Information Sciences with Applications, Vol. 5, 2025

https://doi.org/10.61356/j.iswa.2025.5502/

Claw-free Graph and AT-free Graph in Fuzzy, Neutrosophic, and Plithogenic Graphs

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Abstract: Graph theory studies networks consisting of nodes (vertices) and their connections (edges), with various graph classes being extensively researched. This paper focuses on three specific graph classes: AT-Free Graphs, Claw-Free Graphs, and Triangle-Free Graphs. Additionally, it examines uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs, which are designed to address uncertainty in diverse applications. In this study, we introduce and analyze AT-Free Graphs, Claw-Free Graphs, and Triangle-Free Graphs within the framework of Fuzzy Graphs, investigating their properties and relationships in uncertain graph theory.

Keywords: Neutrosophic graph, Fuzzy graph, Claw-Free Graph, AT-Free Graph, Triangle-Free Graph

1 Introduction

1.1 Graph Theory

Graph theory is the study of networks composed of nodes (vertices) and their connections (edges). It provides the foundation for analyzing the structure, connectivity, and properties of networks [17]. This paper focuses on three specific types of graphs: AT-Free Graphs, Claw-Free Graphs, and Triangle-Free Graphs.

An AT-Free Graph is characterized by the absence of an asteroidal triple, meaning there is no set of three vertices that can be connected without passing through the neighborhood of the third vertex [8, 14, 20, 18, 46]. A Claw-Free Graph is defined as one that does not contain a "claw," a subgraph formed by a vertex that is exclusively connected to three others without any connections among them [56]. Examples of claw-free graphs include line graphs [38], Moser graphs [50], and proper interval graphs [37]. A Triangle-Free Graph is defined as a graph without any cycles of length 3, indicating that no three vertices are mutually connected [54, 10, 47]. These graph types have been extensively studied in various areas, such as recognition algorithms, matching problems, independent set problems, and coloring.

1.2 Uncertain Graphs

This paper also explores various models of uncertain graphs, including Fuzzy [55], Intuitionistic Fuzzy [43, 44, 52], Vague [60, 45, 61, 41], Neutrosophic [34, 22, 29, 28], Turiyam Neutrosophic [27, 35], and Plithogenic Graphs [25, 30, 33, 21, 26]. These models are designed to handle uncertainty in various applications, extending classical graph theory by introducing different levels of uncertainty. Furthermore, extensive research has been conducted on graph classes within these uncertainty-based models [30, 33].

1.3 Our Contribution

Given the significance of research on graph classes and uncertain graphs, this study introduces and analyzes the concepts of AT-Free Graphs, Claw-Free Graphs, and Triangle-Free Graphs within the frameworks of Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. We explore their properties and interrelationships, contributing to a deeper understanding of uncertain graph models and further expanding the theoretical foundation of uncertain graph theory.

2 Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper.

2.1 Basic Graph Concepts

Here, we present some basic concepts of graph theory. For more foundational concepts and notations, please refer to lecture notes, surveys, or introductory texts such as [17].

Definition 1 (Graph)**.** [17] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Subgraph)**.** [17] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 3 (Degree)**.** [17] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 4 (Induced Subgraph)**.** Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset of vertices $S \subseteq V$, the *induced subgraph* of G on S, denoted by $G[S]$, is defined as the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S.

Formally, the induced subgraph $G[S]$ is defined as:

$$G[S] = (S, E_S),$$

where

$$E_S = \{(u, v) \in E \mid u \in S,\ v \in S\}.$$

In other words, $G[S]$ includes all the vertices in S and all the edges of G that connect pairs of vertices in S.

Definition 5 (Path)**.** [17] A *path* in a graph $G = (V, E)$ is an ordered sequence of distinct vertices $(v_0, v_1, \dots, v_k)$ such that:

- $v_i \in V$ for all $0 \le i \le k$, where $k \ge 0$ is the length of the path.
- $(v_i, v_{i+1}) \in E$ for all $0 \le i < k$, i.e., consecutive vertices in the sequence are connected by edges in G.

If $v_0 = v_k$, the path is called a *cycle*.

Definition 6 (Tree)**.** [17] A *tree* is a connected, acyclic graph $T = (V_T, E_T)$, where:

- V_T is the set of vertices.
- E_T is the set of edges.
- T is connected, meaning there exists a path between any two vertices in V_T.
- T contains no cycles, i.e., there is no path in T where the first and last vertices are the same.

Alternatively, a tree can be defined as a graph in which there is exactly one path between any two vertices.

2.2 Fuzzy, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we explore Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The following definitions include related concepts.

Definition 7. (cf.[55]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Definition 8 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs)**.** (cf.[25]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

(1) *Fuzzy Graph* [55]*:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.

(2) *Intuitionistic Fuzzy Graph (IFG)* [2]*:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $v_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $v_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + v_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [42, 5, 59, 39]*:*

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
  - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
  - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
  - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
  - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.

(4) *Turiyam Neutrosophic Graph* [36]*:*

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
  - $t(v) \in [0, 1]$ is the truth value,
  - $iv(v) \in [0, 1]$ is the indeterminacy value,
  - $fv(v) \in [0, 1]$ is the falsity value,
  - $lv(v) \in [0, 1]$ is the liberal state value,
  - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.

- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing "Contradiction" with "Liberal" [63].

2.3 Plithogenic Graphs

Plithogenic Graphs have been introduced as an extension of Fuzzy Graphs and Turiyam Neutrosophic Graphs, broadening the concept to encompass Plithogenic Sets [67, 32, 66]. These graphs have become a prominent subject of ongoing research and development [25, 70]. The formal definition is provided below.

Definition 9. [70] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.
  - Ml is the range of possible attribute values.
  - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
  - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with the edges.
  - Nm is the range of possible attribute values.
  - $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
  - $bCf : Nm \times Nm \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 10. (cf.[30]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph.*
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph.*
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph.*
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graph.*
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graph.*
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Graph.*
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic Graph.*
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Graph.*
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Graph.*

3 Result in this paper

In this section, we present the results of this paper.

3.1 Uncertain AT-Free, Claw-Free, and Triangle-Free Graphs

We examine Uncertain AT-Free, Claw-Free, and Triangle-Free Graphs. The following definitions include the relevant concepts.

Definition 11 (AT-Free, Claw-Free, and Triangle-Free Graphs)**.** [8, 56, 13] Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. We define three specific types of graphs as follows:

(1) *AT-Free Graph:*

A graph G is called an *AT-free graph* if it does not contain any *asteroidal triple* (AT). An asteroidal triple is defined as a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices in this set, there exists a path between them that avoids the closed neighborhood of the third vertex.

Formally, a graph G is AT-free if:

$$\forall \{u, v, w\} \subseteq V, \exists \text{ a path from } u \text{ to } v \text{ avoiding } N[w],$$

where $N[w] = \{w\} \cup \{x \in V \mid (w, x) \in E\}$ is the closed neighborhood of vertex w. In other words, an AT-free graph has no set of three vertices such that any two of them can be connected by a path that does not pass through the neighborhood of the third.

(2) *Claw-Free Graph:*

A graph G is called a *claw-free graph* if it does not contain an induced subgraph that is isomorphic to a *claw.* A claw is defined as a complete bipartite graph $K_{1,3}$, which consists of a central vertex connected to three other vertices, none of which are connected to each other.

Formally, a graph G is claw-free if:

$$\nexists S \subseteq V, |S| = 4 \text{ such that } G[S] \cong K_{1,3},$$

where $G[S]$ denotes the subgraph induced by the vertex set S. In simpler terms, a claw-free graph does not have any four vertices such that one vertex is connected to the other three, with no edges among those three vertices.

(3) *Triangle-Free Graph:*

A graph G is called a *triangle-free graph* if it does not contain any subgraph that forms a cycle of length 3, which is known as a triangle.

Formally, a graph G is triangle-free if:

$$\nexists \{u, v, w\} \subseteq V \text{ such that } (u, v), (v, w), (w, u) \in E.$$

This means that a triangle-free graph has no set of three vertices that are mutually adjacent, i.e., no three vertices form a cycle.

Definition 12 (Fuzzy Triangle-Free Graph). A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy triangle-free graph* if for all distinct vertices $u, v, w \in V$, at least one of the following holds:

$$\mu(u, v) = 0, \quad \mu(v, w) = 0, \quad \mu(w, u) = 0.$$

Definition 13 (Fuzzy Claw-Free Graph). A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy claw-free graph* if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$\mu(v, u_i) > 0 \quad \text{for } i = 1, 2, 3,$$
$$\mu(u_i, u_j) = 0 \quad \text{for all } i \neq j.$$

Definition 14 (Fuzzy AT-Free Graph). A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy AT-free graph* if it does not contain any *fuzzy asteroidal triple*. A fuzzy asteroidal triple is a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices, say u and v, there exists a path P from u to v with positive edge membership degrees, avoiding the *fuzzy closed neighborhood* of the third vertex w, defined as:

$$N[w] = \{x \in V \mid \mu(w, x) > 0 \text{ or } x = w\}.$$

Theorem 15. *Fuzzy AT-Free, Fuzzy Claw-Free, and Fuzzy Triangle-Free Graphs are specific instances of fuzzy graphs.*

Proof: To prove this theorem, we verify that Fuzzy AT-Free, Fuzzy Claw-Free, and Fuzzy Triangle-Free Graphs satisfy the definition of a fuzzy graph. A fuzzy graph $G = (V, \sigma, \mu)$ is defined by its vertex membership function $\sigma : V \to [0, 1]$ and edge membership function $\mu : V \times V \to [0, 1]$, where $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$.

(1) **Fuzzy Triangle-Free Graphs:**

A fuzzy graph $G = (V, \sigma, \mu)$ is said to be triangle-free if for all distinct vertices $u, v, w \in V$:

$$\mu(u, v) = 0, \quad \mu(v, w) = 0, \quad \text{or } \mu(w, u) = 0.$$

Since $\mu(u, v)$ is defined on the edge set of the fuzzy graph and satisfies the edge membership constraint $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$, the fuzzy triangle-free property is consistent with the general fuzzy graph definition. Therefore, a fuzzy triangle-free graph is a fuzzy graph.

(2) **Fuzzy Claw-Free Graphs:**

A fuzzy graph $G = (V, \sigma, \mu)$ is said to be claw-free if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$\mu(v, u_i) > 0 \text{ for all } i = 1, 2, 3, \quad \mu(u_i, u_j) = 0 \text{ for all } i \neq j.$$

Here, the condition $\mu(v, u_i) > 0$ ensures the presence of edges connecting v to u_i, while $\mu(u_i, u_j) = 0$ prevents edges among u_1, u_2, u_3. Since these conditions operate within the framework of edge membership $\mu(u, v)$, which is bounded by vertex membership $\sigma(u)$ and $\sigma(v)$, the fuzzy claw-free property aligns with the definition of a fuzzy graph. Thus, a fuzzy claw-free graph is a fuzzy graph.

(3) **Fuzzy AT-Free Graphs:**

A fuzzy graph $G = (V, \sigma, \mu)$ is said to be AT-free if it does not contain any fuzzy asteroidal triple (FAT). A FAT is defined as a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices u, v, there exists a path P from u to v with positive edge membership values, avoiding the fuzzy closed neighborhood of the third vertex w, where:

$$N[w] = \{x \in V \mid \mu(w, x) > 0 \text{ or } x = w\}.$$

The requirement for positive edge memberships along the path P and avoidance of $N[w]$ ensures that the graph structure adheres to the edge membership and vertex membership constraints of a fuzzy graph. Therefore, a fuzzy AT-free graph is a fuzzy graph.

Fuzzy AT-Free, Fuzzy Claw-Free, and Fuzzy Triangle-Free Graphs satisfy the structural constraints and membership requirements of fuzzy graphs. Hence, they are specific instances of fuzzy graphs. □

Theorem 16. *The concepts of AT-free, claw-free, and triangle-free graphs can be generalized to fuzzy graphs as fuzzy AT-free, fuzzy claw-free, and fuzzy triangle-free graphs, respectively.*

Proof: (1) **Fuzzy Triangle-Free Graph Generalization:**

In a classical graph $G = (V, E)$, triangle-free means that no three vertices $\{u, v, w\}$ form a cycle. For a fuzzy graph $G = (V, \sigma, \mu)$, the condition is generalized by requiring:

$$\mu(u, v) = 0 \text{ or } \mu(v, w) = 0 \text{ or } \mu(w, u) = 0,$$

ensuring no cycle exists with positive membership values. Thus, fuzzy triangle-free graphs directly extend the classical definition by incorporating edge membership.

(2) **Fuzzy Claw-Free Graph Generalization:**

In classical graphs, claw-free implies the absence of $K_{1,3}$ (a vertex connected to three others with no connections between them). For a fuzzy graph $G = (V, \sigma, \mu)$, this is generalized by requiring that for any $v \in V$, and any three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$:

$$\mu(v, u_i) > 0 \text{ for all } i, \quad \mu(u_i, u_j) = 0 \text{ for } i \neq j.$$

This condition ensures no fuzzy claw exists, generalizing the classical claw-free property.

(3) **Fuzzy AT-Free Graph Generalization:**

In classical graphs, AT-free means the absence of an asteroidal triple $\{u, v, w\}$, where paths between any two vertices avoid the neighborhood of the third. For fuzzy graphs, the concept of a *fuzzy asteroidal triple* is defined as:

$$\{u, v, w\} \subseteq V \text{ such that } \exists \text{ a path from } u \text{ to } v \text{ avoiding } N[w],$$

where the fuzzy neighborhood $N[w]$ is defined as:

$$N[w] = \{x \in V \mid \mu(w, x) > 0 \text{ or } x = w\}.$$

If no such fuzzy asteroidal triple exists, G is fuzzy AT-free. This condition extends the classical AT-free property to account for edge membership degrees.

The fuzzy definitions of triangle-free, claw-free, and AT-free graphs are consistent generalizations of their classical counterparts, as they retain the structural constraints while introducing fuzzy membership degrees for vertices and edges. □

Definition 17 (Neutrosophic Triangle-Free Graph)**.** A neutrosophic graph $G = (V, \sigma, \mu)$ is called a *neutrosophic triangle-free graph* if for all distinct vertices $u, v, w \in V$, at least one of the following holds:

$$\mu_T(u, v) = 0, \quad \mu_T(v, w) = 0, \quad \mu_T(w, u) = 0.$$

Definition 18 (Neutrosophic Claw-Free Graph)**.** A neutrosophic graph $G = (V, \sigma, \mu)$ is called a *neutrosophic claw-free graph* if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$\mu_T(v, u_i) > 0 \quad \text{for } i = 1, 2, 3,$$
$$\mu_T(u_i, u_j) = 0 \quad \text{for all } i \neq j.$$

Definition 19 (Neutrosophic AT-Free Graph)**.** A neutrosophic graph $G = (V, \sigma, \mu)$ is called a *neutrosophic AT-free graph* if it does not contain any *neutrosophic asteroidal triple*, defined similarly to the fuzzy case, using the truth-membership degrees $\mu_T(u, v)$.

Definition 20 (Turiyam Neutrosophic Triangle-Free Graph)**.** A Turiyam Neutrosophic graph $G = (V, \sigma, \mu)$ is called a *Turiyam Neutrosophic triangle-free graph* if for all distinct vertices $u, v, w \in V$, at least one of the following holds:

$$t(u, v) = 0, \quad t(v, w) = 0, \quad t(w, u) = 0.$$

Theorem 21. *The concepts of fuzzy AT-free, fuzzy claw-free, and fuzzy triangle-free graphs can be generalized to neutrosophic graphs as neutrosophic AT-free, neutrosophic claw-free, and neutrosophic triangle-free graphs, respectively.*

Proof: (1) **Neutrosophic Triangle-Free Graph Generalization:**

In a fuzzy triangle-free graph $G = (V, \sigma, \mu)$, no three vertices $\{u, v, w\}$ form a triangle if:

$$\mu(u, v) = 0, \quad \mu(v, w) = 0, \quad \text{or } \mu(w, u) = 0.$$

For a neutrosophic graph $G = (V, \sigma, \mu)$, this is generalized by requiring:

$$\mu_T(u, v) = 0, \quad \mu_T(v, w) = 0, \quad \text{or } \mu_T(w, u) = 0,$$

where $\mu_T(u, v)$ represents the truth-membership degree. This ensures that no triangle exists based on truth-membership, extending the fuzzy case.

(2) **Neutrosophic Claw-Free Graph Generalization:**

In a fuzzy claw-free graph, no vertex v is connected to three distinct vertices u_1, u_2, u_3 such that:

$$\mu(v, u_i) > 0 \quad \text{for } i = 1, 2, 3, \quad \mu(u_i, u_j) = 0 \quad \text{for } i \neq j.$$

For a neutrosophic graph $G = (V, \sigma, \mu)$, this is generalized by requiring:

$$\mu_T(v, u_i) > 0 \quad \text{for } i = 1, 2, 3, \quad \mu_T(u_i, u_j) = 0 \quad \text{for } i \neq j,$$

ensuring no claw exists in terms of truth-membership.

(3) **Neutrosophic AT-Free Graph Generalization:**

In a fuzzy AT-free graph, no set of three vertices $\{u, v, w\}$ forms a fuzzy asteroidal triple, where any path between two vertices avoids the fuzzy closed neighborhood of the third. For a neutrosophic graph, this is generalized by requiring that no set $\{u, v, w\} \subseteq V$ forms a *neutrosophic asteroidal triple*, defined as follows:

$$\text{For any two vertices } u, v \in \{u, v, w\}, \text{ there exists a path } P \text{ from } u \text{ to } v,$$

where all edges along P have positive truth-membership degrees μ_T, and P avoids the neutrosophic closed neighborhood of the third vertex w, defined as:

$$N[w] = \{x \in V \mid \mu_T(w, x) > 0 \text{ or } x = w\}.$$

This condition ensures that no neutrosophic asteroidal triple exists, extending the fuzzy AT-free graph concept.

The neutrosophic definitions of AT-free, claw-free, and triangle-free graphs extend their fuzzy counterparts by incorporating neutrosophic truth-membership degrees μ_T, while retaining the structural constraints. □

Definition 22 (Turiyam Neutrosophic Claw-Free Graph)**.** A Turiyam Neutrosophic graph $G = (V, \sigma, \mu)$ is called a *Turiyam Neutrosophic claw-free graph* if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$t(v, u_i) > 0 \quad \text{for } i = 1, 2, 3,$$
$$t(u_i, u_j) = 0 \quad \text{for all } i \neq j.$$

Definition 23 (Turiyam Neutrosophic AT-Free Graph)**.** A Turiyam Neutrosophic graph $G = (V, \sigma, \mu)$ is called a *Turiyam Neutrosophic AT-free graph* if it does not contain any *Turiyam Neutrosophic asteroidal triple*, defined using the truth values $t(u, v)$.

Definition 24 (Plithogenic Triangle-Free Graph)**.** A plithogenic graph $G = (V, adf_V, E, adf_E)$ is called a *plithogenic triangle-free graph* if for all distinct vertices $u, v, w \in V$, at least one of the following holds:

$$adf_E(u, v) = \mathbf{0}, \quad adf_E(v, w) = \mathbf{0}, \quad adf_E(w, u) = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector in $[0, 1]^s$.

Definition 25 (Plithogenic Claw-Free Graph)**.** A plithogenic graph $G = (V, adf_V, E, adf_E)$ is called a *plithogenic claw-free graph* if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$\begin{aligned} adf_E(v, u_i) \neq \mathbf{0} \quad & \text{for } i = 1, 2, 3, \\ adf_E(u_i, u_j) = \mathbf{0} \quad & \text{for all } i \neq j. \end{aligned}$$

Definition 26 (Plithogenic AT-Free Graph)**.** A plithogenic graph $G = (V, adf_V, E, adf_E)$ is called a *plithogenic AT-free graph* if it does not contain any *plithogenic asteroidal triple*. This is defined as a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices, there exists a path P from u to v with edge attribute degrees not equal to $\mathbf{0}$, avoiding the *plithogenic closed neighborhood* of the third vertex w.

Example 27. (cf.[30]) The following examples of Plithogenic Triangle-Free (respectively, AT-Free, Claw-Free) Graph are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 7, t = 1$, PG is called a *Plithogenic Heptapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph*.

Theorem 28. *Plithogenic Triangle-Free, Plithogenic Claw-Free, and Plithogenic AT-Free Graphs are specific instances of Plithogenic Graphs.*

Proof: A Plithogenic Graph $PG = (PM, PN)$ is defined using Plithogenic Vertex Sets $PM = (M, l, Ml, adf, aCf)$ and Plithogenic Edge Sets $PN = (N, m, Nm, bdf, bCf)$, with Degrees of Appurtenance (DAF) and Contradiction (DCF) satisfying specific conditions.

(1) **Plithogenic Triangle-Free Graphs:**

A Plithogenic Graph $G = (V, adf_V, E, adf_E)$ is said to be triangle-free if for any three distinct vertices $u, v, w \in V$:

$$adf_E(u, v) = \mathbf{0}, \quad adf_E(v, w) = \mathbf{0}, \quad \text{or } adf_E(w, u) = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector in $[0, 1]^s$. Since the Degree of Appurtenance Function (DAF) is used to determine edge relationships, this constraint ensures that no triangle exists with non-zero appurtenance, satisfying the Plithogenic Graph conditions.

(2) **Plithogenic Claw-Free Graphs:**

A Plithogenic Graph $G = (V, adf_V, E, adf_E)$ is said to be claw-free if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$adf_E(v, u_i) \neq \mathbf{0} \quad \text{for } i = 1, 2, 3, \quad adf_E(u_i, u_j) = \mathbf{0} \quad \text{for all } i \neq j.$$

This condition ensures that no vertex v serves as a center for a claw (i.e., a $K_{1,3}$ subgraph) based on edge appurtenance degrees, thereby conforming to the Plithogenic Graph definition.

(3) **Plithogenic AT-Free Graphs:**

A Plithogenic Graph $G = (V, adf_V, E, adf_E)$ is said to be AT-free if it does not contain any Plithogenic Asteroidal Triple (PAT). A PAT is defined as a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices u, v:

$$\exists\, P \text{ (path from } u \text{ to } v) \text{ with } adf_E \neq \mathbf{0} \text{ for all edges along } P,$$

avoiding the Plithogenic closed neighborhood of the third vertex w, defined as:

$$N[w] = \{x \in V \mid adf_E(w, x) \neq \mathbf{0} \text{ or } x = w\}.$$

This ensures the absence of a fuzzy-like AT-structure under Plithogenic conditions, thus meeting the criteria for Plithogenic Graphs.

The structural constraints and appurtenance-based definitions of Plithogenic Triangle-Free, Claw-Free, and AT-Free Graphs align with the general framework of Plithogenic Graphs. Therefore, these specific graph types are valid instances of Plithogenic Graphs. □

Theorem 29. *Every Plithogenic Triangle-Free (respectively, AT-Free, Claw-Free) Graph can be transformed into a Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graph, a Turiyam Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graph, and a Classic Triangle-Free (AT-Free, Claw-Free) Graph.*

Proof: Let $G = (V, adf_V, E, adf_E)$ be a Plithogenic Triangle-Free (or AT-Free, or Claw-Free) Graph, where:

- V is the set of vertices.
- $adf_V : V \to [0,1]^s$ assigns an attribute degree function to each vertex.
- $E \subseteq V \times V$ is the set of edges.
- $adf_E : E \to [0,1]^s$ assigns an attribute degree function to each edge.

We consider the following transformations:

(1) *Transformation to Neutrosophic Graph:*

Set $s = 3$ and interpret the attribute degree functions as:

$$adf_V(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \quad adf_E(e) = (\mu_T(e), \mu_I(e), \mu_F(e)),$$

where $\sigma_T(v)$ is the truth-membership degree, $\sigma_I(v)$ is the indeterminacy-membership degree, and $\sigma_F(v)$ is the falsity-membership degree for vertex v. Similarly for edges.

The conditions and definitions of Plithogenic Triangle-Free (AT-Free, Claw-Free) Graphs correspond directly to those of Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graphs when considering the truth-membership degrees. Therefore, G can be regarded as a Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graph.

(2) *Transformation to Turiyam Neutrosophic Graph:*

Set $s = 4$ and interpret the attribute degree functions as:

$$adf_V(v) = (t(v), iv(v), fv(v), lv(v)), \quad adf_E(e) = (t(e), iv(e), fv(e), lv(e)),$$

where $t(v)$ is the truth value, $iv(v)$ is the indeterminacy value, $fv(v)$ is the falsity value, and $lv(v)$ is the liberal state value for vertex v.

Under this interpretation, G satisfies the definitions of Turiyam Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graphs. Therefore, G can be viewed as a Turiyam Neutrosophic Triangle-Free (AT-Free, Claw-Free) Graph.

(3) *Transformation to Classic Graph:*

By considering only the underlying crisp graph (ignoring the attribute degree functions adf_V and adf_E), G becomes a classic graph $G' = (V, E)$. Since the absence of triangles, asteroidal triples, or claws is determined by the structure of the graph (the vertex and edge sets), G' is a Classic Triangle-Free (AT-Free, Claw-Free) Graph.

Therefore, a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph can be transformed into Neutrosophic, Turiyam, and Classic versions by appropriate interpretations of the attribute degree functions and dimensions. □

Theorem 30. *Any induced subgraph of a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph is also a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph.*

Proof: Let $G = (V, adf_V, E, adf_E)$ be a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph, and let $V' \subseteq V$ be a subset of vertices. The induced subgraph $G' = (V', adf_V|_{V'}, E', adf_E|_{E'})$ is defined by:

$$E' = \{(u, v) \in E \mid u, v \in V'\}.$$

Since G does not contain any triangles (or asteroidal triples, claws) involving vertices from V, G' cannot contain such structures within V'. Therefore, G' is also a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph. □

Theorem 31. *The intersection of two Plithogenic Triangle-Free (AT-Free, Claw-Free) Graphs on the same vertex set V is a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph.*

Proof: Let $G_1 = (V, adf_{V1}, E_1, adf_{E1})$ and $G_2 = (V, adf_{V2}, E_2, adf_{E2})$ be two Plithogenic Triangle-Free (AT-Free, Claw-Free) Graphs. Define their intersection $G = (V, adf_V, E, adf_E)$ as:

$$E = E_1 \cap E_2, \quad adf_V = \min\{adf_{V1}, adf_{V2}\}, \quad adf_E = \min\{adf_{E1}, adf_{E2}\},$$

where the minimum is taken component-wise.

Any triangle (or asteroidal triple, claw) in G would have to exist in both G_1 and G_2, but since both are Triangle-Free (AT-Free, Claw-Free), G must also be Triangle-Free (AT-Free, Claw-Free). □

Theorem 32. *Adding an edge with attribute degree function equal to* **0** *to a Plithogenic Triangle-Free (AT-Free, Claw-Free) Graph does not create a triangle (asteroidal triple, claw) with non-zero attribute degrees.*

Proof: An edge with attribute degree function **0** effectively does not contribute to the graph's structure concerning triangles, asteroidal triples, or claws, as its presence does not enable the formation of such structures with non-zero attribute degrees. Therefore, adding such an edge preserves the Triangle-Free (AT-Free, Claw-Free) property. □

3.2 Triangle-Free (respectively, AT-Free, Claw-Free)OffGraph

Recently, OverGraphs [15], UnderGraphs, and OffGraphs [16] have been introduced to graphically represent the concepts of Overset, Underset, and Offset[64, 26]. In this subsection, we focus on the Plithogenic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph. Since the OffGraph is a generalization of OverGraphs and UnderGraphs, we will only consider the OffGraph here. However, OverGraphs and UnderGraphs can also be defined in a similar manner.

Definition 33. (cf.[16]) A *Single-Valued Neutrosophic OffGraph* is a graph $G = (V, E)$ defined over a universe U_{off}, where:

- Each vertex $v \in V$ is assigned degrees $T(v)$, $I(v)$, and $F(v)$, with $T(v) \in [\Psi, \Omega]$ and $F(v) \in [\Psi, \Omega]$, where $\Omega > 1$ and $\Psi < 0$, allowing $T(v) > 1$ and $F(v) < 0$.

- Each edge $e = (u, v) \in E$ is assigned degrees $T(e) \in [\Psi, \Omega]$, $I(e) \in [\Psi, \Omega]$, and $F(e) \in [\Psi, \Omega]$.
- For all $v \in V$, $T(v) + I(v) + F(v) \leq 3\Omega$.

Definition 34 (Plithogenic OffGraph)**.** [26] A *Plithogenic OffGraph* is a Plithogenic Graph where the membership degrees μ_{A_i} can both exceed 1 and be less than 0. That is, $\mu_{A_i}(x) \in [\Psi_i, \Omega_i]$ with $\Psi_i < 0$ and $\Omega_i > 1$ for all attributes A_i and elements $x \in V \cup E$.

Definition 35 (Plithogenic Triangle-Free OffGraph)**.** A Plithogenic OffGraph $G = (V, adf_V, E, adf_E)$ is called a *Plithogenic Triangle-Free OffGraph* if for all distinct vertices $u, v, w \in V$, at least one of the following holds:

$$adf_E(u, v) = \mathbf{0}, \quad adf_E(v, w) = \mathbf{0}, \quad adf_E(w, u) = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector in $[\Psi, \Omega]^s$.

Definition 36 (Plithogenic Claw-Free OffGraph)**.** A Plithogenic OffGraph $G = (V, adf_V, E, adf_E)$ is called a *Plithogenic Claw-Free OffGraph* if there does not exist a vertex $v \in V$ and three distinct vertices $u_1, u_2, u_3 \in V \setminus \{v\}$ such that:

$$\begin{aligned} adf_E(v, u_i) \neq \mathbf{0} &\quad \text{for } i = 1, 2, 3, \\ adf_E(u_i, u_j) = \mathbf{0} &\quad \text{for all } i \neq j. \end{aligned}$$

Definition 37 (Plithogenic AT-Free OffGraph)**.** A Plithogenic OffGraph $G = (V, adf_V, E, adf_E)$ is called a *Plithogenic AT-Free OffGraph* if it does not contain any *Plithogenic Asteroidal Triple*. This is defined as a set of three distinct vertices $\{u, v, w\} \subseteq V$ such that for any two vertices, there exists a path P from u to v with edge attribute degrees not equal to $\mathbf{0}$, avoiding the *Plithogenic Closed Neighborhood* of the third vertex w.

Example 38. (cf.[30]) The following examples of Plithogenic Triangle-Free (respectively, AT-Free, Claw-Free) Graph are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) Graph.*
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 7, t = 1$, PG is called a *Plithogenic Heptapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Triangle-Free (respectively, AT-Free, Claw-Free) OffGraph.*

Theorem 39. *A Plithogenic Triangle-Free OffGraph, Plithogenic Claw-Free OffGraph, and Plithogenic AT-Free OffGraph are all Plithogenic OffGraphs.*

Proof: (1) **Plithogenic Triangle-Free OffGraph:** By definition, a Plithogenic Triangle-Free OffGraph ensures that for any three distinct vertices u, v, w, at least one of the edge membership degrees

$$adf_E(u, v), adf_E(v, w), adf_E(w, u)$$

is $\mathbf{0}$. Since the membership degrees adf_E in a Plithogenic OffGraph are allowed to be in the range $[\Psi, \Omega]$, with $\Psi < 0$ and $\Omega > 1$, this definition directly satisfies the conditions of a Plithogenic OffGraph.

(2) **Plithogenic Claw-Free OffGraph:** For a graph to be a Plithogenic Claw-Free OffGraph, no vertex v can have three distinct neighbors u_1, u_2, u_3 such that $adf_E(v, u_i) \neq \mathbf{0}$ for $i = 1, 2, 3$, while $adf_E(u_i, u_j) = \mathbf{0}$ for all $i \neq j$. This condition is satisfied under the Plithogenic OffGraph framework since the membership degrees adf_E are defined in $[\Psi, \Omega]$, and the claw-free property imposes additional constraints on the connectivity structure without violating the range of adf_E.

(3) **Plithogenic AT-Free OffGraph:** A Plithogenic AT-Free OffGraph prohibits the existence of a Plithogenic Asteroidal Triple $\{u, v, w\}$, where paths between any two vertices u, v avoid the closed neighborhood of the third vertex w. The edge membership degrees adf_E in this structure remain within the range $[\Psi, \Omega]$, and the absence of such triples is a structural constraint that does not conflict with the Plithogenic OffGraph definition. Therefore, a Plithogenic AT-Free OffGraph is also a Plithogenic OffGraph.

Hence, Plithogenic Triangle-Free OffGraphs, Plithogenic Claw-Free OffGraphs, and Plithogenic AT-Free OffGraphs are all valid subclasses of Plithogenic OffGraphs. □

Theorem 40. *If the membership degree ranges* $[\Psi, \Omega]$ *in a Plithogenic Triangle-Free OffGraph, Plithogenic Claw-Free OffGraph, or Plithogenic AT-Free OffGraph are restricted to the interval* $[0, 1]$*, then the resulting structures correspond to a Plithogenic Triangle-Free Graph, Plithogenic Claw-Free Graph, or Plithogenic AT-Free Graph, respectively.*

Proof: (1) **Plithogenic Triangle-Free OffGraph to Plithogenic Triangle-Free Graph:**

A Plithogenic Triangle-Free OffGraph ensures that for any three distinct vertices $u, v, w \in V$, at least one of the edge membership degrees $adf_E(u, v), adf_E(v, w), adf_E(w, u)$ is $\mathbf{0}$, where $adf_E \in [\Psi, \Omega]$. If the membership range is restricted to $[0, 1]$, then the structure satisfies the definition of a Plithogenic Triangle-Free Graph, as the membership degrees now represent valid fuzzy membership values, and the triangle-free property remains unchanged.

(2) **Plithogenic Claw-Free OffGraph to Plithogenic Claw-Free Graph:**

A Plithogenic Claw-Free OffGraph prohibits the existence of a vertex $v \in V$ and three distinct neighbors $u_1, u_2, u_3 \in V \setminus \{v\}$ such that $adf_E(v, u_i) \neq \mathbf{0}$ for $i = 1, 2, 3$, while $adf_E(u_i, u_j) = \mathbf{0}$ for all $i \neq j$, with membership degrees $adf_E \in [\Psi, \Omega]$. Restricting adf_E to $[0, 1]$ retains the claw-free property while ensuring that the structure conforms to the definition of a Plithogenic Claw-Free Graph.

(3) **Plithogenic AT-Free OffGraph to Plithogenic AT-Free Graph:**

A Plithogenic AT-Free OffGraph prohibits the existence of a Plithogenic Asteroidal Triple $\{u, v, w\}$, where paths between any two vertices u, v avoid the closed neighborhood of the third vertex w, and $adf_E \in [\Psi, \Omega]$. When the membership degrees are restricted to $[0, 1]$, this structure satisfies the definition of a Plithogenic AT-Free Graph, as the absence of asteroidal triples is a structural constraint independent of the specific range of membership degrees, provided they are valid fuzzy values in $[0, 1]$.

By restricting the membership degree range $[\Psi, \Omega]$ to $[0, 1]$, the Plithogenic OffGraph definitions align perfectly with the corresponding Plithogenic Graph definitions for triangle-free, claw-free, and AT-free structures. □

4 Future Tasks

This section outlines the future directions of our research. We plan to explore biregular [71], triregular [48], quartic [53], and biquartic graphs [7] within the frameworks of Fuzzy, Neutrosophic, and Plithogenic Graphs.

These graphs extend the conventional definitions of biregular, triregular, quartic, and biquartic graphs by incorporating the structural and uncertainty-based properties of Fuzzy, Neutrosophic, and Plithogenic Graphs.

Additionally, we aim to extend our study to Hypergraphs [11, 19, 9] and other advanced graph structures, including Soft Graphs[40, 6, 62, 12], Bipolar Fuzzy Graphs[1, 49, 4] , Rough Graphs[21, 31, 3, 51], Hypersoft Graphs[58, 65, 57], and SuperHyperGraphs [23, 24, 68, 69], thereby broadening the applicability of these frameworks.

Declarations

Ethics Approval and Consent to Participate

The results, data, and figures presented in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and no permissions are required.

Consent for Publication

This article does not contain any studies involving human participants or animals performed by any of the authors.

Availability of Data and Materials

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in this research.

Funding

This research was not supported by any funding agency or institute.

Author Contribution

All authors contributed equally to this research.

Acknowledgment

The author expresses gratitude to the editorial team and reviewers, as well as the corresponding author, for their valuable advice, assessment, and review during the study period.

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Received: 01 Aug 2024, **Revised:** 02 Feb 2025,
Accepted: 03 Mar 2025, **Available online:** 05 Mar 2025.