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Matrix Theory Approach to Fuzzy Hypersoft Set in Multi-Criteria Airline Selection

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Abstract

When we talk about decision-making then it means we have to tackle the attributes and alternatives, and it becomes difficult when attributes are further subdivided. The hypersoft theory allows us to handle further bifurcated attributes. This paper is aim to provide the legal way to deal with multi-criteria decision-making (MCDM) problems in the sense of further subdividing attributes by applying the concept of fuzzy hypersoft matrices (FHSM's). Since matrices provide us with a fast and reliable way to solve problems. In this paper, we have proposed the concept of FHSM with their operators, theorems, propositions, and decision-making (DM) algorithms. The proposed algorithms have been supported with real-life application of airline selection while you have to take any flight. The concept of further bifurcation gives more accurate and refined results than the existing techniques, along with the implication of FHSM. In the future, the proposed concept can be implemented in other hybrids of hypersoft set theory with DM algorithms. The challenges faced in implication can be dealt with machine learning algorithms for fast and more accurate results.

Keywords: Fuzzy Set; Hypersoft Sets; Matrix Theory; Decision-making; Score Function; Air Line Selection Problem.

1 | Introduction

Uncertainty and vagueness are very difficult to classify when the decision-maker (DM) has incomplete information about the relationship between attributes and parameters whether they are dependent or independent. For example, if DM has to guess the age of Peter, then it will be helpful to collect some information about Peter for accurate age and DM only knows the height (h), weight (w), and body shape (p), and this is not enough to guess the correct age. The parameters h, w, and p increase with the increase of age and it stop at a certain time. Secondly, what is the relation between these parameters and age, this will be hard to guess when we have a piece of incomplete information that's from which geographical area Peter belongs. Since geography plays a vital role in the development of any person along with diet and health conditions. Thus, it is important to know the relationship between the parameters and the further bi-furcation. Since the hypersoft, set structure is based on the further sub-divided parameters, it allows us to deal with the parameters, the relation with the parameters, and how they are closely linked with the problem.

In, this paragraph we present the literature review and link of set theories with this paper. Zadeh [1] presented the concept of discrete and continuous values in the sense of decision making, known as the membership

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values or the concept of truthiness, and it can be measured only between the importance of the parameters means if we know this, then how much it will be going to help us in DM? If we reconsider the example stated before, and DM wants to write how much height increases with age and roughly speaking 20% is the role of height with age, then the membership value will be 0.2, likewise, for weight, we can say 0.3 (means 30%) are the chance that with the information of weight, one can guess a correct age. Ambiguity is another issue in decision-making problems, when DM is not sure about the distinctness of the attribute, then it means there are some chances to take wrong values. To present those chances Atanassov [2] came up with the concept of intuitionistic fuzzy sets (IFs). In this theory, one can present the ambiguity in the form of membership ($\mathcal{T}$) and non-membership values ($\mathcal{F}$) i.e. $0 \leq \mathcal{T} + \mathcal{F} \leq 1$. Researchers, around the globe, presented many theories, algorithms, and operators to solve DM problems. In DM, the concept of soft set (SS) theory has a vital role and Molodtsov [3] proposed the concept of SS theory, theory considers the attributes and alternatives along with crisp values and was further extended by Maji et al. [4-5] and the logical analysis of the soft sets, which included all of the necessary operators and properties. The theory of soft set was further generalized by Majumdar [6], and many new operators were defined by Ali et al. [7]. The concept was extended to interval-valued fuzzy soft set by [8]. The soft set theory has been widely used to solve multi-criteria decision-making (MCDM) problems [9-10]. Khan et al. [11-12] generalized an intuitionistic soft set and presented a decision support system to solve real-life applications. Garg [13] introduced score functions for interval-valued intuitionistic soft sets along with algorithms to solve decision problems. The soft set was further extended to many hybrid set structures like Pythagorean fuzzy soft set and cubic bi-polar soft set [14-15], its aggregate operators, and application in decision support systems were introduced by [16-17]. Cagman [18] presented the concept of a soft matrix in decision-making approaches. The concept of soft matrices was later extended to the existing hybrid set structures of soft sets, fuzzy soft matrices with applications [19-20], soft matrices on fuzzy soft multi-sets [21], T-spherical fuzzy soft matrices [22], and many other decision-making techniques involving the concept of soft matrices.

In 2018, Smarandache [23] proposed the concept of a hypersoft set (HSS), which is the generalization of soft set theory. Hypersoft set (HSS) theory tends to consider further divided attributes or attributes bi-furcation. The theory of HSS can be applied to solve both, MCDM and MADM problems. Another beauty of HSS, it can be molded as per the DM requirements. Basic concepts of HS, such as subset, complement, not HS set, absolute set, union, intersection, AND, OR, limited union, and more, were explained by Saeed et al. [24–25]. By adding operations to this HS theory, such as intersection, relevant complement, restricted difference, restricted symmetric difference, and more, Abass et al. [26] expanded it. Rahman et al. [27–29] defined complex HSS and created the hybrids of the HS set using fuzzy parametrized HSS theory, rough HSS theory, and a complex fuzzy hypersoft set, respectively. [30] proposed the expert HSS theory and its operators, such as subsets, equal sets, null sets, absolute sets, etc. The hypersoft set structure has been extended to a fuzzy hypersoft set (FHSS) with operators and applications by [31-33]. The correlation coefficients and similarity measures under intuitionistic hypersoft set (IHSs) [34], and matrix theory for IHSs with applications to decision-making problems [35] and for neutrosophic hypersoft set (NHSs) were introduced by [36-37]. Using the value matrix of picture fuzzy hypersoft set theory the application of renewable energy source selection was presented by [38].

Conventional models struggle to represent the complexity of imprecise and ambiguous information in an increasingly complex environment. Matrices offer a promising way to improve the accuracy of decision support systems and enable flexible modeling in domains such as optimization and artificial intelligence. Since in matrix notation, it is easy to write any complex form of data sets. When the attributes of the problems are further subdivided, we cannot easily write in the set structure, but in matrix notation, it is easy to present. Thus in the presence of existing limitations of mathematical models; the fuzzy hypersoft matrices' overcome these issues and provide more useful answers in various applications are the driving forces behind this study. Although fuzzy hypersoft matrices have great potential, there are still a lot of unanswered questions about their usage and comprehension. There is an absence of methodical progress in the foundations of theory, and there is a lack of research on useful algorithms for huge sets of data. The further divided attributes may have vagueness and uncertainty, then it is difficult to present with the existing approaches. Since the hypersoft set structure deals with bi-furcated attributes and the fuzzy hypersoft set deals with vagueness, the fuzzy hypersoft matrix has been defined in this paper. Our proposed work aims to close these research gaps by advancing our theoretical knowledge and practical uses of fuzzy hypersoft matrices, which will benefit the decision-support system community as a whole.

- This work aims to develop knowledge and applications of fuzzy hypersoft matrices, primarily by producing, definitions, theorems, and propositions. Our first goal is to provide a solid theoretical framework for fuzzy hypersoft matrices by clarifying their basic concepts and characteristics. Simultaneously, we want to create effective algorithms that can manipulate and compute with these matrices, especially for large-scale datasets in scenarios involving real-time decision-making.

- Second, we want to investigate and validate the implication of fuzzy hypersoft matrices in a variety of domains, including transportation, finance, healthcare, the energy sector, and artificial intelligence. We hope to demonstrate how well these matrices handle uncertainty and imprecision in practical contexts through empirical research and case studies. The case study for the selection of appropriate flights is presented in this study. The selection of appropriate flights is crucial in ensuring the fast, safe, and reliable operation of these vehicles, and their selection depends on factors such as travel time, airfare, stops, destination country, and friendly environment.

- Finally, we aim to positively impact fuzzy hypersoft matrices' uniformity assessment requirements. By creating a set of standards, we want to improve the precision of research and enable comparisons across various approaches, resulting in a more cohesive and cooperative research environment.

The following shows that, how the work has been organized: The fundamental ideas of FHSM are broken down in detail in section 2. In section 3, we present a definition, notions, and examples of FHSM with basic properties and operations. In part 4, an MCDM framework is described for the FHSM with a case study to demonstrate the benefits of the proposed algorithm. The findings of the study have been summarized, along with their significance, in section 5, and the last proposed study has been concluded with future directions.

2 | Preliminaries

This section presents the definitions, which are necessary to understand before the study of the proposed sections.

Definition 1. [3] Soft Set (SS) was proposed by Molodtsov [3] to deal with attributes and alternatives let $\mathcal{Y} = \{\text{p}_1, \text{p}_2, \text{p}_3, \dots \text{p}_s\}$ be the set of alternatives and A be a set of attributes. Let $P(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$ and $\mathcal{A} \subset$. A pair $(\eta, \mathcal{A})$ is called a soft set over $\mathcal{Y}$, where the mapping η is given by

$$\eta: \mathcal{A} \rightarrow P(\mathcal{Y}) \tag{2.1}$$

Definition 2. [21] Smarandache extended soft sets (SSs) to hypersoft sets and dealt with the further bifurcations of attributes and defined them as

Let $\mathcal{Y} = \{\text{p}_1, \text{p}_2, \text{p}_3, \dots \text{p}_s\}$ be the set of alternatives and $\mathcal{A}$ be the set of attributes. Let $P(\mathcal{Y})$ denote the power set of $\mathcal{Y}$. Let $\xi^1, \xi^2, \xi^3 \dots \xi^n$ for $n \geq 1$ be different attributive features, whose corresponding attributive values are the sets $\varsigma^1, \varsigma^2, \varsigma^3, \dots \varsigma^n$ with $\varsigma^{m_i} \cap \varsigma^{n_i} = \emptyset$ for $m \neq n, m_i, n_i = 1,2 \dots n$ respectively. Then, the pair $(\mathcal{G}, \xi^1, \xi^2, \xi^3 \dots \xi^n)$ is said to be hypersoft set over $\mathcal{Y}$, where

$$\mathcal{G}: \xi^1 \times \xi^2 \times \xi^3 \dots \times \xi^n \rightarrow P(\mathcal{Y}) \tag{2.2}$$

Definition 3. [22-25] presented the fuzzy hypersoft set (FHSS) and dealt with the further bifurcations of attributes and defined as

Let $\mathcal{Y} = \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \dots \mathrm{p}_s\}$ be the set of alternatives and $\mathcal{A}$ be the set of attributes. Let $P(\mathcal{Y})$ denote the power set of $\mathcal{Y}$. Let $\xi^1, \xi^2, \xi^3 \dots \xi^n$ for $n \geq 1$ be different attributive features, whose corresponding attributive values are the sets $\varsigma^1, \varsigma^2, \varsigma^3, \dots \varsigma^n$ with $\varsigma^{m_i} \cap \varsigma^{n_i} = \emptyset$ for$m \neq n, m_i, n_i = 1,2 \dots n$ respectively. Then, the pair ($(\xi^1, \xi^2, \xi^3 \dots \xi^n)$ is said to be hypersoft set over$\mathcal{Y}$, where

$$\mathcal{G}: \xi^1 \times \xi^2 \times \xi^3 \dots \times \xi^n \to P(\mathcal{Y}) \tag{2.3}$$

and

$$\mathcal{G}(\xi^1 \times \xi^2 \times \xi^3 \dots \times \xi^n) = \mathcal{G}(\kappa) = \{\prec \mathrm{p}, T(\mathcal{G}(\kappa)), \mathrm{p} \in \mathcal{Y} \succ\} \tag{2.4}$$

Where T is the membership value and $T: \mathcal{Y} \to [0,1]$ with $0 \leq T(\mathcal{G}(\kappa)) \leq 1$, where κ is an n-tuple

3 | Fuzzy Hypersoft Matrix

We have divided this section into two parts. In subsection 3.1, we develop a matrix form of FHSS and will present examples for a better understanding and prove a large number of theorems, propositions, and properties of FHSMs. In subsection 3.2, we present the score function along with its importance in decision-making.

3.1 | Definition and Results on FHSM

Definition 3.1. Let $S = \{s^1, s^2 \dots . s^\alpha\}$, $P(S)$ denotes the power set and universal set and considers the $\mathcal{R}_1, \mathcal{R}_2, \dots . \mathcal{R}_\gamma$ for $\gamma \geq 1, \gamma$ set $\mathcal{R}_1^a, \mathcal{R}_2^b, \dots . \mathcal{R}_\gamma^z$ and its relation $\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z$ are two sets with well-defined attributes with matching attributive values where a,b,c…. z= 1,2,…n then the pair

(F,$\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z$) is called to be Fuzzy Hyper soft set by $\mathcal{R}$ where

$F: (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z) \to P(S)$ And it is defined as:

$F: (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z) \to P(S)) = \{< s, M\varrho(s) > s \in S, \varrho \in (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z)\}$

Let $\mathcal{R}_\varrho = \mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z$ be a relation and the properties of the function are given

$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}: (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z) \to P(S)$ And it is define as

$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho} = \{< s, M\varrho(s) > s \in S, \varrho \in (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z\}$ represented in Table 1.

Table 1. Matrix representation of FHSS.

	$\mathcal{R}_1^a$	$\mathcal{R}_2^b$	$\cdots$	$\mathcal{R}_\gamma^z$
s^1	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^1, \mathcal{R}_1^a)$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^1, \mathcal{R}_2^b)$	$\cdots$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^1, \mathcal{R}_\gamma^z)$
s^2	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^2, \mathcal{R}_1^a)$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^2, \mathcal{R}_2^b)$	$\cdots$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^2, \mathcal{R}_\gamma^z)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
s^α	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^\alpha, \mathcal{R}_1^a)$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^\alpha, \mathcal{R}_2^b)$	$\cdots$	$\mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^\alpha, \mathcal{R}_\gamma^z)$

If $E_{i\,j} = \mathrm{\mathring{A}}_{\mathcal{R}_\varrho}(s^i, \mathcal{R}_j^k)$, where $i = 1,2,3 \dots . \alpha, j = 1,2,3 \dots . \gamma, \dot{k} = a, b, c \dots z$, then a matrix is define as

$$[E_i]_{\alpha \times \gamma} = \begin{pmatrix} E_{1\,1} & E_{1\,2} & \cdots & E_{1\gamma} \\ E_{2\,1} & E_{2\,2} & \dots & E_{2\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ E_{\alpha\,1} & E_{\alpha 2} & \cdots & E_{\alpha\gamma} \end{pmatrix}$$

Where $E_{i\,j} = (M_{\mathcal{R}_j^k}(s^i), s^i \in S$,

$\mathcal{R}_j^k \in (\mathcal{R}_1^a \times \mathcal{R}_2^b \times \dots . \times \mathcal{R}_\gamma^z)) = (M_{i\,j}^E)$

As a result, every Fuzzy Hyper Soft Matrix can be used to represent any Fuzzy Hyper Soft, implying that they are interchangeable.

Example: Let $H = \{\delta^1, \delta^2, \delta^3, \delta^4, \delta^5\}$ be the set of passengers traveling from Lahore to Jeddah. Consider the following attributes:

B_1 =Lounge = {1, 2, 3, 4}

B_2 =Tickets Cost= {45000p, 44000p, 40000p, 50000p}

B_3 =Classes = {Business, Economy}

B_4 =Weight = {37Kg, 43Kg, 47Kg, 50Kg}

The decision-makers will assign a membership value to each attribute, and its sub-attributes are represented in Table 2-5. The final representation of the selected alternatives is presented in Table 6 in the form of FHSs.

Table 2. Relation between lounge and passengers.

$B_1^E(Lounge)$	δ^1	δ^2	δ^3	δ^4	δ^5
No.01	0.5	0.6	0.5	0.3	0.3
No.02	0.3	0.2	0.6	0.3	0.4
No.03	0.2	0.5	0.5	0.2	0.1
No.04	0.3	0.2	0.3	0.4	0.4

Table 3. Relation between ticket prices and passengers.

$B_2^F(Ticket\ Price)$	δ^1	δ^2	δ^3	δ^4	δ^5
45000rup	0.3	0.3	0.7	0.4	0.4
44000rup	0.9	0.4	0.6	0.5	0.4
40000rup	0.4	0.6	0.2	0.5	0.2
50000rup	0.6	0.8	0.4	0.6	0.4

Table 4. Relation between air classes and passengers.

$B_3^G(Classes)$	δ^1	δ^2	δ^3	δ^4	δ^5
Business	0.5	0.8	0.7	0.1	0.4
Economy	0.2	0.4	0.3	0.3	0.2

Table 5. Relation between weights and passengers.

$B_4^H(Weight)$	δ^1	δ^2	δ^3	δ^4	δ^5
42Kg	0.8	0.4	0.3	0.3	0.7
45Kg	0.7	0.3	0.3	0.4	0.6
40Kg	0.5	0.9	0.3	0.1	0.1
43Kg	0.3	0.8	0.4	0.4	0.2

Then fuzzy hypersoft set (FHSS) can be defined as above.

$F: \left(B_1^E, B_2^F, B_3^G, B_4^H\right) =$ F (3, 44000p, Economy, 43kg) =

$$\{\langle\delta^1, (\ 3\ (0.2), 44000p\ (0.9), \bar{E}co\ (0.2), 43Kg\ (0.3)\rangle,$$

$$\langle\delta^3, (\ 3\ (0.5), 44000p\ (0.6), \bar{E}co\ (0.3), 43Kg(0.4)\rangle,$$

$$\langle\delta^4, (\ 3\ (0.2), 44000p\ (0.5), \bar{E}co\ (0.3), 43Kg\ (0.4)\rangle,$$

$$\langle\delta^5, (3\ (0.1), 44000p\ (0.4), \bar{E}co\ (0.5), 43Kg(0.2)\rangle\}$$

Where $F: (B_1^E, B_2^F, B_3^G, B_4^H) \to P(U)$, and $U = \{\delta^1, \delta^2, \delta^3, \delta^4, \delta^5\}$

Table 6. Tabular form of FHS (FHSM).

	B_1^E	B_2^F	B_3^G	B_4^H
δ^1	3 (0.2)	44000p (0.9)	Eco (0.2)	43Kg (0.3)
δ^3	3 (0.5)	44000p (0.6)	Eco (0.3)	43Kg (0.4)
δ^4	3 (0.2)	44000p (0.5)	Eco (0.3)	43Kg (0.4)
δ^5	3 (0.1)	44000p (0.4)	Eco (0.2)	43Kg (0.2)

And its matrix is defined as:

$$[E]_{4\times4} = \begin{bmatrix} 3(0.2) & 44000p(0.9) & \bar{E}co(0.2) & 43kg(0.3) \\ 3(0.5) & 44000p(0.6) & \bar{E}co(0.3) & 43kg(0.4) \\ 3(0.2) & 44000p(0.5) & \bar{E}co(0.3) & 43kg(0.4) \\ 3(0.1) & 44000p(0.4) & \bar{E}co(0.5) & 43kg(0.2) \end{bmatrix}$$

Definition 3.2. Square FHSM Let $E = [E_{ij}]$ be the Fuzzy hypersoft matrix (FHSM) of order$\alpha \times \gamma$, where $E_{ij} = \left(M^E_{i\,jk}\right)$ Also, E is considered to be square (FHSM) if it has the same number of columns (alternatives) and of rows (attributes).

Definition. 3.3: Transpose of Square FHSM

Let $E = [E_{ij}]$ be the (FHSM) of order $\alpha \times \gamma$ where $E_{ij} = \left(M^E_{i\,jk}\right)$then $E^t$is said to be the transpose of square (FHSM) if rows and columns of E are Interchange. It is denoted as.

$$E^t = [E_{i\,j}]^t = [M^E_{i\,j\,k}]^t = [M^E_{j\,k\,i}] = [M_{j\,i}]$$

Example: Transpose of the matrix defined Example above is given as

$$[E]^t{}_{4\times4} = \begin{bmatrix} 3(0.2) & 3(0.5) & 3(0.2) & 3(0.1) \\ 44000p(0.9) & 44000p(0.6) & 44000p(0.5) & 44000p(0.4) \\ \bar{E}co(0.2) & \bar{E}co(0.3) & \bar{E}co(0.3) & \bar{E}co(0.5) \\ 43kg(0.3) & 43kg(0.4) & 43kg(0.4) & 43kg(0.2) \end{bmatrix}$$

Proposition (P-1):

Let $S = \left[S_{ij}\right]$ and $A = \left[A_{ij}\right]$ be two FHSMs where $S_{ij} = \left(M^s_{ijk}\right)$ and$A_{ij} = \left(M^A_{ijk}\right)$. For two scalars $p, t \in [0,1]$ then

i. $p(tS) = (pt)S$

ii. If $p < t$ then $pS < tS$

iii. If $S \subseteq A$ then $pS \subseteq pA$

Proof:

i). $p(tS) = s\left[tS_{ij}\right]$

$$= p\left[\left(tM^S_{ijk}\right)\right] = \left[\left(ptM^S_{ijk}\right)\right]$$

$$= pt\left[\left(M^S_{ijk}\right)\right] = pt\left[M^S{}_{ij}\right]$$

$$= (pt)S.$$

ii). Since $M^S_{ijk} \in [0,1]$ so $pM^S_{ijk} \leq tM^S_{ijk}$

Now $pS = \left[pS_{ij}\right]$

$$= \left[\left(pM^S_{ijk}\right)\right] \leq \left[\left(tM^S_{ijk}\right)\right] = \left[tS_{ij}\right] = tS$$

iii). $S \subseteq A \Rightarrow [S_{ij}] \subseteq [A_{ij}]$

$$\Rightarrow M_{ijk}^{S} \leq M_{ijk}^{A}$$

$$\Rightarrow pM_{ijk}^{S} \leq pM_{ijk}^{A}$$

$$\Rightarrow p[S_{ij}] \subseteq p[A_{ij}]$$

$$\Rightarrow pS \subseteq pA$$

Theorem (TH-1)

Let $S = [S_{ij}]$ be the FHSM of order $\alpha \times \gamma$, where $S_{ij} = (M_{ijk}^{S})$. Then,

i). $(pS)^t = pS^t$ where $p \in [0,1]$.

ii). $(S^t)^t = S$.

iii). If $S = [S_{ij}]$ is the upper triangular and S^t is the lower triangular FHSM and vice versa.

Proof:

i). Here $(pS)^t, pS^t \in FHSM_{\alpha\times\gamma}$, so

$$(pS)^t = [(pM_{ijk}^{S})]^t$$

$$= [(pM_{jki}^{S})]$$

$$= p[(M_{jki}^{S})]$$

$$= p[(M_{ijk}^{S})]^t = pS^t.$$

ii). Since $S^t \in FHSM_{\alpha\times\gamma}\ so\ (S^t)^t \in FHSM_{\alpha\times\gamma}$. Now,

$$(S^t)^t = \left([(M_{ijk}^{S})]^t\right)^t$$

$$= ([(M_{jki}^{S})])^t$$

$$= [(M_{ijk}^{S})] = S$$

iii). Straight forward-proof.

Definition 3.4. Trace of FHSM

Let $S = [S_{ij}]$ be the square FHSM of order $\alpha \times \gamma$, where $S_{ij} = (M_{ijk}^{S})$ and $\alpha = \gamma$. Then, the trace of FHSM is denoted as $tr(S)$ and is defined as

$tr(S) = \sum_{i=1,k=a}^{\alpha,z}[M_{iik}^{S}]$.

Example: Let us consider a FHSM $[E]_{4\times4}$

$$[E]_{4\times4} = \begin{bmatrix} 3(0.2) & 44000p(0.9) & \bar{E}co(0.2) & 43kg(0.3) \\ 3(0.5) & 44000p(0.6) & \bar{E}co(0.3) & 43kg(0.4) \\ 3(0.2) & 44000p(0.5) & \bar{E}co(0.3) & 43kg(0.4) \\ 3(0.1) & 44000p(0.4) & \bar{E}co(0.5) & 43kg(0.2) \end{bmatrix}$$

Then $tr(O) = 0.2 + 0.6 + 0.3 + 0.2 = 1.3$

Proposition (P-2):

Let $S = [S_{i\,j}]$ be the square FHSM of order $\alpha \times \gamma$, where $S_{i\,j} = (M^{s}_{i\,j\,k})$ and $\alpha \times \gamma$. P be any scalar then $tr(pS) =$

$p\ tr(S)$.

Proof:

$tr(\mathcal{s}O) = \sum_{i=1,k=a}^{\alpha,z}[pM^{s}_{iik}] = p\sum_{i=1,k=a}^{\alpha,z}[M^{s}_{iik}]= p\ tr(S)$

Definition 3.5. Max-Min Product of FHSM:

Let $S = [S_{ij}]$ and $A = [A_{jm}]$ Be two FHSMs where $S_{ij} = (M^{s}_{ijk})$ and$A_{jm} = (M^{s}_{jkm})$. Then, if the dimensions of S and A are equal (the number of columns in S equals the number of rows in A), they are said to be conformable. If $S = [S_{ij}]_{\alpha\times\beta}$and $A = [A_{jm}]_{\beta\times\gamma}$ then $S\otimes\ A = [\mathcal{S}_{im}]_{\alpha\times\gamma}$ where

$[\mathcal{S}_{im}] = (max_{jk}\min(M^{S}_{ijk}, M^{A}_{jkm}))$

Theorem (TH-2):

Let $S = [S_{ij}]_{\alpha\times\beta}$and $A = [A_{jm}]_{\beta\times\gamma}$ be two FHSMs where $S_{ij} = (M^{s}_{ijk})$ and$A_{jm} = (M^{A}_{jkm})$. Then, $(S\otimes A)^{t} = A^{t} \otimes\ S^{t}$

Proof:

Let $S\otimes\ A = [\mathcal{S}_{im}]_{\alpha\times\gamma}$ then $(S\otimes A)^{t} = [\mathcal{S}_{mi}]_{\gamma\times\alpha}, S^{t} = [S_{ji}]_{\beta\times\alpha}, A^{t} = [A_{mj}]_{\gamma\times\beta}$

Now $(\dot{\boldsymbol{S}}\otimes A)^{t} = (M^{p}_{kmi})_{\gamma\times\alpha}$

$$= (max_{jk}\min(M^{A}_{mjk}, M^{s}_{jki}))_{\gamma\times\alpha}$$

$$= (M^{A}_{mjk},)_{\gamma\times\beta} \otimes (M^{s}_{jki})_{\beta\times\alpha}$$

$$= A^{t} \otimes\ \dot{\boldsymbol{S}}^{t}$$

Definition 3.6. Operators of FHSMs

Let $S = [\dot{\boldsymbol{S}}_{ij}]$ and $A = [A_{ij}]$ be two FHSMs where $S_{ij} = (M^{s}_{ijk})$ and$A_{ij} = (M^{A}_{ijk},)$. Then,

i. **Union:**

$S \cup A = p$ Where

$M^{p}_{ijk} = \max(M^{s}_{ijk}, M^{A}_{ijk})$.

ii. **Intersection:**

$\dot{S} \cap A = p$ Where

$M^{p}_{ijk} = \min(M^{s}_{ijk}, M^{A}_{ijk})$.

iii. **Arithmetic Mean:**

$S\oplus A = p$ Where

$M^{p}_{ijk} = \frac{(M^{s}_{ijk}+M^{A}_{ijk})}{2}$.

iv. **Weighted Arithmetic Mean:**

$\dot{S} \oplus^{w} A = p$

Where $M_{ijk}^{p}=\frac{\left(w^1M_{ijk}^{s}+w^2M_{ijk}^{A}\right)}{w^1+w^2}\ .w^1,w^2>0$

v. Geometric Mean:

$\dot{S}\odot A=p$ Where

$$M_{ijk}^{p}=\sqrt{M_{ijk}^{S}\,.\,M_{ijk}^{A}}.$$

vi. Weighted Geometric Mean:

$\dot{S}\odot^{w}A=p$

Where

$$M_{ijk}^{p}=\sqrt[w^1+w^2]{(M_{ijk}^{s})^{w^1}\,.\,(M_{ijk}^{A})^{w^2}},\ w^1,w^2>0$$

vii. Harmonic Mean:

$\dot{S}\oslash A=p$ Where

$$M_{ijk}^{p}=\frac{2M_{ijk}^{s}\,M_{ijk}^{A}}{M_{ijk}^{s}+M_{ijk}^{A}},.$$

viii. Weighted Harmonic Mean:

$\dot{S}\oslash^{w}A=p$ Where

$$M_{ijk}^{p}=\frac{w^1+w^2}{\frac{w^1}{M_{ijk}^{s}}+\frac{w^2}{M_{ijk}^{A}}}$$

Proposition (P-3):

Let $S=\left[S_{ij}\right]$ and $A=\left[A_{ij}\right]$ be two FHSMs where $S_{ij}=\left(M_{ijk}^{s}\right)$ and $A_{ij}=\left(M_{ijk}^{A}\right)$. Then,

i. $(S\cup A)^t=S^t\cup A^t$

ii. $(S\cap A)^t=S^t\cap A^t$

iii. $(S\oplus A)^t=S^t\oplus A^t$

iv. $(S\oplus^{w}A)^t=S^t\oplus A^t$

v. $(S\odot A)^t=S^t\odot A^t$

vi. $(S\odot^{w}A)^t=S^t\odot^{w}A^t$

vii. $(S\oslash A)^t=S^t\oslash A^t$

viii. $(S\oslash^{w}A)^t=S^t\oslash^{w}A^t$

Proof:

i). $(S\cup A)^t=\left[\left(\max\left(M_{ijk}^{s},M_{ijk}^{A}\right)\right)\right]^t$

$$=\left[\left(\max\left(M_{jki}^{s},M_{jki}^{A}\right)\right)\right]$$

$$=\left[\left(M_{jki}^{s}\right)\right]\cup\left[\left(M_{jki}^{A}\right)\right]$$

$$=\left[\left(M_{ijk}^{s}\right)\right]^t\cup\left[\left(M_{ijk}^{A}\right)\right]^t$$

$$=S^t\cup A^t$$

R.H.S $(S \cap A)^t = \left[\left(\min\left(M^{s}_{ijk}, M^{A}_{ijk}\right)\right)\right]^t$

$$= \left[\left(\min\left(M^{s}_{jki}, M^{A}_{jki}\right)\right)\right]$$

$$= \left[\left(M^{s}_{jki}\right)\right] \cap \left[\left(M^{A}_{jki}\right)\right]$$

$$= \left[\left(M^{s}_{ijk}\right)\right]^t \cap \left[\left(M^{A}_{ijk}\right)\right]^t$$

$$= S^t \cap A^t$$

ii). $(S \oplus A)^t = \left[\left(\frac{\left(M^{S}_{ijk} + M^{S}_{ijk}\right)}{2}\right)\right]^t$

$$= \left[\left(\frac{\left(M^{S}_{jki} + M^{S}_{jki}\right)}{2}\right)\right]$$

$$= \left[\left(M^{S}_{jki}\right)\right] \oplus \left[\left(M^{A}_{jki}\right)\right]$$

$$= \left[\left(M^{S}_{ijk}\right)\right]^t \oplus \left[\left(M^{A}_{ijk}\right)\right]^t$$

$$= S^t \oplus A^t$$

iii). $(S \oplus^w A)^t = \left[\left(\frac{\left(w^1 M^{S}_{ijk} + w^2 M^{A}_{ijk}\right)}{w^1 + w^2}\right)\right]^t \quad w^1, w^2 > 0$

$$= \left[\left(\frac{\left(w^1 M^{S}_{jki} + w^2 M^{A}_{jki}\right)}{w^1 + w^2}\right)\right]$$

$$= \left[\left(M^{S}_{jki}\right)\right] \oplus^w \left[\left(M^{A}_{jki}\right)\right]$$

$$= \left[\left(M^{S}_{ijk}\right)\right]^t \oplus^w \left[\left(M^{A}_{ijk}\right)\right]^t$$

$$= S^t \oplus^w A^t$$

iv). $(S \odot A)^t = \left[\left(\left(\sqrt{M^{S}_{ijk} \, . \, M^{A}_{ijk}}\right)\right)\right]^t$

$$= \left[\left(\left(\sqrt{M^{S}_{jki} \, . \, M^{A}_{jki}}\right)\right)\right]$$

$$= \left[\left(M^{S}_{jki}\right)\right] \odot \left[\left(M^{A}_{jki}\right)\right]$$

$$= \left[\left(M^{S}_{ijk}\right)\right]^t \odot \left[\left(M^{A}_{ijk}\right)\right]^t$$

$$= Q^t \odot R^t$$

v). $(S \odot^w A)^t = \left[\left(\left(\sqrt[w^1+w^2]{(M^{S}_{ijk})^{w^1} . \, (M^{A}_{ijk})^{w^2}}\right)\right)\right]^t \quad w^1, w^2 > 0$

$$= \left[\left(\left(\sqrt[w^1+w^2]{(M^{S}_{jki})^{w^1} . \, (M^{A}_{jki})^{w^2}}\right)\right)\right]$$

$$= \left[\left(M^{S}_{jki}\right)\right] \odot^w \left[\left(M^{A}_{jki}\right)\right]$$

$$= \left[\left(M^{S}_{ijk}\right)\right]^t \odot^w \left[\left(M^{A}_{ijk}\right)\right]^t$$

$$= S^t \odot^w A^t$$

vi). $(S \oslash A)^t = \left[\left(\left(\frac{2M_{ijk}^S M_{ijk}^A}{M_{ijk}^S + M_{ijk}^A}\right)\right)\right]^t$

$$= \left[\left(\left(\frac{2M_{jki}^S M_{jki}^A}{M_{jki}^S + M_{jki}^A}\right)\right)\right]$$

$$= \left[\left(M_{jki}^S\right)\right] \oslash \left[\left(M_{jki}^A\right)\right]$$

$$= \frac{\left[\left(M_{ijk}^S\right)\right]^t}{\left[\left(M_{ijk}^A\right)\right]^t} = S^t \oslash A^t$$

vii). $(S \oslash^w A)^t = \left[\left(\left(\frac{w^1 + w^2}{\frac{w^1}{M_{ijk}^S} + \frac{w^2}{M_{ijk}^A}}\right)\right)\right]^t, w^1, w^2 > 0$

$$= \left[\left(\left(\frac{w^1 + w^2}{\frac{w^1}{T_{jki}^S} + \frac{w^2}{T_{jki}^A}}\right)\right)\right]$$

$$= \left[\left(M_{jki}^S\right)\right] \oslash^w \left[\left(M_{jki}^A\right)\right]$$

$$= \left[\left(M_{ijk}^S\right)\right]^t \oslash^w \left[\left(M_{ijk}^A\right)\right]^t$$

$$= S^t \oslash^w A^t$$

Theorem (TH-3):

Let $S = [S_{ij}]$ and $A = [A_{ij}]$ be two upper triangular FHSMs where $S_{ij} = (M_{ijk}^s)$ and $A_{ij} = (M_{ijk}^A)$. Then,$(S \cup A)$,$(S \cap A)$, $(S \oplus A)$, $(S \oplus^w A)$, $(S \odot A)$ and $(S \odot^w A)$ are all upper triangular FHSM and vice versa.

Theorem (TH-4):

Let $S = [S_{ij}]$ and $A = [A_{ij}]$ be two FHSM where $S = (M_{ijk}^s)$ and $A_{ij} = (M_{ijk}^A)$. Then,

i. $(S \cup A)^\diamond = S^\diamond \cap A^\diamond$

ii. $(S \cap A)^\diamond = S^\diamond \cup A^\diamond$

iii. $(S \oplus A)^\diamond = S^\diamond \oplus A^\diamond$

iv. $(S \oplus^w A)^\diamond = S^\diamond \oplus A^\diamond$

v. $(S \odot A)^\diamond = S^\diamond \odot A^\diamond$

vi. $(S \odot^w A)^\diamond = S^\diamond \odot^w A^\diamond$

vii. $(S \oslash A)^\diamond = S^\diamond \oslash A^\diamond$

viii. $(S \oslash^w A)^\diamond = S^\diamond \oslash^w A^\diamond$

Proof:

L.H.S

i). $(S \cup A)^{\diamond} = \left[\left(\max\left(M_{ijk}^{s}, M_{ijk}^{A}\right)\right)\right]^{\diamond}$

$$= 1 - \left(\max\left(M_{ijk}^{s}, M_{ijk}^{A}\right)\right)$$

$$= \min\left(1 - M_{ijk}^{s}, 1 - M_{ijk}^{A}\right)$$

Case 1: If $M_{ijk}^{s} > M_{ijk}^{A}$ then

$$1 - M_{ijk}^{s} < 1 - M_{ijk}^{A}$$

$$\Rightarrow (S \cup A)^{\diamond} = 1 - M_{ijk}^{s}$$

Case 2: If $M_{ijk}^{s} < M_{ijk}^{A}$ then

$$1 - M_{ijk}^{s} > 1 - M_{ijk}^{A}$$

$$\Rightarrow (S \cup A)^{\diamond} = 1 - M_{ijk}^{A}$$

R.H.S

$$S^{\diamond} = 1 - M_{ijk}^{s}$$

$$A^{\diamond} = 1 - M_{ijk}^{A}$$

$$S^{\diamond} \cap A^{\diamond} = \min\left(1 - M_{ijk}^{s}, 1 - M_{ijk}^{A}\right)$$

Case 1: If $M_{ijk}^{s} > M_{ijk}^{A}$ then

$$1 - M_{ijk}^{s} < 1 - M_{ijk}^{A}$$

$$\Rightarrow S^{\diamond} \cap A^{\diamond} = 1 - M_{ijk}^{s}$$

Case 2: If $M_{ijk}^{s} < M_{ijk}^{A}$ then

$$1 - M_{ijk}^{s} > 1 - M_{ijk}^{A}$$

$$\Rightarrow S^{\diamond} \cap A^{\diamond} = 1 - M_{ijk}^{A}$$

ii). $(S \cap A)^{\diamond} = \left[\left(\min\left(M_{ijk}^{S}, M_{ijk}^{A}\right)\right)\right]^{\diamond}$

$$= \left[\left(M_{ijk}^{A}\right)\right] \cup \left[\left(M_{ijk}^{S}\right)\right]$$

$$= \left[\left(M_{ijk}^{S}\right)\right]^{\diamond} \cup \left[\left(M_{ijk}^{A}\right)\right]^{\diamond}$$

$$= S^{\diamond} \cup A^{\diamond}$$

iii). $(S \oplus A)^{\diamond} = \left[\left(\frac{\left(M_{ijk}^{S} + M_{ijk}^{A}\right)}{2}\right)\right]^{\diamond}$

$$= \left[\left(\frac{\left(M_{ijk}^{A} + M_{ijk}^{S}\right)}{2}\right)\right]$$

$$= \left[\left(M_{ijk}^{A}\right)\right] \oplus \left[\left(M_{ijk}^{S}\right)\right]$$

$$= \left[\left(M_{ijk}^{S}\right)\right]^{\diamond} \oplus \left[\left(M_{ijk}^{A}\right)\right]^{\diamond}$$

$$= S^{\diamond} \oplus A^{\diamond}$$

iv). $(S \oplus^{w} A)^{\diamond} = \left[\left(\frac{\left(w^1 M_{ijk}^{S} + w^2 M_{ijk}^{A}\right)}{w^1 + w^2}\right)\right]^{\diamond} \; w^1, w^2 > 0$

$$= \left[\left(\frac{\left(w^1 M_{ijk}^{A} + w^2 M_{ijk}^{S}\right)}{w^1 + w^2}\right)\right]$$

$$= \left[\left(M_{ijk}^{A}\right)\right] \oplus^{w} \left[\left(M_{ijk}^{S}\right)\right]$$

$$= \left[\left(, M_{ijk}^{S}\right)\right]^{\diamond} \oplus^{w} \left[\left(M_{ijk}^{A}\right)\right]^{\diamond}$$

$$= S^{\diamond} \oplus^{w} A^{\diamond}$$

Theorem (TH-5):

Let $S = \left[S_{ij}\right]$ and $A = \left[A_{ij}\right]$ Be two FHSMs where $S_{ij} = \left(M_{ijk}^{s}\right)$ and $A_{ij} = \left(M_{ijk}^{A}\right)$. Then,

i). $(S \cup A) = (A \cup S)$

$$= \left[\left(M_{ijk}^{A}\right)\right] \odot^{w} \left[\left(M_{ijk}^{S}\right)\right]$$

$$= (A \odot^{w} S)$$

ii). $(S \oslash A) = \left[\left(\left(\frac{2 M_{ijk}^{S} M_{ijk}^{A}}{M_{ijk}^{S} + M_{ijk}^{A}}\right)\right)\right]$

$$= \left[\left(\left(\frac{2 M_{ijk}^{A} M_{ijk}^{S}}{M_{ijk}^{A} + M_{ijk}^{S}}\right)\right)\right]$$

$$= \left[\left(M_{ijk}^{A}\right)\right] \oslash \left[\left(M_{ijk}^{S}\right)\right]$$

$$= (A \oslash S)$$

iii). $(S \oslash^{w} A) = \left[\left(\left(\frac{w^1 + w^2}{\frac{w^1}{M_{ijk}^{S}} + \frac{w^2}{M_{ijk}^{A}}}\right)\right)\right] w^1, w^2 > 0$

$$= \left[\left(\left(\frac{w^1 + w^2}{\frac{w^1}{M_{ijk}^{A}} + \frac{w^2}{M_{ijk}^{S}}}\right)\right)\right]$$

$$= \left[\left(M_{ijk}^{A}\right)\right] \oslash^{w} \left[\left(M_{ijk}^{S}\right)\right]$$

$$= (A \oslash^{w} S)$$

4.5 Theorem:

Let $= \left[S_{ij}\right]$, $A = \left[A_{ij}\right]$ and $C = \left[C_{ij}\right]$ be FHSM where $S_{ij} = \left(M_{ijk}^{s}\right)$, $A_{ij} = \left(M_{ijk}^{A}\right)$ and $C_{ij} = \left(M_{ijk}^{c}\right)$ .Then,

i. $(S \cup A) \cup C = S \cup (A \cup C)$

ii. $(S \cap A) \cap C = S \cap (A \cap C)$

iii. $(S \oplus A) \oplus C \neq S \oplus (A \oplus C)$

iv. $(S \odot A) \odot C \neq S \odot (A \odot C)$

v. $(S \oslash A) \oslash C \neq S \oslash (A \oslash C)$

Proof:

i). $(S \cup A) \cup C$

$$= \left[\left(\max\left(M^{s}_{ijk}, M^{A}_{ijk}\right)\right)\right] \cup \left[\left(M^{c}_{ijk}\right)\right]$$
$$= \left[\left(\max\left(M^{s}_{ijk}, M^{A}_{ijk}, M^{c}_{ijk}\right)\right)\right]$$
$$= \left(M^{s}_{ijk}\right) \cup \left[\left(\max\left(M^{A}_{ijk}, M^{c}_{ijk}\right)\right)\right]$$
$$= \left(M^{s}_{ijk}\right) \cup \left(\left(M^{A}_{ijk}\right) \cup \left(M^{C}_{ijk}\right)\right)$$
$$= S \cup (A \cup C)$$

ii). $(S \cap A) \cap C$

$$= \left[\left(\min\left(M^{s}_{ijk}, M^{A}_{ijk}\right)\right)\right] \cap \left[\left(M^{c}_{ijk}\right)\right]$$
$$= \left[\left(\min\left(M^{s}_{ijk}, M^{A}_{ijk}, M^{c}_{ijk}\right)\right)\right]$$
$$= \left(M^{s}_{ijk}\right) \cap \left[\left(\min\left(M^{A}_{ijk}, M^{c}_{ijk}\right)\right)\right]$$
$$= \left(M^{s}_{ijk}\right) \cap \left(\left(M^{A}_{ijk}\right) \cap \left(M^{C}_{ijk}\right)\right)$$
$$= S \cap (A \cap C)$$

The remaining components are proven similarly.

Theorem (TH-6):

Let $= \left[S_{ij}\right]$, $A = \left[A_{ij}\right]$ and $C = \left[C_{ij}\right]$ be FHSM where $S_{ij} = \left(M^{s}_{ijk}\right)$, $A_{ij} = \left(M^{A}_{ijk}\right)$ and $C_{ij} = \left(M^{c}_{ijk}\right)$ .Then,

i. $S \cap (A \oplus C) = (S \cap A) \oplus (S \cap C)$

ii. $(S \oplus A) \cap C = (S \cap C) \oplus (A \cap C)$

iii. $S \cup (A \oplus C) = (S \cup A) \oplus (S \cup C)$

iv. $(S \oplus A) \cup C = (S \cup C) \oplus (A \cup C)$

Proof:

i). $S \cap (A \oplus C) = \left(M^{s}_{ijk}\right) \cap \left[\left(\frac{\left(M^{s}_{ijk} + M^{c}_{ijk}\right)}{2}\right)\right]$

$$= \left[\left(\min\left(M^{s}_{ijk}, \frac{\left(M^{A}_{ijk} + M^{c}_{ijk}\right)}{2}\right)\right)\right]$$
$$= \left[\left(\min\left(\frac{\left(M^{s}_{ijk} + M^{A}_{ijk}\right)}{2}, \frac{\left(M^{s}_{ijk} + M^{c}_{ijk}\right)}{2}\right)\right)\right]$$
$$= \left[\left(\min\left(M^{S}_{ijk}, M^{A}_{ijk}\right)\right)\right] \oplus \left[\left(\min\left(M^{s}_{ijk}, M^{c}_{ijk}\right)\right)\right]$$
$$= \left[\left(M^{s}_{ijk}\right) \cap \left(M^{A}_{ijk}\right)\right] \oplus \left[\left(M^{s}_{ijk}\right) \cap \left(M^{c}_{ijk}\right)\right]$$
$$= (S \cap A) \oplus (S \cap C)$$

ii). $(S \oplus A) \cap C = \left[\left(\frac{\left(M^{s}_{ijk} + M^{A}_{ijk}\right)}{2}\right)\right] \cap \left(M^{C}_{ijk}\right)$

$$= \left[\left(\min\left(\frac{(M^S_{ijk} + M^A_{ijk})}{2}, M^C_{ijk}\right)\right)\right]$$

$$= \left[\left(\min\left(\frac{(M^S_{ijk}+M^C_{ijk})}{2}, \frac{(M^A_{ijk}+M^C_{ijk})}{2}\right)\right)\right]$$

$$= [(\min(M^S_{ijk}, M^C_{ijk}))] \oplus [(\min(M^A_{ijk}, M^c_{ijk}))]$$

$$= [(M^s_{ijk}) \cap (M^C_{ijk})] \oplus [(M^A_{ijk}) \cap (M^c_{ijk})]$$

$$= (S \cap C) \oplus (A \cap C)$$

iii). $S \cup (A \oplus C) = (M^s_{ijk}) \cup \left[\left(\frac{(M^s_{ijk}+M^c_{ijk})}{2}\right)\right]$

$$= \left[\left(\max\left(M^s_{ijk}, \frac{(M^A_{ijk} + M^c_{ijk})}{2}\right)\right)\right]$$

$$= \left[\left(\max\left(\frac{(M^s_{ijk}+M^A_{ijk})}{2}, \frac{(M^s_{ijk}+M^c_{ijk})}{2}\right)\right)\right]$$

$$= [(\max(M^S_{ijk}, M^A_{ijk}))] \oplus [(\max(M^s_{ijk}, M^c_{ijk}))]$$

$$= [(M^s_{ijk}) \cup (M^A_{ijk})] \oplus [(M^s_{ijk}) \cup (M^c_{ijk})]$$

$$= (S \cup A) \oplus (S \cup C)$$

iv). $(S \oplus A) \cup C = \left[\left(\frac{(M^s_{ijk}+M^A_{ijk})}{2}\right)\right] \cup (M^C_{ijk})$

$$= \left[\left(\max\left(\frac{(M^S_{ijk} + M^A_{ijk})}{2}, M^C_{ijk}\right)\right)\right]$$

$$= \left[\left(\max\left(\frac{(M^s_{ijk}+M^C_{ijk})}{2}, \frac{(M^A_{ijk}+M^C_{ijk})}{2}\right)\right)\right]$$

$$= [(\max(M^S_{ijk}, M^C_{ijk}))] \oplus [(\max(M^A_{ijk}, M^c_{ijk}))]$$

$$= [(M^s_{ijk}) \cup (M^C_{ijk})] \oplus [(M^A_{ijk}) \cup (M^c_{ijk})]$$

$$= (S \cup C) \oplus (A \cup C)$$

3.2 | Score Function

In the context of matrix theory, the Score Function is essential for improving decision-making since it offers a methodical and impartial way to assess options. The Score Function is a quantitative metric used in matrix-based decision models to evaluate each option's performance or appropriateness to predetermined criteria. The Score Function converts qualitative data into a standardized and comparable format by giving scores to various features or criteria, enabling a more thorough study.

3.2.1 | Score Function on FHSM

Let us examine a decision matrix $S = [S_{ij}]$ In which several criteria are used to analyze the choices. Let S represent the decision matrix with α criteria and γ choices. Every $[S_{ij}]$ Entry in the matrix denotes how well option α performed concerning criteria γ. The value of matrix S is then indicated by V(S) and it is defined as $\mathcal{V}(S) = [\mathcal{V}^S_{ij}]$ of order $\alpha \times \gamma$ Each object's overall score in the universal set is $\left|\sum_{j=1}^{n} \mathcal{S}_{ij}\right|$.

The Score of two FHSM $S = [A_{ij}]$ and $A = [B_{ij}]$ of order, $\alpha \times \gamma$ is given as $\mathcal{S}(A, B) = \mathcal{V}(A) + \mathcal{V}(B)$ and where $\mathcal{S}_{ij} = \mathcal{V}_{ij}^{S} + \mathcal{V}_{ij}^{A}$.

The Score Function essentially serves as a link between qualitative factors and mathematical frameworks, providing decision-makers with an understandable and straightforward method for navigating intricate matrices and reaching well-informed conclusions. It is significant because it makes decision-making procedures in matrix theory more objective and transparent, allowing for a more effective and efficient assessment of options.

4 | Description of Proposed Algorithm

In this section, we present an MCDM that utilizes the basics of FHSM and score function. This algorithm is further illustrated by solving a case study of the airline selection problem associated with pilgrim travelers.

4.1 | Algorithm

The stepwise procedure of the algorithm is presented in Table 7. The pictorial representation of the flowchart is presented in Figure 1.

Step 1: Construct an FHSM as defined in 3.1.

Step 2: Calculate the FHSM value matrix $S = [S_{ij}]$ of order $\alpha \times \gamma$, where $S_{ij} = (M_{ijk}^{S})$. The value of matrix S is therefore denoted by V(S) and defined as V(S) $= [\mathcal{V}_{ij}^{S}]$ of order $\alpha \times \gamma$, where $\mathcal{V}_{ij}^{S} = M_{ijk}^{S}$.

Step 3: The Score of two FHSM $S = [S_{ij}]$ and $A = [A_{ij}]$ of order, $\alpha \times \gamma$ is given as $\mathcal{S}(S, A) = \mathcal{V}(S) + \mathcal{V}(A)$ and $\mathcal{S}(S, A) = [\mathcal{S}_{ij}]$ where $\mathcal{S}_{ij} = \mathcal{V}_{ij}^{S} + \mathcal{V}_{ij}^{A}$, value matrices are used to calculate the score matrix.

Step 4: Calculate the overall score using the score matrix. Each object's overall score in the universal set is $\left|\sum_{j=1}^{n} \mathcal{S}_{ij}\right|$.

Step 5: Select an object with the highest score from the total score matrix to find the best solution.

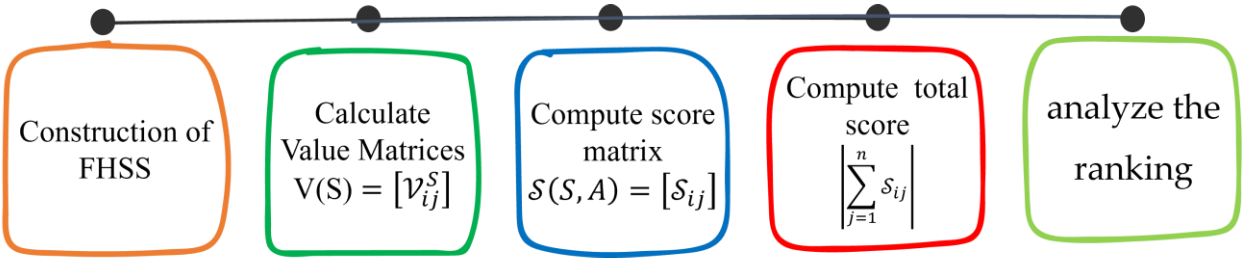

Figure 1. Graphical representation of the proposed algorithm.

Table 7

FHSM-based MCDM algorithm		
Input Goal, Criteria's $\boldsymbol{\mathcal{C}_i}$, and Alternative's $\boldsymbol{\mathcal{A}_j}$		
Output Ranking and final prioritization		
1	**for** $\mathcal{A}_j \leftarrow 1$ to α **for** $\mathcal{C}_i \leftarrow 1$ to γ	
2	**Read** [$\mathcal{V}_{ij}^{S} = M_{ijk}^{S}$ for each alternative depending on DM choice]	
3		**for** V(S) $\leftarrow \alpha \times \gamma$
4		**for** $\mathcal{V}_{ij} \leftarrow$ k
5	**Output** $\leftarrow$ $[\mathcal{V}_{ij}^{S}] = M_{ijk}^{s}$	
	endfor	
6		**for** $S = [S_{ij}] \leftarrow \mathcal{V}(S) + \mathcal{V}(A)$
7	**Output** $\leftarrow \mathcal{S}_{ij} = \mathcal{V}_{ij}^{S} + \mathcal{V}_{ij}^{A}$	
	endfor	
8		**for** $\mathcal{S}(S, A)$
9	**Output** $\leftarrow$ $\left\lvert \sum_{j=1}^{n} \mathcal{S}_{ij} \right\rvert$	
	endfor	
10	**Ouput** Rank or goal $\boldsymbol{R_p}$ [$\boldsymbol{R_p}$: $p = 1,2, \dots, n$]	
11	**Read** R_p	
12	**Output =** Rank the alternatives	
	endfor	

4.2 | Case Study

A passenger (Adeel) is planning a trip from Lahore to Jeddah for Umrah. He is trying to select the best airline based on factors such as cost, duration of the flight, comfort, and loyalty programs. Adeel needs to find an airline that offers a good balance between cost and convenience, as well as other factors such as comfort and loyalty programs. Adeel begins his research by searching for flights on various airline websites and comparing their prices and schedules. He also reads online reviews and ratings to get an idea of the quality of service provided by each airline. Adeel shortlists three airlines based on their cost, duration, and overall customer satisfaction ratings. Airline A offers the cheapest fare, but the flight duration is the longest among the three airlines. Airline B offers a mid-range fare with a slightly shorter flight duration, but it has lower customer satisfaction ratings. Airline C is the most expensive, but it has the shortest flight duration and the highest customer satisfaction ratings. Adeel weighs the pros and cons of each airline based on her priorities. He wants to save money but also values her time and comfort. He decided to choose Airline C, despite its higher fare, because it offers the shortest flight duration and the highest level of customer satisfaction, which is important to him. Adeel's decision to choose Airline C demonstrates the importance of considering multiple factors when selecting an airline. By evaluating cost, duration, comfort, and other factors, he was able to find the airline that best met her needs and priorities. This case study shows that a well-informed and thoughtful decision-making process can result in a positive travel experience. We'll discuss the same case by applying FHSMs.

Problem Statement: Adeel needs to find an airline that offers a good balance between cost and convenience, as well as other factors such as comfort and loyalty programs.

Methodology: Adeel begins his research by searching for flights on various airline websites and comparing their prices and schedules. He also reads online reviews and ratings to get an idea of the quality of service provided by each airline.

Results: Adeel shortlists three airlines based on their cost, duration, and overall customer satisfaction ratings. Airline A offers the cheapest fare, but the flight duration is the longest among the three airlines. Airline B offers a mid-range fare with a slightly shorter flight duration, but it has lower customer satisfaction ratings. Airline C is the most expensive, but it has the shortest flight duration and the highest customer satisfaction ratings.

Analysis: Adeel weighs the pros and cons of each airline based on her priorities. He wants to save money, but also values her time and comfort. He decides to choose Airline C, despite its higher fare, because it offers the shortest flight duration and the highest level of customer satisfaction, which is important to him.

Conclusion: Adeel's decision to choose Airline C demonstrates the importance of considering multiple factors when selecting an airline. By evaluating cost, duration, comfort, and other factors, he was able to find the airline that best met her needs and priorities. This case study shows that a well-informed and thoughtful decision-making process can result in a positive travel experience.

Mathematically, one can consider an appropriate airline by using a weighted fuzzy decision matrix, where each airline is evaluated based on a set of criteria and given a score for each criterion. The scores are then weighted based on the importance of each criterion, and the airline with the highest total score is considered the most appropriate.

4.3 | Construction of Case Study in terms of FHSM

Let $\delta = \{\delta^1, \delta^2, \delta^3, \delta^4, \delta^5, \delta^6, \delta^7, \delta^8, \delta^9, \delta^{10}, \delta^{11}, \delta^{12}, \delta^{13}, \delta^{14}, \delta^{15}\}$ is the set of different airlines running their operations from Lahore to Jeddah, the passenger $\mathcal{P}$ wants to purchase a ticket by keeping in mind, the parameters $B = \{B_1 = lounge, B_2 = cost, B_3 = class,\ B_4 = weight\ allowed\}$. A decision-maker (D) determines δ airline by applying a proposed decision-making technique. This analysis can be used to determine whether δ airline is suitable for travel for passengers or not.

The bi-furcation of these attributes is classified as;

B_1^E =Lounge = {1, 2, 3, 4}

B_2^F =Tickets Cost= {45000p, 44000p, 40000p, 50000p}

B_3^G = Classes = { Business , Economy }

B_4^H = Weight = {37Kg, 46Kg, 47Kg, 50Kg}

Function: $F: (B_1^E, B_2^F, B_3^G, B_4^H) \to P(U)$ is

Keeping in mind the parameter requirement of the passenger, we have the possibility of two sets A, and B, such that. A = F (3, 44000p, Economy, 43kg) = $\{\langle \delta^1, (3\ (0.3), 44000p\ (0.4), \bar{E}co\ (0.6), 43Kg\ (0.4)\rangle,$

$$\langle \delta^3, (3\ (0.2), 44000p\ (0.5), \bar{E}co\ (0.8), 43Kg(0.5)\rangle,$$

$$\langle \delta^9, (3\ (0.3), 44000p\ (0.4), \bar{E}co\ (0.5), 43Kg\ (0.2)\rangle,$$

$\langle \delta^{15}, (3\ (0.4), 44000p\ (0.6), \bar{E}co\ (0.2), 43Kg(0.5)\rangle\}$

B = F(3, 44000p, Business,37kg) =

$$\{\langle \delta^1, (3\ (0.1), 44000p\ (0.4), \bar{E}co\ (0.4), 37Kg\ (0.3)\rangle,$$

$$\langle \delta^3, (3\ (0.2), 44000p\ (0.6), \bar{E}co\ (0.9), 37Kg(0.6)\rangle,$$

$$\langle \delta^9, (3(0.2), 44000p\ (0.7), \bar{E}co\ (0.6), 37Kg\ (0.5)\rangle,$$

$$\langle \delta^{15}, (3\ (0.6), 44000p\ (0.4), \bar{E}co\ (0.4), 37Kg(0.4)\rangle\}$$

Applying the algorithm for the calculation of score values:

Step 1: The above two sets of FHSSs are given in the form of FHSMs.

$$[A] = \begin{bmatrix} 3(0.3) & 44000p(0.4) & \bar{E}co(0.6) & 43kg(0.4) \\ 3(0.2) & 44000p(0.5) & \bar{E}co(0.8) & 43kg(0.5) \\ 3(0.3) & 44000p(0.4) & \bar{E}co(0.5) & 43kg(0.2) \\ 3(0.4) & 44000p(0.6) & \bar{E}co(0.2) & 43kg(0.5) \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3(0.1) & 44000p(0.4) & Bus(0.4) & 37kg(0.3) \\ 3(0.2) & 44000p(0.6) & Bus(0.9) & 37kg(0.6) \\ 3(0.2) & 44000p(0.7) & Bus(0.6) & 37kg(0.5) \\ 3(0.6) & 44000p(0.4) & Bus(0.4) & 37kg(0.4) \end{bmatrix}$$

Step 2: Now calculate the values matrices of FHSMs defined in Step 1.

$$[v((A)] = \begin{bmatrix} 3(0.3) & 44000p(0.4) & \bar{E}co(0.6) & 43kg(0.4) \\ 3(0.2) & 44000p(0.5) & \bar{E}co(0.8) & 43kg(0.5) \\ 3(0.3) & 44000p(0.4) & \bar{E}co(0.5) & 43kg(0.2) \\ 3(0.4) & 44000p(0.6) & \bar{E}co(0.2) & 43kg(0.5) \end{bmatrix}$$

$$[v(B)] = \begin{bmatrix} 3(0.1) & 44000p(0.4) & Bus(0.4) & 37kg(0.3) \\ 3(0.2) & 44000p(0.6) & Bus(0.9) & 37kg(0.6) \\ 3(0.2) & 44000p(0.7) & Bus(0.6) & 37kg(0.5) \\ 3(0.6) & 44000p(0.4) & Bus(0.4) & 37kg(0.4) \end{bmatrix}$$

Step 3: Compute the score matrix by combining the value matrices generated in steps 2 and 3.

$$[S((A,B)] = \begin{bmatrix} 3(0.4) & 44000p(0.8) & \bar{E}co(1.0) & 43kg(0.7) \\ 3(0.4) & 44000p(1.1) & \bar{E}co(1.7) & 43kg(1.1) \\ 3(0.5) & 44000p(1.1) & \bar{E}co(1.1) & 43kg(0.7) \\ 3(1.0) & 44000p(1.0) & \bar{E}co(0.6) & 43kg(0.9) \end{bmatrix}$$

Step 4: The computed, total score.

$$Total\ Score = \begin{bmatrix} 2.9 \\ 4.3 \\ 3.4 \\ 3.5 \end{bmatrix}$$

Step 5: We calculated the score function presented in Figure 2, which shows that the airline $D2 = \delta^2$ will be the best to travel the passenger $\mathcal{P} = Adeel$.

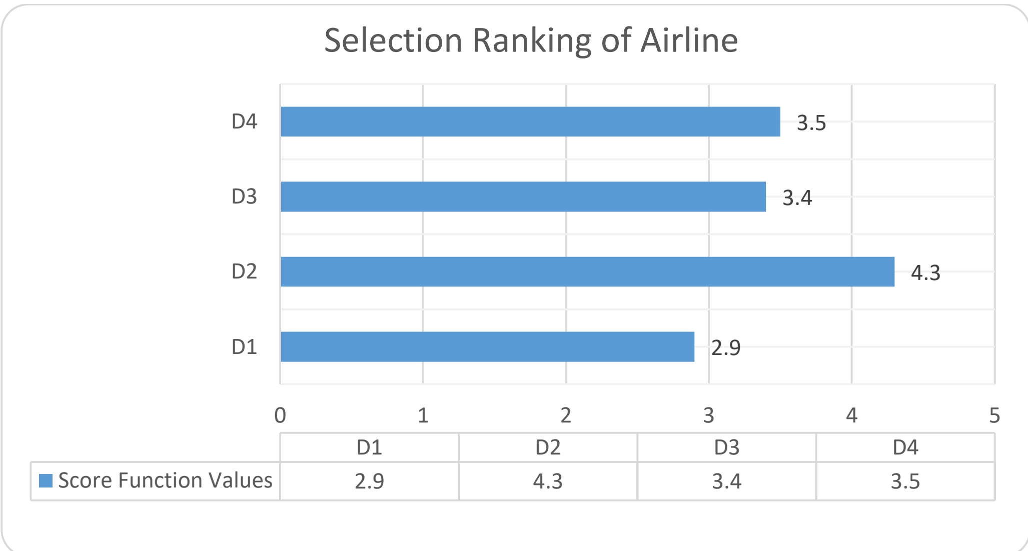

Figure 2. Ranking of airlines for travelers.

Using the Score Function technique based on Fuzzy Hypersoft Set Matrices (FHSMs) provided a solid framework for choosing an airline in Adeel's case study. FHSMs effectively handled the ambiguous nature of factors including price, length of travel, comfort, and loyalty programs, offering a sophisticated comprehension of Adeel's preferences. Adeel's choice to take Airline B, even though it was more expensive fits with FHSM's ability to manage ambiguous preferences and combine several factors. Adeel wanted a complex balance between time, money, and qualitative aspects, and the model was able to capture it, demonstrating how flexible FHSM is in real-world decision-making situations.

5 | Result Discussion and Comparison

In comparison, the intricacy of Adeel's criteria may be too much for conventional methods of decision-making. The strength of FHSM is its capacity to combine qualitative and quantitative aspects seamlessly, providing a flexible and organized model for decision-making. The investigation demonstrates how well FHSM navigates the ambiguity and dependency of criteria in judgments with several facets. A more nuanced perspective of Adeel's decision-making process is offered by the complete FHSM-based approach, which highlights the significance of a holistic model in efficiently handling a variety of, occasionally competing, factors. The final ranking and the graphical representation are also given in Figure 2. The applicability and superiority of the proposed technique have been presented in Table 8. FHSM tends to solve the decision-making problems of both types of MCDM. It is suggested that the efficiency of the proposed algorithm is due to bifurcated sub-attributes and this proposed technique is more reliable.

Table 8. Comparison of the current and prior studies.

Researcher	Set structure	Membership value	Score Function	Bi-furcated attributes
Cagman [18]	FSM	Yes	Yes	No
Yang & Chenli [19]	FSM	Yes	Yes	No
Warrier et al. [21]	FSM	Yes	Yes	No
Bajaj & Guleria [22]	TSFSM	Yes	Yes	No
Jafar et al. [35]	IFHSM	Yes	Yes	No
Dhumras & Bajaj [38]	PFHSM	Yes	Yes	No
Proposed	FHSM	Yes	Yes	Yes

Adeel's case study's FHSM-based analysis demonstrates how well the model works to solve the difficulties associated with making decisions in the real world. Fuzzy logic is included in the selection criteria in a way that both supports Adeel's choice and provides more context for understanding his thinking. Because of its versatility and subjectivity, the FHSM is a valuable tool for decision-makers who have to make difficult decisions, like choosing an airline. The flexible and methodical approach of the FHSM contributes significantly to the field as choice landscapes become more complex, demonstrating its potential to improve decision-making processes in a variety of contexts. Adeel's thoughtful choice demonstrates how FHSMs may capture the nuances of decision criteria and produce favorable results in real-world circumstances. The fuzzy hypersoft matrix (FHSM) theory is efficient and reliable to employ in cases of decision-making when attributes are further subdivided. The advantages of the proposed theory are the proposed score function can be applied to solve both MCDM and MADM problems. As it has a specialty, that can consider a large number of decision-makers very quickly along with a simple computing approach. The proposed operations and operators are very accurate and comparable to the existing approaches in fuzzy context, demonstration, and applicability. The recommended method analyses the interrelationships of attributes in real-life case studies while existing approaches cannot. Finally, the suggested approach for MCDM issues in this work can take into account additional attribute correlations, resulting in increased accuracy and reference value.

6 | Conclusion

In this proposed work, we have presented the definition of FHSM, notion, operations, properties, theorems, and algorithms. FHSM theory is a very efficient tool to represent and investigate MCDM issues. This paper makes a substantial contribution by introducing a score function approach to analyze MCDM challenges. The technique was applied in a real-world case study, with an emphasis on airline selection, which demonstrated the usefulness of the suggested algorithms. Specifically, these algorithms enable dynamic interactions between FHSS qualities, offering insightful information on decision-maker (DM) attitude aspects and allowing a categorization system to be applied to varying FHSM levels. Not only do the suggested algorithms allow for interaction between attributes of FHSS, but they also contribute to the investigation of DM attitude features and make it possible to apply a categorization system to the degrees of the FHSM. The suggested method's effectiveness, speed, and dependability are validated by comparison with previous research, especially when features are further subdivided. But the adventure doesn't stop here.

- Subsequent investigations may concentrate on the study of FHS hybrids, investigating their possible benefits and uses in various contexts involving decision-making.
- Furthermore, the FHSM framework's incorporation of well-known MCDM methods like TOPSIS, VIKOR, and AHP opens up an intriguing new field for research.
- This extension can improve the decision-making toolset by providing more thorough answers and insights for a wider range of intricate choice situations.

In conclusion, this study's investigation of FHSM establishes a solid basis for further research avenues. The suggested approaches provide opportunities for creativity, cooperation, and application across a range of fields, indicating future developments in the field of multiple-criteria decision-making.

Author Contributions

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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