, for each pair $(u, v) \in V \times V$, the edge (u, v) exists in the fuzzy intersection graph with the membership degree:

$$\mu(u, v) = \min(\sigma(u), \sigma(v))$$

if the Euclidean distance between the corresponding points of u and v satisfies the condition for intersection, and $\mu(u, v) = 0$ otherwise.

In this way, the fuzzy intersection graph generalizes the concept of an intersection graph by incorporating fuzzy set theory, allowing for partial membership and gradual relationships between vertices and edges.

The following theorem is well-known in the context of fuzzy intersection graphs.

Theorem 49. [37] *Any undirected fuzzy graph $G = (V, \sigma, \mu)$ can be represented as a fuzzy intersection graph.*

We will explore the relationship between Fuzzy Interval Graphs and Fuzzy Intersection Graphs. The theorem is presented as follows.

Theorem 50. *A Fuzzy Interval Graph is a Fuzzy Intersection Graph.*

Proof: Let $G = (V, \mu_V, \rho)$ be a Fuzzy Interval Graph where:

- V is the set of vertices.
- $\mu_V(v_i)$ represents the membership function for each vertex $v_i \in V$.
- $\rho(v_i, v_j)$ represents the fuzzy adjacency relation between vertices v_i and v_j.

In a Fuzzy Interval Graph, each vertex v_i is associated with a fuzzy interval $\mu_i : \mathbb{R} \to [0,1]$ on the real line. The fuzzy adjacency relation $\rho(v_i, v_j)$ measures the degree of overlap between the fuzzy intervals μ_i and μ_j corresponding to the vertices v_i and v_j. Specifically, for any two distinct vertices v_i and v_j, the adjacency relation is given by:

$$\rho(v_i, v_j) = \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\},$$

which calculates the maximum degree of overlap between the intervals μ_i and μ_j. The function $\min\{\mu_i(x), \mu_j(x)\}$ captures the intersection of the two intervals, as it represents the smallest membership value at any point $x \in \mathbb{R}$ common to both intervals.

Next, consider a *Fuzzy Intersection Graph* $G' = (V, E, \sigma, \mu)$, defined as follows:

- V is the set of vertices.
- $\sigma : V \to [0,1]$ is a membership function that assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0,1]$ is the fuzzy relation representing the strength of the connection between any two vertices $v_i, v_j \in V$, defined as:

$$\mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j)).$$

To prove that a fuzzy interval graph is a fuzzy intersection graph, we need to show that the fuzzy adjacency relation $\rho(v_i, v_j)$ in the fuzzy interval graph is equivalent to the edge membership function $\mu(v_i, v_j)$ in the fuzzy intersection graph.

By the construction of a fuzzy interval graph, the adjacency relation $\rho(v_i, v_j)$ is based on the intersection of the fuzzy intervals μ_i and μ_j, as given by the formula:

$$\rho(v_i, v_j) = \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}.$$

This represents the highest degree of intersection between the two fuzzy intervals. On the other hand, in a fuzzy intersection graph, the relation $\mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j))$ directly uses the membership degrees $\sigma(v_i)$ and $\sigma(v_j)$, which can be interpreted as the maximum heights (i.e., supremum values) of the corresponding fuzzy intervals μ_i and μ_j.

Thus, by setting $\sigma(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x)$ for each vertex $v_i \in V$, we ensure that the fuzzy relation in the fuzzy intersection graph is exactly the same as the fuzzy adjacency relation in the fuzzy interval graph:

$$\rho(v_i, v_j) = \mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j)) = \sup_{x \in \mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}.$$

□

Corollary 51. *A Fuzzy Proper Interval Graph is a Fuzzy Intersection Graph.*

Proof: It can be proven in the same way as above. □

4 Future Perspectives of This Research

In this section, we briefly outline the future prospects of this study. Potential directions include exploring the Neutrosophic Interval OffGraph [38, 102], the Neutrosophic Proper Interval OffGraph, and the Interval SuperHyperGraph (cf. [104, 42, 41, 35, 106]). These avenues offer promising opportunities for further development and applications of the concepts introduced in this research.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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