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Permutation Graphs in Fuzzy and Neutrosophic Graphs

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Abstract

Graph theory is a fundamental branch of mathematics that examines networks composed of nodes (vertices) and connections (edges). This paper explores the concepts of permutation graphs within the frameworks of fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, all of which handle uncertainty in graph structures. We define permutation and bipartite permutation graphs in each context and investigate their properties. While permutation graphs have been studied extensively in classical graph theory, there has been limited exploration in fuzzy and neutrosophic settings.

Keywords: Neutrosophic graphs, Permutation graphs, Fuzzy graphs, Intersection graphs

1 | Introduction

1.1 | Permutation graphs

Graph theory is a fundamental branch of mathematics that examines networks composed of nodes (vertices) and connections (edges), which are essential for analyzing the structure, paths, and properties of these networks [1]. One important example in graph theory is the intersection graph, where vertices correspond to sets, and edges are drawn between vertices if their corresponding sets intersect [2]. Many related graph classes have been extensively researched, such as interval graphs [3], proper interval graphs [4], weighted interval graphs [5], semi-proper interval graphs [6], mixed interval graphs [7], unit disk graphs [8], circular arc graphs [9], and polygon-circle graphs [10].

In this paper, we focus on permutation graphs [11], a specific type of intersection graph that has garnered significant attention due to its practical applications and importance in the study of various graph classes. Permutation graphs are defined such that vertices represent elements of a permutation, and edges connect pairs of vertices if their corresponding elements in the permutation are reversed in order.

The following properties and statements hold.

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Theorem 1. *The following are known classes of graphs recognized as permutation graphs:*

- *Comparability graphs*[12]*: Graphs where the vertices represent elements and edges exist if they are comparable in a partial order.*
- *chordal comparability graphs*[13]*: Chordal graph for comparability graphs.*
- *Co-comparability graphs*[14]*: Graphs where the complement is a comparability graph, meaning vertices are non-adjacent if they are comparable.*
- *Trapezoid graphs*[15]*: Intersection graphs of trapezoids between two parallel lines, generalizing interval graphs.*

Theorem 2. *The following are known as generalized or related graph classes of permutation graphs:*

- *Circular permutation graphs*[16]*: Circular permutation graphs are intersection graphs derived from circular permutation diagrams, where edges represent intersecting chords between two circles.*
- *Bipartite permutation graphs* [17]*: A bipartite permutation graph is both bipartite and a permutation graph, offering efficient solutions for certain NP-complete problems.*
- *Random permutation graphs* [18]*: A random permutation graph is formed by connecting two vertices if their permutation order and index difference have opposite signs.*
- *Functi graphs* [19]*: Functigraphs generalize permutation graphs by connecting two disjoint copies of a graph with additional edges defined by a function between their vertices.*
- *Split permutation graphs* [20]*: Split permutation graphs are graphs that belong to both split and permutation graph classes, combining properties of both.*
- *Probe permutation graphs* [21]*: Probe permutation graphs are permutation graphs where vertices are partitioned into probes and nonprobes, with additional edges only between certain nonprobes.*
- *Polar permutation graphs* [22]*: Polar permutation graphs are permutation graphs where the vertex set can be partitioned into two: one part forms a complete multipartite graph, the other forms disjoint complete graphs.*
- *Connected permutation graphs*[23]*: Connected graph of permutation graphs.*
- *Double-threshold permutation graphs*[24]*: Double-threshold graphs are defined by two thresholds, where vertex adjacency is based on the sum of their ranks falling in a specific "YES" region.*
- *cycle permutation graphs*[25]*: Cycle permutation graphs represent graphs formed by cyclic permutations, where vertices correspond to elements, and edges represent a specific cyclic permutation of these elements.*
- *weighted permutation graphs*[26]*: Weighted version of permutation graphs.*
- *π-Permutation Graphs*[27]*: A π-permutation graph is formed by connecting two disjoint copies of a graph via a matching determined by a permutation π.*
- *Balanced Permutation Graphs*[28]*: A balanced permutation graph is a graph where vertices i and j are adjacent if and only if $i + j = \pi(i) + \pi(j)$, based on a given permutation π.*
- *Permutation hypergraphs*[29]*: Hypergraph version of Permutation hypergraphs.*

1.2 | Fuzzy Graphs and Neutrosophic Graphs

In this paper, we examine Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. These graph concepts were developed to address uncertainty in real-world applications.

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, reflecting the degree of uncertainty or imprecision [30]. In essence, fuzzy graphs serve as graphical representations of fuzzy sets [31], and are frequently applied in areas such as social networks, decision-making, and transportation systems, where relationships are often uncertain or ambiguous [30]. Intuitionistic Fuzzy Graphs expand upon fuzzy graphs by introducing both membership and non-membership degrees for vertices and edges, thus further capturing the uncertainty inherent in relationships [32]. Neutrosophic Graphs [33], derived from neutrosophic set theory [34, 35], extend classical and fuzzy logic by incorporating three components: truth, indeterminacy, and falsity, providing a more flexible approach to managing uncertainty. Turiyam Neutrosophic Graphs, introduced as a further extension of neutrosophic and fuzzy graphs, assign four attributes—truth, indeterminacy, falsity, and a liberal state—to each vertex and edge [36]. Further extensions, such as Plithogenic Graphs, have also been studied [37].

Despite substantial advancements in the study of fuzzy and neutrosophic graphs, including their intersection variants (e.g., fuzzy intersection graphs [38] and neutrosophic intersection graphs [39]), there has been relatively little exploration of permutation graphs within the frameworks of fuzzy, neutrosophic, and Turiyam Neutrosophic graphs.

1.3 | Our Contribution

As mentioned above, extensive research has been conducted on Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Similarly, there has been considerable work on permutation graphs in the context of classic graph theory. In this paper, we define permutation graphs and bipartite permutation graphs within the frameworks of Fuzzy Graphs (cf. [40]), Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs, and examine their properties and relationships.

2 | Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs.

2.1 | Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [1].

Definition 3 (Graph)**.** [1] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 4 (Degree)**.** [1] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 5 (Subgraph)**.** [1] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 6 (Induced subgraph)**.** [41, 42] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Definition 7 (Complete Graph)**.** (cf.[43, 44]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

Definition 8 (Bipartite Graph)**.** (cf.[45]) A *bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- Every edge in E connects a vertex from V_1 to a vertex from V_2. In other words, there are no edges connecting two vertices within the same subset V_1 or V_2.

Formally, $G = (V, E)$ is bipartite if there exists a partition (V_1, V_2) such that for every edge $e = \{u, v\} \in E$, either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

A graph G is bipartite if and only if it contains no odd-length cycles.

Definition 9 (Complete Bipartite Graph)**.** (cf.[46]) A *complete bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- There is an edge between every vertex in V_1 and every vertex in V_2.
- There are no edges between vertices within the same subset V_1 or V_2.

The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. It has the following properties:

- The number of vertices is $|V| = m + n$.
- The number of edges is $|E| = m \times n$.
- Each vertex in V_1 has degree n, and each vertex in V_2 has degree m.

Definition 10 (homomorphic)**.** (cf.[47]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

2.2 | Intersection graph and permutation graphs

In this paper, we focus on permutation graphs, which are known as intersection graphs. Intersection graphs have been extensively studied[2]. The definition is provided below[2].

Definition 11 (Intersection graph)**.** [2] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \dots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

The definitions of permutation graphs and bipartite permutation graphs are provided below. As mentioned in the introduction, extensive research has been conducted on these graphs [48, 49].

Definition 12. [11] A graph $G = (V, E)$ is called a *permutation graph* if there exists a permutation π of the set $\{1, 2, \ldots, n\}$, where $n = |V|$, such that for any two distinct vertices u and $v \in V$, the edge $(u, v) \in E$ exists if and only if the indices of u and v in π are reversed in order. Formally, for a permutation π, if vertices $u = i$ and $v = j$ satisfy $i < j$ and $\pi(i) > \pi(j)$, then there exists an edge $(u, v) \in G$.

In other words, a permutation graph is the intersection graph of line segments joining pairs of points on two parallel lines, where each vertex corresponds to a line segment, and two vertices are adjacent if their corresponding line segments intersect.

Example 13. Consider the set $V = \{1, 2, 3, 4\}$ and the permutation $\pi = (3, 1, 4, 2)$, which maps the elements of V as follows:

$$\pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 4, \quad \pi(4) = 2$$

The permutation graph $G = (V, E)$ is formed by adding edges between vertices i and j if the indices are reversed in π. That is, $(i, j) \in E$ if $i < j$ and $\pi(i) > \pi(j)$.

From the permutation $\pi = (3, 1, 4, 2)$, we add edges between the following vertices:

$$(1, 2) \text{ because } 1 < 2 \text{ and } \pi(1) = 3 > \pi(2) = 1,$$

$$(1, 4) \text{ because } 1 < 4 \text{ and } \pi(1) = 3 > \pi(4) = 2,$$

$$(3, 4) \text{ because } 3 < 4 \text{ and } \pi(3) = 4 > \pi(4) = 2.$$

Thus, the edge set E is:

$$E = \{(1, 2), (1, 4), (3, 4)\}$$

The resulting graph is the permutation graph corresponding to $\pi = (3, 1, 4, 2)$.

Definition 14. [17, 50] A graph $G = (V, E)$ is called a *bipartite permutation graph* if G is both a *bipartite graph* and a *permutation graph*. Specifically, G is bipartite if its vertex set can be partitioned into two independent sets V_1 and V_2, where there are no edges between vertices within the same set. The edges of G are defined based on the adjacency condition of a permutation graph.

Formally, $G = (V_1 \cup V_2, E)$ is a bipartite permutation graph if there exists a permutation π of the vertices in $V_1 \cup V_2$ such that for any two vertices $u \in V_1$ and $v \in V_2$, there is an edge $(u, v) \in E$ if and only if $\pi(u)$ and $\pi(v)$ intersect in the corresponding permutation diagram. Thus, bipartite permutation graphs combine the structure of bipartite graphs with the properties of permutation graphs.

Example 15. Consider a bipartite graph $G = (V_1 \cup V_2, E)$, where $V_1 = \{1, 2\}$ and $V_2 = \{3, 4\}$. Let the permutation $\pi = (4, 3, 2, 1)$ represent the mapping of the vertices $1, 2, 3, 4$.

In this graph, edges are formed between the vertices in V_1 and V_2 based on the permutation condition $(i, j) \in E$ if $\pi(i)$ and $\pi(j)$ are reversed. For $\pi = (4, 3, 2, 1)$:

$$(1, 3) \text{ because } \pi(1) = 4 > \pi(3) = 2,$$

$$(2, 4) \text{ because } \pi(2) = 3 > \pi(4) = 1.$$

Thus, the bipartite permutation graph has the edge set:

$$E = \{(1, 3), (2, 4)\}$$

This graph satisfies the bipartite and permutation graph conditions.

2.3 | Permutation graph in Fuzzy Graphs

Now, we explore permutation graph and bipartite permutation graphs within the context of fuzzy graphs.

Fuzzy sets extend classical sets by allowing elements to have varying degrees of membership, represented by values between 0 and 1, rather than a binary classification of membership or non-membership[31]. Similarly, fuzzy graphs generalize classical graph theory by incorporating the principles of fuzzy sets [31, 51]. Due to their flexibility and practical utility, fuzzy graphs have been widely studied [52, 30]. They find applications in fields such as decision-making [53, 54], Disaster management[55, 56], and neural networks [57, 58], among others.

The definition of a fuzzy graph is provided below.

Definition 16 (Fuzzy Graph)**.** [30] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ consists of:

- V is a set of vertices.
- $\sigma : V \to [0, 1]$ is a function that assigns a membership degree to each vertex $v \in V$, representing the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

These functions satisfy the following conditions:

(1) $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$,

(2) $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$ (symmetry),

(3) $\mu(v, v) = 0$ for all $v \in V$ (no self-loops).

The definitions of Fuzzy Permutation Graph and Fuzzy Bipartite Permutation Graph are provided below.

Definition 17 (Fuzzy Permutation Graph)**.** (cf. [40]) Let $V = \{1, 2, \dots, n\}$ be a finite set of vertices, and let π be a permutation of V. A *fuzzy permutation graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph where:

- $\sigma : V \to (0, 1]$ assigns a positive membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is defined as:

$$\mu(u, v) = \begin{cases} \min\{\sigma(u), \sigma(v)\}, & \text{if } u < v \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$

The underlying crisp graph $G = (V, E)$, where $E = \{(u, v) \mid \mu(u, v) > 0\}$, is a permutation graph corresponding to the permutation π.

Definition 18 (Fuzzy Bipartite Permutation Graph)**.** Let $V = V_1 \cup V_2$ be a finite set of vertices partitioned into two disjoint independent sets V_1 and V_2, and let π be a permutation of V. A *fuzzy bipartite permutation graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph where:

- $\sigma : V \to (0, 1]$ assigns positive membership degrees to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is defined as:

$$\mu(u, v) = \begin{cases} \min\{\sigma(u), \sigma(v)\}, & \text{if } u \in V_1, v \in V_2, \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$

The underlying crisp graph $G = (V, E)$, where $E = \{(u, v) \mid \mu(u, v) > 0\}$, is a bipartite permutation graph corresponding to the permutation π.

2.4 | Permutation graph in Intuitionistic fuzzy Graphs

Next, we examine permutation graphs within the framework of intuitionistic fuzzy graphs. Intuitionistic fuzzy graphs are an extended form of fuzzy graphs and have been the focus of extensive research for over 15 years [32]. They are closely related to the concept of intuitionistic fuzzy sets [59, 60].

Much like fuzzy-related concepts, the study of intuitionistic fuzzy graphs and their related ideas has also seen significant progress [61, 62]. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy permutation graphs are presented below.

Definition 19 (Intuitionistic Fuzzy Graph (IFG))**.** [63] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

(1) (μ_A, v_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $v_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + v_A(x) \leq 1$$

The value $1 - \mu_A(x) - v_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

(2) (μ_B, v_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $v_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + v_B(x, y) \leq 1$$

Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$v_B(x, y) \leq v_A(x) \vee v_A(y)$$

In this definition:

- $\mu_A(x)$ and $v_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $v_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $v_A(x) = 0$ and $v_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Next, we define the Intuitionistic Fuzzy Permutation Graph and Intuitionistic Fuzzy Bipartite Permutation Graph.

Definition 20 (Intuitionistic Fuzzy Permutation Graph)**.** An *Intuitionistic Fuzzy Permutation Graph* is an extension of the permutation graph into the intuitionistic fuzzy domain, where each vertex and edge is assigned both a membership degree and a non-membership degree, representing the uncertainty of their presence. Let $G = (V, E)$ be a classical permutation graph and π be a permutation of the set $\{1, 2, \ldots, n\}$, where $n = |V|$. The graph is defined as follows:

- V is the set of vertices.
- For each vertex $v \in V$, the degree of membership $\mu_V(v) \in [0, 1]$ and the degree of non-membership $\nu_V(v) \in [0, 1]$ satisfy:

  $$\mu_V(v) + \nu_V(v) \leq 1$$

  The value $1 - \mu_V(v) - \nu_V(v)$ represents the hesitancy or uncertainty regarding the membership of the vertex v.

- For each pair of vertices $u, v \in V$, the degree of membership $\mu_E(u, v) \in [0, 1]$ and the degree of non-membership $\nu_E(u, v) \in [0, 1]$ for the edge (u, v) satisfy:
$$\mu_E(u, v) + \nu_E(u, v) \leq 1$$
The edge membership degree $\mu_E(u, v)$ is determined as:
$$\mu_E(u, v) = \begin{cases} \min(\mu_V(u), \mu_V(v)), & \text{if } u < v \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$
Similarly, the non-membership degree $\nu_E(u, v)$ is:
$$\nu_E(u, v) = \begin{cases} \max(\nu_V(u), \nu_V(v)), & \text{if } u < v \text{ and } \pi(u) > \pi(v), \\ 1, & \text{otherwise.} \end{cases}$$

The underlying crisp graph corresponds to the permutation graph associated with the permutation π.

Definition 21 (Intuitionistic Fuzzy Bipartite Permutation Graph)**.** An *Intuitionistic Fuzzy Bipartite Permutation Graph* extends both the concepts of bipartite graphs and permutation graphs into the intuitionistic fuzzy framework. Let $V = V_1 \cup V_2$ be a finite set of vertices, partitioned into two independent sets V_1 and V_2, with π as a permutation on $V_1 \cup V_2$. The graph $G = (V, E, \mu_V, \nu_V, \mu_E, \nu_E)$ is defined as follows:

- V_1 and V_2 are independent sets of vertices.
- For each vertex $v \in V$, the degree of membership $\mu_V(v) \in [0, 1]$ and the degree of non-membership $\nu_V(v) \in [0, 1]$ satisfy:
$$\mu_V(v) + \nu_V(v) \leq 1$$
The value $1 - \mu_V(v) - \nu_V(v)$ represents the uncertainty regarding the membership of the vertex v.
- For each edge $(u, v) \in V_1 \times V_2$, the degree of membership $\mu_E(u, v) \in [0, 1]$ and the degree of non-membership $\nu_E(u, v) \in [0, 1]$ satisfy:
$$\mu_E(u, v) + \nu_E(u, v) \leq 1$$
The membership degree of the edge (u, v) is:
$$\mu_E(u, v) = \begin{cases} \min(\mu_V(u), \mu_V(v)), & \text{if } u \in V_1, v \in V_2 \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$
The non-membership degree $\nu_E(u, v)$ is:
$$\nu_E(u, v) = \begin{cases} \max(\nu_V(u), \nu_V(v)), & \text{if } u \in V_1, v \in V_2 \text{ and } \pi(u) > \pi(v), \\ 1, & \text{otherwise.} \end{cases}$$

This definition ensures that the intuitionistic fuzzy bipartite permutation graph maintains both the bipartite and permutation graph structures while incorporating intuitionistic fuzzy membership and non-membership degrees.

2.5 | Permutation graph in Neutrosophic Graphs

As noted in the introduction, neutrosophic graphs are an extension of fuzzy graphs and intuitionistic fuzzy graphs. A neutrosophic graph assigns truth, indeterminacy, and falsity membership degrees to each vertex and edge, representing uncertainty. Similar to fuzzy graphs, neutrosophic graphs have been extensively studied [64, 65, 66].

Neutrosophic graphs are closely linked to the concept of neutrosophic sets [67].

Beyond graphs and sets, research on neutrosophic theory has paralleled the depth of studies on fuzzy theory, with numerous contributions [68, 69].

The formal definition is provided below [70, 71]

Definition 22. [71] A neutrosophic graph $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a graph where:

- $\sigma : V \to [0,1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.
- $\mu : E \to [0,1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.
- For every edge $e = v_i v_j \in E$, the following condition holds:

$$\mu_T(e) \leq \min(\sigma_T(v_i), \sigma_T(v_j)).$$

(1) σ is called the *neutrosophic vertex set*.

(2) μ is called the *neutrosophic edge set*.

(3) The number of vertices $|V|$ is the *order* of G, denoted by $O(G)$.

(4) The sum of the truth values over all vertices, $\sum_{v \in V} \sigma_T(v)$, is the *neutrosophic order* of G, denoted by $On(G)$.

(5) The number of edges $|E|$ is the *size* of G, denoted by $S(G)$.

(6) The sum of the truth values over all edges, $\sum_{e \in E} \mu_T(e)$, is the *neutrosophic size* of G, denoted by $Sn(G)$.

Next, we define the neutrosophic permutation graph and neutrosophic Bipartite Permutation Graph. The neutrosophic permutation graph extends the structure of the permutation graph by degrees of truth, indeterminacy, and falsity for each vertex and edge.

Definition 23 (Neutrosophic Permutation Graph)**.** A *Neutrosophic Permutation Graph* is an extension of the classical permutation graph into the neutrosophic domain, where each vertex and edge is assigned truth, indeterminacy, and falsity membership degrees. Let $G = (V, E)$ be a permutation graph with a permutation π of the set $\{1, 2, \dots, n\}$, where $n = |V|$. The neutrosophic permutation graph $G_N = (V, E, \sigma, \mu)$ is defined as follows:

- $\sigma : V \to [0,1]^3$ assigns to each vertex $v \in V$ a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees, respectively.
- $\mu : E \to [0,1]^3$ assigns to each edge $e \in E$ a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity membership degrees, respectively.
- For each edge $e = (u, v) \in E$, the truth membership degree $\mu_T(u, v)$ is determined as:

$$\mu_T(u, v) = \begin{cases} \min(\sigma_T(u), \sigma_T(v)), & \text{if } u < v \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$

  The indeterminacy $\mu_I(u, v)$ and falsity $\mu_F(u, v)$ degrees are defined similarly.

Definition 24 (Neutrosophic Bipartite Permutation Graph)**.** A *Neutrosophic Bipartite Permutation Graph* is an extension of the bipartite permutation graph into the neutrosophic domain, where truth, indeterminacy, and falsity membership degrees are assigned to vertices and edges. Let $G = (V_1 \cup V_2, E)$ be a bipartite permutation graph with vertex sets V_1 and V_2, and let π be a permutation on $V_1 \cup V_2$. The neutrosophic bipartite permutation graph $G_N = (V_1 \cup V_2, E, \sigma, \mu)$ is defined as follows:

- $\sigma : V \to [0,1]^3$ assigns to each vertex $v \in V_1 \cup V_2$ a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees, respectively.
- $\mu : E \to [0,1]^3$ assigns to each edge $e = (u, v) \in E$ a triple $(\mu_T(u, v), \mu_I(u, v), \mu_F(u, v))$, representing the truth, indeterminacy, and falsity membership degrees, respectively.
- The truth membership degree $\mu_T(u, v)$ for each edge $e = (u, v)$ is defined as:

$$\mu_T(u, v) = \begin{cases} \min(\sigma_T(u), \sigma_T(v)), & \text{if } u \in V_1, v \in V_2 \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$

  The indeterminacy $\mu_I(u, v)$ and falsity $\mu_F(u, v)$ degrees are defined similarly.

2.6 | Permutation graph in Turiyam Neutrosophic Graph

We explore Permutation graphs within the context of Turiyam Neutrosophic Graphs. Turiyam Neutrosophic Graph extends classical graphs by assigning four values—truth, indeterminacy, falsity, and liberal state—to each vertex and edge. Research on Turiyam Neutrosophic Graphs, which introduce parameters to extend Neutrosophic Graphs, is actively ongoing [72, 73]. These graphs serve as graphical representations of the Turiyam Neutrosophic Set [74]. The formal definition is provided below.

Definition 25 (Turiyam Neutrosophic Graph)**.** [72, 75] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \dots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \dots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

(1) *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$

where:

- $t(v_i)$ is the truth value (tv) of the vertex v_i,
- $iv(v_i)$ is the indeterminacy value (iv) of v_i,
- $fv(v_i)$ is the falsity value (fv) of v_i,
- $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

for all $v_i \in V$, such that the following condition holds for each vertex:

$$0 \leq t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4.$$

(2) *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

Next, we define the Turiyam Neutrosophic Permutation Graph and Turiyam Neutrosophic Bipartite Permutation Graph. The resulting graph combines the structure of the permutation graph with the Turiyam Neutrosophic values for truth, indeterminacy, falsity, and liberal state.

Definition 26 (Turiyam Neutrosophic Permutation Graph)**.** A *Turiyam Neutrosophic Permutation Graph* extends the classical permutation graph into the Turiyam Neutrosophic framework, where each vertex and edge is assigned four values: truth, indeterminacy, falsity, and liberal state. Let $G = (V, E)$ be a classical permutation graph and π be a permutation of the set $\{1, 2, \dots, n\}$, where $n = |V|$. The Turiyam Neutrosophic permutation graph $G^T = (V^T, E^T, t, iv, fv, lv)$ is defined as follows:

- For each vertex $v \in V$, the Turiyam Neutrosophic graph assigns the following values:
$$t(v), iv(v), fv(v), lv(v) : V \to [0,1],$$
where $t(v)$, $iv(v)$, $fv(v)$, and $lv(v)$ represent the truth value, indeterminacy value, falsity value, and liberal value of the vertex v, respectively, satisfying:
$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$
- For each edge $e = (u,v) \in E$, the Turiyam Neutrosophic graph assigns the following values:
$$t(u,v), iv(u,v), fv(u,v), lv(u,v) : E \to [0,1],$$
where $t(u,v)$, $iv(u,v)$, $fv(u,v)$, and $lv(u,v)$ represent the truth value, indeterminacy value, falsity value, and liberal value of the edge (u,v), respectively, satisfying:
$$0 \leq t(u,v) + iv(u,v) + fv(u,v) + lv(u,v) \leq 4.$$
- The edges in the graph are defined based on the permutation π. There is an edge between vertices u and v if $u < v$ and $\pi(u) > \pi(v)$.

Definition 27 (Turiyam Neutrosophic Bipartite Permutation Graph)**.** A *Turiyam Neutrosophic Bipartite Permutation Graph* extends both bipartite graphs and permutation graphs into the Turiyam Neutrosophic framework, where vertices and edges are assigned truth, indeterminacy, falsity, and liberal values. Let $G = (V_1 \cup V_2, E)$ be a bipartite permutation graph, where V_1 and V_2 are independent sets, and π is a permutation on $V_1 \cup V_2$. The Turiyam Neutrosophic bipartite permutation graph $G^T = (V^T, E^T, t, iv, fv, lv)$ is defined as follows:

- For each vertex $v \in V_1 \cup V_2$, the Turiyam Neutrosophic graph assigns the following values:
$$t(v), iv(v), fv(v), lv(v) : V \to [0,1],$$
where $t(v)$, $iv(v)$, $fv(v)$, and $lv(v)$ represent the truth value, indeterminacy value, falsity value, and liberal value of the vertex v, respectively, satisfying:
$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$
- For each edge $e = (u,v) \in E$, where $u \in V_1$ and $v \in V_2$, the Turiyam Neutrosophic graph assigns the following values:
$$t(u,v), iv(u,v), fv(u,v), lv(u,v) : E \to [0,1],$$
where $t(u,v)$, $iv(u,v)$, $fv(u,v)$, and $lv(u,v)$ represent the truth value, indeterminacy value, falsity value, and liberal value of the edge (u,v), respectively, satisfying:
$$0 \leq t(u,v) + iv(u,v) + fv(u,v) + lv(u,v) \leq 4.$$
- The edges between V_1 and V_2 are defined based on the permutation π, such that there is an edge between $u \in V_1$ and $v \in V_2$ if $\pi(u) > \pi(v)$.

3 | Result in this paper

In this section, we present the results of this paper.

3.1 | Property of Neutrosophic Permutation graph

We examine the properties of the Neutrosophic Permutation Graph. Similar properties are also observed in the Fuzzy Permutation Graph, Intuitionistic Fuzzy Permutation Graph, and Turiyam Neutrosophic Permutation Graph.

Theorem 28. *Every Neutrosophic Permutation Graph can be transformed into a classical Permutation Graph.*

Proof: Let
$$G_N = (V, E, \sigma, \mu)$$
be a Neutrosophic Permutation Graph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set.
- $\sigma : V \to [0,1]^3$ assigns to each vertex $v \in V$ a triple
$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$$
representing the truth, indeterminacy, and falsity membership degrees.
- $\mu : E \to [0,1]^3$ assigns to each edge $e = (u,v) \in E$ a triple
$$\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$$
satisfying the usual neutrosophic condition
$$\mu_T(e) \leq \min\{\sigma_T(u), \sigma_T(v)\}.$$

Assume that the graph is equipped with a permutation π on V such that an edge $(v_i, v_j) \in E$ exists if and only if $i < j$ and $\pi(v_i) > \pi(v_j)$. Define the *underlying crisp graph* $G = (V, E')$ by setting
$$E' = \{(v_i, v_j) \mid (v_i, v_j) \in E \text{ and } \mu_T(v_i, v_j) > 0\}.$$
Because the existence of an edge in G_N is dictated by the permutation condition (i.e., $i < j$ and $\pi(v_i) > \pi(v_j)$) and a positive truth value, it follows that G is exactly the classical permutation graph associated with π. In other words, by "forgetting" the neutrosophic membership degrees, we recover a classical permutation graph. This completes the proof. □

Corollary 29. *Every Neutrosophic Bipartite Permutation Graph can be transformed into a classical Bipartite Permutation Graph.*

Proof: Let
$$G_N = (V_1 \cup V_2, E, \sigma, \mu)$$
be a Neutrosophic Bipartite Permutation Graph, where the vertex set is partitioned into two disjoint sets V_1 and V_2, and a permutation π is defined on $V_1 \cup V_2$ so that for every edge $(u,v) \in E$ with $u \in V_1$ and $v \in V_2$, the condition $\pi(u) > \pi(v)$ holds.

Construct the crisp graph $G = (V_1 \cup V_2, E')$ by
$$E' = \{(u,v) \mid (u,v) \in E \text{ and } \mu_T(u,v) > 0\}.$$
Since the permutation condition is preserved and every edge in E' satisfies the bipartite condition, G is a classical Bipartite Permutation Graph. □

Corollary 30. *Every Fuzzy Permutation Graph, Intuitionistic Fuzzy Permutation Graph, and Turiyam Neutrosophic Permutation Graph can be transformed into a classical Permutation Graph.*

Proof: In each of these graph models, the existence of an edge is defined by the permutation condition on the vertex set, in conjunction with a positive membership (or truth) value. By constructing the underlying crisp graph via
$$E' = \{(u,v) \mid (u,v) \in E \text{ and the corresponding membership degree is positive}\},$$
we recover a classical permutation graph. The argument follows identically to that in Theorem 28. □

Corollary 31. *Every Fuzzy Bipartite Permutation Graph, Intuitionistic Fuzzy Bipartite Permutation Graph, and Turiyam Neutrosophic Bipartite Permutation Graph can be transformed into a classical Bipartite Permutation Graph.*

Proof: The transformation is analogous to that in Corollary 29. In each case, by retaining only the edges with positive membership degrees and preserving the permutation condition within the bipartition $V_1 \cup V_2$, the resulting crisp graph is a classical Bipartite Permutation Graph. □

Theorem 32. *Every Neutrosophic Graph whose underlying crisp graph is a permutation graph can be represented as a Neutrosophic Permutation Graph.*

Proof: Let

$$G_N = (V, E, \sigma, \mu)$$

be an arbitrary Neutrosophic Graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Our goal is to construct a permutation π on V such that the edge set of G_N conforms to the permutation condition.

Step 1: Constructing the Permutation. Assign to each vertex $v \in V$ its truth degree $\sigma_T(v)$. Order the vertices in non-increasing order with respect to $\sigma_T(v)$. (If two vertices have identical truth values, use a secondary criterion, such as their indeterminacy values $\sigma_I(v)$.) This ordering defines a permutation π on V.

Step 2: Reconstructing the Edge Set. For any two vertices v_i and v_j with $i < j$ (according to the ordering induced by π), include an edge (v_i, v_j) in the Neutrosophic Permutation Graph if and only if G_N originally contains an edge between v_i and v_j. The neutrosophic membership of the edge is preserved.

Step 3: Verification. By construction, the permutation π guarantees that an edge (v_i, v_j) satisfies $i < j$ and $\pi(v_i) > \pi(v_j)$ if it exists. Furthermore, the membership conditions

$$\mu_T(v_i, v_j) \leq \min\{\sigma_T(v_i), \sigma_T(v_j)\}$$

remain valid. Therefore, the graph G_N is represented as a Neutrosophic Permutation Graph. □

Theorem 33. *Every subgraph of a Neutrosophic Permutation Graph is itself a Neutrosophic Permutation Graph.*

Proof: Let

$$G_N = (V, E, \sigma, \mu)$$

be a Neutrosophic Permutation Graph with associated permutation π, and let

$$G'_N = (V', E', \sigma', \mu')$$

be a subgraph with $V' \subseteq V$ and $E' \subseteq E$. Define the restricted permutation π' on V' as the restriction of π to V'. For any two vertices $u, v \in V'$ with $u < v$ in the order induced by π', if $(u, v) \in E'$ then by the definition of G_N it must be that $\pi(u) > \pi(v)$. Furthermore, the membership functions σ' and μ' are the restrictions of σ and μ to V' and E'. Thus, G'_N satisfies all conditions of a Neutrosophic Permutation Graph. □

3.2 | Relation of Neutrosophic Bipartite Graphs

Next, we examine the relationship between Neutrosophic Bipartite Graphs and Neutrosophic Bipartite Permutation Graphs. The definition of Neutrosophic Bipartite Graph is provided below.

Definition 34 (Neutrosophic Bipartite Graph)**.** A *Neutrosophic Bipartite Graph* is an extension of the classical bipartite graph into the neutrosophic domain, where truth, indeterminacy, and falsity membership degrees are assigned to each vertex and edge.

Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph. The graph G is said to be a *Neutrosophic Bipartite Graph* if there exists a partition of the vertex set V into two disjoint sets V_1 and V_2, such that:

$$V = V_1 \cup V_2 \quad \text{and} \quad V_1 \cap V_2 = \emptyset,$$

and for each edge $e = (u, v) \in E$, one vertex $u \in V_1$ and the other vertex $v \in V_2$, i.e., every edge connects a vertex in V_1 to a vertex in V_2.

Furthermore, the following condition holds for each edge $e = (u, v) \in E$:

$$\mu_T(e) \leq \min(\sigma_T(u), \sigma_T(v)),$$

where $\sigma_T(u)$ and $\sigma_T(v)$ represent the truth membership degrees of vertices $u \in V_1$ and $v \in V_2$, respectively.

Theorem 35. *A Neutrosophic Bipartite Graph can be transformed into a Bipartite Graph.*

Proof: Let

$$G_N = (V, E, \sigma, \mu)$$

be a Neutrosophic Bipartite Graph, where the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 (i.e., $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$). By definition, every edge $e = (u, v) \in E$ satisfies $u \in V_1$ and $v \in V_2$ (or vice versa). In addition, each vertex $v \in V$ is assigned a neutrosophic membership triple

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)),$$

and each edge $e \in E$ is assigned a neutrosophic membership triple

$$\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$

To obtain the underlying crisp graph, define the edge set E' as follows:

$$E' = \{(u, v) \in E \mid \mu_T(u, v) > 0\}.$$

That is, we retain only those edges for which the truth membership degree is strictly positive. The resulting crisp graph is then

$$G = (V, E').$$

Since G_N is bipartite with partition V_1 and V_2, and the transformation does not alter the vertex set or the bipartite nature of the connections (it only filters edges based on a membership criterion), the crisp graph G is also bipartite with the same vertex partition.

Thus, every Neutrosophic Bipartite Graph can be transformed into a (crisp) Bipartite Graph by this procedure. □

Theorem 36. *A Neutrosophic Bipartite Graph is a Neutrosophic Graph.*

Proof: By definition, a Neutrosophic Graph is a graph

$$G_N = (V, E, \sigma, \mu)$$

in which each vertex $v \in V$ is assigned a neutrosophic membership triple

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)),$$

and each edge $e \in E$ is assigned a neutrosophic membership triple

$$\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)),$$

satisfying the condition

$$\mu_T(e) \leq \min\{\sigma_T(u), \sigma_T(v)\} \quad \text{for every } e = (u, v) \in E.$$

A Neutrosophic Bipartite Graph is defined as a Neutrosophic Graph with the additional property that the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge $e = (u, v) \in E$ connects a vertex in V_1 to a vertex in V_2. This bipartite condition is an extra structural constraint, but it does not violate any of the requirements of being a Neutrosophic Graph.

Therefore, since a Neutrosophic Bipartite Graph satisfies all the conditions of a Neutrosophic Graph (with the extra bipartite partition imposed on V), it is indeed a Neutrosophic Graph. □

Theorem 37. *Every Neutrosophic Bipartite Permutation Graph is a Neutrosophic Graph.*

Proof: By definition, a Neutrosophic Bipartite Permutation Graph is a Neutrosophic Graph $G_N = (V, E, \sigma, \mu)$ with the additional structure that V is partitioned into V_1 and V_2 and that there exists a permutation π on V satisfying $\pi(u) > \pi(v)$ for every edge (u, v) with $u \in V_1$ and $v \in V_2$. This additional condition does not violate any properties required by the definition of a Neutrosophic Graph. Hence, every Neutrosophic Bipartite Permutation Graph is, in particular, a Neutrosophic Graph. □

Theorem 38. *Every Neutrosophic Bipartite Graph whose underlying crisp graph is a bipartite permutation graph can be represented as a Neutrosophic Bipartite Permutation Graph.*

Proof: Let

$$G_N = (V_1 \cup V_2, E, \sigma, \mu)$$

be a Neutrosophic Bipartite Graph. We construct a permutation π on $V_1 \cup V_2$ as follows:

(1) Order the vertices in V_1 in non-increasing order with respect to their truth degrees $\sigma_T(v)$ and order the vertices in V_2 in non-decreasing order.

(2) Define the permutation π on $V_1 \cup V_2$ so that for every edge $(u, v) \in E$ with $u \in V_1$ and $v \in V_2$, we have $\pi(u) > \pi(v)$.

Then, form the Neutrosophic Bipartite Permutation Graph

$$G_N^\pi = (V_1 \cup V_2, E', \sigma, \mu'),$$

where

$$E' = \{(u, v) \mid u \in V_1,\ v \in V_2,\ (u, v) \in E \text{ and } \pi(u) > \pi(v)\}.$$

For each edge (u, v), define

$$\mu_T(u, v) = \min\{\sigma_T(u), \sigma_T(v)\},$$

(with analogous definitions for the indeterminacy and falsity degrees). Since the permutation π is constructed to satisfy the required condition, G_N^π is indeed a Neutrosophic Bipartite Permutation Graph representing the original G_N. □

4 | Conclusion

In this paper, we introduced permutation graphs and bipartite permutation graphs within the frameworks of Fuzzy Graphs (cf. [40]), Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs, and then examine their properties and interrelationships. As a future direction of this research, we aim to explore the practical applications of these graphs in real-life settings.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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