Information Sciences with Applications, Vol. 6, 2025

https://doi.org/10.61356/j.iswa.2025.6546/

A Unified Mathematical Framework for Fuzzy Customer Relationship Management (FCRM) and Neutrosophic Human Resource Management (NHRM)

Takaaki Fujita[1*]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan; t171d603@gunma-u.ac.jp.

Abstract: Numerous frameworks have been developed to address uncertainty in various domains. Among the most prominent are Fuzzy Sets, Rough Sets, Hyperrough Sets, Vague Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, Plithogenic Sets, as well as other emerging theories that continue to be actively explored.

Customer Relationship Management (CRM) is a structured methodology for managing and analyzing interactions with current and prospective customers, placing strong emphasis on data-driven insights to improve loyalty, satisfaction, and profitability. Human Resource Management (HRM), on the other hand, focuses on organizing, developing, and optimizing employee performance, well-being, and interpersonal dynamics within an organization.

In this paper, we introduce rigorous Mathematical Frameworks for Fuzzy Customer Relationship Management (FCRM), Neutrosophic Customer Relationship Management (NCRM), Fuzzy Human Resource Management (FHRM), and Neutrosophic Human Resource Management (NHRM), which integrate uncertainty-oriented paradigms with established CRM and HRM practices to enhance decision-making, adaptability, and overall organizational efficiency.

Keywords: Fuzzy set, Neutrosophic Set, Customer Relationship Management, CRM, Human Resource Management, HRM

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

1.1 Fuzzy Set and Neutrosophic Set

A fuzzy set assigns to each element a membership degree in the interval $[0, 1]$, thereby capturing uncertainty through more granular membership levels rather than a strict binary classification [1, 2, 3, 4, 5, 6, 7]. Below, we present the relevant definitions, including those for these extended frameworks.

Definition 1 (Set). [8] A *set* is a well-defined collection of distinct elements or objects. If a is an element of a set A, we write $a \in A$; otherwise, we write $a \notin A$.

Definition 2 (Subset). [8] Let A and B be sets. A is called a *subset* of B, denoted $A \subseteq B$, if every element of A is also an element of B. If $A \subseteq B$ but $A \neq B$, then A is called a *proper subset* of B, denoted $A \subset B$.

Definition 3 (Empty Set). [8] The *empty set*, denoted by $\emptyset$, is the unique set containing no elements. Formally, for any set A, $\emptyset \subseteq A$.

Definition 4 (Universal Set)**.** A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

Definition 5 (Fuzzy Set)**.** [1, 9] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 6 (Real-World Example of a Fuzzy Set)**.** Let Y be the set of room temperatures (in °C). Define the fuzzy set "Comfortable" by

$$\mu_{\text{Comfort}}(t) = \begin{cases} 0, & t \leq 16, \\ \frac{t-16}{6}, & 16 < t < 22, \\ 1, & 22 \leq t \leq 26, \\ \frac{32-t}{6}, & 26 < t < 32, \\ 0, & t \geq 32. \end{cases}$$

Then, for example,

$$\mu_{\text{Comfort}}(18) = \tfrac{2}{6} \approx 0.33, \quad \mu_{\text{Comfort}}(24) = 1, \quad \mu_{\text{Comfort}}(30) = \tfrac{2}{6} \approx 0.33.$$

This models the gradual transition between "too cold," "comfortable," and "too warm."

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, thereby addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Their definitions are presented below.

Definition 7 (Neutrosophic Set)**.** [20, 10] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 8 (Real-World Example of a Neutrosophic Set)**.** (cf.[21, 22]) Let X be a set of patients undergoing a diagnostic test for disease D. Define a neutrosophic set A of "Likely D" by assigning for each patient $x \in X$:

$$T_A(x) = \text{degree test indicates } D, \quad I_A(x) = \text{degree of inconclusive result}, \quad F_A(x) = \text{degree test indicates no } D.$$

For instance, for three patients:

$$\begin{aligned} x_1 : &\quad (T_A, I_A, F_A) = (0.8, 0.1, 0.1), \\ x_2 : &\quad (T_A, I_A, F_A) = (0.4, 0.5, 0.1), \\ x_3 : &\quad (T_A, I_A, F_A) = (0.1, 0.2, 0.7). \end{aligned}$$

Here x_1 is likely diseased, x_2 is largely indeterminate, and x_3 is likely healthy.

2 Result of This Paper

The results of this paper are presented as follows.

2.1 Mathematical Model of Customer Relationship Management (CRM)

Customer Relationship Management (CRM) is a strategy to manage customer interactions, improve satisfaction, and drive business growth through data[23, 24, 25, 26]. This can be deliberately defined in mathematical terms as follows.

Definition 9 (Concrete Mathematical Model of Customer Relationship Management (CRM))**.** A CRM system is modeled by the septuple

$$\mathrm{CRM} = (C,\, P,\, T,\, I,\, Q_p,\, Q_s,\, f_w),$$

where

- C is a finite set of *customers*, e.g. $C = \{\mathrm{Alice}, \mathrm{Bob}\}$;
- P is a finite set of *products or services*, e.g. $P = \{\mathrm{Basic}, \mathrm{Premium}\}$;
- T is a totally ordered set of discrete *time–stamps*, e.g. $T = \{2025{-}04{-}01,\ 2025{-}04{-}15,\ 2025{-}04{-}20\}$;
- $I \subseteq C \times P \times T$ is the *interaction log*; each $(c, p, t) \in I$ records that customer c interacted with product p at time t;
- $Q_p : I \to \mathbb{N}$ is the *purchase quantity function* (number of units bought in that interaction);
- $Q_s : I \to [0, 1]$ is the *support-satisfaction function* (customer's post-interaction satisfaction rating from support, normalized to $[0, 1]$);
- $f_w : I \to \mathbb{R}_{\geq 0}$ is a *weighted-score function* defined by

  $$f_w(c, p, t) = w_p\, Q_p(c, p, t) \;+\; w_s\, Q_s(c, p, t),$$

  where $w_p, w_s \geq 0$ are preassigned weights with $w_p + w_s = 1$.

For each customer $c \in C$, their *aggregate satisfaction* is

$$S(c) \;=\; \frac{1}{\big|\{(c, p, t) \in I\}\big|} \sum_{(c,p,t)\in I} f_w(c, p, t).$$

Theorem 10 (Weighted-Average Representation of Aggregate Satisfaction)**.** *Let*

$$I_c = \{(c, p, t) \in I\} \quad \textit{and} \quad n_c = |I_c|.$$

Define the average purchase quantity and average satisfaction for customer c by

$$A_p(c) = \frac{1}{n_c} \sum_{(c,p,t)\in I_c} Q_p(c, p, t), \qquad A_s(c) = \frac{1}{n_c} \sum_{(c,p,t)\in I_c} Q_s(c, p, t).$$

Then the aggregate satisfaction $S(c) = \frac{1}{n_c} \sum_{(c,p,t)\in I_c} f_w(c, p, t)$ can be written as

$$S(c) \;=\; w_p\, A_p(c) \;+\; w_s\, A_s(c).$$

Proof: By definition,

$$S(c) = \frac{1}{n_c} \sum_{(c,p,t)\in I_c} f_w(c, p, t) = \frac{1}{n_c} \sum_{(c,p,t)\in I_c} \big(w_p\, Q_p(c, p, t) + w_s\, Q_s(c, p, t)\big).$$

Since w_p, w_s are constants with $w_p + w_s = 1$, we may distribute:

$$S(c) = \; w_p \frac{1}{n_c} \sum_{(c,p,t)\in I_c} Q_p(c, p, t) \;+\; w_s \frac{1}{n_c} \sum_{(c,p,t)\in I_c} Q_s(c, p, t) = w_p\, A_p(c) + w_s\, A_s(c),$$

as claimed. □

Theorem 11 (Monotonicity)**.** *Suppose two CRM interaction-logs I^1 and I^2 for the same customer c satisfy*

$$Q_p^1(c, p, t) \geq Q_p^2(c, p, t) \quad \textit{and} \quad Q_s^1(c, p, t) \geq Q_s^2(c, p, t) \quad \forall\ (c, p, t) \in I^1 \cap I^2.$$

Then the corresponding aggregate satisfactions obey $S^1(c) \geq S^2(c)$.

Proof: From the weighted-average form,

$$S^i(c) = w_p\, A_p^i(c) + w_s\, A_s^i(c), \quad i = 1, 2.$$

By hypothesis $A_p^1(c) \ge A_p^2(c)$ and $A_s^1(c) \ge A_s^2(c)$, and since $w_p, w_s \ge 0$, it follows that

$$S^1(c) - S^2(c) = w_p\big(A_p^1(c) - A_p^2(c)\big) + w_s\big(A_s^1(c) - A_s^2(c)\big) \ge 0,$$

hence $S^1(c) \ge S^2(c)$. □

Theorem 12 (Bounds on Aggregate Satisfaction)**.** *Let* $m_c = \min_{(c,p,t)\in I_c} f_w(c,p,t)$ *and* $M_c = \max_{(c,p,t)\in I_c} f_w(c,p,t)$. *Then the aggregate satisfaction* $S(c)$ *satisfies*

$$m_c \;\le\; S(c) \;\le\; M_c.$$

Proof: Since $S(c)$ is the arithmetic mean of the real values $\{f_w(c,p,t)\}_{(c,p,t)\in I_c}$, the well-known inequality for means gives

$$\min_{(c,p,t)\in I_c} f_w(c,p,t) \;\le\; \frac{1}{n_c} \sum_{(c,p,t)\in I_c} f_w(c,p,t) \;\le\; \max_{(c,p,t)\in I_c} f_w(c,p,t),$$

i.e. $m_c \le S(c) \le M_c$. □

Example 13. Let

$$C = \{\text{Alice, Bob}\}, \quad P = \{\text{Basic, Premium}\}, \quad T = \{2025{-}04{-}01,\ 2025{-}04{-}15,\ 2025{-}04{-}20\},$$

and record the following interactions:

$$I = \big\{(\text{Alice}, \text{Basic}, 2025{-}04{-}01),\ (\text{Alice}, \text{Basic}, 2025{-}04{-}15),\ (\text{Bob}, \text{Premium}, 2025{-}04{-}20)\big\}.$$

Suppose

$$Q_p(\text{Alice}, \text{Basic}, 2025{-}04{-}01) = 2, \quad Q_s(\text{Alice}, \text{Basic}, 2025{-}04{-}01) = 0.7,$$

$$Q_p(\text{Alice}, \text{Basic}, 2025{-}04{-}15) = 1, \quad Q_s(\text{Alice}, \text{Basic}, 2025{-}04{-}15) = 0.9,$$

$$Q_p(\text{Bob}, \text{Premium}, 2025{-}04{-}20) = 3, \quad Q_s(\text{Bob}, \text{Premium}, 2025{-}04{-}20) = 0.6.$$

Choose weights

$$w_p = 0.6, \quad w_s = 0.4.$$

Then

$$f_w(\text{Alice}, \text{Basic}, 2025{-}04{-}01) = 0.6 \cdot 2 + 0.4 \cdot 0.7 = 1.2 + 0.28 = 1.48,$$

$$f_w(\text{Alice}, \text{Basic}, 2025{-}04{-}15) = 0.6 \cdot 1 + 0.4 \cdot 0.9 = 0.6 + 0.36 = 0.96,$$

$$f_w(\text{Bob}, \text{Premium}, 2025{-}04{-}20) = 0.6 \cdot 3 + 0.4 \cdot 0.6 = 1.8 + 0.24 = 2.04.$$

Therefore

$$S(\text{Alice}) = \frac{1.48 + 0.96}{2} = 1.22, \qquad S(\text{Bob}) = \frac{2.04}{1} = 2.04.$$

Interpretation: Alice's average interaction score is $1.22, while Bob's is $2.04 under this CRM model.

2.2 Fuzzy Customer Relationship Management (Fuzzy CRM)

The definition of Fuzzy Customer Relationship Management (Fuzzy CRM) is presented as follows. There are several studies in which Customer Relationship Management and fuzzy logic are investigated together, and based on these observations, it is considered natural to define Fuzzy CRM in a mathematical manner (e.g., [27, 28, 29]).

Definition 14 (Fuzzy Customer Relationship Management (Fuzzy CRM))**.** Let

$$C,\ P,\ T$$

be finite sets of customers, products (or services), and discrete time-points respectively. A *Fuzzy CRM system* is a septuple

$$\mathrm{FCRM} = (C,\ P,\ T,\ \mu_I,\ Q_p,\ Q_s,\ w_p, w_s),$$

equipped with:

- a *fuzzy interaction relation* $\mu_I : C \times P \times T \longrightarrow [0,1]$, where $\mu_I(c,p,t)$ denotes the degree to which customer c's interaction with product p at time t is "relevant";
- a *purchase-quantity function* $Q_p : C \times P \times T \longrightarrow \mathbb{N}$, giving the number of units purchased in that interaction;
- a *support-satisfaction function* $Q_s : C \times P \times T \longrightarrow [0,1]$, giving the customer's satisfaction rating (normalized) after that interaction;
- nonnegative *weights* w_p, w_s with $w_p + w_s = 1$.

Define the *normalized interaction score*

$$\tilde{f}(c,p,t) \;=\; w_p\, \frac{Q_p(c,p,t)}{Q_{\max}} \;+\; w_s\, Q_s(c,p,t),$$

where $Q_{\max} = \max_{c,p,t} Q_p(c,p,t)$. Then the *fuzzy satisfaction degree* of $c \in C$ is the fuzzy average

$$S_{\mathrm{FCRM}}(c) \;=\; \frac{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I(c,p,t)\ \tilde{f}(c,p,t)}{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I(c,p,t)} \;\in [0,1].$$

Hence S_{FCRM} is a fuzzy set on C.

Theorem 15. *The classical (crisp) Mathematical Model of CRM is obtained as a special case of Fuzzy CRM by*

$$\mu_I(c,p,t) \in \{0,1\}, \quad Q_{\max} = 1,$$

and identifying $\tilde{f}(c,p,t)$ with the original weighted score $f_w(c,p,t)$. In particular, when μ_I is the indicator of a crisp interaction log $I \subseteq C \times P \times T$, then

$$S_{\mathrm{FCRM}}(c) = \frac{1}{|\{(c,p,t) \in I\}|} \sum_{(c,p,t)\in I} f_w(c,p,t)$$

coincides with the classical satisfaction average.

Proof: Assume $\mu_I(c,p,t) = 1$ exactly when $(c,p,t) \in I$, and $Q_{\max} = 1$. Then by definition

$$\tilde{f}(c,p,t) = w_p\, Q_p(c,p,t) + w_s\, Q_s(c,p,t) = f_w(c,p,t)$$

and the denominator $\sum_{p,t} \mu_I(c,p,t) = |\{(c,p,t) \in I\}|$. Hence

$$S_{\mathrm{FCRM}}(c) = \frac{\sum_{p,t} \mu_I(c,p,t)\, \tilde{f}(c,p,t)}{\sum_{p,t} \mu_I(c,p,t)} = \frac{1}{|\{(c,p,t) \in I\}|} \sum_{(c,p,t)\in I} f_w(c,p,t),$$

which is exactly the classical CRM satisfaction score. This shows the crisp model embeds into Fuzzy CRM. Moreover, since $S_{\mathrm{FCRM}} : C \to [0,1]$, it endows C with a fuzzy-set structure. □

Theorem 16 (Decomposition into Component Averages)**.** *Define, for each $c \in C$,*

$$A_p(c) = \frac{\displaystyle\sum_{p\in P}\sum_{t\in T}\mu_I(c,p,t)\,\frac{Q_p(c,p,t)}{Q_{\max}}}{\displaystyle\sum_{p\in P}\sum_{t\in T}\mu_I(c,p,t)}, \qquad A_s(c) = \frac{\displaystyle\sum_{p\in P}\sum_{t\in T}\mu_I(c,p,t)\,Q_s(c,p,t)}{\displaystyle\sum_{p\in P}\sum_{t\in T}\mu_I(c,p,t)}.$$

Then the fuzzy satisfaction degree admits the linear decomposition

$$S_{\mathrm{FCRM}}(c) = w_p\,A_p(c) \;+\; w_s\,A_s(c).$$

Proof: By definition,

$$S_{\mathrm{FCRM}}(c) = \frac{\sum_{p,t}\mu_I(c,p,t)\,\tilde{f}(c,p,t)}{\sum_{p,t}\mu_I(c,p,t)} = \frac{\sum_{p,t}\mu_I(c,p,t)\,(w_p\,\frac{Q_p(c,p,t)}{Q_{\max}} + w_s\,Q_s(c,p,t))}{\sum_{p,t}\mu_I(c,p,t)}.$$

Since w_p, w_s are constants with $w_p + w_s = 1$, split the sum:

$$\begin{aligned} S_{\mathrm{FCRM}}(c) &= \;w_p\,\frac{\sum_{p,t}\mu_I(c,p,t)\,\frac{Q_p(c,p,t)}{Q_{\max}}}{\sum_{p,t}\mu_I(c,p,t)} \;+\; w_s\,\frac{\sum_{p,t}\mu_I(c,p,t)\,Q_s(c,p,t)}{\sum_{p,t}\mu_I(c,p,t)} \\ &= \;w_p\,A_p(c) \;+\; w_s\,A_s(c), \end{aligned}$$

as claimed. □

Theorem 17 (Bounds on Fuzzy Satisfaction)**.** *Let*

$$m(c) = \min_{p\in P,\; t\in T}\{\tilde{f}(c,p,t)\}, \qquad M(c) = \max_{p\in P,\; t\in T}\{\tilde{f}(c,p,t)\},$$

where the minima and maxima are taken over all (p,t) with $\mu_I(c,p,t) > 0$. Then

$$m(c) \;\le\; S_{\mathrm{FCRM}}(c) \;\le\; M(c).$$

Proof: Set

$$N(c) = \sum_{p,t}\mu_I(c,p,t)\,\tilde{f}(c,p,t), \qquad D(c) = \sum_{p,t}\mu_I(c,p,t).$$

Then

$$S_{\mathrm{FCRM}}(c) = \frac{N(c)}{D(c)}.$$

Since $\mu_I(c,p,t) \ge 0$, we have for each (p,t)

$$m(c)\,\mu_I(c,p,t) \;\le\; \mu_I(c,p,t)\,\tilde{f}(c,p,t) \;\le\; M(c)\,\mu_I(c,p,t).$$

Summing over all (p,t) yields

$$m(c)\,D(c) \;\le\; N(c) \;\le\; M(c)\,D(c).$$

Dividing through by $D(c) > 0$ gives

$$m(c) \;\le\; \frac{N(c)}{D(c)} \;\le\; M(c),$$

i.e. $m(c) \le S_{\mathrm{FCRM}}(c) \le M(c)$. □

Theorem 18 (Monotonicity in Interaction Scores)**.** *Let two FCRM systems share the same C, P, T, μ_I, w_p, w_s but have score functions $\tilde{f}^1, \tilde{f}^2$. If*

$$\tilde{f}^1(c,p,t) \;\ge\; \tilde{f}^2(c,p,t) \quad \textit{for all } (c,p,t),$$

then their fuzzy satisfaction degrees satisfy

$$S^1_{\mathrm{FCRM}}(c) \;\ge\; S^2_{\mathrm{FCRM}}(c) \quad \textit{for every } c \in C.$$

Proof: Write

$$N^i(c) = \sum_{p,t} \mu_I(c,p,t)\, \tilde{f}^i(c,p,t), \qquad D(c) = \sum_{p,t} \mu_I(c,p,t),$$

so $S^i(c) = N^i(c)/D(c)$, $i = 1,2$. Since $\mu_I \geq 0$ and $\tilde{f}^1 \geq \tilde{f}^2$, term by term $\mu_I\, \tilde{f}^1 \geq \mu_I\, \tilde{f}^2$. Summing gives

$$N^1(c) \;\geq\; N^2(c).$$

Dividing by the common positive denominator $D(c)$ yields

$$S^1_{\mathrm{FCRM}}(c) = \frac{N^1(c)}{D(c)} \;\geq\; \frac{N^2(c)}{D(c)} = S^2_{\mathrm{FCRM}}(c),$$

as required. □

Example 19. Let

$$C = \{\text{Alice}, \text{Bob}\}, \quad P = \{\text{Basic}, \text{Premium}\}, \quad T = \{\,\text{2025-04-01},\ \text{2025-04-15}\},$$

and suppose

$$Q_{\max} = 3, \quad w_p = 0.6, \quad w_s = 0.4.$$

Define

$$\mu_I(\text{Alice}, \text{Basic}, 2025-04-01) = 0.8, \quad Q_p = 2,\ Q_s = 0.7;$$
$$\mu_I(\text{Alice}, \text{Basic}, 2025-04-15) = 0.9, \quad Q_p = 1,\ Q_s = 0.9;$$
$$\mu_I(\text{Bob}, \text{Premium}, 2025-04-15) = 0.6, \quad Q_p = 3,\ Q_s = 0.6.$$

Then the normalized scores are

$$\tilde{f}(\text{Alice}, \text{Basic}, 1) = 0.6 \cdot \frac{2}{3} + 0.4 \cdot 0.7 = 0.68, \quad \tilde{f}(\text{Alice}, \text{Basic}, 2) = 0.6 \cdot \frac{1}{3} + 0.4 \cdot 0.9 = 0.56,$$
$$\tilde{f}(\text{Bob}, \text{Premium}, 2) = 0.6 \cdot 1 + 0.4 \cdot 0.6 = 0.84.$$

Thus

$$S_{\mathrm{FCRM}}(\text{Alice}) = \frac{0.8 \cdot 0.68 + 0.9 \cdot 0.56}{0.8 + 0.9} = \frac{0.544 + 0.504}{1.7} \approx 0.616, \quad S_{\mathrm{FCRM}}(\text{Bob}) = \frac{0.6 \cdot 0.84}{0.6} = 0.84.$$

We obtain fuzzy satisfaction degrees 0.616 for Alice and 0.84 for Bob.

2.3 Neutrosophic Customer Relationship Management (Neutrosophic CRM)

The definition of Neutrosophic Customer Relationship Management (Neutrosophic CRM) is presented as follows.

Definition 20 (Neutrosophic Customer Relationship Management (Neutrosophic CRM))**.** Let

$$C,\ P,\ T$$

be finite sets of customers, products (or services), and discrete time-points respectively. A *Neutrosophic CRM system* is a decuple

$$\mathrm{NCRM} = (C,\ P,\ T,\ \mu_I^T,\ \mu_I^I,\ \mu_I^F,\ Q_p,\ Q_s,\ w_p,\ w_s),$$

equipped with:

- three *neutrosophic interaction relations*
$$\mu_I^T,\ \mu_I^I,\ \mu_I^F : C \times P \times T \longrightarrow [0,1],$$
where for each interaction (c,p,t), $\mu_I^T(c,p,t)$, $\mu_I^I(c,p,t)$, $\mu_I^F(c,p,t)$ are the degrees of truth, indeterminacy, and falsity of that interaction;
- a *purchase-quantity function* $Q_p : C \times P \times T \longrightarrow \mathbb{N}$, giving the number of units purchased;
- a *support-satisfaction function* $Q_s : C \times P \times T \longrightarrow [0,1]$, giving the customer's normalized satisfaction after support;
- nonnegative *weights* $w_p, w_s \in [0,1]$ with $w_p + w_s = 1$.

Define the *normalized interaction score*

$$\tilde{f}(c,p,t) \;=\; w_p\,\frac{Q_p(c,p,t)}{Q_{\max}} \;+\; w_s\,Q_s(c,p,t),$$

where $Q_{\max} = \max\limits_{c,p,t} Q_p(c,p,t)$. Then for each $c \in C$ the *neutrosophic satisfaction degrees* are

$$S_T(c) = \frac{\sum_{p,t} \mu_I^T(c,p,t)\,\tilde{f}(c,p,t)}{\sum_{p,t} \mu_I^T(c,p,t)}, \quad S_I(c) = \frac{\sum_{p,t} \mu_I^I(c,p,t)\,\tilde{f}(c,p,t)}{\sum_{p,t} \mu_I^I(c,p,t)}, \quad S_F(c) = \frac{\sum_{p,t} \mu_I^F(c,p,t)\,\tilde{f}(c,p,t)}{\sum_{p,t} \mu_I^F(c,p,t)}.$$

These three numbers form the neutrosophic set $S_{\mathrm{NCRM}}(c) = (S_T(c),\, S_I(c),\, S_F(c)) \in [0,1]^3$.

Theorem 21. *The Fuzzy CRM model is recovered from Neutrosophic CRM by setting $\mu_I^I \equiv 0$ and $\mu_I^F(c,p,t) = 1 - \mu_I^T(c,p,t)$. In that case*

$$S_T(c) = S_{\mathrm{FCRM}}(c), \quad S_I(c) = 0, \quad S_F(c) = 1 - S_{\mathrm{FCRM}}(c).$$

Proof: If $\mu_I^I = 0$ and $\mu_I^F = 1 - \mu_I^T$, then the truth-component aggregator $S_T(c)$ coincides with the fuzzy-CRM satisfaction degree $S_{\mathrm{FCRM}}(c)$ since $\sum \mu_I^T = \sum \mu_I$ and $\tilde{f}$ is identical. Moreover $\sum \mu_I^I = 0$ implies $S_I(c)$ is undefined in numerator and denominator but is naturally taken as 0, and $\sum \mu_I^F = \sum (1 - \mu_I^T)$ yields $S_F(c) = 1 - S_T(c)$. Hence the classical fuzzy-CRM embedding holds. □

Theorem 22 (Component-wise Decomposition)**.** *For each $X \in \{T, I, F\}$ and $c \in C$, define*

$$A_p^X(c) = \frac{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I^X(c,p,t)\,\frac{Q_p(c,p,t)}{Q_{\max}}}{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I^X(c,p,t)}, \quad A_s^X(c) = \frac{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I^X(c,p,t)\,Q_s(c,p,t)}{\displaystyle\sum_{p\in P}\sum_{t\in T} \mu_I^X(c,p,t)}.$$

Then

$$S_X(c) = \; w_p\,A_p^X(c) \;+\; w_s\,A_s^X(c).$$

Proof: By definition,

$$S_X(c) = \frac{\sum_{p,t} \mu_I^X(c,p,t)\,\tilde{f}(c,p,t)}{\sum_{p,t} \mu_I^X(c,p,t)} = \frac{\sum_{p,t} \mu_I^X(c,p,t)\,(w_p\,\frac{Q_p}{Q_{\max}} + w_s\,Q_s)}{\sum_{p,t} \mu_I^X(c,p,t)}.$$

Since $w_p + w_s = 1$ are constants, split the sum:

$$\begin{aligned} S_X(c) &= w_p\,\frac{\sum \mu_I^X\,\frac{Q_p}{Q_{\max}}}{\sum \mu_I^X} + w_s\,\frac{\sum \mu_I^X\,Q_s}{\sum \mu_I^X} \\ &= w_p\,A_p^X(c) \;+\; w_s\,A_s^X(c). \end{aligned}$$

□

Theorem 23 (Bounds on Neutrosophic Satisfaction)**.** *For each $X \in \{T, I, F\}$ and $c \in C$, let*

$$m_X(c) = \min_{\substack{p\in P,\,t\in T \\ \mu_I^X(c,p,t)>0}} \tilde{f}(c,p,t), \quad M_X(c) = \max_{\substack{p\in P,\,t\in T \\ \mu_I^X(c,p,t)>0}} \tilde{f}(c,p,t).$$

Then

$$m_X(c) \;\le\; S_X(c) \;\le\; M_X(c).$$

Proof: Set

$$N_X(c) = \sum_{p,t} \mu_I^X(c,p,t)\,\tilde{f}(c,p,t), \quad D_X(c) = \sum_{p,t} \mu_I^X(c,p,t),$$

so $S_X(c) = N_X(c)/D_X(c)$. Since $\mu_I^X \ge 0$ and

$$m_X(c)\,\mu_I^X(c,p,t) \le \mu_I^X(c,p,t)\,\tilde{f}(c,p,t) \le M_X(c)\,\mu_I^X(c,p,t),$$

summing yields

$$m_X(c)\, D_X(c) \leq N_X(c) \leq M_X(c)\, D_X(c),$$

and dividing by $D_X(c) > 0$ gives the result. $\square$

Theorem 24 (Monotonicity). *Let two NCRM systems share the same $C, P, T, Q_p, Q_s, w_p, w_s$ but have scores $\tilde{f}^1$ and $\tilde{f}^2$. If*

$$\tilde{f}^1(c,p,t) \;\geq\; \tilde{f}^2(c,p,t) \quad \textit{for all } (c,p,t),$$

then for each $X \in \{T, I, F\}$ and $c \in C$,

$$S_X^1(c) \;\geq\; S_X^2(c).$$

Proof: For $i = 1, 2$, set

$$N_X^i(c) = \sum \mu_I^X(c,p,t)\, \tilde{f}^i(c,p,t), \quad D_X(c) = \sum \mu_I^X(c,p,t).$$

Since $\tilde{f}^1 \geq \tilde{f}^2$ and $\mu_I^X \geq 0$, term-wise $\mu_I^X\, \tilde{f}^1 \geq \mu_I^X\, \tilde{f}^2$, so $N_X^1(c) \geq N_X^2(c)$. Dividing by the common $D_X(c) > 0$ gives $S_X^1(c) \geq S_X^2(c)$. $\square$

Theorem 25 (Idempotence). *If for some constant $k \in [0,1]$ we have $\tilde{f}(c,p,t) = k$ whenever $\mu_I^X(c,p,t) > 0$, then*

$$S_X(c) = k \quad \textit{for each } X \in \{T, I, F\}.$$

Proof: Under the assumption,

$$N_X(c) = \sum_{p,t} \mu_I^X(e,r,t)\, k = k \sum_{p,t} \mu_I^X(e,r,t) = k\, D_X(c),$$

hence $S_X(c) = N_X(c)/D_X(c) = k$. $\square$

Example 26. Let

$$C = \{\text{Alice}, \text{Bob}\}, \quad P = \{\text{Basic}, \text{Premium}\}, \quad T = \{2025\text{—}04\text{—}01,\ 2025\text{—}04\text{—}15\},$$

with $Q_{\max} = 3,\ w_p = 0.6,\ w_s = 0.4$. Define for Alice's Basic interactions:

$$\mu_I^T = 0.8,\ \mu_I^I = 0.1,\ \mu_I^F = 0.1, \quad Q_p = 2,\ Q_s = 0.7$$

at 2025-04-01, and

$$\mu_I^T = 0.9,\ \mu_I^I = 0.0,\ \mu_I^F = 0.1, \quad Q_p = 1,\ Q_s = 0.9$$

at 2025-04-15. For Bob's Premium at 2025-04-15:

$$\mu_I^T = 0.6,\ \mu_I^I = 0.1,\ \mu_I^F = 0.3, \quad Q_p = 3,\ Q_s = 0.6.$$

Compute normalized scores:

$$\tilde{f}(\text{Alice}, \text{Basic}, 1) = 0.6 \cdot \frac{2}{3} + 0.4 \cdot 0.7 = 0.68, \quad \tilde{f}(\text{Alice}, \text{Basic}, 2) = 0.6 \cdot \frac{1}{3} + 0.4 \cdot 0.9 = 0.56,$$

$$\tilde{f}(\text{Bob}, \text{Premium}, 2) = 0.6 \cdot 1 + 0.4 \cdot 0.6 = 0.84.$$

Then

$$S_T(\text{Alice}) = \frac{0.8 \cdot 0.68 + 0.9 \cdot 0.56}{0.8 + 0.9} \approx 0.616, \quad S_I(\text{Alice}) = \frac{0.1 \cdot 0.68 + 0 \cdot 0.56}{0.1} = 0.68,$$

$$S_F(\text{Alice}) = \frac{0.1 \cdot 0.68 + 0.1 \cdot 0.56}{0.1 + 0.1} = 0.624;$$

$$S_T(\text{Bob}) = \frac{0.6 \cdot 0.84}{0.6} = 0.84, \quad S_I(\text{Bob}) = \frac{0.1 \cdot 0.84}{0.1} = 0.84, \quad S_F(\text{Bob}) = \frac{0.3 \cdot 0.84}{0.3} = 0.84.$$

Thus Alice's neutrosophic satisfaction is (0.616, 0.68, 0.624) and Bob's is (0.84, 0.84, 0.84).

2.4 Mathematical Model of Human Resource Management (HRM)

Human Resource Management (HRM) involves recruiting, developing, and managing people within an organization to optimize performance and employee satisfaction [30, 31, 32, 33]. The definition of Human Resource Management (HRM) is presented as follows.

Definition 27 (Mathematical Model of Human Resource Management (HRM))**.** A Human Resource Management system is formalized as a sextuple

$$\mathrm{HRM} = (E,\, R,\, T,\, I,\, f,\, g),$$

where

- E is a finite set of *employees*, e.g. $E = \{\text{Alice}, \text{Bob}\}$;
- R is a finite set of *roles* or *tasks*, e.g. $R = \{\text{Developer}, \text{Manager}\}$;
- T is a totally ordered set of discrete *time-points*, e.g. $T = \{2025-04-01,\, 2025-04-15\}$;
- $I \subseteq E \times R \times T$ is the *assignment log*, each $(e, r, t) \in I$ indicating that employee e performed role r at time t;
- $f : I \to \mathbb{R}_{\geq 0}$ is a *performance–rating function* assigning to each logged assignment a nonnegative score;
- $g : \bigsqcup_{e \in E} \{\, f(e, r, t) \mid (e, r, t) \in I\} \;\to\; \mathbb{R}_{\geq 0}$ is an *aggregate–performance function* which for each $e \in E$ computes a summary (e.g. arithmetic mean) of all f–values corresponding to e.

The *aggregate performance* of employee $e \in E$ is then

$$P(e) \;=\; g(\{\, f(e, r, t) \mid (e, r, t) \in I\}).$$

Theorem 28 (Arithmetic-Mean Representation)**.** *Suppose the aggregate–performance function g is the arithmetic mean; that is, for each $e \in E$,*

$$g(\{\, f(e, r, t) \mid (e, r, t) \in I\}) = \frac{1}{|I_e|} \sum_{(e,r,t) \in I_e} f(e, r, t),$$

where $I_e = \{(e, r, t) \in I\}$. Then

$$P(e) = \frac{1}{|I_e|} \sum_{(e,r,t) \in I_e} f(e, r, t).$$

Proof: By definition of P and the assumption on g,

$$P(e) = g(\{f(e, r, t) \mid (e, r, t) \in I\}) = \frac{1}{|I_e|} \sum_{(e,r,t) \in I_e} f(e, r, t),$$

as required. □

Theorem 29 (Bounds on Aggregate Performance)**.** *Let*

$$m_e = \min_{(e,r,t) \in I_e} f(e, r, t), \quad M_e = \max_{(e,r,t) \in I_e} f(e, r, t).$$

If g is the arithmetic mean as above, then

$$m_e \;\leq\; P(e) \;\leq\; M_e.$$

Proof: Since $P(e)$ is the average of the finitely many values $\{f(e, r, t)\}_{(e,r,t) \in I_e}$, the standard inequality for arithmetic means gives

$$\min_{(e,r,t) \in I_e} f(e, r, t) \;\leq\; \frac{1}{|I_e|} \sum_{(e,r,t) \in I_e} f(e, r, t) \;\leq\; \max_{(e,r,t) \in I_e} f(e, r, t),$$

i.e. $m_e \leq P(e) \leq M_e$. □

Theorem 30 (Monotonicity)**.** *Let two HRM models share the same sets E, R, T and assignment log I, and let their performance ratings be f^1 and f^2. If*

$$f^1(e,r,t) \;\geq\; f^2(e,r,t) \quad \textit{for all } (e,r,t) \in I,$$

then their aggregate performances satisfy

$$P^1(e) \;\geq\; P^2(e) \quad \textit{for every } e \in E.$$

Proof: For each e, write

$$P^i(e) = \frac{1}{|I_e|} \sum_{(e,r,t)\in I_e} f^i(e,r,t), \quad i = 1,2.$$

Since $f^1(e,r,t) \geq f^2(e,r,t)$ for each term and the denominator $|I_e| > 0$ is common, summing yields

$$\sum_{(e,r,t)\in I_e} f^1(e,r,t) \;\geq\; \sum_{(e,r,t)\in I_e} f^2(e,r,t),$$

and dividing by $|I_e|$ gives $P^1(e) \geq P^2(e)$. □

Theorem 31 (Idempotence)**.** *If an employee $e \in E$ has a constant performance rating*

$$f(e,r,t) = k \quad \textit{for all } (e,r,t) \in I_e,$$

then

$$P(e) = k.$$

Proof: Under the arithmetic-mean assumption,

$$P(e) = \frac{1}{|I_e|} \sum_{(e,r,t)\in I_e} f(e,r,t) = \frac{1}{|I_e|} \sum_{(e,r,t)\in I_e} k = \frac{k\,|I_e|}{|I_e|} = k.$$

□

Example 32. Let

$$E = \{\text{Alice, Bob}\}, \quad R = \{\text{Developer, Manager}\},$$

$$T = \{2025-04-01,\ 2025-04-15\},$$

and record these assignments:

$$I = \big\{(\text{Alice}, \text{Developer}, 2025-04-01),\ (\text{Alice}, \text{Developer}, 2025-04-15),$$

$$(\text{Bob}, \text{Manager}, 2025-04-15)\big\}.$$

Define the performance–rating function f by

$$f(\text{Alice}, \text{Developer}, 2025-04-01) = 85, \quad f(\text{Alice}, \text{Developer}, 2025-04-15) = 92,$$

$$f(\text{Bob}, \text{Manager}, 2025-04-15) = 78,$$

and let g compute the arithmetic mean. Then

$$P(\text{Alice}) = \frac{85+92}{2} = 88.5, \qquad P(\text{Bob}) = \frac{78}{1} = 78.$$

Thus under this HRM model, Alice's aggregate performance score is 88.5 and Bob's is 78.

2.5 Fuzzy Human Resource Management (Fuzzy HRM)

The definition of Fuzzy Human Resource Management (HRM) is presented as follows. There are several studies in which Human Resource Management (HRM) and fuzzy logic are investigated together, and based on these observations, it is considered natural to define Fuzzy HRM in a mathematical manner (e.g., [34, 35, 36]).

Definition 33 (Fuzzy Human Resource Management (Fuzzy HRM))**.** Let

$$E,\ R,\ T$$

be finite sets of employees, roles (or tasks), and discrete time-points respectively. A *Fuzzy HRM system* is an octuple

$$\mathrm{FHRM} = (E,\ R,\ T,\ \mu_A,\ Q_r,\ Q_p,\ w_r,\ w_p),$$

equipped with:

- a *fuzzy assignment relation* $\mu_A : E \times R \times T \longrightarrow [0,1]$, where $\mu_A(e,r,t)$ denotes the degree to which employee e performs role r at time t;
- a *role-proficiency function* $Q_r : E \times R \times T \longrightarrow [0,1]$, measuring normalized proficiency of e in role r at t;
- a *performance-rating function* $Q_p : E \times R \times T \longrightarrow [0,1]$, giving stakeholder satisfaction rating after that performance;
- nonnegative *weights* $w_r, w_p \in [0,1]$ with $w_r + w_p = 1$.

Define the *normalized performance score*

$$\tilde{f}(e,r,t) \;=\; w_r\, Q_r(e,r,t) \;+\; w_p\, Q_p(e,r,t).$$

Then the *fuzzy aggregate performance* of $e \in E$ is

$$P_{\mathrm{FHRM}}(e) \;=\; \frac{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)\, \tilde{f}(e,r,t)}{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)} \;\in [0,1].$$

Hence P_{FHRM} is a fuzzy set on E.

Theorem 34. *The crisp Mathematical Model of HRM is obtained as a special case of Fuzzy HRM by setting $\mu_A(e,r,t) \in \{0,1\}$ (indicator of a crisp assignment log) and choosing $Q_r(e,r,t) = 1$, $Q_p(e,r,t) = f(e,r,t)/f_{\max}$, $w_r = 1$, $w_p = 0$. Then*

$$P_{\mathrm{FHRM}}(e) = \frac{1}{|\{(e,r,t) \in I\}|} \sum_{(e,r,t)\in I} f(e,r,t)$$

recovers the classical aggregate performance $P(e)$ up to normalization by $f_{\max}$.

Proof: If μ_A is the indicator of $I \subseteq E \times R \times T$, then the denominator $\sum \mu_A = |I_e|$, the number of assignments of e. With $Q_r \equiv 1$, $w_r = 1$, $w_p = 0$, we have $\tilde{f}(e,r,t) = 1$, and choosing $Q_p(e,r,t) = f(e,r,t)/f_{\max}$ with $w_p = 0$ makes the numerator $\sum_{(e,r,t)\in I} f(e,r,t)/f_{\max}$. Hence

$$P_{\mathrm{FHRM}}(e) = \frac{1}{|I_e|} \sum_{(e,r,t)\in I} \frac{f(e,r,t)}{f_{\max}} = \frac{P(e)}{f_{\max}},$$

which coincides with the crisp HRM score up to the constant $1/f_{\max}$. This completes the embedding. □

Theorem 35 (Decomposition into Role-Proficiency and Performance Components)**.** *Define, for each $e \in E$,*

$$A_r(e) = \frac{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)\, Q_r(e,r,t)}{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)}, \quad A_p(e) = \frac{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)\, Q_p(e,r,t)}{\displaystyle\sum_{r\in R}\sum_{t\in T} \mu_A(e,r,t)}.$$

Then the fuzzy aggregate performance decomposes as

$$P_{\mathrm{FHRM}}(e) = w_r\, A_r(e) \;+\; w_p\, A_p(e).$$

Proof: By definition,

$$P_{\mathrm{FHRM}}(e)=\frac{\sum_{r,t}\mu_A(e,r,t)\,\tilde{f}(e,r,t)}{\sum_{r,t}\mu_A(e,r,t)}=\frac{\sum_{r,t}\mu_A(e,r,t)\,(w_r\,Q_r(e,r,t)+w_p\,Q_p(e,r,t))}{\sum_{r,t}\mu_A(e,r,t)}.$$

Since w_r, w_p are constants with $w_r + w_p = 1$, split the sum:

$$\begin{aligned}P_{\mathrm{FHRM}}(e)&=w_r\,\frac{\sum_{r,t}\mu_A(e,r,t)\,Q_r(e,r,t)}{\sum_{r,t}\mu_A(e,r,t)}+w_p\,\frac{\sum_{r,t}\mu_A(e,r,t)\,Q_p(e,r,t)}{\sum_{r,t}\mu_A(e,r,t)}\\&=w_r\,A_r(e)\;+\;w_p\,A_p(e),\end{aligned}$$

as required. □

Theorem 36 (Bounds on Fuzzy Aggregate Performance). *Let*

$$m_e=\min_{\substack{r\in R,\,t\in T\\ \mu_A(e,r,t)>0}}\tilde{f}(e,r,t),\quad M_e=\max_{\substack{r\in R,\,t\in T\\ \mu_A(e,r,t)>0}}\tilde{f}(e,r,t).$$

Then

$$m_e\;\le\;P_{\mathrm{FHRM}}(e)\;\le\;M_e.$$

Proof: Write

$$N(e)=\sum_{r,t}\mu_A(e,r,t)\,\tilde{f}(e,r,t),\quad D(e)=\sum_{r,t}\mu_A(e,r,t),$$

so $P_{\mathrm{FHRM}}(e) = N(e)/D(e)$. Since $\mu_A(e,r,t) \geq 0$ and

$$m_e\,\mu_A(e,r,t)\le\mu_A(e,r,t)\,\tilde{f}(e,r,t)\le M_e\,\mu_A(e,r,t)$$

for each (r,t), summing gives

$$m_e\,D(e)\le N(e)\le M_e\,D(e).$$

Dividing by $D(e) > 0$ yields the desired bounds. □

Theorem 37 (Monotonicity in Score Functions). *Let two FHRM systems share the same E, R, T, μ_A, w_r, w_p but have scores $\tilde{f}^1$ and $\tilde{f}^2$. If*

$$\tilde{f}^1(e,r,t)\;\ge\;\tilde{f}^2(e,r,t)\quad\textit{for all }(e,r,t),$$

then

$$P^1_{\mathrm{FHRM}}(e)\;\ge\;P^2_{\mathrm{FHRM}}(e)\quad\textit{for every }e\in E.$$

Proof: For $i = 1, 2$ set

$$N^i(e)=\sum_{r,t}\mu_A(e,r,t)\,\tilde{f}^i(e,r,t),\quad D(e)=\sum_{r,t}\mu_A(e,r,t).$$

Since $\tilde{f}^1 \geq \tilde{f}^2$ and $\mu_A \geq 0$, term-wise $\mu_A\,\tilde{f}^1 \geq \mu_A\,\tilde{f}^2$, so $N^1(e) \geq N^2(e)$. Dividing by the common $D(e) > 0$ gives

$$P^1_{\mathrm{FHRM}}(e)=\frac{N^1(e)}{D(e)}\ge\frac{N^2(e)}{D(e)}=P^2_{\mathrm{FHRM}}(e).$$

□

Theorem 38 (Idempotence). *If for some constant $k \in [0,1]$ one has $\tilde{f}(e,r,t) = k$ whenever $\mu_A(e,r,t) > 0$, then*

$$P_{\mathrm{FHRM}}(e)=k.$$

Proof: Under this assumption,

$$N(e) = \sum_{r,t} \mu_A(e,r,t)\, k = k \sum_{r,t} \mu_A(e,r,t) = k\, D(e),$$

hence

$$P_{\mathrm{FHRM}}(e) = \frac{N(e)}{D(e)} = \frac{k\, D(e)}{D(e)} = k,$$

as claimed. □

Example 39. Let

$$E = \{\text{Alice}, \text{Bob}\}, \quad R = \{\text{Dev}, \text{Mgr}\}, \quad T = \{\, 2025{-}04{-}01,\ 2025{-}04{-}15\},$$

and choose weights $w_r = 0.7,\ w_p = 0.3$. Suppose:

$$\mu_A(\text{Alice}, \text{Dev}, 1) = 0.9,\ Q_r = 0.8,\ Q_p = 0.6; \quad \mu_A(\text{Alice}, \text{Dev}, 2) = 0.7,\ Q_r = 0.9,\ Q_p = 0.8;$$

$$\mu_A(\text{Bob}, \text{Mgr}, 2) = 0.8,\ Q_r = 0.7,\ Q_p = 0.9.$$

Compute normalized scores:

$$\tilde{f}(\text{Alice}, \text{Dev}, 1) = 0.7 \cdot 0.8 + 0.3 \cdot 0.6 = 0.74, \quad \tilde{f}(\text{Alice}, \text{Dev}, 2) = 0.7 \cdot 0.9 + 0.3 \cdot 0.8 = 0.87,$$

$$\tilde{f}(\text{Bob}, \text{Mgr}, 2) = 0.7 \cdot 0.7 + 0.3 \cdot 0.9 = 0.76.$$

Thus

$$P_{\mathrm{FHRM}}(\text{Alice}) = \frac{0.9 \cdot 0.74 + 0.7 \cdot 0.87}{0.9 + 0.7} = \frac{0.666 + 0.609}{1.6} \approx 0.803,$$

$$P_{\mathrm{FHRM}}(\text{Bob}) = \frac{0.8 \cdot 0.76}{0.8} = 0.76.$$

We obtain fuzzy performance degrees 0.803 for Alice and 0.76 for Bob.

2.6 Neutrosophic Human Resource Management (Neutrosophic HRM)

The definition of Neutrosophic Human Resource Management (HRM) is presented as follows. There are several studies in which Human Resource Management (HRM) and Neutrosophic logic are investigated together, and based on these observations, it is considered natural to define Neutrosophic HRM in a mathematical manner (e.g., [37, 38, 39]).

Definition 40 (Neutrosophic Human Resource Management (Neutrosophic HRM))**.** Let

$$E,\ R,\ T$$

be finite sets of employees, roles (or tasks), and discrete time-points respectively. A *Neutrosophic HRM system* is a decuple

$$\mathrm{NHRM} = (E,\, R,\, T,\, \mu_A^T,\ \mu_A^I,\ \mu_A^F,\ Q_r,\ Q_p,\ w_r,\ w_p),$$

equipped with:

- three *neutrosophic assignment relations*

  $$\mu_A^T,\ \mu_A^I,\ \mu_A^F : E \times R \times T \longrightarrow [0,1],$$

  where for each (e,r,t), $\mu_A^T(e,r,t)$, $\mu_A^I(e,r,t)$, $\mu_A^F(e,r,t)$ are the degrees of truth, indeterminacy, and falsity of employee e performing role r at time t;
- a *role-proficiency function* $Q_r : E \times R \times T \longrightarrow [0,1]$, measuring normalized proficiency of e in r at t;
- a *performance-rating function* $Q_p : E \times R \times T \longrightarrow [0,1]$, giving stakeholder satisfaction after that performance;
- nonnegative *weights* $w_r, w_p \in [0,1]$ with $w_r + w_p = 1$.

Define the *normalized performance score*

$$\tilde{f}(e,r,t) \;=\; w_r\,Q_r(e,r,t) \;+\; w_p\,Q_p(e,r,t).$$

Then for each $e \in E$ the *neutrosophic aggregate performance* is given by

$$P_T(e) = \frac{\sum_{r,t}\mu_A^T(e,r,t)\,\tilde{f}(e,r,t)}{\sum_{r,t}\mu_A^T(e,r,t)}, \quad P_I(e) = \frac{\sum_{r,t}\mu_A^I(e,r,t)\,\tilde{f}(e,r,t)}{\sum_{r,t}\mu_A^I(e,r,t)}, \quad P_F(e) = \frac{\sum_{r,t}\mu_A^F(e,r,t)\,\tilde{f}(e,r,t)}{\sum_{r,t}\mu_A^F(e,r,t)}.$$

These three components form the neutrosophic satisfaction $S_{\mathrm{NHRM}}(e) = (P_T(e),\, P_I(e),\, P_F(e)) \in [0,1]^3$.

Theorem 41. *By setting $\mu_A^I \equiv 0$ and $\mu_A^F = 1 - \mu_A^T$, the Neutrosophic HRM reduces to the Fuzzy HRM model, with $\mu_A^T = \mu_A$. In particular,*

$$P_T(e) = P_{\mathrm{FHRM}}(e), \quad P_I(e) = 0, \quad P_F(e) = 1 - P_{\mathrm{FHRM}}(e).$$

Proof: If $\mu_A^I(e,r,t) = 0$ and $\mu_A^F(e,r,t) = 1 - \mu_A^T(e,r,t)$, then

$$P_T(e) = \frac{\sum \mu_A^T\,\tilde{f}}{\sum \mu_A^T} \equiv \frac{\sum \mu_A\,\tilde{f}}{\sum \mu_A} = P_{\mathrm{FHRM}}(e),$$

while $\sum \mu_A^I = 0$ gives $P_I(e) = 0$, and $\sum \mu_A^F = \sum(1 - \mu_A^T)$ yields $P_F(e) = 1 - P_T(e)$. Thus the claimed specialization holds. □

Theorem 42 (Component-wise Decomposition)**.** *For each $X \in \{T, I, F\}$ and $e \in E$, define*

$$A_r^X(e) = \frac{\displaystyle\sum_{r\in R}\sum_{t\in T}\mu_A^X(e,r,t)\,Q_r(e,r,t)}{\displaystyle\sum_{r\in R}\sum_{t\in T}\mu_A^X(e,r,t)}, \quad A_p^X(e) = \frac{\displaystyle\sum_{r\in R}\sum_{t\in T}\mu_A^X(e,r,t)\,Q_p(e,r,t)}{\displaystyle\sum_{r\in R}\sum_{t\in T}\mu_A^X(e,r,t)}.$$

Then

$$P_X(e) = \frac{\displaystyle\sum_{r,t}\mu_A^X(e,r,t)\,\tilde{f}(e,r,t)}{\displaystyle\sum_{r,t}\mu_A^X(e,r,t)} \;=\; w_r\,A_r^X(e) \;+\; w_p\,A_p^X(e).$$

Proof: By definition,

$$P_X(e) = \frac{\sum_{r,t}\mu_A^X(e,r,t)\,\bigl(w_r\,Q_r(e,r,t) + w_p\,Q_p(e,r,t)\bigr)}{\sum_{r,t}\mu_A^X(e,r,t)}.$$

Since w_r, w_p are constants with $w_r + w_p = 1$, split the sum:

$$\begin{aligned} P_X(e) &= w_r\,\frac{\sum \mu_A^X\,Q_r}{\sum \mu_A^X} + w_p\,\frac{\sum \mu_A^X\,Q_p}{\sum \mu_A^X} \\ &= w_r\,A_r^X(e) \;+\; w_p\,A_p^X(e), \end{aligned}$$

as required. □

Theorem 43 (Bounds on Neutrosophic Performance)**.** *For each $X \in \{T, I, F\}$ and $e \in E$, let*

$$m_X(e) = \min_{\substack{r\in R,t\in T\\ \mu_A^X(e,r,t)>0}} \tilde{f}(e,r,t), \quad M_X(e) = \max_{\substack{r\in R,t\in T\\ \mu_A^X(e,r,t)>0}} \tilde{f}(e,r,t).$$

Then

$$m_X(e) \;\le\; P_X(e) \;\le\; M_X(e).$$

Proof: Set

$$N_X(e) = \sum_{r,t} \mu_A^X(e,r,t)\, \tilde{f}(e,r,t), \quad D_X(e) = \sum_{r,t} \mu_A^X(e,r,t).$$

Since $\mu_A^X \geq 0$ and

$$m_X(e)\, \mu_A^X(e,r,t) \;\leq\; \mu_A^X(e,r,t)\, \tilde{f}(e,r,t) \;\leq\; M_X(e)\, \mu_A^X(e,r,t),$$

summing over r, t yields

$$m_X(e)\, D_X(e) \leq N_X(e) \leq M_X(e)\, D_X(e),$$

and dividing by $D_X(e) > 0$ gives the desired inequality. □

Theorem 44 (Monotonicity)**.** *Let two NHRM systems share the same $E, R, T, Q_r, Q_p, w_r, w_p$ but have scores $\tilde{f}^1$ and $\tilde{f}^2$. If*

$$\tilde{f}^1(e,r,t) \;\geq\; \tilde{f}^2(e,r,t) \quad \forall\,(e,r,t) \in E \times R \times T,$$

then for each $X \in \{T, I, F\}$ and $e \in E$,

$$P_X^1(e) \;\geq\; P_X^2(e).$$

Proof: For $i = 1, 2$, let

$$N_X^i(e) = \sum_{r,t} \mu_A^X(e,r,t)\, \tilde{f}^i(e,r,t), \quad D_X(e) = \sum_{r,t} \mu_A^X(e,r,t).$$

Since $\tilde{f}^1 \geq \tilde{f}^2$ and $\mu_A^X \geq 0$, term-wise $\mu_A^X\, \tilde{f}^1 \geq \mu_A^X\, \tilde{f}^2$, so $N_X^1(e) \geq N_X^2(e)$. Dividing by the common $D_X(e) > 0$ gives

$$P_X^1(e) = \frac{N_X^1(e)}{D_X(e)} \geq \frac{N_X^2(e)}{D_X(e)} = P_X^2(e).$$

□

Theorem 45 (Idempotence)**.** *If for some constant $k \in [0,1]$ one has $\tilde{f}(e,r,t) = k$ whenever $\mu_A^X(e,r,t) > 0$, then for each $X \in \{T, I, F\}$,*

$$P_X(e) = k.$$

Proof: Under this assumption,

$$N_X(e) = \sum_{r,t} \mu_A^X(e,r,t)\, k = k \sum_{r,t} \mu_A^X(e,r,t) = k\, D_X(e),$$

hence $P_X(e) = N_X(e)/D_X(e) = k.$ □

Example 46. Let

$$E = \{\text{Alice}, \text{Bob}\},\ R = \{\text{Dev}, \text{Mgr}\},\ T = \{2025{-}04{-}01, 2025{-}04{-}15\},$$

with weights $w_r = 0.7,\ w_p = 0.3$. Define:

Interaction	(μ^T, μ^I, μ^F)	(Q_r, Q_p)
$(\text{Alice}, \text{Dev}, 1)$	$(0.9, 0.05, 0.05)$	$(0.8, 0.6)$
$(\text{Alice}, \text{Dev}, 2)$	$(0.7, 0.2, 0.1)$	$(0.9, 0.85)$
$(\text{Bob}, \text{Mgr}, 2)$	$(0.6, 0.3, 0.1)$	$(0.7, 0.8)$

Compute normalized scores:

$$\tilde{f}(\text{Alice}, 1) = 0.7 \cdot 0.8 + 0.3 \cdot 0.6 = 0.74,$$
$$\tilde{f}(\text{Alice}, 2) = 0.7 \cdot 0.9 + 0.3 \cdot 0.85 = 0.885,$$
$$\tilde{f}(\text{Bob}, 2) = 0.7 \cdot 0.7 + 0.3 \cdot 0.8 = 0.73.$$

Then for Alice:

$$P_T = \frac{0.9 \cdot 0.74 + 0.7 \cdot 0.885}{0.9 + 0.7} \approx 0.8034, \quad P_I = \frac{0.05 \cdot 0.74 + 0.2 \cdot 0.885}{0.05 + 0.2} \approx 0.856,$$

$$P_F = \frac{0.05 \cdot 0.74 + 0.1 \cdot 0.885}{0.05 + 0.1} \approx 0.8367.$$

For Bob:

$$P_T = \frac{0.6 \cdot 0.73}{0.6} = 0.73, \quad P_I = \frac{0.3 \cdot 0.73}{0.3} = 0.73,$$

$$P_F = \frac{0.1 \cdot 0.73}{0.1} = 0.73.$$

Hence $S_{\mathrm{NHRM}}(\mathrm{Alice}) = (0.8034, 0.856, 0.8367)$ and $S_{\mathrm{NHRM}}(\mathrm{Bob}) = (0.73, 0.73, 0.73)$.

3 Conclusion

In this paper, we have introduced rigorous mathematical frameworks for Fuzzy Customer Relationship Management (FCRM), Neutrosophic Customer Relationship Management (NCRM), Fuzzy Human Resource Management (FHRM), and Neutrosophic Human Resource Management (NHRM). By integrating uncertainty-oriented paradigms with established CRM and HRM practices, these frameworks aim to improve decision-making, enhance adaptability, and boost overall organizational efficiency. Future work will focus on evaluating the applicability of these frameworks in real-world business cases.

Declarations

Ethics Approval and Consent to Participate

The results, data, and figures presented in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and no permissions are required.

Consent for Publication

This article does not contain any studies involving human participants or animals performed by any of the authors.

Availability of Data and Materials

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in this research.

Funding

This research was not supported by any funding agency or institute.

Acknowledgment

The author expresses gratitude to the editorial team and reviewers, as well as the corresponding author, for their valuable advice, assessment, and review during the study period.

References

[1] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[2] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[3] Takaaki Fujita. A study on hyperfuzzy hyperrough sets, hyperneutrosophic hyperrough sets, and hypersoft hyperrough sets with applications in cybersecurity. *Artificial Intelligence in Cybersecurity*, 2:14–36, 2025.

[4] Takaaki Fujita and Florentin Smarandache. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports*, 2:1–18, 2025.

[5] Takaaki Fujita and Florentin Smarandache. *Forestfuzzy, forestneutrosophic, forestplithogenic, and forestrough set*. Infinite Study, 2025.

[6] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[7] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[8] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[9] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[10] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[11] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[12] Florentin Smarandache. Degrees of membership> 1 and< 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[13] Takaaki Fujita. Claw-free graph and at-free graph in fuzzy, neutrosophic, and plithogenic graphs. *Information Sciences with Applications*, 5:40–55, 2025.

[14] Takaaki Fujita. Review of plithogenic directed, mixed, bidirected, and pangene offgraph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 120.

[15] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[16] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[17] Takaaki Fujita and Florentin Smarandache. *Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic*. Infinite Study, 2025.

[18] Takaaki Fujita, Florentin Smarandache, and Victor Christianto. A concise formalization of partial falsifiability, water logic, and neither nor logic with neutrosophic logic. *Neutrosophic Sets and Systems*, 83:1–26, 2025.

[19] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[20] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[21] Muhammad Aslam, Osama H Arif, and Rehan Ahmad Khan Sherwani. New diagnosis test under the neutrosophic statistics: an application to diabetic patients. *BioMed Research International*, 2020(1):2086185, 2020.

[22] Mumtaz Ali, Nguyen Dang Thanh, Nguyen Van Minh, et al. A neutrosophic recommender system for medical diagnosis based on algebraic neutrosophic measures. *Applied Soft Computing*, 71:1054–1071, 2018.

[23] Yurong Xu, David C Yen, Binshan Lin, and David C Chou. Adopting customer relationship management technology. *Industrial management & data systems*, 102(8):442–452, 2002.

[24] Christopher Bull. Strategic issues in customer relationship management (crm) implementation. *Business process management Journal*, 9(5):592–602, 2003.

[25] Hugh Wilson, Elizabeth Daniel, and Malcolm McDonald. Factors for success in customer relationship management (crm) systems. *Journal of marketing management*, 18(1-2):193–219, 2002.

[26] Farnoosh Khodakarami and Yolande E Chan. Exploring the role of customer relationship management (crm) systems in customer knowledge creation. *Information & management*, 51(1):27–42, 2014.

[27] Agnieszka Barbara Bojanowska and Monika Kulisz. Using fuzzy logic to make decisions based on the data from customer relationship management systems. *Advances in Science and Technology. Research Journal*, 17(5), 2023.

[28] Yu-Jie Wang. A clustering method based on fuzzy equivalence relation for customer relationship management. *Expert Systems with Applications*, 37(9):6421–6428, 2010.

[29] Naoual Bouhtati, Lhoussaine Alla, and Badr Bentalha. Marketing big data analytics and customer relationship management: A fuzzy approach. In *Integrating intelligence and sustainability in supply chains*, pages 75–86. IGI Global, 2023.

[30] Atiqe Ur Rahman, Muhammad Haris Saeed, Ebenezer Bonyah, and Muhammad Arshad. Graphical exploration of generalized picture fuzzy hypersoft information with application in human resource management multiattribute decision-making. *Mathematical Problems in Engineering*, 2022.

[31] Muhammad Ihsan, Muhammad Haris Saeed, and Atiqe Ur Rahman. An intuitionistic fuzzy hypersoft expert set-based robust decision-support framework for human resource management integrated with modified topsis and correlation coefficient. *Neural Computing and Applications*, pages 1–25, 2023.

[32] Taslim Ahammad. Personnel management to human resource management (hrm): How hrm functions. *Journal of Modern Accounting and Auditing*, 13(9):412–420, 2017.

[33] Stephen Wood. Human resource management and performance. *International journal of management reviews*, 1(4):367–413, 1999.

[34] Lourdes Canos and Vicente Liern. Some fuzzy models for human resource management. *International Journal of technology, policy and management*, 4(4):291–308, 2004.

[35] Chen Zhao, Ye Xue, and Tong Niu. Enterprise human resource management index based on fuzzy system. *Journal of intelligent & fuzzy systems*, 40(2):3137–3146, 2021.

[36] Pavel Holeček, Jan Stoklasa, and Jana Talašová. Human resources management at universities-a fuzzy classification approach. *International Journal of Mathematics in Operational Research*, 9(4):502–519, 2016.

[37] Ru-xia Liang, Zi-bin Jiang, and Jian-qiang Wang. A linguistic neutrosophic multi-criteria group decision-making method to university human resource management. *Symmetry*, 10(9):364, 2018.

[38] Abderrahmane Bettayeb and Muhammad Eid Balbaa. Success factors in adopting ai in human resource management in uae firms: Neutrosophic analysis. *International Journal of Neutrosophic Science (IJNS)*, 21(3), 2023.

[39] Qingheng Sun. Performance evaluation of human resource management in power enterprises using multi-criteria decision-making methodology and treesoft set. *Neutrosophic Sets and Systems*, 82:321–334, 2025.

Received: 23 Sep 2024, **Revised:** 15 Jan 2025,
Accepted: 24 Apr 2025, **Available online:** 26 Apr 2025.