Multicriteria Algorithms with Applications

Journal Homepage: sciencesforce.com/mawa

Multicriteria Algo. Appl. Vol. 6 (2025) 1–8

Paper Type: Original Article

The Fuzzy Subgroups for the *p*-groups of an *n* Power Order of Four for Any Integer *n* not Less than Three

Sunday Adesina Adebisi [1,*], Mike Ogiugo [1] and Michael Enioluwafe [2]

[1] Department of Mathematics, Faculty of Science, University of Lagos, Nigeria.
Emails: adesinasunday@yahoo.com; ekpenogiugo@gmail.com.
[2] Department of Mathematics, Faculty of Science, University of Ibadan, Nigeria; michael.enioluwafe@gmail.com.

Received: 13 Sep 2024 **Revised:** 01 Dec 2024 **Accepted:** 28 Dec 2024 **Published:** 01 Jan 2025

Abstract

In this paper, the explicit formulae as well as the full structural expression is given for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group with a cyclic group both of which possess the order of *n* power of two respectively which results in a *p*-group of an *n* power order of four for any integer *n* not less than three.

Keywords: Finite *p*-Groups; Nilpotent Group; Fuzzy Subgroups; Dihedral Group; Inclusion-Exclusion Principle; Maximal Subgroups.

1 | Introduction

The theory of fuzzy sets possesses so many forms of applications. Of such and of course, without restriction is that of fuzzy groups. Part of its applications is to provide formalized tools for dealings with the imprecision intrinsic to many problems. Denote the number of chains of subgroups of a finite group *G* which ends in *G* by h(G). The method of computing h(G) is based on the application of the Inclusion-Exclusion Principle. In this context, h(G) is referred to as the number of distinct fuzzy subgroups for the finite nilpotent p-group. This work is therefore designed as part of classifying the nilpotent groups formed from the Cartesian products of p-groups through their computations [2-4].

2 | Methodology

At this juncture, we shall introduce the technique which was used in the course of our processes. Hence, are going to adopt a method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G. That is the number of fuzzy subgroups of a finite group G which end in G. This is denoted by $h(G)$, and it is the number of distinct fuzzy subgroups for the finite nilpotent group. The expression was derived in [6] as follows: (our esteemed readers can also consult [5, 7-9] for more details.

Suppose that G is a finite nilpotent group, in which $N_1, N_2, \ldots, N_t$ are the maximal subgroups of G. Let us represent these number of chains of subgroups of G by the symbol $h(G)$. These chains are known to terminate at G. Now, for the singular purpose of computing somewhat the exact value of $h(G)$, we are going

Corresponding Author: adesinasunday@yahoo.com
https://doi.org/10.61356/j.mawa.2025.6441

to base our technique on the proper application of the Inclusion-Exclusion Principle. This particular method has been discussed more extensively and in detail in [1] and [6]. Here in particular, given that A is the set of chains in G which are of the type given by: $C_1 \subset C_2 \subset \cdots \subset C_r = G$, and A' represents the set of chains in G of types $C_1 \subset C_2 \subset \cdots \subset C_r \neq G$, and let C_r Be the set of chains of A' Which are contained in $N_r, r = 1, \dots, t$. Then we have:

$$|A| = 1 + |A'| = 1 + \left|\bigcup_{r=1}^{t} C_r\right|$$
$$= 1 + \sum_{r=1}^{t} |C_r| - \sum_{1 \le r_1 \le r_2 \le t} |C_{r_1} \cap C_{r_2}| + \cdots + (-1)^{t-1}\left|\bigcap_{r=1}^{t} C_r\right|$$

Here, it should be strictly noted that, for every $1 \le w \le t$ and $1 \le r_1 < r_2 < \cdots < r_w \le t$, the set $\bigcap_{i=1}^{w} C_{r_i}$ must consist mainly and in all cases of all chains of A'. Such chains ought to be included in $\bigcap_{i=1}^{w} N_{r_i}$. And so, we are going to have that.

$$\left|\bigcap_{i=1}^{w} C_{r_i}\right| = 2h\left(\bigcap_{i=1}^{w} N_{r_i}\right) - 1$$
$$\therefore |A| = 1 + \sum_{r=1}^{t} (2h(N_r) - 1) - \sum_{1 \le r_1 < r_2 \le t} \big(2h(N_{r_1} \cap N_{r_2}) - 1\big)$$
$$+ \cdots + (-1)^{t-1}\left(2h\left(\bigcap_{r=1}^{t} N_r\right) - 1\right)$$
$$= 2\left(\sum_{r=1}^{t} h(N_r) - \sum_{1 \le r_1 < r_2 \le t} h\big(N_{r_1} \cap N_{r_2}\big) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} N_r\right)\right) + C$$

And

$$C = 1 + \sum_{r=1}^{t} (-1) - \sum_{1 \le r_1 < r_2 \le t} (-1) + \cdots + (-1)^{t-1}(-1)$$
$$= (1-1)^t = 0$$

we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(N_r) - \sum_{1 \le r_1 < r_2 \le t} h\big(N_{r_1} \cap N_{r_2}\big) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} N_r\right)\right) \tag{1}$$

Definition: $h(G)$ given in (1) can be defined as the number of fuzzy subgroups. This particular number is unique for the finite group G. It should also be carefully noted and categorically stated here that, they are also very distinct (please, see [1, 5-6] for more details on this. In [6], (1) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.Here D_{2n} denotes the dihedral group of order $2n$. Now, if $n = p^m$, then,

$$h(D_{2n}) = \frac{2^m}{p-1}(p^{m+1} + p - 2) \tag{2}$$

From here, $2n = 2p^m \cdot p = 2 \Rightarrow 2n = 2.2^m = 2^{m+1}$

$\therefore h(D_{2n}) = h(D_{2^m+1}) = \frac{2^m}{2-1}(2^{m+1} + 2 - 2) = 2^m(2^{m+1})$.

Hence, by putting $n = m + 1 \Rightarrow m = n - 1 \Rightarrow h(D_{2^n}) = 2^{n-1}.2^n = 2^{2n-1}$

$$h(D_{2^n}) = 2^{2n-1} \quad (3)$$

So, if the given subgroup structure of a finite group G which has been certified to have possessed as it were, in each of the respectful cases the maximality condition, their types are then determined, and their intersections computed by using GAP [17], then we are going to have it after some calculations and simplifications Eq. (1) given above results in some recurrence relations which allows the value of $h(G)$ to be explicitly determined [12, 15].

Theorem [1, 6]: Every finite p-group possessing an order which equals p^n And having a cyclic maximal subgroup must have its number of fuzzy subgroups which are distinct to be given by:

i). $h(C_{p^n}) = 2^n$,

ii). $h(C_p \times C_{p^{n-1}}) = 2^{n-1}[2 + (n-1)p]$ We are going to apply this theorem at some point or the other in our computational processes.

Proposition [13]: Given that $G = C_4 \times C_{2^n}, n \geq 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$

Corollary: Following the last proposition, $h(C_4 \times C_{2^5}), h(C_4 \times C_{2^6})$, $h(C_4 \times C_{2^7})$ and $h(C_4 \times C_{2^8}) = 1536, 4096, 10496,$ and 26112 respectively.

Theorem [15]: Suppose that $G = D_{2^n} \times C_2$. Here, G happens to be the nilpotent group which has been formed through the proceeds of getting the Cartesian product for the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of fuzzy subgroups for G which are distinct can be given by: $h(G) = 2^{2n}(2n + 1) - 2^{n+1}, n > 3$

Proposition [12]: Let $G = D_{2^n} \times C_4$. Then, the number of fuzzy subgroups for G which are distinct can be given by :

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

Proposition [10]: Suppose that G is a p-group having the abelian properties and is also of the type given by: $C_p \times C_p \times C_{p^n}$, such that p is a prime number and $n \geq 1$. The number of fuzzy subgroups for G which are distinct is given by $h(C_p \times C_p \times C_{p^n}) = 2^n p(p+1)(n-1)(3 + np + 2p) + (2^n - 2)p^3 - 2^{n+1}(n - 1)p^3 + 2^n[p^3 + 4(1 + p + p^2)]$.

Corollary: From (3) above, observe that we are going to have that:

$$h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 2^{n+1}[18n^2 + 9n + 26] - 54$$

Similarly, suppose that we have $p = 5$, and by following the procedure above as well, we are going to have that:

$$\begin{aligned} h(C_5 \times C_5 \times C_{5^n}) =& 2[30h(C_5 \times C_{5^n}) + h(C_5 \times C_5 \times C_{5^{n-1}}) \\ & -p^3 h(C_{5^n}) - 30h(C_{5^{n-1}}) + 125] \end{aligned}$$

Also, if $p = 7, h(C_7 \times C_7 \times C_{7^n}) = 2[56h(C_7 \times C_{7^n}) + h(C_7 \times C_7 \times C_{7^{n-1}}) - 343h(C_{7^n}) - 56h(C_{7^{n-1}}) + 343]$ If this process continues this way, we are definitely going to have it in general that: $h(C_p \times C_p \times C_{p^{n-2}}) = 2^{n-2}[4 + (3n-5)p$ $+(n^2 - 5)p^2 + (n^2 - 5n + 8)p^3] - 2p^2$. This is definitely true for any given prime p.

Proposition [14]: If G is a group and such that: $G = (D_{2^3} \times C_{2^m})$ and for $m \geq 3$. Then, $h(G) = 89m - 23m^2 + (85)2^{m+3} - 124$

Proof:

$$h(D_{2^3} \times C_{2^m}) = (46m-3) \cdot 2^{m+1} + 2^6 + (46m-49)2^{m+1} + 2^7 + (46m-95)2^{m+1} + 2^8 + 2^3 h(D_{2^3} \times C_{2^{m-3}})$$

$$= 2^{m+1} \cdot [(46m-3) + (46m-49) + (46m-95)] + 2^6 + 2^7 + 2^8 + 2^3 h(D_{2^3} \times C_2^{m-3}) =$$

$$+2^{m+1} \cdot [46mk + \underbrace{2^6 + 2^7 + 2^8 + \cdots + 2^{5+k}}_{\text{series (1)}} \underbrace{(-3-49-95\cdots(-3-46(k-1)))}_{\text{series (2)}}]$$

$$+2^k h(D_{2^3} \times C_{2^m{}_{-k}}), k \in \{1,2.3 \cdots n \in N\}$$

Observe that in the series (1), we have that, $V_n = 2^6.2^{n-1} = 2^{5+t}, n+5 = t+5, \Rightarrow n = t$. So that $S_{n=t} = 2^6 \left[\frac{2^t-1}{2-1}\right] = 2^6(2^t-1)$

Also, note that in the second series (2), we should have it that, $U_n = -3 + (n-1)(-46) = -3 - 46(t-1) \Rightarrow \quad n-1 = t-1, n = t$ Whence, $S_n = t = \frac{t}{2}[2(-3) + (t-1)(-46)] = \frac{t}{2}(-6 - 46t + 46) = \frac{t}{2}(40 - 46t)$. Hence, we are going to have that: $h(D_{2^3} \times C_{2^m}) = \frac{t}{2}(40-46t) + 2^6(2^k - 1) + 2^k h(D_3 \times C_{2^m{}_{-t}}$. By setting $m = t$ we have that $t = m - 3$. Hence, $h(D_{2^3} \times C_{2^m}) = (m-3)(20 - 23m) + 2^6(2^{m-3} - 1) + 2^m - 3h(D_3 \times C_{2^3})$ $h(G) = (m-3)(20-23m) + 2^6(2^{m-3}-1) + 2^{m-3}(5376) = (m-3)(20-23m) + 2^{m-3} - 2^6 + 2^{m+5}(21)$ $= 20m - 23m^2 - 60 + 69m + 2^{m+3} - 2^6 + (21)2^{m+5} = (89m - 23m^2 - 60) + 2^{m+3} - 2^6 + (21)2^{m+5} = m(89-23m) - 124 + (85)2^{m+3}$

Theorem [11]: Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

Proposition (see [16]: Suppose that $G = D_{2^n} \times C_8$. Then, the number of fuzzy subgroups of G which are distinct is going to be equal to:

$$2^{2(n-1)}(6n+113) + 2^n \left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)\right]$$

$$+\frac{1}{3}(2^{n+2})\left[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]\right]$$

Theorem: Let $G = D_{2^4} \times C_{2^4}$. Then, $h(G) = 61384$

3 | The Number of Fuzzy Subgroups for $G = D_{2^4} \times C_{2^n}, n \geq 4$

Which are distinct. Our computation on the algebraic fuzzy structure given actually has an outcome that involves multiple sums.

Proof:

The maximal subgroups are:

$(D_{2^4} \times C_{2^{n-1}}), 2(D_{2^3} \times C_{2^n}), 2(D_{2^n} \times C_{2^2}), (D_{2^n} \times C_{2^3})$ and (C_{2^n}).

We have that : $\frac{1}{2}h(G) = h(D_{2^4} \times C_{n-1}) + 2h(D_{2^3} \times C_n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) + h(C_{2^n}) - 6h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 6h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^n}) + 2h(D_{2^3} \times C_{2^{n-1}}) + 28h(C_{2^{n-1}} \times C_{2^n}) + h(C_{2^{n-1}} \times C_{2^3}) + 2h(C_{2^n} \times C_{2^2}) + 2h(\mathbb{Z}_{2^n}) - 35h(C_{2^{n-1}} \times C_{2^2}) + 21h(C_{2^{n-1}} \times C_{2^2}) - 7h(C_{2^{n-1}} \times C_{2^2}) + h(C_{2^{n-1}} \times C_{2^2}) = h(D_{2^4} \times C_{2^{n-1}}) + 2h(D_{2^3} \times C_2^n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) - 4h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^n})$

$$\frac{1}{2}h(G) = h\big(D_{2^4} \times \mathbb{Z}_{2^{n-k}}\big) + 2h(D_{2^3} \times \mathbb{Z}_{2^n}) - 4h\big(D_{2^3} \times \mathbb{Z}_{2^{n-k}}\big) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2})$$

$$-2h\big(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^3}\big) + 8h\big(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^2}\big) + \sum_{j=1}^{k} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}\big) + 2\sum_{j=1}^{k} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}\big) - 3\sum_{j=1}^{k} h\big(\mathbb{Z}_{2^{n+1-j}}\big)$$

$$-2\sum_{j=1}^{k-1} h\big(D_{2^3} \times \mathbb{Z}_{2^{n-j}}\big) + 4\sum_{j=1}^{k-1} h\big(D_{2^{n-j}} \times \mathbb{Z}_{2^2}\big) - 2\sum_{j=1}^{k-1} h\big(D_{2^n{}_{-j}} \times \mathbb{Z}_{2^3}\big)$$

whence, $n - k = 4, \Rightarrow k = n - 4. \therefore h(G) = 2h(D_{2^4} \times \mathbb{Z}_{2^4}) + 4h(D_{2^3} \times \mathbb{Z}_{2^n}) - 8h(D_{2^3} \times \mathbb{Z}_{2^4}) - 8h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 16h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) +$

$$2\sum_{j=1}^{n-4} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}\big) + 4\sum_{j=1}^{n-4} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}\big) - 6\sum_{j=1}^{n-4} h\big(\mathbb{Z}_{2^{n+1-j}}\big)$$

$$-4\sum_{j=1}^{n-5} h\big(D_{2^3} \times \mathbb{Z}_{2^{n-j}}\big) + 8\sum_{j=1}^{n-5} h\big(D_{2^n{}_{-j}} \times \mathbb{Z}_{2^2}\big) - 4\sum_{j=1}^{n-5} h\big(D_{2^{n-j}} \times \mathbb{Z}_{2^3}\big)$$

$$\therefore h(G) = 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + 2\sum_{j=1}^{n-4} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}\big)$$

$$+4\sum_{j=1}^{n-4} h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}\big) - 6\sum_{j=1}^{n-4} h\big(\mathbb{Z}_{2^{n+1-j}}\big) - 4\sum_{j=1}^{n-5} h\big(D_{2^3} \times \mathbb{Z}_{2^{n-j}}\big) + 8\sum_{j=1}^{n-5} h\big(D_{2^n{}_{-j}} \times \mathbb{Z}_{2^2}\big) - 4\sum_{j=1}^{n-5} h\big(D_{2^{n-j}} \times \mathbb{Z}_{2^3}\big)$$

$$= 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + \sum_{j=1}^{n-4}\big[2h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}\big) + 4h\big(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}\big) - 6h\big(\mathbb{Z}_{2^{n+1-j}}\big)\big]$$

$$-\sum_{j=1}^{n-5}\big[4h\big(D_{2^3} \times \mathbb{Z}_{2^{n-j}}\big) - 8h\big(D_{2^n{}_{-j}} \times \mathbb{Z}_{2^2}\big) + 4h\big(D_{2^{n-j}} \times \mathbb{Z}_{2^3}\big)\big]$$

Hence, proven as required.

Applications

The computations so far by the use of GAP (General Algorithm Algorithms and Programming) and the Inclusion-Exclusion Principle can actually be confirmed here as being very useful in the computations of the number of fuzzy subgroups for the finite p-groups which are district [17].

4 | Determining the Fuzzy Subgroups for $G = D_{2^n} \times C_{2^4}, n \geq 4$

If $G = (D_{2^n} \times C_{2^4})$, then,

$\frac{1}{2}h(G) = h(D_{2^n} \times C_{2^3}) + 2h(C_{2^{n-1}} \times C_{2^4}) + 2h(D_{2^4} \times C_{2^{n-2}}) + h(D_{2^4} \times C_{2^{n-1}}) - 4h(D_{2^{n-1}} \times C_{2^3}) - 4h(C_{2^4} \times C_{2^{n-2}}) - 2h(C_{2^3}) \times C_{2^{n-1}}) + 8h(C_{2^3} \times C_{2^{n-2}}) - 3h(C_{2^4})$

So, $h(G) = 2h(D_{2^n} \times C_{2^3}) + 4h(D_{2^{n-1}} \times C_{2^4}) + 4h(C_{2^4} \times C_{2^{n-2}}) + 2h(D_{2^4} \times C_{2^{n-1}})$

$-8h(D_{2^{n-1}} \times C_2^3) - 8h(C_{2^4} \times C_{2^{n-2}}) - 4h(C_{2^3} \times C_{2^{n-1}}) + 16h(C_{2^3} \times C_{2^5}) - 2h(C_{2^4}) \times C_{2^4}) +$

$8h(C_{2^3} \times C_{2^{n-2}}) - 6h(C_{2^4})$

$= 2^{2n-8}h(D_{2^4} \times C_{2^4}) - 3(10 + 2^6 + 2^8 + \cdots 2^{2n-8})h(C_{2^4}) + 2h(D_{2^n} \times C_{2^3}) + 12h(D_{2^4} \times C_{2^{n-2}}) + 2h(D_{2^4} \times C_{2^{n-1}}) - 32h(C_{2^4} \times C_{2^{n-3}}) - 8h(C_{2^4} \times C_{2^{n-2}})$

$$-4h(C_{2^3} \times C_2^{n-1}) - 2^7h(C_{2^4} \times C_{2^{n-4}}) + 48h(D_{2^4} \times C_{2^{n-3}})$$

$$+ 3\sum_{j=1}^{2n-8} 2^{n-4}h\left(D_{2^4} \times \mathbb{Z}_{2^{n+1-j}}\right) + \sum_{j=1}^{2n-7} 2^{2n+9}h\left(D_{2^4} \times \mathbb{Z}_{2^{n+1-j}}\right)$$

As required.

5 | Finding the Number of Fuzzy Subgroups for the Group $G = D_{2^n} \times C_{2^m}$, for $3 \leq n \leq m$

Suppose that $G = (D_{2^n} \times C_{2^m})$ for $3 \leq n \leq m$.

Then,

$\frac{1}{2}h(G) = h(D_{2^n} \times C_{2^{m-1}}) + 2h(D_{2^{n-1}} \times C_{2^m}) + 2h(D_{2^m} \times C_{2^{n-2}}) + h(D_{2^m} \times C_{2^{n-1}}) - 4h(D_{2^{n-1}} \times C_{2^{m-1}}) - 4h(C_{2^{n-2}} \times C_{2^m}) - 2h(C_{2^{n-1}}) \times C_{2^{m-1}}) + 8h(C_{2^{n-2}} \times C_{2^{m-2}}) - 3h(C_{2^m})$

So, $h(G) = 2h(D_{2^n} \times C_{2^{m-1}}) + 4h(D_{2^{n-1}} \times C_{2^m}) + 4h(D_{2^m} \times C_{2^{n-2}}) + 2h(D_{2^m} \times C_{2^{n-1}}) - 8h(D_{2^{n-1}} \times C_{2^{m-1}}) - 8h(C_{2^{n-2}} \times C_{2^m}) - 4h(C_{2^{n-1}}) \times C_{2^{m-1}}) + 16h(C_{2^{n-2}} \times C_{2^{m-1}}) - 6h(C_{2^m})$

Now, suppose for instance, that $m = n = 5$. Then, we have that: $h(G) = 2h(D_{2^5} \times C_{2^4}) + 4h(D_{2^4} \times C_5) + 4h(D_5 \times C_{2^4}) + 2h(D_{2^5} \times C_{2^4}) - 8h(D_{2^4} \times C_{2^4}) - 8h(C_{2^3} \times C_{2^5}) - 4h(C_{2^4}) \times C_{2^4}) + 16h(C_{2^3} \times C_{2^4}) - 3.2^6$.

Instances: We have the following examples as parts surfacing from our computations so far. We urge our esteemed readers to consider the examples given below in tabular format.

Example: Now, since the given stated condition that $m \geq 3$ must be fulfilled then the readers may consider the examples below in for simple illustration.

Table 1. Summarizing some Number of Distinct Fuzzy Subgroups of $(D_{2^4} \times C_{2^n})$ FOR $n \geq 4$.

S/N for the Number of n	4	5	6
$h(G) = (D_{2^4} \times C_{2^n}), n \geq 4$	20,200	375,648	3,893,800

6 | The Number of Fuzzy Subgroups which are Distinct for the Group $G = D_{2^n} \times C_{2^n}$, for $n \geq 4$

Suppose that $G = (D_{2^n} \times C_{2^n})$ for $n \geq 4$. Then ,

$h(G) = 2h(D_{2^n} \times C_{2^{n-1}}) + 4h(D_{2^{n-1}} \times C_{2^n}) + 4h(D_{2^n} \times C_{2^{n-2}}) + 2h(D_{2^n} \times C_{2^{n-1}}) - 8h(D_{2^{n-1}} \times C_{2^{n-1}}) - 8h(C_{2^{n-2}} \times C_{2^n}) - 4h(C_{2^{n-1}}) \times C_{2^{n-1}}) + 16h(C_{2^{n-2}} \times C_{2^{n-1}}) - 6h(C_{2^n})$

$= 4h(D_{2^n} \times C_{2^{n-1}}) + 4h(D_{2^{n-1}} \times C_{2^n}) + 4h(D_{2^n} \times C_{2^{n-2}}) - 8h(D_{2^{n-1}} \times C_{2^{n-1}}) - 8h(C_{2^{n-2}} \times C_{2^n}) - 4h(C_{2^{n-1}}) \times C_{2^{n-1}}) + 16h(C_{2^{n-2}} \times C_{2^{n-1}}) - 6h(C_{2^n})$

Now , if $n = 4$. Then , we have that : $h(G) = 4h(D_{2^4} \times C_{2^3}) + 4h(D_{2^4} \times C_2) + 4h(D_{2^3} \times C_{2^4}) - 8h(D_{2^3} \times C_{2^3}) - 8h(C_{2^4} \times C_{2^2}) - 4h(C_{2^3}) \times C_{2^3}) + 16h(C_{2^3} \times C_{2^2}) - 3.2^5$ □

Example: If we set $\{g_i\}_{\{i=1,2\}} = h\left(D_{2^4} \times C_{2^{4-i}}\right)\{g_j\}_{\{j=3,4\}} = h\left(D_{2^3} \times C_{2^{7-j}}\right)$

$\{g_k\}_{\{k=5\}} = h\left(C_{2^{9-k}} \times C_{2^{7-k}}\right)\{g_l\}_{\{l=6,7\}} = h\left(C_{2^3} \times C_{2^{9-l}}\right)\{g_h\}_{\{h=8\}} = h\left(C_{2^{12-i}}\right)$,

Then, the following table emerges

Table 2. Summarizing some Number of Distinct Fuzzy Subgroups of $(D_{2^n} \times C_{2^n})$ FOR ≥ 3.

g_i	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
$h(g_i)$	14848	7200	10744	5376	544	864	176	16
$\alpha h(g_i)$	$+4h(g_1)$	$+4h(g_2)$	$+4h(g_3)$	$-8h(g_4)$	$-8h(g_5)$	$-4h(g_6)$	$+16h(g_7$	$-6h(g_8)$
value	+59392	+28800	+42976	-43008	-4352	-3456	+2816	-96
Total = $83,072$								

7 | Conclusion

The discoveries from our studies so far have helped to observe that any given product of the nilpotent group also gives a nilpotent result. Furthermore, the problem of classifying the fuzzy subgroups for groups that are known to be finite has experienced very rapid progress. Tables 1 and 2 summarize some details concerning this. Finally, this particular method can also be applied in further and subsequent computations up to the generalizations of both similar as well as the s other given structures.

Acknowledgments

The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

Author Contributions

All authors contributed equally to this work.

Funding

This research has no funding source.

Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] M.Mashinchi and M.Mukaidono(1992).Aclassificationoffuzzysubgroups.NinthFuzzySystem Symposium, Sapporo, Japan, 649-652.

[2] M. Mashinchi, M. Mukaidono .(1993). On fuzzy subgroups classification, Research Reports of Meiji Univ. (9) , 31-36.

[3] V. Murali and B. B. Makamba (2003). On an equivalence of Fuzzy Subgroups III, Int. J. Math. Sci. 36, 2303-2313.

[4] Odilo Ndiweni (2014). The classification of Fuzzy subgroups of the Dihedral Group D_n, for n, a product of distinct primes. A Ph.D. thesis, Univ. of Fort Hare, Alice, S.A.

[5] M. Tarnauceanu (2009). The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, European J. Combin. (30), 283-289, doi: 10.1016/j.ejc.2007.12.005.

[6] M. Tarnauceanu (2011). Classifying fuzzy subgroups for a class of finite p-groups. "ALL CUZa" Univ. Iasi, Romania.

[7] M. Tarnauceanu (2012). Classifying fuzzy subgroups of finite nonabelian groups. Iran.J.Fussy Systems. (9) 33-43

[8] M. Tarnauceanu, L. Bentea (2008). A note on the number of fuzzy subgroups of finite groups, Sci. An. Univ. "ALL.Cuza" Iasi, Matt., (54) 209-220.

[9] M. Tarnauceanu, L. Bentea (2008). On the number of fuzzy subgroups of finite abelian groups, Fuzzy Sets and Systems (159), 1084-1096, doi:10.1016/j.fss.2017.11.014

[10] S. A. ADEBISI and M. EniOluwafe (2020) The Abelian Subgroup: $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}, p$ is Prime and $n \geq 1$. Progress in Nonlinear Dynamics and Chaos Vol. 7, No. 1, 2019. http://dx.doi.org/10.22457/pindac.80v7n1a343

[11] S.A. Adebisi, M. Ogiugo And M. Enioluwafe (2020) The Fuzzy Subgroups For The Abelian Structure: $h(\mathbb{Z}_8 \times \mathbb{Z}_{2^n})$ for $n > 2$. Journal of the Nigerian Mathematical Society, Vol. 39, Issue 2, pp. 167-171.

[12] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe(2020)Computing the Number of Distinct Fuzzy Subgroups for the Nilpotent p-Group of $D_{2^n} \times C_4$ International J.Math.Combin.1(2020),86-89.

[13] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe (2020) Determining The Number Of Distinct Fuzzy Subgroups For The Abelian Structure : $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}, n > 2$.Transactions of the Nigerian Association of Mathematical Physics Vol 11,, pp 5 - 6

[14] S. A. Adebisi, M. Ogiugo And M. Enioluwafe (2022)The Fuzzy Subgroups For The Nilpotent (p-Group) of $(D_{2^3} \times C_{2^m})$ For $m \geq 3$ Journal of Fuzzy Extension And Applications.

[15] S. A. ADEBISI and M. EniOluwafe (2020) An explicit formula for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group of order 2 n with a cyclic group of order 2 Universal Journal of Mathematics and Mathematical Sciences.http://www.pphmj.com http://dx.doi.org/10.17654/UM013010001 Vol 13, no1, 2020, Pages 1-7

[16] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe(2020) On the p-Groups of the Algebraic Structure of $D_{2^n} \times C_8$ International J.Math. Combin. Vol.3(2020), 100-103

[17] The GAP Group (2020). GAP - Groups, Algorithms, and Programming, Version 4.11.0, https://www.gapsystem.org