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Soft Theta-product: A New Product for Soft Sets with Its Decision-Making

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Abstract

For handling uncertainty, the theory of soft sets provides a thorough mathematical foundation. Soft set operations are significant concepts in soft set theory, as they offer new approaches to problems involving parametric data. In this context, we introduce a new product operation for soft sets, called "soft theta-product" and investigate its whole algebraic properties in terms of different types of soft subsets and soft equalities. Additionally, we explore the relations of this soft product with other soft set operations by investigating the distributions of the soft theta-product over them. In conclusion, using the uni-int decision function for the soft-theta product together with the uni-int operator for the uni-int decision-making method, which chooses a collection of optimal elements from the alternatives, we provide an example demonstrating how the technique may be effectively applied in a range of areas. As the theoretical underpinnings of soft computing techniques are drawn from purely mathematical concepts, this study is an essential contribution to the literature on soft sets.

Keywords: Soft Set; Soft Theta-product; Soft Subset; Soft Equal Relations; Decision-making.

1 | Introduction

For formal modeling, reasoning, and computing, most of the conventional tools are precise and crisp. Nonetheless, uncertainty is a factor in several intricate issues in the domains of economics, engineering, environmental research, social science, medicine, and others. The theory of probability, theory of fuzzy sets [1], theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets are theoretical approaches to uncertainty. In 1999, Molodstov [2] noted that these concepts had limitations of their own. Molodtsov [2] continued by suggesting that these limitations may be the result of an inadequate parameterization tool in the theory. In this context, Molodstov's soft set theory differs significantly from the above-mentioned theories. Soft set theory is very helpful, readily applicable, and flexible since it does not impose any limitations on the approximate description. After Maji et al. [3] applied soft set theory to a decision-making problem, other researchers [4–10] developed the first innovative soft set-based decision-making techniques. *Uni-int* decision-making is a trumpeted soft set-based decision-making technique that was established by Çağman and Enginoğlu [11]. Additionally, the concept of the soft matrix was proposed by Çağman and Enginoğlu [12], who also developed decision-making techniques for OR, AND, AND-NOT, and

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OR-NOT products of the soft matrices and applied them to real-world problems with uncertainties. Since then, soft set theory has been widely and successfully used to solve decision-making problems [13–24].

A thorough theoretical study of soft sets, including soft subsets and supersets, equality of soft sets, and soft set operations like union, intersection, AND, and OR-product was provided by Maji et al. [25]. Pei and Miao [26] investigated the relationship between soft sets and information systems and redefined the terms intersection and soft set subset. Novel operations including the restricted union, restricted intersection, restricted difference, and extended intersection of soft sets were proposed and explained by Ali et al. [27]. The authors [28-41] then identified several conceptual misunderstandings about the fundamentals of soft set theory seen in the literature, proposed improved and novel methods, and examined the algebraic structure of the set of all soft sets. In recent years, research on soft sets has made significant strides. Eren and Çalışıcı [42] defined a new kind of difference operation of soft sets. Stojanovic [43] defined the extended symmetric difference of soft sets and addressed its fundamental properties. Additionally, new types of soft set operations have been proposed and their whole characteristics have been explored [44-49].

In the context of soft set theory, soft equal relations and soft subsets are crucial ideas. A somewhat accurate concept of soft subsets was first used by Maji et al. [25]. The concept of soft subsets was expanded upon by Pei and Miao [26] and Feng et al. [29], which might be considered a generalization of Maji's earlier definition [25]. Qin and Hong [50] created two new types of soft equal relations and congruence relations on soft sets. To modify Maji's soft distributive laws, Jun and Yang [51] looked into a wider variety of soft subsets and used generalized soft equal relations, which we call J-soft equal relations for consistency's sake. In [51], Jun and Yang explored more about the generalized soft distributive laws of soft product operations. To provide a concise research note on soft L-subsets and soft L-equal relations, Liu, Feng, and Jun [52] were motivated by the novel ideas of Jun and Yang [51] and proved that distributive rules do not encompass all of the soft equality mentioned in the literature.

As a consequence, Feng et al. [53] extended the study specified in [52] by focusing on soft subsets and the soft products proposed in [25]. In contrast to the notes [52], Feng et al. [53] focused on different types of soft subsets and the algebraic features of soft product operations. They covered commutative laws, association rules, and other fundamental features, as well as distribution laws, which were extensively investigated by several researchers. They also provided theoretical research on soft products, namely the AND-product and OR-product using soft L-subsets, as well as some related subjects. They completed various partial conclusions about soft product operations that had previously been published in the literature, as well as thoroughly analyzed the algebraic properties of soft product operations in terms of J-equality and L-equality. Soft L-equal relations were shown to be congruent on free soft algebras and their corresponding quotient structures, which constitute commutative semigroups. (For further information on soft equal relations, see [54-58]).

Çağman and Enginoğlu [11] redefined Molodtsov's soft sets' definition and operations to be more useful. They also proposed four types of products in soft set theory: AND-product, OR-product, AND-NOT product, and OR-NOT-product and uni-int decision function. By using these new definitions, they proposed a unique decision-making method that picks the optimal components from the options. Sezgin et al. [59] studied the AND-product of soft sets, which has long served as the foundation and has been used by decision-makers in decision-making problems from a theoretical view. Although many academics investigated the AND-product and its features in connection to many sorts of soft equalities, including soft L-equality and soft J-equality, in [59] the authors investigated the entire algebraic properties of the AND-product in detail, including commutative laws, associative laws, idempotent laws, and other fundamental properties, and compared them to previously obtained properties in terms of soft F-subsets, soft M-equality, soft L-equality, and soft J-equality. Furthermore, by establishing some new results on the distributive properties of AND-product over restricted, extended, and soft binary piecewise soft set operations, they showed that the set of all soft sets over the universe together with restricted/extended union and AND-product, is a commutative hemiring with identity in the sense of L-equality and the set of all the soft sets over the universe, coupled with restricted/extended symmetric difference and AND-product, forms a commutative hemiring with identity in the sense of soft L-equality.

In this study, we first introduce a new product for soft sets, called soft theta-product by using the soft set definition of Molodtsov. We give its example and examine its whole algebraic properties in detail as regards different types of soft subsets and soft equalities such as M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. We also obtain the distributions of soft theta-product over other certain types of soft set operations. Finally, we apply the soft decision-making approach that picks optimal elements from alternatives without requiring rough sets or fuzzy soft sets and provides an example demonstrating how the approach may be effectively applied to various fields. In this context, this paper aims to add to the soft set literature as soft sets are a powerful mathematical tool for identifying uncertain objects and the theoretical foundations of soft computing approaches are derived from purely mathematical principles.

The following is how this document is structured. In Section 2, we go over the basic concepts of soft set theory. Section 3 proposes the soft theta-product and discusses its algebraic properties in terms of different types of soft subsets and soft equalities. Section 4 explores the distributions of the soft theta product over other certain types of soft set operations. Section 5 applies the *uni-int* decision function for soft theta-product to solve a problem including uncertainty. The conclusion section includes a brief deduction.

2 | Preliminary

Definition 2.1 [1] Let U be the universal set, E be the parameter, $\mathrm{P(U)}$ be the power set of U and $\mathcal{M} \subseteq \mathrm{E}$. A pair $(℧, \mathcal{M})$ is called a soft set over U where $℧$ is a set-valued function such that $℧: \mathcal{M} \rightarrow \mathrm{P(U)}$.

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of soft set in our work. Throughout this paper, the collections of all the soft sets defined over U is designated as $\mathrm{S_E(U)}$. Let $\mathcal{M}$ be a fixed subset of E and $\mathrm{S}_{\mathcal{M}}(\mathrm{U})$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{M}$. That is, while in the set $\mathrm{S}_{\mathcal{M}}(\mathrm{U})$, there are only soft sets whose parameter sets are $\mathcal{M}$; in the set $\mathrm{S_E(U)}$, there are soft sets whose parameter sets may be any set. From now on, while soft sets will be designated by SS and parameters set by PS; soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2.2 [27] Let $(℧, \mathcal{M})$ be an SS over U. $(℧, \mathcal{M})$ is called a relative null SS (with respect to the PS $\mathcal{M}$), denoted by $\emptyset_{\mathcal{M}}$, if $℧(m) = \emptyset$ for all $m \in \mathcal{M}$ and $(℧, \mathcal{M})$ is called a relative whole SS (with respect to the PS $\mathcal{M}$), denoted by $\mathrm{U}_{\mathcal{M}}$ if $℧(m) = \mathrm{U}$ for all $m \in \mathcal{M}$. The relative whole SS $\mathrm{U_E}$ with respect to the universe's set of parameters E is called the absolute SS over U.

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe U, unless otherwise stated.

Definition 2.3 [25] Let $(℧, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. $(℧, \mathcal{M})$ is called a soft M-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(℧, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $℧(m) = \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(℧, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft M-equal, denoted by $(℧, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$, if $(℧, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{M}} (℧, \mathcal{M})$.

Definition 2.4 [26] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be two SSs over U. $(\mho,\mathcal{M})$ is called a soft F-subset of $(\mathfrak{F},\mathcal{D})$ denoted by $(\mho,\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F},\mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) \subseteq \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ are said to be soft F-equal, denoted by $(\mho,\mathcal{M}) =_F (\mathfrak{F},\mathcal{D})$, if $(\mho,\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F},\mathcal{D})$ and $(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_F (\mho,\mathcal{M})$.

It should be noted that the definitions of soft F-subset and soft F-equal were initially provided by Pei and Miao in [26]. However, some SS papers regarding soft subsets and soft equalities claimed that Feng et al. provided these definitions first in [29]. As a result, the letter "F" is used to denote this connection.

It was demonstrated in [52] that the soft equal relations $=_M$ and $=_F$ coincide. In other words, $(\mho,\mathcal{M}) =_M (\mathfrak{F},\mathcal{D}) \Leftrightarrow (\mho,\mathcal{M}) =_F (\mathfrak{F},\mathcal{D})$. Since they share the same set of parameters and approximation function, two SSs that meet this soft equivalence are truly identical [52], hence $(\mho,\mathcal{M}) =_M (\mathfrak{F},\mathcal{D})$ means $(\mho,\mathcal{M}) = (\mathfrak{F},\mathcal{D})$. Jun and Yang [51] extended the ideas of F-soft subsets and soft F-equal relations by loosening the restrictions on PSs. We refer to them as soft J-subsets and soft J-equal relations, the initial letter of Jun, even though in [51] they are named generalized soft subset and generalized soft equal relation.

Definition 2.5 [51] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be two SSs over U. $(\mho,\mathcal{M})$ is called a soft J-subset of $(\mathfrak{F},\mathcal{D})$ denoted by $(\mho,\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F},\mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) \subseteq \mathfrak{F}(d)$. Two SSs $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ are said to be soft J-equal, denoted by $(\mho,\mathcal{M}) =_J (\mathfrak{F},\mathcal{D})$, if $(\mho,\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F},\mathcal{D})$ and $(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_J (\mho,\mathcal{M})$.

In [52] and [53], it was shown that $(\mho,\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F},\mathcal{D})$, but the converse may not be true. Liu, Feng, and Jun [52] also presented the following new kind of soft subsets (henceforth referred to as soft L-subsets and soft L-equality) that generalize both soft M-subsets and ontology-based soft subsets, inspired by the ideas of soft J-subset [51] and ontology-based soft subsets [30].

Definition 2.6 [52] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be two SSs over U. $(\mho,\mathcal{M})$ is called a soft L-subset of $(\mathfrak{F},\mathcal{D})$ denoted by $(\mho,\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F},\mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) = \mathfrak{F}(d)$. Two SSs $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ are said to be soft L-equal, denoted by $(\mho,\mathcal{M}) =_L (\mathfrak{F},\mathcal{D})$, if $(\mho,\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F},\mathcal{D})$ and $(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_J (\mho,\mathcal{M})$.

As regards the relations regarding certain types of soft subsets and soft qualities, $(\mho,\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F},\mathcal{D})$ and $(\mho,\mathcal{M}) =_M (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) =_L (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) =_J (\mathfrak{F},\mathcal{D})$ [52]. However, the converses may not be true. Also, it is well-known that $(\mho,\mathcal{M}) =_M (\mathfrak{F},\mathcal{D})$ if and only if $(\mho,\mathcal{M}) =_F (\mathfrak{F},\mathcal{D})$.

We may thus deduce that soft M-equality (and so soft F-equality) is the strictest sense, whereas soft J-equality is the weakest soft equal connection. In the middle of these is the idea of the soft L-equal connection [52].

Example 2.7. Let E=$\{c_1,c_2,c_3,c_4,c_5\}$ be the PS, $\mathcal{M}=\{c_1,c_4\}$ and $\mathcal{D}=\{c_1,c_4,c_5\}$ be the subsets of E and U=$\{z_1,z_2,z_3,z_4,z_5\}$ be the initial universe set. Let

$$(\mho,\mathcal{M})=\{(c_1,\{z_1,z_3\}),(c_4,\{z_2,z_3,z_5\})\},\ (\mathfrak{F},\mathcal{D})=\{(c_1,\{z_1,z_3\}),(c_4,\{z_2,z_3\}),(c_5,\{z_1,z_2,z_3,z_5\})\}.$$

$$(\mathcal{V},\mathcal{D})=\{(c_1,\{z_2,z_3,z_5\}),(c_4,\{z_1,z_3\}),(c_5,\{z_1,z_2,z_3,z_5\})\}$$

Since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (and also $\mho(c_1) \subseteq \mathfrak{F}(c_5)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \cong\!\subseteq_J (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_4) \neq \mathfrak{F}(c_1)$, $\mho(c_4) \neq \mathfrak{F}(c_4)$, and $\mho(c_4) \neq \mathfrak{F}(c_5)$, we can deduce that $(\mho, \mathcal{M})$ is not a soft L-subset of $(\mathfrak{F}, \mathcal{D})$. Moreover, as $\mho(c_4) \neq \mathfrak{F}(c_4)$, $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$.

Now, since, $\mho(c_1) = \mathcal{V}(c_4)$ and $\mho(c_4) = \mathcal{V}(c_1)$, it is obvious that $(\mho, \mathcal{M}) \cong\!\subseteq_L (\mathcal{V}, \mathcal{D})$. However, as $\mho(c_1) \neq \mathcal{V}(c_1)$, $\mho(c_4) \neq \mathcal{V}(c_4)$, $(\mho, \mathcal{M})$is not again a soft M-subset of $(\mathcal{V}, \mathcal{D})$.

Example 2.8. Let E=$\{c_1, c_2, c_3, c_4, c_5\}$ be the PS, $\mathcal{M} = \{c_1, c_4\}$ and $\mathcal{D} = \{c_1, c_4, c_5\}$ be the subsets of E and U=$\{z_1, z_2, z_3, z_4, z_5\}$ be the initial universe set. Let

$$(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\})\}, \quad (\mathfrak{F}, \mathcal{D}) = \{(c_1, \{z_1, z_2, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\}), (c_5, \{z_1\})\}.$$

Since $\mho(c_1) \neq \mathfrak{F}(c_1)$, $\mho(c_1) \neq \mathfrak{F}(c_4)$ and $\mho(c_1) \neq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \neq_L (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (moreover $\mho(c_1) \subseteq \mathfrak{F}(c_4)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_4)$, we can deduce that $(\mho, \mathcal{M}) \cong\!\subseteq_J (\mathfrak{F}, \mathcal{D})$. Moreover, since $\mathfrak{F}(c_1) \subseteq \mho(c_4)$ and $\mathfrak{F}(c_4) \subseteq \mho(c_4)$, and $\mathfrak{F}(c_5) \subseteq \mho(c_1)$, we can deduce that $(\mathfrak{F}, \mathcal{D}) \cong\!\subseteq_J (\mho, \mathcal{M})$. Therefore, $(\mho, \mathcal{M}) =_J (\mathfrak{F}, \mathcal{D})$. As $\mho(c_1) \neq \mathfrak{F}(c_1)$ and $\mho(c_4) \neq \mathfrak{F}(c_4)$, it is obvious that $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$.

For more on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and some other existing definitions of soft subsets and soft equal relations in the literature, we refer to [50-58].

Definition 2.9. [27] Let $(\mho, \mathcal{M})$ be an SS over U. The relative complement of an SS Let $(\mho, \mathcal{M})$, denoted by $(\mho, \mathcal{M})^r$, is defined by $(\mho, \mathcal{M})^r = (\mho^r, \mathcal{M})$, where $\mho^r : \mathcal{M} \to P(U)$ is a mapping given by $\mho^r(m) = U \setminus \mho(m)$ for all $m \in \mathcal{M}$. From now on, $U \setminus \mho(m) = [\mho(m)]'$ is designated by $\mho'(m)$ for the sake of designation.

Definition 2.10. [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The AND-product (∧-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} x \mathcal{D})$, where for all $(m, d) \in \mathcal{M} x \mathcal{D}$, $\mho(m, d) = \mho(m) \cap \mathfrak{F}(d)$.

Definition 2.11. [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The OR-product (∨-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} x \mathcal{D})$, where for all $(m, d) \in \mathcal{M} x \mathcal{D}$, $\mho(m, d) = \mho(m) \cup \mathfrak{F}(d)$.

Çağman [60] defined inclusive complement and exclusive complements as a novel idea in set theory and investigated the connections between these two by contrasting them. In [60], these new concepts were also applied to group theory.

Definition 2.12. [60] Let A and B be two subsets of the universe. Then, the B-exclusive complement of A is defined by $A - B := A' \cap B = A' \cap B'$.

To avoid confusion with different operations, the B-exclusive complement of A was denoted by AθB by Sezgin et al. [61]. Then, theta operation was applied to soft set theory [62-65] to propose new soft set operations.

Let "⊛" be used to stand for set operations like ∩, ∪, \,∆. The following definitions are given for restricted, extended, and soft binary piecewise operations.

Definition 2.13. [27] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The restricted $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_R (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_R (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cap \mathcal{D}$ and if $\mathcal{J} \neq \emptyset$, then for all $j \in \mathcal{J}$, $\mho(j) = \mho(j) \circledast \mathfrak{F}(j)$; if $\mathcal{J} = \emptyset$, then $(\mho, \mathcal{M}) \circledast_R (\mathfrak{F}, \mathcal{D}) = \emptyset_\emptyset$.

Definition 2.14. [27,43,62] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The extended $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_\varepsilon (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_\varepsilon (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cup \mathcal{D}$ and then for all $j \in \mathcal{J}$,

$$\mho(j) = \begin{cases} \mho(j), & j \in \mathcal{M} \backslash \mathcal{D} \\ \mathfrak{F}(j), & j \in \mathcal{D} \backslash \mathcal{M} \\ \mho(j) \circledast \mathfrak{F}(j), & j \in \mathcal{M} \cap \mathcal{D} \end{cases}$$

Definition 2.15. [44, 65] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over U. The extended $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \widetilde{\circledast} (\mathfrak{F}, \mathcal{D})$, is defined by $(\mho, \mathcal{M}) \widetilde{\circledast} (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M})$, where for all $j \in \mathcal{M}$,

$$\mho(j) = \begin{cases} \mho(j), & j \in \mathcal{M} \backslash \mathcal{D}, \\ \mho(j) \circledast \mathfrak{F}(j), & j \in \mathcal{M} \cap \mathcal{D} \end{cases}$$

For more about soft algebraic structures of soft sets, we refer to [66-91], and [92,93] to picture fuzzy soft sets and their product operations with soft decision-making and picture fuzzy soft measure and their applications to multi-criteria decision-making.

3 | Soft Theta-product and Its Algebraic Properties

In this subsection, we introduce a new product for soft sets, called soft theta-product. We give its example and examine its algebraic properties in detail as regards certain types of soft subsets and soft equalities.

Definition 3.1. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U. The soft theta-product of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \Lambda_\theta (\mathfrak{F}, \mathcal{D})$, is defined by $(\mho, \mathcal{M}) \Lambda_\theta (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} x \mathcal{D})$, where for all $(m, d) \in \mathcal{M} x \mathcal{D}$,

$$\mho(m, d) = \mho(m) \theta \mathfrak{F}(d)$$

Here, $\mho(m) \theta \mathfrak{F}(d) = \mho'(m) \cap \mathfrak{F}'(d)$.

Example 3.2. Assume that $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the PS, $\mathcal{M} = \{e_1, e_2, e_3\}$ and $\mathcal{D} = \{e_1, e_4, e_5\}$, be the subsets of E, $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the universal set, the SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be over U such that

$$(\mho, \mathcal{M}) = \{(e_1, \{h_1, h_2, h_3, h_5\}), (e_2, \{h_1, h_2, h_3\}), (e_3, \{h_4, h_5, h_6\})\}$$

$$(\mathfrak{F}, \mathcal{D}) = \{(e_1, \{h_6\}), (e_4, \{h_2, h_3, h_5\}), (e_5, \{h_2\})\}$$

Let $(\mho, \mathcal{M}) \Lambda_\theta (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} x \mathcal{D})$. Then,

$$(\mho, \mathcal{M} x \mathcal{D})$$
$$= \{((e_1, e_1), \{h_4\}), ((e_1, e_4), \{h_4, h_6\}), ((e_1, e_5), \{h_4, h_6\}), ((e_2, e_1), \{h_4, h_5\}), ((e_2, e_4), \{h_4, h_6\}),$$

Here, the table method can be used as it is more convenient than writing in the list method format:

Table 1. The table designation of the soft theta-product's result of the soft sets in Example 3.2

$(\mho,\mathcal{M})\ \Lambda_\theta\ (\mathfrak{F},\mathcal{D})$	e_1	e_4	e_5
e_1	$\{h_4\}$	$\{h_2,h_6\}$	$\{h_4,h_6\}$
e_2	$\{h_4,h_5\}$	$\{h_4,h_6\}$	$\{h_4,h_5,h_6\}$
e_3	$\{h_1,h_2,h_3\}$	$\{h_1\}$	$\{h_1,h_3\}$

Proposition 3.3. Λ_θ-product is closed in $S_E(U)$.

Proof: It is obvious that Λ_θ-product is a binary operation in $S_E(U)$. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then,

$$\Lambda_\theta: S_E(U)\ \text{x}\ S_E(U) \longrightarrow S_E(U)$$

$$\big((\mho,\mathcal{M}),(\mathfrak{F},\mathcal{D})\big) \longrightarrow (\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\text{x}\mathcal{D}) = (\mathcal{V},\mathcal{J})$$

That is, $(\mathcal{V},\mathcal{J})$ is an SS over U, since the set $S_E(U)$ contains all the SS over U. Here, note that the set $S_{\mathcal{M}}(U)$ is not closed under Λ_θ-product, since if $(\mho,\mathcal{M}),(\mathfrak{F},\mathcal{M})$ are the elements of $S_{\mathcal{M}}(U)$, $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{M})$ is an element of $S_{\mathcal{M}\text{x}\mathcal{M}}(U)$ not $S_{\mathcal{M}}(U)$.

Proposition 3.4. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then,

$$(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})] \neq_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J})$$

Thus, Λ_θ -product is not associative in $S_E(U)$.

Proof: In order to show that Λ_θ -product is not associative in $S_E(U)$, we provide an example: Let $E=\{e_1,e_2,e_3,e_4\}$ be the PS, $\mathcal{M}=\{e_2,e_3\}$, $\mathcal{D}=\{e_1\}$ and $\mathcal{J}=\{e_4\}$ be the subsets of E, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universal set, $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U such that $(\mho,\mathcal{M})=\{(e_2,\{h_3,h_4\}),(e_3,\{h_1\})\}$, $(\mathfrak{F},\mathcal{D})=\{(e_1,\emptyset)\}$ ans $(\mathcal{V},\mathcal{J})=\{(e_4,\{h_1,h_3,h_5\})\}$.

We show that $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})] \neq_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J})$. Let $(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J}) = (\Upsilon,\mathcal{D}\text{x}\mathcal{J})$. Then,

$$(\Upsilon,\mathcal{D}\text{x}\mathcal{J}) = \{((e_1,e_4),\{h_2,h_4\})\}$$

and let $(\mho,\mathcal{M})\ \Lambda_\theta\ (\Upsilon,\mathcal{D}\text{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big)$. Thus,

$$\big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) = \{((e_2,(e_1,e_4)),\{h_1,h_5\}),((e_3,(e_1,e_4)),\{h_3,h_5\})\}$$

Assume that $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) = (\mathfrak{z},\mathcal{M}\text{x}\mathcal{D})$. Thereby,

$$(\mathfrak{z},\mathcal{M}\text{x}\mathcal{D}) = \{\big((e_2,e_1),\{h_1,h_2,h_5\}\big),((e_3,e_1),\{h_2,h_3,h_4,h_5\})\}$$

Suppose that $(\mathfrak{z},\mathcal{M}\text{x}\mathcal{D})\ \Lambda_\theta\ (\mathcal{V},\mathcal{J}) = \big(\wp,(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big)$. Therefore,

$$\big(\wp,(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big) = \{(((e_2,e_1),e_4),\{h_4\}),(((e_3,e_1),e_4),\emptyset)\}$$

It is seen that $(\mathfrak{X}, \mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})) \neq_{\mathrm{M}} (\wp, (\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J})$. It is also obvious that.

$$(\mathfrak{X}, \mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})) \neq_{\mathrm{L}} (\wp, (\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}) \text{ and } (\mathfrak{X}, \mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})) \neq_{\mathrm{J}} (\wp, (\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}\mathcal{J}).$$

Proposition 3.5. Let $(\mathfrak{F}, \mathcal{D})$ and $(\mathcal{V}, \mathcal{J})$ be SSs over U. Then, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) \neq_{\mathrm{M}} (\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})$. That is, Λ_θ-product is not commutative in $S_E(U)$.

Proof: Let $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) = (\varsigma, \mathcal{D}\mathrm{x}\mathcal{J})$ and $(\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})=(\mathfrak{X}, \mathcal{J}\mathrm{x}\mathcal{D})$. Since $\mathcal{D}\mathrm{x}\mathcal{J} \neq \mathcal{J}\mathrm{x}\mathcal{D}$, the rest of the proof is obvious.

Proposition 3.6. Let $(\mathfrak{F}, \mathcal{D})$ and $(\mathcal{V}, \mathcal{J})$ be SSs over U. Then, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) =_{\mathrm{L}} (\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})$. That is, Λ_θ-product is commutative in $S_E(U)$ under L-equality.

Proof: Let $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) = (\mho, \mathcal{D}\mathrm{x}\mathcal{J})$ and $(\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D}) = (\mathfrak{X}, \mathcal{J}\mathrm{x}\mathcal{D})$. Thus, for all $(d, \mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $\mho(d, \mathrm{j}) = \mathfrak{F}'(d) \cap \mathcal{V}'(\mathrm{j})$ and for all $(\mathrm{j}, d) \in \mathcal{J}\mathrm{x}\mathcal{D}$, $\mathfrak{X}(\mathrm{j}, d) = \mathcal{V}'(\mathrm{j}) \cap \mathfrak{F}'(d)$. Since for all $(d, \mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, there exists $(\mathrm{j}, d) \in \mathcal{J}\mathrm{x}\mathcal{D}$ such that $\mathfrak{F}'(d) \cap \mathcal{V}'(\mathrm{j}) = \mathcal{V}'(\mathrm{j}) \cap \mathfrak{F}'(d)$, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) \subseteq_{\mathrm{L}} (\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})$. Similarly, since for all $(\mathrm{j}, d) \in \mathcal{J}\mathrm{x}\mathcal{D}$, there exists $(d, \mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$ such that $\mathcal{V}'(\mathrm{j}) \cap \mathfrak{F}'(d) = \mathfrak{F}'(d) \cap \mathcal{V}'(\mathrm{j})$, $(\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D}) \subseteq_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J})$. Therefore, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) =_{\mathrm{L}} (\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})$, Moreover, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{J}) =_{\mathrm{J}} (\mathcal{V}, \mathcal{J})\Lambda_\theta(\mathfrak{F}, \mathcal{D})$.

Proposition 3.7. Let $(\mathfrak{F}, \mathcal{D})$ be an SS over U. Then, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta\emptyset_\emptyset =_{\mathrm{M}} \emptyset_\emptyset\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_\emptyset$. That is, $\emptyset_\emptyset$-the empty SS-is the absorbing element of Λ_θ-product in $S_E(U)$.

Proof: Let $\emptyset_\emptyset = (\varsigma, \emptyset)$ and $(\mathfrak{F}, \mathcal{D})\Lambda_\theta\emptyset_\emptyset = (\mathfrak{F}, \mathcal{D})\Lambda_\theta(\varsigma, \emptyset) = (\wp, \mathcal{D}\mathrm{x}\emptyset) = (\wp, \emptyset)$. Since the only SS whose PS is $\emptyset_\emptyset$, $(\wp, \emptyset) = \emptyset_\emptyset$. One can similarly show that $\emptyset_\emptyset\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_\emptyset$.

Proposition 3.8. Let $(\mathfrak{F}, \mathcal{D})$ be an SS over U. Then, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta\emptyset_\mathcal{D} =_{\mathrm{M}} \emptyset_\mathcal{D}\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$.

Proof: Let $\emptyset_\mathcal{D} = (\varsigma, \mathcal{D})$. Then, for all $d \in \mathcal{D}, \varsigma(d) = \emptyset$. Let $(\mathfrak{F}, \mathcal{D})\Lambda_\theta\emptyset_\mathcal{D} = (\mathfrak{F}, \mathcal{D})\Lambda_\theta(\varsigma, \mathcal{D}) = (\wp, \mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d, m) \in \mathcal{D}\mathrm{x}\mathcal{D}, \wp(d, m) = \mathfrak{F}'(d) \cap \varsigma'(m) = \mathfrak{F}'(d) \cap \emptyset' = \mathfrak{F}'(d) \cap U = \mathfrak{F}'(d)$, implying that $(\wp, \mathcal{D}\mathrm{x}\mathcal{D}) = (\mathfrak{F}, \mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$. Similarly, $\emptyset_\mathcal{D}\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$ is obtained.

Proposition 3.9. Let $(\mathfrak{F}, \mathcal{D})$ an SS over U. Then, $(\mathfrak{F}, \mathcal{D})\Lambda_\theta U_\mathcal{D} =_{\mathrm{M}} U_\mathcal{D}\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$.

Proof: Let $U_\mathcal{D} = (\mathcal{V}, \mathcal{D})$. Then, for all $d \in \mathcal{D}, \mathcal{V}(d) = U$. Let $(\mathfrak{F}, \mathcal{D})\Lambda_\theta U_\mathcal{D} = (\mathfrak{F}, \mathcal{D})\Lambda_\theta(\mathcal{V}, \mathcal{D}) = (\mathfrak{X}, \mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d, m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\mathfrak{X}(d, m) = \mathfrak{F}'(d) \cap \mathcal{V}'(m) = \mathfrak{F}'(d) \cap U' = \mathfrak{F}'(d) \cap \emptyset = \emptyset$, implying that $(\mathfrak{X}, \mathcal{D}\mathrm{x}\mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$. Similarly, $U_\mathcal{D}\Lambda_\theta(\mathfrak{F}, \mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$ is obtained.

Proposition 3.10. Let $(\mho, \mathcal{M})$ be an SS over U. Then, $(\mho, \mathcal{M})\Lambda_\theta(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})^{\mathrm{r}}$. That is, Λ_θ-product is not idempotent in $S_E(U)$ under J-equality.

Proof: Let $(\mho, \mathcal{M})\Lambda_\theta(\mho, \mathcal{M}) = (\mathfrak{F}, \mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{F}(m, d) = \mho'(m) \cap \mho'(d)$. Since for all $(m, d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{F}(m, d) = \mho'(m) \cap \mho'(d) \subseteq \mho'(m)$, $(\mathfrak{F}, \mathcal{M}\mathrm{x}\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})^{\mathrm{r}}$ is obtained.

Proposition 3.11. Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over U, Then, $(\mho, \mathcal{M})\Lambda_\theta(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})^{\mathrm{r}}$ and $(\mho, \mathcal{M})\Lambda_\theta(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})^{\mathrm{r}}$.

Proof: Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$. Since for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, there exists $d\in\mathcal{D}$ such that $\mho'(m)\cap\mathfrak{F}'(d)\subseteq\mathfrak{F}'(d)$, $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})\mathrel{\tilde{\subseteq}_J}(\mathfrak{F},\mathcal{D})^r$. Similarly, since for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, there exists $m\in\mathcal{M}$ such that $\mho'(m)\cap\mathfrak{F}'(d)\subseteq\mho'(m)$, $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})\mathrel{\tilde{\subseteq}_J}(\mho,\mathcal{M})^r$.

Proposition 3.12. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $[(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]^r=(\mho,\mathcal{M})^r\mathrm{V}_*(\mathfrak{F},\mathcal{D})^r$.

Proof: Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$. Thus, $\mathcal{V}'(m,d)=\mho(m)\cup\mathfrak{F}(d)=(\mho')'(m)\cup(\mathfrak{F}')'(d)$. Hence, $(\mathcal{V}',\mathcal{M}\mathrm{X}\mathcal{D})=(\mho,\mathcal{M})^r\,\mathrm{V}_*(\mathfrak{F},\mathcal{D})^r$. (For V_*-product, please see [94].)

Proposition 3.13. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})\mathrm{V}_*(\mathfrak{F},\mathcal{D})$.

Proof: Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M})\mathrm{V}_*(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$ and for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m,d)=\mho'(m)\cup\mathfrak{F}'(d)$. Thus, for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)\subseteq\mho'(m)\cup\mathfrak{F}'(d)=\mho(m,d)$. This completes the proof.

Proposition 3.14. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. If $(\mho,\mathcal{M})^r\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})^r$, then $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})$.

Proof: Let $(\mho,\mathcal{M})^r\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})^r$. Then, $\mathcal{M}\subseteq\mathcal{D}$ and for all $m\in\mathcal{M}$, $\mho'(m)\subseteq\mathfrak{F}'(m)$. Thus, $\mathcal{M}\ \mathrm{x}\ \mathcal{J}\subseteq\mathcal{D}\ \mathrm{x}\ \mathcal{J}$ and for all $(m,\mathrm{j})\in\mathcal{M}\ \mathrm{x}\ \mathcal{J}$, $\mho'(m)\cap\mathcal{V}'(\mathrm{j})\subseteq\mathfrak{F}'(m)\cap\mathcal{V}'(\mathrm{j})$. This completes the proof.

Proposition 3.15. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, $(\mathcal{V},\mathcal{J})$ and $(\mho,\mathfrak{X})$ be SSs over U. If $(\mho,\mathcal{M})^r\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})^r$ and $(\mathcal{V},\mathcal{J})^r\mathrel{\tilde{\subseteq}_F}(\mho,\mathfrak{X})^r$, then $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mho,\mathfrak{X})$ and $(\mathcal{V},\mathcal{J})\Lambda_\theta(\mho,\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mho,\mathfrak{X})\Lambda_\theta(\mathfrak{F},\mathcal{D})$.

Proof: Let $(\mho,\mathcal{M})^r\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})^r$ ve $(\mathcal{V},\mathcal{J})^r\mathrel{\tilde{\subseteq}_F}(\mho,\mathfrak{X})^r$. Then, $\mathcal{M}\subseteq\mathcal{D}$, $\mathcal{J}\subseteq\mathfrak{X}$, for all $m\in\mathcal{M}$, $\mho'(m)\subseteq\mathfrak{F}'(m)$ and for all $\mathrm{j}\in\mathcal{J}$, $\mathcal{V}'(\mathrm{j})\subseteq\mho'(\mathrm{j})$. Thus, $\mathcal{M}\mathrm{x}\mathcal{J}\subseteq\mathcal{D}\mathrm{x}\mathfrak{X}$, for all $(m,\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$, $\mho'(m)\cap\mathcal{V}'(\mathrm{j})\subseteq\mathfrak{F}'(m)\cap\mho'(\mathrm{j})$ and for all $(\mathrm{j},m)\in\mathcal{J}\mathrm{x}\mathcal{M}$, $\mathcal{V}'(\mathrm{j})\cap\mho'(m)\subseteq\mho'(\mathrm{j})\cap\mathfrak{F}'(m)$. This completes the proof.

Proposition 3.16. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{M})$, $(\mho,\mathcal{M})$ and $(\mathcal{V},\mathcal{M})$ be SSs over U. If $(\mho,\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{M})$ and $(\mathcal{V},\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})$, then $(\mathfrak{F},\mathcal{M})\Lambda_\theta(\mho,\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{M})$.

Proof: Let $(\mho,\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{M})$ and $(\mathcal{V},\mathcal{M})\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})$. Thus, for all $m\in\mathcal{M}$, $\mho(m)\subseteq\mathfrak{F}(m)$ and for all $\mathrm{j}\in\mathcal{M}$, $\mathcal{V}(\mathrm{j})\subseteq\mho(\mathrm{j})$. Hence, for all $(m,\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{F}'(m)\cap\mho'(\mathrm{j})\subseteq\mho'(m)\cap\mathcal{V}'(\mathrm{j})$. This completes the proof.

Proposition 3.17. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}x\mathcal{D}}\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})$ and $\emptyset_{\mathcal{D}x\mathcal{M}}\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mho,\mathcal{M})$.

Proof: Let $\emptyset_{\mathcal{M}x\mathcal{D}}=(\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(\mathfrak{z},\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\emptyset$ and for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathfrak{z}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$. Since $\mathcal{M}\mathrm{x}\mathcal{D}\subseteq\mathcal{M}\mathrm{x}\mathcal{D}$ and for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d)=\emptyset\subseteq\mho'(m)\cap\mathfrak{F}'(d)=\mathfrak{z}(m,d)$, $\emptyset_{\mathcal{M}x\mathcal{D}}\mathrel{\tilde{\subseteq}_F}(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})$ is obtained. Similarly, $\emptyset_{\mathcal{D}x\mathcal{M}}\mathrel{\tilde{\subseteq}_F}(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mho,\mathcal{M})$ can be shown.

Proposition 3.18. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$, $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$.

Proof: Let $\emptyset_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) = (\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{Z}(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$. Since for all $m \in \mathcal{M}$, there exists $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$ such that $\mathcal{V}(m) = \emptyset \subseteq \mho'(m) \cap \mathfrak{F}'(d) = \mathcal{Z}(m,d)$, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$ is obtained. One can similarly show that $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D})$.

Proposition 3.19. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ ve SSs over U. Then, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}\text{x}\mathcal{D}}$ and $(\mathfrak{F},\mathcal{D})\Lambda_{\theta}(\mho,\mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}\text{x}\mathcal{M}}$.

Proof: Let $U_{\mathcal{M}\text{x}\mathcal{D}} = (\mathcal{V},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) = (\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{V}(m,d) = U$ and for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{Z}(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$. Since $\mathcal{M}\text{x}\mathcal{D} \subseteq \mathcal{M}\text{x}\mathcal{D}$ and for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{Z}(m,d) = \mho'(m) \cap \mathfrak{F}'(d) \subseteq U = \mathcal{V}(m,d)$, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}\text{x}\mathcal{D}}$ is obtained. One can similarly show that $(\mathfrak{F},\mathcal{D})\Lambda_{\theta}(\mho,\mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}\text{x}\mathcal{M}}$.

Proposition 3.20. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$.

Proof: Let $U_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) = (\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = U$ and for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{Z}(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$. Since for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, there exists $m \in \mathcal{M}$ such that $\mathcal{Z}(m,d) = \mho'(m) \cap \mathfrak{F}'(d) \subseteq U = \mathcal{V}(m)$, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$ is obtained. Similarly, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$ can be shown.

Proposition 3.21. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{M} U_{\mathcal{M}\text{x}\mathcal{D}}$ if and only if $(\mho,\mathcal{M}) =_{M} \emptyset_{\mathcal{M}}$ and $(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\mathcal{D}}$.

Proof: Let $U_{\mathcal{M}\text{x}\mathcal{D}} = (\mathcal{O},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\text{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{O}(m,d) = U$ and for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$. Let $(\mathcal{O},\mathcal{M}\text{x}\mathcal{D}) = (\mathcal{V},\mathcal{M}\text{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mho'(m) \cap \mathfrak{F}'(d) = U$. Thus, for all $m \in \mathcal{M}$, $\mho'(m) = U$ and for all $d \in \mathcal{D}$, $\mathfrak{F}'(d) = U$. Thereby, $(\mho,\mathcal{M}) = \emptyset_{\mathcal{M}}$ and $(\mathfrak{F},\mathcal{D}) = \emptyset_{\mathcal{D}}$.

Conversely, let $(\mho,\mathcal{M}) =_{M} \emptyset_{\mathcal{M}}$ and $(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\mathcal{D}}$. Then, for all $m \in \mathcal{M}$, $\mho(m) = \emptyset$ and for all $d \in \mathcal{D}$, $\mathfrak{F}(d) = \emptyset$. Thus, for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho'(m) \cap \mathfrak{F}'(d) = U \cap U = U$, implying that $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{M} U_{\mathcal{M}\text{x}\mathcal{D}}$.

Proposition 3.23. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. Then, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$ if and only if $(\mho,\mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$.

Proof: Let $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$. Then, $\mathcal{M}\text{x}\mathcal{D} = \emptyset$, and so $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$ Since $\emptyset_{\emptyset}$ is the only SS with the empty PS, $(\mho,\mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$.

Conversely, let $(\mho,\mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$. Thus, $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$, implying that $\mathcal{M}\text{x}\mathcal{D} = \emptyset$ and $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{M} \emptyset_{\emptyset}$.

4 | Distributions of Soft Theta-product over Other Certain Types of Soft Set Operations

In this section, we explore the distributions of soft theta-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

Theorem 4.1. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft theta-product over restricted intersection and union operations:

i) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\cup_R(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\cap_R[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

ii) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\cap_R(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\cup_R[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iii) $[(\mho,\mathcal{M})\cap_R(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J})=_M[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]\cup_R[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iv) $[(\mho,\mathcal{M})\cup_R(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J})=_M[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]\cap_R[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$

Proof: (i) The PS of the left-hand side (LHS) is $\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})$, and the PS of the right-hand side (RHS) is $(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})$. Since $\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})=(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})$, the first condition of the M-equality is satisfied. Let $(\mathfrak{F},\mathcal{D})\cup_R(\mathcal{V},\mathcal{J})=(\mathfrak{X},\mathcal{D}\cap\mathcal{J})$, where for all $\varphi\in\mathcal{D}\cap\mathcal{J}$, $\mathfrak{X}(\varphi)=\mathfrak{F}(\varphi)\cup\mathcal{V}(\varphi)$. Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{X},\mathcal{D}\cap\mathcal{J})=(\text{ƻ},\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J}))$, where for all $(m,\varphi)\in\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})$, $\text{ƻ}(m,\varphi)=\mho'(m)\cap\mathfrak{X}'(\varphi)$. Thus,

$$\text{ƻ}(m,\varphi)=\mho'(m)\cap[\mathfrak{F}(\varphi)\cup\mathcal{V}(\varphi)]'=\mho'(m)\cap[\mathfrak{F}'(\varphi)\cap\mathcal{V}'(\varphi)]$$

Suppose that $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})=(\text{ლ},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$, $\text{Ŧ}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$ and $(m,j)\in\mathcal{M}\text{x}\mathcal{J}$, $\text{ლ}(m,j)=\mho'(m)\cap\mathcal{V}'(j)$. Let $(\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})\cap_R(\text{ლ},\mathcal{M}\text{x}\mathcal{J})=(\text{☼},(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J}))$, where for all $(m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})$,

$$\text{☼}(m,\varphi)=\text{Ŧ}(m,\varphi)\cap\text{ლ}(m,\varphi)=[\mho'(m)\cap\mathfrak{F}'(\varphi)]\cap[\mho'(m)\cap\mathcal{V}'(\varphi)]$$

Thus, $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\cup_R(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\cap_R[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

Here, if $\mathcal{D}\cap\mathcal{J}=\emptyset$, then $\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})=(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\emptyset$. Since the only soft set with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})$, if $(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\emptyset$, then $\mathcal{M}=\emptyset$ or $\mathcal{D}\cap\mathcal{J}=\emptyset$. By assumption, $\mathcal{M}\neq\emptyset$. Thus, $(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\emptyset$ implies that $\mathcal{D}\cap\mathcal{J}=\emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$.

(iii) The PS of the LHS is $(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}$, and the PS of the RHS is $(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})$, and since $(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}=(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})$, the first condition of M-equality is satisfied. Let $(\mho,\mathcal{M})\cap_R(\mathfrak{F},\mathcal{D})=(\mathfrak{X},\mathcal{M}\cap\mathcal{D})$, where for all $\varphi\in\mathcal{M}\cap\mathcal{D}$, $\mathfrak{X}(\varphi)=\mho(\varphi)\cap\mathfrak{F}(\varphi)$. Let $(\mathfrak{X},\mathcal{M}\cap\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})=(\text{ƻ},(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J})$, where for all $(\varphi,j)\in(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}$, $\text{ƻ}(\varphi,j)=\mathfrak{X}'(\varphi)\cap\mathcal{V}'(j)$. Thus,

$$\text{ƻ}(\varphi,j)=[\mho(\varphi)\cap\mathfrak{F}(\varphi)]'\cap\mathcal{V}'(j)=[\mho'(\varphi)\cup\mathfrak{F}'(\varphi)]\cap\mathcal{V}'(j)$$

Assume that $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J}) = (\wp,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $Ŧ(m,\mathrm{j}) = \mho'(m) \cap \mathcal{V}'(\mathrm{j})$ and $(d,\mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $\wp(d,\mathrm{j}) = \mathfrak{F}'(d) \cap \mathcal{V}'(\mathrm{j})$. Let $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J}) \cup_R (\wp,\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathfrak{D},(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(\varphi,\mathrm{j}) \in (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$,

$$\mathfrak{D}(\varphi,\mathrm{j}) = Ŧ(\varphi,\mathrm{j}) \cup \wp(\varphi,\mathrm{j}) = [\mho'(\varphi) \cap \mathcal{V}'(\mathrm{j})] \cup [\mathfrak{F}'(\varphi) \cap \mathcal{V}'(\mathrm{j})]$$

Thus, $[(\mho,\mathcal{M}) \cap_R (\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J}) =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})] \cup_R [(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

Here, if $\mathcal{M} \cap \mathcal{D} = \emptyset$, then $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = \emptyset$. Since the only soft set with the empty parameter set is $\emptyset_\emptyset$, both sides of the equality are $\emptyset_\emptyset$. Moreover, since $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, if $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})=\emptyset$, then $\mathcal{M} \cap \mathcal{D} = \emptyset$ or $\mathcal{J}=\emptyset$. By assumption, $\mathcal{J} \neq \emptyset$. Thus, $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = \emptyset$ implies that $\mathcal{M} \cap \mathcal{D} = \emptyset$. Hence, under this condition, both sides of the equality are again $\emptyset_\emptyset$.

Note 4.2. The restricted soft set operation can not distribute over soft theta-product as the intersection does not distribute over cartesian product and two SSs must be M-equal so that their PS should be the same.

Theorem 4.3. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over **U.** Then, we have the following distributions of soft theta-product over extended intersection and union operations:

i) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D}) \cap_\varepsilon (\mathcal{V},\mathcal{J})] =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})] \cup_\varepsilon [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

ii) $[(\mho,\mathcal{M}) \cap_\varepsilon (\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J}) =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})] \cup_\varepsilon [(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iii) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D}) \cup_\varepsilon (\mathcal{V},\mathcal{J})] =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})] \cap_\varepsilon [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iv) $[(\mho,\mathcal{M}) \cup_\varepsilon (\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J}) =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})] \cap_\varepsilon [(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

Proof: (i) The PS of the LHS is $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$, and the PS of the RHS is $(\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J})$. Since $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J})$, the first condition of the M-equality is satisfied. As $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, $\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J}) \neq \emptyset$ and $(\mathcal{M}\mathrm{x}\mathcal{D}) \cup (\mathcal{M}\mathrm{x}\mathcal{J}) \neq \emptyset$. Thus, no side may be equal to an empty soft set. Let $(\mathfrak{F},\mathcal{D}) \cap_\varepsilon (\mathcal{V},\mathcal{J}) = (\mathfrak{X},\mathcal{D} \cup \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cup \mathcal{J}$,

$$\mathfrak{X}(\varphi)=\begin{cases} \mathfrak{F}(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{J} \\ \mathcal{V}(\varphi), & \varphi\in\mathcal{J}\text{-}\mathcal{D} \\ \mathfrak{F}(\varphi) \cap \mathcal{V}(\varphi), & \varphi\in\mathcal{D} \cap \mathcal{J} \end{cases}$$

Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{X},\mathcal{D} \cup \mathcal{J}) = (\mathfrak{Z},\mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J}))$, where for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$, $\mathfrak{Z}(m,\varphi) = \mho'(m) \cap \mathfrak{X}'(\varphi)$. Thus, for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cup \mathcal{J})$,

$$\mathfrak{Z}(m,\varphi)=\begin{cases} \mho'(m) \cap \mathfrak{F}'(\varphi), & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}) \\ \mho'(m) \cap \mathcal{V}'(\varphi), & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{J}\text{-}\mathcal{D}) \\ \mho'(m) \cap [\mathfrak{F}'(\varphi) \cup \mathcal{V}'(\varphi)], & (m,\varphi)\in\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Now let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J}) = (\wp,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $Ŧ(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$ and $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $\wp(m,\mathrm{j}) = \mho'(m) \cap \mathcal{V}'(\mathrm{j})$. Assume that

$(Ŧ, \mathcal{M}x\mathcal{D}) \cup_{\varepsilon} (ღ, \mathcal{M}x\mathcal{J}) = (☼, (\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J}))$, where for all $(m, \varphi) \in (\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cup \mathcal{J})$,

$$☼(m, \varphi)=\begin{cases} Ŧ(m, \varphi), & (m, \varphi)\in(\mathcal{M}x\mathcal{D})\text{-}(\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D}\text{-}\mathcal{J}) \\ ღ(m, \varphi), & (m, \varphi)\in(\mathcal{M}x\mathcal{J})\text{-}(\mathcal{M}x\mathcal{D}) = \mathcal{M}x(\mathcal{J}\text{-}\mathcal{D}) \\ Ŧ(m, \varphi) \cup ღ(m, \varphi), & (m, \varphi)\in(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Thus,

$$☼(m, \varphi)=\begin{cases} Ʊ'(m) \cap 𝔉'(\varphi), & (m, \varphi)\in(\mathcal{M}x\mathcal{D})\text{-}(\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D}\text{-}\mathcal{J}) \\ Ʊ'(m) \cap ℣'(\varphi), & (m, \varphi)\in(\mathcal{M}x\mathcal{J})\text{-}(\mathcal{M}x\mathcal{D}) = \mathcal{M}x(\mathcal{J}\text{-}\mathcal{D}) \\ [Ʊ'(m) \cap 𝔉'(\varphi)] \cup [Ʊ'(m) \cap ℣'(\varphi)], & (m, \varphi)\in(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Hence, $(Ʊ, \mathcal{M})\Lambda_{\theta}[(𝔉, \mathcal{D}) \cap_{\varepsilon} (℣, \mathcal{J})] =_{M} [(Ʊ, \mathcal{M})\Lambda_{\theta}(𝔉, \mathcal{D})] \cup_{\varepsilon} [(Ʊ, \mathcal{M})\Lambda_{\theta}(℣, \mathcal{J})]$.

(iii) The PS of the LHS is $(\mathcal{M} \cup \mathcal{D})x\mathcal{J}$, and the PS of the RHS is $(\mathcal{M}x\mathcal{J}) \cup (\mathcal{D}x\mathcal{J})$. Since $(\mathcal{M} \cup \mathcal{D})x\mathcal{J} = (\mathcal{M}x\mathcal{J}) \cup (\mathcal{D}x\mathcal{J})$, the first condition of the M-equality is satisfied. By assumption, $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$, and $\mathcal{J} \neq \emptyset$. Thus, $(\mathcal{M} \cup \mathcal{D})x\mathcal{J} \neq \emptyset$ and $(\mathcal{M}x\mathcal{J}) \cup (\mathcal{D}x\mathcal{J}) \neq \emptyset$. Thereby, no sides may be equal to an empty soft set. Let $(Ʊ, \mathcal{M}) \cup_{\varepsilon} (𝔉, \mathcal{D}) = (Ӿ, \mathcal{M} \cup \mathcal{D})$, where for all $\varphi \in \mathcal{M} \cup \mathcal{D}$,

$$Ӿ(\varphi)=\begin{cases} Ʊ(\varphi), & \varphi\in\mathcal{M}\text{-}\mathcal{D} \\ 𝔉(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{M} \\ Ʊ(\varphi) \cup 𝔉(\varphi), & \varphi\in\mathcal{M} \cap \mathcal{D} \end{cases}$$

Assume that $(Ӿ, \mathcal{M} \cup \mathcal{D})\Lambda_{\theta}(℣, \mathcal{J}) = (ƻ, (\mathcal{M} \cup \mathcal{D})x\mathcal{J})$, where for all $(\varphi, j) \in (\mathcal{M} \cup \mathcal{D})x\mathcal{J}$, $ƻ(\varphi, j) = Ӿ'(\varphi) \cap ℣'(j)$. Thus, for all $(\varphi, j) \in (\mathcal{M} \cup \mathcal{D})x\mathcal{J}$,

$$ƻ(\varphi, j)=\begin{cases} Ʊ'(\varphi) \cap ℣'(j), & (\varphi, j)\in(\mathcal{M}\text{-}\mathcal{D})x\mathcal{J} \\ 𝔉'(\varphi) \cap ℣'(j), & (\varphi, j)\in(\mathcal{D}\text{-}\mathcal{M})x\mathcal{J} \\ [Ʊ'(\varphi) \cap 𝔉'(\varphi)] \cap ℣'(j), & (\varphi, j)\in(\mathcal{M} \cap \mathcal{D})x\mathcal{J} \end{cases}$$

Now let $(Ʊ, \mathcal{M})\Lambda_{\theta}(℣, \mathcal{J}) = (Ŧ, \mathcal{M}x\mathcal{J})$ ve $(𝔉, \mathcal{D})\Lambda_{\theta}(℣, \mathcal{J}) = (ღ, \mathcal{D}x\mathcal{J})$, where for all $(m, j) \in \mathcal{M}x\mathcal{J}$, $Ŧ(m, j) = Ʊ'(m) \cap ℣'(j)$ and $(d, j) \in \mathcal{D}x\mathcal{J}$, $ღ(d, j) = 𝔉'(d) \cap ℣'(j)$. Let $(Ŧ, \mathcal{M}x\mathcal{J}) \cap_{\varepsilon} (ღ, \mathcal{D}x\mathcal{J}) = (☼, (\mathcal{M}x\mathcal{J}) \cup (\mathcal{D}x\mathcal{J}))$, where for all $(\varphi, j) \in (\mathcal{M}x\mathcal{J}) \cup (\mathcal{D}x\mathcal{J}) = (\mathcal{M} \cup \mathcal{D})x\mathcal{J}$,

$$☼(\varphi, j)=\begin{cases} Ŧ(\varphi, j), & (\varphi, j)\in(\mathcal{M}x\mathcal{J})\text{-}(\mathcal{D}x\mathcal{J}) = (\mathcal{M}\text{-}\mathcal{D})x\mathcal{J} \\ ღ(\varphi, j), & (\varphi, j)\in(\mathcal{D}x\mathcal{J})\text{-}(\mathcal{M}x\mathcal{J}) = (\mathcal{D}\text{-}\mathcal{M})x\mathcal{J} \\ Ŧ(\varphi, j) \cap ღ(\varphi, j), & (\varphi, j)\in(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})x\mathcal{J} \end{cases}$$

Thus,

$$☼(\varphi, j)=\begin{cases} Ʊ'(\varphi) \cap ℣'(j), & (\varphi, j)\in(\mathcal{M}x\mathcal{J})\text{-}(\mathcal{D}x\mathcal{J}) = (\mathcal{M}\text{-}\mathcal{D})x\mathcal{J} \\ 𝔉'(\varphi) \cap ℣'(j), & (\varphi, j)\in(\mathcal{D}x\mathcal{J})\text{-}(\mathcal{M}x\mathcal{J}) = (\mathcal{D}\text{-}\mathcal{M})x\mathcal{J} \\ [Ʊ'(\varphi) \cap ℣'(j)] \cap [𝔉'(\varphi) \cap ℣'(j)], & (\varphi, j)\in(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})x\mathcal{J} \end{cases}$$

Hence, $[(Ʊ, \mathcal{M}) \cup_{\varepsilon} (𝔉, \mathcal{D})]\Lambda_{\theta}(℣, \mathcal{J}) =_{M} [(Ʊ, \mathcal{M})\Lambda_{\theta}(℣, \mathcal{J})] \cap_{\varepsilon} [(𝔉, \mathcal{D})\Lambda_{\theta}(℣, \mathcal{J})]$.

Note 4.4. The extended soft set operation can not distribute over soft theta-product as the union operation does not distribute over cartesian product and two SSs must be M-equal that their PS should be the same.

Theorem 4.5. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over **U**. Then, we have the following distributions of soft theta-product over soft binary piecewise intersection and union operations:

i) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J})] =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

ii) $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\,\tilde{\cup}\,(\mathcal{V},\mathcal{J})] =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\,\tilde{\cap}\,[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iii) $[(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J}) =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]\,\tilde{\cap}\,[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

iv) $[(\mho,\mathcal{M})\,\tilde{\cap}\,(\mathfrak{F},\mathcal{D})]\Lambda_\theta(\mathcal{V},\mathcal{J}) =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]\,\tilde{\cup}\,[(\mathfrak{F},\mathcal{D})\Lambda_\theta(\mathcal{V},\mathcal{J})]$.

Proof: (i) Since the PS of the SSs of both sides are $\mathcal{M}\text{x}\mathcal{D}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{M}\neq\emptyset$ and $\mathcal{D}\neq\emptyset$ by assumption, $\mathcal{M}\text{x}\mathcal{D}\neq\emptyset$. Thus, no side may be equal to an empty soft set. Let $(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J}) = (\mathfrak{X},\mathcal{D})$, where for all $d\in\mathcal{D}$,

$$\mathfrak{X}(d)=\begin{cases}\mathfrak{F}(d), & d\in\mathcal{D}\text{-}\mathcal{J}\\ \mathfrak{F}(d)\cap\mathcal{V}(d), & d\in\mathcal{D}\cap\mathcal{J}\end{cases}$$

Assume that $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{X},\mathcal{D}) = (\text{ღ},\mathcal{M}\text{x}\mathcal{D})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$, $\text{ღ}(m,d)=\mho'(m)\cap\mathfrak{X}'(d)$. Thus,

$$\text{ღ}(m,d)=\begin{cases}\mho'(m)\cap\mathfrak{F}'(d), & (m,d)\in\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ \mho'(m)\cap[\mathfrak{F}'(d)\cup\mathcal{V}'(d)], & (m,d)\in\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Suppose that $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) = (\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J}) = (\text{ƻ},\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$, $\text{Ŧ}(m,d)=\mho'(m)\cap\mathfrak{F}'(d)$ and $(m,\text{j})\in\mathcal{M}\text{x}\mathcal{J}$, $\text{ƻ}(m,\text{j})=\mho'(m)\cap\mathcal{V}'(\text{j})$. Let $(\text{Ŧ},\mathcal{M}\text{x}\mathcal{D})\,\tilde{\cup}\,(\text{ƻ},\mathcal{M}\text{x}\mathcal{J}) = (\text{☼},\mathcal{M}\text{x}\mathcal{D})$, where $(m,d)\in\mathcal{M}\text{x}\mathcal{D}$,

$$\text{☼}(m,d)=\begin{cases}\text{Ŧ}(m,d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ \text{Ŧ}(m,d)\cup\text{ƻ}(m,d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Thus,

$$\text{☼}(m,d)=\begin{cases}\mho'(m)\cap\mathfrak{F}'(d), & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J})\\ [\mho'(m)\cap\mathfrak{F}'(d)]\cup[\mho'(m)\cap\mathcal{V}'(d)], & (m,d)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Hence, $(\mho,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J})] =_M [(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\mho,\mathcal{M})\Lambda_\theta(\mathcal{V},\mathcal{J})]$. Since $\mathcal{M}\neq\mathcal{M}\text{x}\mathcal{M}$, the soft binary piecewise operations do not distribute over soft theta-product operations.

(iii) Since the PS of the SSs of both sides is $\mathcal{M}\text{x}\mathcal{J}$, and the first condition of the M-equality is satisfied. Moreover since $\mathcal{M}\neq\emptyset$ and $\mathcal{J}\neq\emptyset$ by assumption, $\mathcal{M}\text{x}\mathcal{J}\neq\emptyset$. Thus, no side may be an empty soft set. Let $(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D}) = (\mathfrak{X},\mathcal{M})$, where for all $m\in\mathcal{M}$,

$$\mathfrak{X}(m)=\begin{cases}\mho(m), & m\in\mathcal{M}\text{-}\mathcal{D}\\ \mho(m)\cup\mathfrak{F}(m), & m\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

Assume that $(Ӿ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J}) = (ღ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $ღ(m,\mathrm{j}) = Ӿ'(m) \cap ꝟ'(\mathrm{j})$. Thus,

$$Ӿ(m,\mathrm{j})=\begin{cases} Ʊ'(m) \cap ꝟ'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J} \\ [Ʊ'(m) \cap \mathfrak{F}'(m)] \cap ꝟ'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} \end{cases}$$

Suppose that $(Ʊ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\Lambda_\theta(ꝟ,\mathcal{J}) = (ƻ,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $Ŧ(m,\mathrm{j}) = Ʊ'(m) \cap ꝟ'(\mathrm{j})$ and $(d,\mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $ƻ(d,\mathrm{j}) = \mathfrak{F}'(d) \cap ꝟ'(\mathrm{j})$. Let $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J}) \tilde{\cap} (ƻ,\mathcal{D}\mathrm{x}\mathcal{J}) = (☼,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$,

$$☼(m,\mathrm{j})=\begin{cases} Ŧ(m,\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J} \\ Ŧ(m,\mathrm{j}) \cap ƻ(m,\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} \end{cases}$$

Thereby,

$$☼(m,\mathrm{j})=\begin{cases} Ʊ'(m) \cap ꝟ'(\mathrm{j}), & (m,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J}) =(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J} \\ [Ʊ'(m) \cap ꝟ'(\mathrm{j})] \cap [\mathfrak{F}'(m) \cap ꝟ'(\mathrm{j})], & (m,\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} \end{cases}$$

Hence, $[(Ʊ,\mathcal{M}) \tilde{\cup} (\mathfrak{F},\mathcal{D})]\Lambda_\theta(ꝟ,\mathcal{J}) =_{\mathrm{M}} [(Ʊ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J})] \tilde{\cap} [(\mathfrak{F},\mathcal{D})\Lambda_\theta(ꝟ,\mathcal{J})]$.

Proposition 4.6. Let $(Ʊ,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(ꝟ,\mathcal{J})$ be SSs over U. Then,

(i) $(Ʊ,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\Lambda(ꝟ,\mathcal{J})] \tilde{\subseteq}_{\mathrm{L}} [(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\mathrm{V}[(Ʊ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J})]$,

(ii) $(Ʊ,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\mathrm{V}(ꝟ,\mathcal{J})] \tilde{\subseteq}_{\mathrm{L}} [(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]\Lambda[(Ʊ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J})]$.

Proof: (i) Let $(\mathfrak{F},\mathcal{D})\Lambda(ꝟ,\mathcal{J}) = (Ʊ,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(d,\mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $Ʊ(d,\mathrm{j}) = \mathfrak{F}(d) \cap ꝟ(\mathrm{j})$. Let $(Ʊ,\mathcal{M})\Lambda_\theta(Ʊ,\mathcal{D}\mathrm{x}\mathcal{J}) = (Ӿ,\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(m,(d,\mathrm{j})) \in \mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})$,

$$Ӿ\big(m,(d,\mathrm{j})\big) = Ʊ'(m) \cap [\mathfrak{F}(d) \cap ꝟ(\mathrm{j})]' = Ʊ'(m) \cap [\mathfrak{F}'(d) \cup ꝟ'(\mathrm{j})]$$

Assume that $(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) = (☼,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(Ʊ,\mathcal{M})\Lambda_\theta(ꝟ,\mathcal{J}) = (ƻ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $☼(m,d) = Ʊ'(m) \cap \mathfrak{F}'(d)$ and $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $ƻ(m,\mathrm{j}) = Ʊ'(m) \cap ꝟ'(\mathrm{j})$. Let $(☼,\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{V}(ƻ,\mathcal{M}\mathrm{x}\mathcal{J}) = (ღ,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J}))$, where for all $\big((m,d),(m,\mathrm{j})\big) \in (\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J})$,

$$ღ\big((m,d),(m,\mathrm{j})\big) = [Ʊ'(m) \cap \mathfrak{F}'(d)] \cup [Ʊ'(m) \cap ꝟ'(\mathrm{j})]$$

Here, for all $(m,(d,\mathrm{j})) \in \mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})$, there exists $((m,d),(m,\mathrm{j})) \in (\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J})$ such that

$$\begin{aligned} Ӿ\big(m,(d,\mathrm{j})\big) &= Ʊ'(m) \cap [\mathfrak{F}'(d) \cup ꝟ'(\mathrm{j})] = [Ʊ'(m) \cap \mathfrak{F}'(d)] \cup [Ʊ'(m) \cap ꝟ'(\mathrm{j})] \\ &= ღ\big((m,d),(m,\mathrm{j})\big) \end{aligned}$$

This completes the proof. It is obvious that the L-subset in Proposition 4.6. can not be L-equality with the following example:

Example 4.7. Let $\mathrm{E} = \{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $\mathcal{M} = \{e_1,e_5\}$, $\mathcal{D} = \{e_3\}$ and $\mathcal{J} = \{e_2\}$ be subsets of E, $\mathrm{U} = \{h_1,h_2,h_3,h_4,h_5,h_6\}$ be the universal set and the SSs $(Ʊ,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(ꝟ,\mathcal{J})$ over U be as follows: $(Ʊ,\mathcal{M})=\{(e_1,\{h_1,h_6\}),(e_5,\{h_2,h_4,h_5\})\}$, $(\mathfrak{F},\mathcal{D})=\{(e_3,\{h_1,h_3,h_4\})\}$ and $(Ꮙ,\mathcal{J})=\{(e_2,\{h_1,h_4,h_5\})\}$. We show that

$$(Ʊ,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\Lambda(Ꮙ,\mathcal{J})]\neq_L[(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]V[(Ʊ,\mathcal{M})\Lambda_\theta(Ꮙ,\mathcal{J})]$$

Let $(\mathfrak{F},\mathcal{D})\Lambda(Ꮙ,\mathcal{J})=(℧,\mathcal{D}\mathrm{x}\mathcal{J})$, where

$$(℧,\mathcal{D}\mathrm{x}\mathcal{J})=\{((e_3,e_2),\{h_1,h_4\})\}$$

Assume that $(Ʊ,\mathcal{M})\ \Lambda_\theta\ (℧,\mathcal{D}\mathrm{x}\mathcal{J})=\big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big)$, where

$$\big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big)=\{((e_1,(e_3,e_2)),\{h_2,h_3,h_5\}),((e_5,(e_3,e_2)),\{h_3,h_6\})\}$$

Let $(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})=(₴,\mathcal{M}\mathrm{x}\mathcal{D})$, where

$$(₴,\mathcal{M}\mathrm{x}\mathcal{D})=\{\big((e_1,e_3),\{h_2,h_5\}\big),((e_5,e_3),\{h_6\})\}$$

Suppose that $(Ʊ,\mathcal{M})\Lambda_\theta(Ꮙ,\mathcal{J})=(☼,\mathcal{M}\mathrm{x}\mathcal{J})$, where

$$(☼,\mathcal{M}\mathrm{x}\mathcal{J})\ =\{\big((e_1,e_2),\{h_2,h_3\}\big),((e_5,e_2),\{h_3,h_6\})\}$$

Let $(₴,\mathcal{M}\mathrm{x}\mathcal{D})V(☼,\mathcal{M}\mathrm{x}\mathcal{J})=(ღ,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J}))$. Then,

$$\big(ღ,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J})\big)=\{\big(((e_1,e_3),(e_1,e_2)),\{h_2,h_3,h_5\}\big),\big(((e_1,e_3),(e_5,e_2)),\{h_2,h_3,h_5,h_6\}\big),$$

$$\big(((e_5,e_3),(e_1,e_2)),\{h_2,h_3,h_6\}\big),\big(((e_5,e_3),(e_5,e_2)),\{h_3,h_6\}\big)\}$$

Thereby, $ღ\big((e_1,e_3),(e_5,e_2)\big)\neq\mathfrak{X}(e_1,(e_3,e_2))$, $ღ\big((e_1,e_3),(e_5,e_2)\big)\neq\mathfrak{X}((e_5,(e_3,e_2))$, $ღ\big((e_5,e_3),(e_1,e_2)\big)\neq\mathfrak{X}(e_1,(e_3,e_2))$ and $ღ\big((e_5,e_3),(e_1,e_2)\big)\neq\mathfrak{X}((e_5,(e_3,e_2))$. Thus, $(ღ,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J}))\ \tilde{\nsubseteq}_L\ \big(\mathfrak{X},\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})\big)$, implying that $(Ʊ,\mathcal{M})\Lambda_\theta[(\mathfrak{F},\mathcal{D})\Lambda(Ꮙ,\mathcal{J})]\neq_L[(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})]V[(Ʊ,\mathcal{M})\Lambda_\theta(Ꮙ,\mathcal{J})]$.

5 | *uni-int* Decision-Making Method Applied to Soft Theta-product

In this section, the *uni-int* operator and *uni-int* decision function defined by Çağman and Enginoğlu [11] are applied for the soft theta-product for *uni-int* decision-making method. A set is reduced to its subset using this method based on the parameters of the decision-makers. Consequently, decision-makers focus on a limited number of options rather than a huge number.

Throughout this section, all the soft theta-products (Λ_θ) of the SSs over U are assumed to be contained in the set $\Lambda_\theta(\mathrm{U})$, and the approximation function of the soft theta-product of $(Ʊ,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, that is, $(Ʊ,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D})$ is

$$Ʊ_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}:\mathcal{M}\mathrm{x}\mathcal{D}\rightarrow \mathrm{P}(\mathrm{U}),$$

where $\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}(m,d) = \mho'(m) \cap \mathfrak{F}'(d)$ for all $(m,d) \in \mathcal{M}x\mathcal{D}$.

Definition 5.1. Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SS over U. Then, *uni-int* operators for soft theta-product, denoted by $uni_x int_y$ and $uni_y int_x$ are defined respectively as

$uni_x int_y$: $\Lambda_\theta \to P(U)$, $uni_x int_y(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}) = \cup_{m\in\mathcal{M}}(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}(m,d)))$

$uni_y int_x$: $\Lambda_\theta \to P(U)$, $uni_y int_x(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}) = \cup_{d\in\mathcal{D}}(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}(m,d)))$

Definition 5.2. [11] Let $(\mho,\mathcal{M})\Lambda_\theta(\mathfrak{F},\mathcal{D}) \in \Lambda_\theta(U)$. Then, *uni-int* decision function for soft theta-product, denoted by *uni-int* are defined by

uni-int: $\Lambda_\theta \to P(U)$, $uni\text{-}int(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}) = uni_x int_y(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}) \cup uni_y int_x(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}})$

that reduces the size of the universe U. Hence, the values *uni-int*($\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}$) is a subset of U called *uni-int* decision set of $\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}}$.

Assume that a set of parameters and a set of options are provided. Next, using the *uni-int* decision-making approach, which is structured as follows, a set of optimal options is chosen considering the issue:

Step 1: Select workable subsets from the parameter collection,
Step 2: Build the SSs for every parameter set.
Step 3: Determine the SSs' soft theta-product,
Step 4: Determine the product's *uni-int* decision set.

We can now illustrate how soft set theory is used in the *uni-int* decision-making problem for the soft theta-product.

Example 5.3. After a challenging university entrance exam process, the ÇAM family entered a decision-making process to determine the private university preferences of their twin daughters, Derya and Deniz. In this process, the parents and the twin girls will consider the parameters they do not want in their preferred private university, and they will make their decision using the soft teta-product's *uni-int* decision-making method. The set of private universities in the ÇAM family's preference list, which they previously had the chance to visit, is represented by $U = \{z_1, z_2, \ldots, z_{17}\}$. The set of parameters used to determine the preferred university is:
$E = \{c_1, c_2, \ldots, c_8\}$, where:

- c_1 = "Has a long distance from home"
- c_2= "Has high tuition fees"
- c_3= "Has an unsafe campus environment"
- c_4= "Has an insufficient teaching staff"
- c_5= "Has limited social activities"
- c_6= "Has low student satisfaction"
- c_7= "Has insufficient technological infrastructure"
- c_8= "Has low cafeteria quality"

Since there are four decision-makers (the parents and the twin daughters), the soft set for the parents will be determined first, followed by the soft set for Derya and Deniz. Afterward, the *uni-int* decision-making method on the soft theta-product will be applied, and the common values emerging from both sets will determine the family's final decision democratically. This process ensures a balanced, democratic decision that takes into account both the parents' and the twins' preferences for selecting the best university. If these steps were to be expressed as an algorithm, they would proceed as follows:

Step 1: Determining the Sets of Parameters

The decision makers' parameter sets are defined. Helin, the sister of the house, asks each decision maker to select the parameters that represent the characteristics they absolutely DO NOT want in the university they choose. These sets are defined as follows:

- For the mother ($\mathcal{M}$): $\mathcal{M} = \{c_1, c_8\}$, meaning the mother does not want universities that are far from home or have low-quality cafeterias.
- For the father ($\mathcal{D}$): $\mathcal{D} = \{c_2, c_3\}$, meaning the father does not want universities with high tuition fees or unsafe campus surroundings.
- For Derya ($\mathcal{J}$): $\mathcal{J} = \{c_4, c_7\}$, meaning Derya does not want universities with low-quality teaching staff or insufficient technological infrastructure.
- For Deniz ($\mathcal{H}$): $\mathcal{H} = \{c_5, c_6\}$ meaning Deniz does not want universities with limited social activities or low student satisfaction"

These parameters represent undesirable properties that make a university unsuitable for selection.

Step 2: Constructing the SSs by using the PSs Determined in Step 1.

- First, the parents' SSs are determined by evaluating the parameters they DO NOT want to be present in the university they select.
- Then, the twins' SSs are determined by evaluating the parameters they DO NOT want in their preferred university.

These SS are as follows $(\mho, \mathcal{M})$, $(\mathfrak{F}, \mathcal{D})$, $(\text{♈}, \mathcal{J})$ ve $(\text{ƶ}, \mathcal{H})$, respectively:

$$(\mho, \mathcal{M}) = \{(c_1, \{z_2, z_6, z_7, z_8, z_{11}, z_{17}\}), (c_8, \{z_1, z_3, z_5, z_7, z_{12}, z_{15}\})$$

$$(\mathfrak{F}, \mathcal{D}) = \{(c_2, \{z_3, z_7, z_9, z_{13}, z_{15}, z_{16}\}), (c_3, \{z_1, z_5, z_6, z_7, z_{11}, z_{12}, z_{15}\})$$

$$(\text{♈}, \mathcal{J}) = \{(c_4, \{z_4, z_7, z_9, z_{11}, z_{13}, z_{17}\}), (c_7, \{z_1, z_3, z_5, z_7, z_{12}, z_{15}\})$$

$$(\text{ƶ}, \mathcal{H}) = \{(c_5, \{z_2, z_5, z_8, z_{10}, z_{16}, z_{17}\}), (c_6, \{z_2, z_4, z_7, z_{10}, z_{11}, z_{13}\})$$

$(\mho, \mathcal{M})$ is an SS representing the universities to be eliminated due to undesirable parameters in $\mathcal{M}$ according to the mother, $(\mathfrak{F}, \mathcal{D})$ is an SS representing the universities to be eliminated due to undesirable parameters in $\mathcal{D}$ according to the father, $(\text{♈}, \mathcal{J})$ is an SS representing the universities to be eliminated due to undesirable parameters in $\mathcal{J}$ according to Derya, $(\text{ƶ}, \mathcal{H})$ is an SS representing the universities to be eliminated due to undesirable parameters in $\mathcal{H}$ according to Deniz. Note that the family aims to select the universities, not to eliminate them.

Step 3: Determine the Λ_θ-product of soft sets:

$$\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}} = \{((c_1,c_2),\{z_1,z_4,z_5,z_{10},z_{12},z_{14}\}),((c_1,c_3),\{z_3,z_4,z_9,z_{10},z_{13},z_{14},z_{16}\}),$$

$$((c_8,c_2),\{z_2,z_4,z_6,z_8,z_{10},z_{11},z_{14},z_{17}\}),((c_8,c_3),\{z_2,z_4,z_8,z_9,z_{10},z_{13},z_{14},z_{16}\})\}$$

$$\mathcal{V}_{\mathcal{J}}\Lambda_{\theta}\text{ƶ}_{\mathcal{H}} = \{((c_4,c_5),\{z_1,z_3,z_6,z_{12},z_{14},z_{15}\}),((c_4,c_6),\{z_1,z_3,z_5,z_6,z_8,z_{12},z_{14},z_{15},z_{16}\}),$$

$$((c_7,c_5),\{z_4,z_6,z_9,z_{11},z_{13},z_{14}\}),((c_7,c_6),\{z_6,z_8,z_9,z_{14},z_{16},z_{17}\})\}.$$

Step 4: First, the set of *uni-int*($\mho_{\mathcal{M}}\Lambda_{\theta}\mho_{\mathcal{D}}$) is determined, then the set of *uni-int*($\mathcal{V}_{\mathcal{J}}\Lambda_{\theta}\text{ƶ}_{\mathcal{H}}$) and in the final stage of the democratic decision-making process, *uni-int*($\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}$) ∩*uni-int*($\mathcal{V}_{\mathcal{J}}\Lambda_{\theta}\text{ƶ}_{\mathcal{H}}$) is calculated.

$$uni_m - int_d(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}) = \cup_{m\in\mathcal{M}}(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d)))$$

We first determine $\cap_{d\in\mathcal{D}}((\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d))$:

$$(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_1,c_2) \cap (\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_1,c_3) = \{z_1,z_4,z_5,z_{10},z_{12},z_{14}\} \cap \{z_3,z_4,z_9,z_{10},z_{13},z_{14},z_{16}\} =$$

$$= \{z_4,z_{10},z_{14}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_8,c_2) \cap (\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_8,c_3) = \{z_2,z_4,z_6,z_8,z_{10},z_{11},z_{14},z_{17}\} \cap \{z_2,z_4,z_8,z_9,z_{10},z_{13},z_{14},z_{16}\} = \{z_2,z_4,z_8,z_{10},z_{14}\}$$

Thus,

$$uni_m - int_d(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}) = \cup_{m\in\mathcal{M}}\left(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d)\right) = \{z_4,z_{10},z_{14}\} \cup \{z_2,z_4,z_8,z_{10},z_{14}\} = \{z_2,z_4,z_8,z_{10},z_{14}\}$$

is obtained.

$$uni_d - int_m(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}) = \cup_{d\in\mathcal{D}}\left(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d)\right)$$

We now determine $\left(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d)\right)$:

$$(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_1,c_2) \cap (\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_8,c_2) = \{z_1,z_4,z_5,z_{10},z_{12},z_{14}\} \cap \{z_2,z_4,z_6,z_8,z_{10},z_{11},z_{14},z_{17}\} = \{z_4,z_{10},z_{14}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_1,c_3) \cap (\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(c_8,c_3) = \{z_3,z_4,z_9,z_{10},z_{13},z_{14},z_{16}\} \cap \{z_2,z_4,z_8,z_9,z_{10},z_{13},z_{14},z_{16}\} = \{z_4,z_9,z_{10},z_{13},z_{14},z_{16}\}$$

Thereby,

$$uni_d - int_m(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}) = \cup_{d\in\mathcal{D}}\left(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})(m,d)\right) = \{z_4,z_{10},z_{14}\} \cup \{z_4,z_9,z_{10},z_{13},z_{14},z_{16}\} = \{z_4,z_9,z_{10},z_{13},z_{14},z_{16}\}$$

Hence,

uni-int$(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}) = [uni_m - int_d(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}})] \cup [uni_d - int_m(\mho_{\mathcal{M}}\Lambda_{\theta}\mathfrak{F}_{\mathcal{D}}0)] = \{z_2, z_4, z_8, z_{10}, z_{14}\} \cup \{z_4, z_9, z_{10}, z_{13}, z_{14}, z_{16}\} = \{z_2, z_4, z_8, z_9, z_{10}, z_{13}, z_{14}, z_{16}\}$.

Therefore, the private universities that align with the parents' preferences are $\{z_2, z_4, z_8, z_9, z_{10}, z_{13}, z_{14}, z_{16}\}$. Since

$$uni_j - int_h(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}) = \cup_{j\in\mathcal{J}}\left(\cap_{h\in\mathcal{H}}\left((\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(j,h)\right)\right)$$

First, we determine $\cap_{h\in\mathcal{H}}\left((\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(j,h)\right)$:

$$(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_4,c_5) \cap (\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_4,c_6) = \{z_1, z_3, z_6, z_{12}, z_{14}, z_{15}\} \cap \{z_1, z_3, z_5, z_6, z_8, z_{12}, z_{14}, z_{15}, z_{16}\} = \{z_1, z_3, z_6, z_{12}, z_{14}, z_{15}\}$$

$$(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_7,c_5) \cap (\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_7,c_6) = \{z_4, z_6, z_9, z_{11}, z_{13}, z_{14}\} \cap \{z_6, z_8, z_9, z_{14}, z_{16}, z_{17}\} = \{z_6, z_9, z_{14}\}$$

Thus,

$$uni_j - int_{h)}(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}) = \cup_{j\in\mathcal{J}}\left(\cap_{h\in\mathcal{H}}\left(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}(j,h)\right)\right) = \{z_1, z_3, z_6, z_{12}, z_{14}, z_{15}\} \cup \{z_6, z_9, z_{14}\}$$

$$= \{z_1, z_3, z_6, z_9, z_{12}, z_{14}, z_{15}\}$$

We now determine $\left(\cap_{j\in\mathcal{J}}\left(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}(j,h)\right)\right)$:

$$(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_4,c_5) \cap (\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_7,c_5) = \{z_1, z_3, z_6, z_{12}, z_{14}, z_{15}\} \cap \{z_4, z_6, z_9, z_{11}, z_{13}, z_{14}\} = \{z_6, z_{14}\}$$

$$(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_4,c_6) \cap (\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})(c_7,c_6) = \{z_1, z_3, z_5, z_6, z_8, z_{12}, z_{14}, z_{15}, z_{16}\} \cap \{z_6, z_8, z_9, z_{14}, z_{16}, z_{17}\} = \{z_6, z_8, z_{14}, z_{16}\}$$

Hence,

$$uni_h - int_j(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}) = \cup_{h\in\mathcal{H}}\left(\cap_{j\in\mathcal{J}}\left(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}(j,h)\right)\right) = \{z_6, z_{14}\} \cup \{z_6, z_8, z_{14}, z_{16}\} = \{z_6, z_8, z_{14}, z_{16}\}$$

Thus,

uni-int$(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}}) = [uni_j - int_h(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})] \cup [uni_h - int_j(\mho_{\mathcal{J}}\Lambda_{\theta}\mathfrak{Z}_{\mathcal{H}})] = \{z_1, z_3, z_6, z_9, z_{12}, z_{14}, z_{15}\} \cup \{z_6, z_8, z_{14}, z_{16}\} = \{z_1, z_3, z_6, z_8, z_9, z_{12}, z_{14}, z_{15}, z_{16}\}$

Therefore, the private universities that match Derya and Deniz's preferences are $\{z_1, z_3, z_6, z_8, z_9, z_{12}, z_{14}, z_{15}, z_{16}\}$.

Finally, for the democratic decision-making process, let's calculate *uni-int*$(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}})$ ∩ *uni-int*$(\Upsilon_{\mathcal{J}}\Lambda_\theta\mathfrak{Z}_{\mathcal{H}})$

$$[\textit{uni-int}(\mho_{\mathcal{M}}\Lambda_\theta\mathfrak{F}_{\mathcal{D}})] \cap [\textit{uni-int}(\Upsilon_{\mathcal{J}}\Lambda_\theta\mathfrak{Z}_{\mathcal{H}})] = \{z_2, z_4, z_8, z_9, z_{10}, z_{13}, z_{14}, z_{16}\} \cap \{z_1, z_3, z_6, z_8, z_9, z_{12}, z_{14}, z_{15}, z_{16}\} = \{z_8, z_9, z_{14}, z_{16}\}$$

Therefore, in the private university preference process for the ÇAM family's twin daughters, Derya and Deniz, the most suitable private universities for the mother, father, Derya, and Deniz are the set. $\{z_8, z_9, z_{14}, z_{16}\}$. This represents the universities that meet the preferences of both the parents and Derya and Deniz. The result of this intersection will show the universities that align with the preferences of all decision-makers involved, thus this process ensures a balanced and democratic decision is made by considering both the parents' and the twins' preferences to select the best university.

6 | Conclusion

In this paper, using the definition of Molodtsov's soft set, we first proposed a novel product for soft sets, which we term the soft theta-product. We provided its example and thoroughly analyzed its algebraic features concerning several soft subsets and soft equality types, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. The distributions of soft theta-product over various specific kinds of soft set operations were also obtained. Finally, we applied the soft decision-making method that selects the best elements from options without the need for fuzzy soft sets or rough sets and gave an example that shows how the method may be successfully used in a variety of sectors. This research will lay the foundation for a wide range of applications, such as novel cryptography techniques based on soft sets and new decision-making approaches. In future studies, some new soft product operations may be proposed and basic properties concerning certain kinds of soft equal relations may be further examined to contribute to the soft set literature from both a theoretical and a practical standpoint.

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Author Contribution

Conceptualization, A.S.; Methodology, A.S. and N.H.Ç.; Validation, A.S. and N.H.Ç.;writing-creating the initial design, N.H.Ç.; writing-reviewing and editing, N.H.Ç. and A.S.; visualization, A.S. and N.H.Ç. All authors have read and agreed to the published version of the manuscript.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

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