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Characteristics Neutrosophic Subgroups of Axiomatic Neutrosophic Groups

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Abstract

This paper aims to introduce the characteristics of neutrosophic subgroups. Neutrosophic laws defined by the partial algebra are totally (100%) true. This work completes our previous work on the axiomatic neutrosophic group. Our work treatments with neutrosophic set $G[I] = \{a + bI: a, b \in G\}$ of type-one and $G[I] = \{aI \cup \{a\}: a \in G\}$ of type-two. Neutrosophic groups were introduced in 2006 by Kandasamy and Smarandache, in this paper we studied the properties of neutrosophic subgroups.

Keywords: Neutrosophic-groups; Neutrosophic-subgroups; Properties of Neutrosophic-subgroups.

1 | Introduction

Neutrosophic algebra is a branch of Neutrosophy, Neutrosophy is a branch of new philosophy was proposed by Smarandache with developed and extension intuitionistic fuzzy into the neutrosophic set that associative with neutrosophic logic, for more information we refer to [1-4]. In 2004 [5], Kandasamy and Smarandache published their book about Basic Neutrosophic Algebraic Structures and Their Application to Fuzzy and Neutrosophic Models; as an introduction of neutrosophic theory to representation to indeterminates vis. determinates. The concept of uncertainty or indeterminacy viz. the concept of certainty or determinacy in philosophy. The concept of indeterminacy "I", where $I^2 = I$ vis. the concept of imagery in complex numbers $j^2 = -1$, and consequently, I can not be defined. In 2006 [6, 7], and [8], Kandasamy published Smarandache neutrosophic algebraic structures including neutrosophic-group and their properties, and other structures of neutrosophic algebras; and presented neutrosophic-group and studied Neutrosophic Bi-groups with their properties, Neutrosophic N-groups with their properties, and other structures of Neutrosophic algebra with Smarandache. After that, they studied neutrosophic rings and their properties; and neutrosophic group rings and their generalizations. Next later, other researchers joined them to study the subject. In 2012 [9], Agboola, Akwu, and Oyebo studied neutrosophic groups and subgroups, and presented the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup. Once again, in 2020 [10], Agboola presented the concept of neutrosophic group by considering three neutrosophic axioms and presented some results about neutrosophic groups, neutrosophic subgroups, neutrosophic cyclic groups, neutrosophic quotient groups, and neutrosophic homeomorphism of groups. In 2023 [11], some results about the neutrosophic group

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according to the degree of neutrosophic membership function. In this article, we introduce our contribution to neutrosophic subgroups to complete the work of axiomatic neutrosophic groups in [12], and our contribution in the field of neutrosophic algebraic relevant to [13-15].

2 |Axiomatic of Neutrosophic Groups

In this section, we review some concepts of neutrosophic groups and their properties.

Definition 2.1. [6, 7] Let $(G,*)$ be any group, and $\langle G \cup I\rangle$ is given by: $\langle G \cup I\rangle = \{a + bI: a, b \in G\}$, then the neutrosophic algebra structure $N(G) = \{\langle G \cup I\rangle, *\}$ is called the neutrosophic group which is generated by I and G under $*$.

Theorem 2.1. [6, 7] Let $(G,*)$ be a group, $N(G) = \{\langle G \cup I\rangle, *\}$ be the neutrosophic group, then:

1. $N(G)$ in general, is not a group, and
2. $N(G)$ always contains a group.

Definition 2.2. [12] Let $G \neq \emptyset$ be any non-empty set. A neutrosophic set (NS) is defined by:

$G[I] = \{a + bI: a, b \in G\}$, where I is an indeterminacy concept.

Definition 2.3. [12] The order pair $NG = \langle G[I], *\rangle$ consists of the neutrosophic set $G[I]$, with a binary operation $*$ defined on $G[I]$ is called neutrosophic-group (NG); if it satisfies the following axioms:

NG$_1$: $(\forall\, x, y, z \in G[I])$, $(x * y) * z = x * (y * z)$, associative law";

NG$_2$: $(\exists 0_{NG} \in G[I])$, $(x * 0_{NG} = 0_{NG} * x = 0_{NG}, \forall\, x \in G[I])$, " existence of an identity", 0_{NG} is a just notation depending on the type of structure of the neutrosophic set and;

NG$_3$: $(\forall\, x \in G[I])$, $(\exists\, x^{-1} \in G[I], x * x^{-1} = x^{-1} * x = 0_{NG})$ "the existence of an inverse. " Thus, the neutrosophic group is a neutrosophic mathematical system. $NG = \langle G[I], *\rangle$ satisfying the axioms **NG$_1$** to **NG$_3$**. Otherwise is called a neutrosophic algebra structure.

Theorem 2.2 [12]Let $G \neq \emptyset$ be any non-empty set; $G[I] = \{a + bI: a, b \in G\}$ be the neutrosophic set (NS); and $N(G) = \langle G[I], *\rangle$ be a neutrosophic group.

1. There exists a unique element $0_{NG} = 0_G + 0_G I \in G[I]$, such that $x * 0_{NG} = x = 0_{NG} * x$, for all $x \in G[I]$.
2. There exists a unique $y \in N(G)$ such that $x * y = y * x = 0_{NG}$.

Theorem 2.3 [12] Let $G \neq \emptyset$ be any non-empty set; $G[I] = \{a + bI: a, b \in G\}$ be the neutrosophic set (NS); and $N(G) = \langle G[I], *\rangle$ be a neutrosophic group.

1. $((a + bI)^{-1})^{-1} = (a + bI)$, for all $(a + bI) \in G[I]$.
2. $\left((a + bI) * (c + dI)\right)^{-1} = (c + dI)^{-1} * (a + bI)^{-1}$, for all $(a + bI), (c + dI) \in G[I]$.
3. For all $x, y, z \in N(G)$, if either $x * z = y * z$ or $z * x = z * y$, then $x = y$, this is called the Cancelation law in the neutrosophic group.
4. For all $x, y \in G[I]$, the unique solution $x * z = y$ has a unique solution in $N(G)$ for z .

Definition 2.4. [6, 7] Let $\mathbb{Z}$ be a set of integer numbers and $\mathbb{Z}[\mathrm{I}] = \{a + bI: a, b \in \mathbb{Z}\}$ be a neutrosophic integer set, where $a + bI$ is a neutrosophic integer number.

Theorem 2.4. [12] Let $\mathbb{Z}[I] = \{a + bI: a, b \in \mathbb{Z}\}$ be the set of neutrosophic integer numbers. Then the neutrosophic structure $N(\mathbb{Z}) = \langle \mathbb{Z}[I], +\rangle$ under usual neutrosophic addition forms a commutative neutrosophic integer group.

Definition 2.5. Let $G \neq \emptyset$ be any non-empty set; $G[I] = \{aI \cup \{a\}: a \in G\}$ be the neutrosophic set (NS).

Example 2.5. Consider the set $G = \{1, -1\}$ under multiplication of complex numbers is a croup in a classical group. Consider the neutrosophic set,$G[I] = \{a + bI: a, b \in G\} = \{1 + I, 1 - I, -1 + I, -1 - I\}$. Under usual neutrosophic multiplication. We note that $(1 + I).(1 + I) = 1 + 3I \notin G[I]$ is not closed, hence, $N(G) = \langle G[I], \bullet\rangle$ is not a neutrosophic group. If we consider neutrosophic-set: $G[I] = \{aI \cup \{a\}: a \in G\} = \{1, -1, \mathrm{I}, -\mathrm{I}\}$. from the Table 1. we see that $N(G) = \langle G[I], \bullet\rangle$ closed, associative, and the neutrosophic identity.

Table 1. $N(G) = \langle G[I], \bullet\rangle$ closed, associative, and the neutrosophic identity.

$\bullet$	1	-1	I	$-I$
1		-1	I	$-I$
-1	-1	1	$-I$	I
I	I	$-I$	I	$-I$
$-I$	$-I$	I	$-I$	I

Example 2.6 Consider the set $G = \{1,2\}$ under multiplication of modulo 3 is a group in classical croup. Consider the neutrosophic set. $G[I] = \{a + bI: a, b \in G\} = \{1 + I, 1 + 2I, 2 + I, 2 + 2I\}$. Under usual neutrosophic multiplication. We see that $(1 + I).(1 + I) = 1 \notin G[I]$, hence the binary operation is not closed, therefore $N(G) = \langle G[I], \bullet\rangle$ is not a neutrosophic group. If we consider neutrosophic-set: $G[I] = \{aI \cup \{a\}: a \in G\} = \{12, \mathrm{I}, 2\mathrm{I}\}$. from the Table 2. we see that $N(G) = \langle G[I], \bullet\rangle$ neutrosophic-monoid.

Table 2. $N(G) = \langle G[I], \bullet\rangle$ neutrosophic-monoid.

$\bullet$	1	2	I	$2I$
1		2	I	$2I$
2	2	1	$2I$	I
I	I	$2I$	I	$2I$
$2I$	$2I$	I	$2I$	I

3 | Neutrosophic-Subgroups and their Properties

Definition 3.1. [6,7] Let $N(G) = \langle G \cup I\rangle$ be a neutrosophic group generated by G and I. A proper subset $P(G)$ is said to be a neutrosophic subgroup if $P(G)$ is a neutrosophic group i.e. $P(G)$ must contain a (sub) group.

Definition 3.2. [11,12,14] Let $N(G) = \langle G \cup I\rangle$ be a neutrosophic group generated by G and I. A proper subset $P(G)$ is said to be a neutrosophic subgroup if $P(G)$ is a neutrosophic group i.e. $P(G)$ must contain a (sub) group.

Definition 3.3. [6,7] A pseudo neutrosophic group is a neutrosophic group, which does not contain a proper subset which is a group. Pseudo-neutrosophic subgroups can be found as a substructure of neutrosophic groups.

Thus, a pseudo neutrosophic group though has a group structure is not a neutrosophic group and a neutrosophic group cannot be a pseudo neutrosophic group. Both concepts are different.

Theorem 3.1. Let $NG = \langle G[I], *\rangle$ be a neutrosophic group, and $H[I]$ be a non-empty neutrosophic subset of $G[I]$. Then the binary operation $*$ on $H[I]$ is said to be closed if $x * y \in H[I]$ for all $x, y \in H[I]$.

Proof. Suppose that $H[I]$ is closed under binary operation $*$. Then the restriction $*$ from $H[I] \times H[I]$ into $H[I]$ is a mapping defined on $G[I]$ induces with a binary operation $*$. Conversely, is evident■.

Definition 3.4. Let $NG = \langle G[I], * \rangle$ be a neutrosophic group, and $H[I]$ be a non-empty neutrosophic-subset of $G[I]$. Then $NH = \langle H[I], * \rangle$ is a neutrosophic subgroup (written, $H[I] \underset{N}{\leqslant} G[I]$) of $NG = \langle G[I], * \rangle$, if $NH = \langle H[I], * \rangle$ is a neutrosophic group.

Theorem 3.2. Let $G[I]$ be called the neutrosophic set (NS), $NG = \langle G[I], * \rangle$ be a neutrosophic group, and $H[I]$ be a non-empty neutrosophic-subset of $G[I]$. Then $NH = \langle H[I], * \rangle$ is called a neutrosophic subgroup of $NG = \langle G[I], * \rangle$, if $NH = \langle H[I], * \rangle$ is a neutrosophic group.

Proof. Assume that $NH = \langle H[I], * \rangle$ is a neutrosophic subgroup of $NG = \langle G[I], * \rangle$ of type-1

Let $e_{NH} = 0_H + 0_H I \in H[I]$ be the identity and $e_{NG} = 0_G + 0_G I \in H[I]$; we obtaining, $e_{NH} * e_{NH} = e_{NH} = e_{NH} * e_{NG}$, then by theorem 2.3[ ], we have $e_{NH} = e_{NG}$. Finally, Let $x = (x_1 + x_2 I) \in H[I]$, and suppose that $x' = (x'_1 + x'_2 I)$ is the inverse of x in $H[I]$, and $x^{-1} = (x_1{}^{-1} + x_2{}^{-1} I)$ is the inverse of x in $G[I]$, to show that $x' = x^{-1}$.

$$
\begin{aligned}
x' = x' * e_{NH} &= (x'_1 + x'_2 I) * (e_H + e_H I) \\
&= (x'_1 + x'_2 I) * (e_G + e_G I) \\
&= (x'_1 + x'_2 I) * \left((x_1 + x_2 I) * (x_1{}^{-1} + x_2{}^{-1} I)\right) \\
&= \left((x'_1 + x'_2 I) * (x_1 + x_2 I)\right) * (x_1{}^{-1} + x_2{}^{-1} I) \\
&= (x'_1 * x_1 + x'_2 * x_2 I) * (x_1{}^{-1} + x_2{}^{-1} I) \\
&= (e_H + e_H I) * (x_1{}^{-1} + x_2{}^{-1} I) \\
&= (e_G + e_G I) * (x_1{}^{-1} + x_2{}^{-1} I) \\
&= e_{NG} * x^{-1} \\
&= x^{-1}, \text{ therefore } NH = \langle H[I], * \rangle \text{ is a neutrosophic-group■.}
\end{aligned}
$$

Theorem 3.3. Let $G[I]$ be called the neutrosophic set (NS), $NG = \langle G[I], * \rangle$ be a neutrosophic group, and $H[I]$ be a non-empty neutrosophic-subset of $G[I]$. Then $NH = \langle H[I], * \rangle$ is called a neutrosophic subgroup of $NG = \langle G[I], * \rangle$, if $x, y \in H[I]$, then $x * y^{-1} \in H[I]$.

Proof. Let $NG = \langle G[I], * \rangle$ be a neutrosophic group, and $H[I] \subset G[I]$ and suppose that

$NH = \langle H[I], * \rangle$ is a neutrosophic subgroup of $NG = \langle G[I], * \rangle$. Consider $x, y \in H[I]$, y proposition 2.2.

$NH = \langle H[I], * \rangle$ is a neutrosophic group; hence, $y^{-1} \in H[I]$, and consequently $x * y^{-1} \in H[I]$ by proposition 2.1. Conversely, assume that $H[I] \subset G[I]$and $x, y \in H[I]$, implies that $x * y^{-1} \in H[I]$. To show that

$NH = \langle H[I], * \rangle$ is a neutrosophic-group, because, $H[I] \neq \emptyset \Rightarrow \exists\, x = (x_1 + x_2 I) \in H[I]$, hence

$x * x^{-1} = e_{NH} \in H[I]$. For all $y \in H[I]$, we have $y^{-1} = (y_1{}^{-1} + y_2{}^{-1} I) = (e_H + e_H I) * \left((y_1{}^{-1} + y_2{}^{-1} I)\right) \in H[I]$, so inverse neutrosophic element exists in $H[I]$and for all $x, y \in H[I]$, implies that $x * y^{-1} \in H[I]$.

In addition, $x * y = (x_1 + x_2 I) * (y_1{}^{-1} + y_2{}^{-1} I)^{-1} = (x_1 + x_2 I) * ((y_1{}^{-1})^{-1} + (y_1{}^{-1})^{-1} I) \in H[I]$.

That is $*$ is closure. Moreover, it is associative, therefore $NH = \langle H[I], * \rangle$ is a neutrosophic group, by proposition 2.2. $NH = \langle H[I], * \rangle$ is a neutrosophic- subgroup of $NG = \langle G[I], * \rangle$■.

Definition 3.5. Let $N(G) = \langle G[I], * \rangle$ be a neutrosophic-group, and $y \in G[I]$,then:

$C\big(N(G)\big) = \{x \in G[I]: yx = xy, \in G[I]\ \}$ is called the center of the neutrosophic- group or neutrosophic centralizer of x.

Theorem 3.4. Let $N(G) = \langle G[I],*\rangle$ be a neutrosophic group, then $C(N(G)) \lneqq_N \langle G[I],*\rangle$ is a commutative neutrosophic subgroup.

Proof. Let $N(G) = \langle G[I],*\rangle$ be a neutrosophic group. Since $ex = xe, \forall\, x \in G[I]$, then $e \in C(N(G)) \neq \emptyset$. Suppose that $x, y \in C(N(G))$; since, $y \in C(N(G)) \Rightarrow zy = yz, \forall\, z \in G[I]$

$$\Rightarrow (z_1 + z_2I)(y_1 + y_2I) = (y_1 + y_2I)\,(z_1 + z_2I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1 + y_2I) = ({y_1}^{-1} + {y_2}^{-1}I)(y_1 + y_2I)\,(z_1 + z_2I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1 + y_2I) = ({y_1}^{-1}y_1 + {y_2}^{-1}y_2\,I)\,(z_1 + z_2I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1 + y_2I) = (e + eI)\,(z_1 + z_2I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1 + y_2I) = (z_1 + z_2I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1 + y_2I)({y_1}^{-1} + {y_2}^{-1}I) = (z_1 + z_2I)({y_1}^{-1} + {y_2}^{-1}I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(y_1{y_1}^{-1} + y_2{y_2}^{-1}I) = (z_1 + z_2I)({y_1}^{-1} + {y_2}^{-1}I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I)(e + eI) = (z_1 + z_2I)({y_1}^{-1} + {y_2}^{-1}I), \forall\, z \in G[I]$$
$$\Rightarrow ({y_1}^{-1} + {y_2}^{-1}I)(z_1 + z_2I) = (z_1 + z_2I)({y_1}^{-1} + {y_2}^{-1}I), \forall\, z \in G[I]$$

$\Rightarrow y^{-1}z = zy^{-1}, \forall\, z \in G[I] \Rightarrow y^{-1} \in C(N(G))$. By Proposition 2.3. We deduce that $C(N(G)) \lneqq_N \langle G[I],*\rangle$, and it is a commutative by its definition ■.

Example 3.1. $\langle \mathbb{Z}[I], +\rangle \lneqq_N \langle \mathbb{Q}[I], +\rangle \lneqq_N \langle \mathbb{R}[I], +\rangle \lneqq_N \langle \mathbb{C}[I], +\rangle$ under the addition of neutrosophic element $x = x_1 + x_2I$; with respect to the neutrosophic integer, irrational, real, and complex numbers respectively.

Example 3.2. $\langle \mathbb{Z}[I],\bullet\rangle \lneqq_N \langle \mathbb{Q}[I],\bullet\rangle \lneqq_N \langle \mathbb{R}[I],\bullet\rangle \lneqq_N \langle \mathbb{C}[I],\bullet\rangle$ under multiplication of neutrosophic element $x = x_1 + x_2I$; with respect to the neutrosophic integer, irrational, real, and complex numbers respectively.

Example 3.3. Consider $\langle \mathbb{Z}[I], +\rangle$ is a neutrosophic integer group and $\langle \mathbb{Z}_{even}[I], +\rangle$ neutrosophic set of even integers, then $\langle \mathbb{Z}_{even}[I], +\rangle \lneqq_N \langle \mathbb{Z}[I], +\rangle$. Under usual addition.

Example 3.4. Consider $\langle \mathbb{Z}_n[I],\oplus_n\rangle$ is the commutative neutrosophic-integer modulo $n \in \mathbb{Z}^+$. Then the chain: $\langle \mathbb{Z}_2[I],\oplus_2\rangle \lneqq_N \langle \mathbb{Z}_3[I],\oplus_3\rangle \lneqq_N \cdots \lneqq_N \langle \mathbb{Z}_{n-1}[I],\oplus_{n-1}\rangle \lneqq_N \langle \mathbb{Z}_n[I],\oplus_n\rangle$

Example 3.5. Consider $\langle \mathbb{Z}_2[I],\oplus_2\rangle$ is the commutative neutrosophic-integer modulo $2 \in \mathbb{Z}^+$, where $\mathbb{Z}_2[I] = \{0,1,I,1+I\}$; then, $H_1[I] = \{0\} \lneqq_N \mathbb{Z}_2[I]$, and $\mathbb{Z}_2[I] \lneqq_N \mathbb{Z}_2[I]$ are two trivial neutrosophic-subgroups of $\mathbb{Z}_2[I]$, in addition, $H_5[I] = \{0,1\} \lneqq_N \mathbb{Z}_2[I]$, $H_6[I] = \{0, I\} \lneqq_N \mathbb{Z}_2[I]$, and

$H_7[I] = \{0,1+I\} \lneqq_N \mathbb{Z}_2[I]$ are non-trivial neutrosophic-subgroups of $\mathbb{Z}_2[I]$.

Observation. According to the argument of Authors [11,12,14] $H_6[I] = \{0, I\}$ and

$H_7[I] = \{0,1+I\}$ are pseudo neutrosophic groups for they do not have a proper subset which is a group by the following argument.

Theorem 3.5. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group and $H[I]$ be a finite non-empty neutrosophic subset of $G[I]$. Then $NH = \langle H[I],*\rangle$ is a neutrosophic subgroup of $NG = \langle G[I],*\rangle$, if for all $x, y \in H[I]$, then $x * y \in H[I]$.

Proof. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group and $H[I] \subset G[I]$. Suppose that $H[I] \lneqq_N G[I])$ such that for all $x, y \in H[I]$, then $x * y \in H[I]$. Conversely, suppose that for all $x, y \in H[I]$, then $x * y \in H[I]$. To show that $H[I] \lneqq_N G[I])$. Consider $x \in H[I]$, since $H[I]$ is finite, then $y, y^2, y^3, \dots, y^n, \dots \subseteq H[I]$all elements in the neutrosophic set of $\{y, y^2, y^3, \dots, y^n, \dots\}$ cannot be distinct; thus, $\exists\, r, s \in \mathbb{Z}^+$ such that $0 \leq r < s$ such that,

$y^r = y^s$, therefore $e_{NH} = y^{s-r} \in H[I]$, but $s - r \geq 1$, implies that $e_{NH} = yy^{s-r-1}$, therefore,

$y^{-1} = y^{s-r-1} \in H[I]$, and consequently, $x * y^{-1} \in H[I]$, by theorem 3.3. Thus, $H[I] \lneqq_N G[I])$■.

Remark. Let $NG = \langle G[I],*\rangle$ be an infinite neutrosophic group and let $H[I]$be an infinite neutrosophic set; if $H[I]$ is a closed under a binary operation $*$, then $H[I]$is not necessarily to be a neutrosophic-subgroup of $\langle G[I],*\rangle$.

Example 3.6. Let $N\mathbb{Z} = \langle \mathbb{Z}[I], +\rangle$ be a neutrosophic group; and, $\mathbb{N}[I] = \{a + bI: a, b \in \mathbb{N}\}$, where, $N\mathbb{N} = \langle \mathbb{N}[I], +\rangle$ is closed under $+$, and $\mathbb{N}[I] \subset \mathbb{Z}[I]$ but, $N\mathbb{N} = \langle \mathbb{N}[I], +\rangle$ is not neutrosophic-subgroup of $N\mathbb{Z} = \langle \mathbb{Z}[I], +\rangle$.

Theorem 3.6. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group;$NH_1 = \langle H_1[I],*\rangle$, and $NH_2 = \langle H_2[I],*\rangle$ are two neutrosophic groups of $NG = \langle G[I],*\rangle$. Then: $NH_1 \cap NH_2 \lneqq_N NG$ $\left(or H_1[I] \cap H_2[I] \lneqq_N G[I]\right)$.

Proof. Since $e_{NH} \in H_1[I]$ and $e_{NH} \in H_2[I]$, then $H_1[I] \cap H_2[I] \neq \emptyset$. Let $x, y \in H_1[I] \cap H_2[I]$,

$$\Rightarrow x, y \in H_1[I] \wedge x, y \in H_2[I]$$

$$\Rightarrow x * y \in H_1[I] \wedge x * y \in H_2[I]$$

$$\Rightarrow x * y^{-1} \in H_1[I] \wedge x * y^{-1} \in H_2[I]$$

$$\Rightarrow x * y^{-1} \in (H_1[I] \cap H_2[I])$$

$$\Rightarrow \in (H_1[I] \cap H_2[I]) \lneqq_N G[I] \blacksquare.$$

Remark. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group;$NH_1 = \langle H_1[I],*\rangle$, and $NH_2 = \langle H_2[I],*\rangle$ are two neutrosophic groups of $NG = \langle G[I],*\rangle$. Then$H_1[I] \cup H_2[I]$ may not be the neutrosophic subgroup of $G[I]$.

Example 3.7. Let $N\mathbb{Z} = \langle \mathbb{Z}[I], +\rangle$ be a neutrosophic group; $2\mathbb{Z}[\mathrm{I}] = \{\mathrm{a} + \mathrm{bI}: \mathrm{a}, \mathrm{b} \in 2\mathbb{Z}\}$, and

$3\mathbb{Z}[\mathrm{I}] = \{\mathrm{a} + \mathrm{bI}: \mathrm{a}, \mathrm{b} \in 3\mathbb{Z}\}$ are two neutrosophic groups of $\mathbb{Z}[I]$. We see that,

$$(2 + 4I) + (3 + 9I) = 5 + 13I \notin 2\mathbb{Z}[\mathrm{I}] \cup 3\mathbb{Z}[\mathrm{I}]$$

4 | Product of Neutrosophic Subgroups

Definition 4.1. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group, $H_1[I] \subset G[I]$ and $H_2[I] \subset [I]$are two non-empty neutrosophic sets of $G[I]$. Then the neutrosophic product of $H_1[I]$ and $H_2[I]$ is defined to be the neutrosophic set: $H_1[I]H_2[I] = \{xy: x \in H_1[I], y \in H_2[I]\}$; in general, the finite neutrosophic product is defined by: $H_1[I]H_2[I] \ldots H_n[I] = \{x_1.x_2, \ldots, x_n: x_i \in H_i[I], 1 \leq i \leq n\}$.

Example 4.1. Let $N(\mathbb{Z}) = \langle \mathbb{Z}[I], +\rangle$ As usual, neutrosophic addition forms a commutative neutrosophic-integer group. Consider $H_1[I] = \{1 + I, 1 + 3I, 3 + I, 3 + 2I\}$, where

$H_1 = \{1,3\}$, and $H_2[I] = \{2 + 2I, 2 + 4I, 4 + 2I, 4 + 4I\}$, where $H_2 = \{2,4\}$ are two non-empty neutrosophic-set of $\mathbb{Z}[\boldsymbol{I}]$. Then:

$$H_1[I]H_2[I] = \begin{Bmatrix} (1 + I) + (2 + 2I), (1 + I) + (2 + 4I), (1 + I) + (4 + 2I), (1 + I) + (4 + 4I) \\ (1 + 3I) + (2 + 2I), (1 + 3I) + (2 + 4I), (1 + 3I) + (4 + 2I), (1 + 3I) + (4 + 4I) \\ (3 + I) + (2 + 2I), (3 + \boldsymbol{I}) + (2 + 4I), (3 + \boldsymbol{I}) + (4 + 2I), (3 + \boldsymbol{I}) + (4 + 4I) \\ (3 + 2I) + (2 + 2I), (3 + 2I) + (2 + 4I), (3 + 2I) + (4 + 2I), (3 + 2I) + (4 + 4I) \end{Bmatrix}$$

$$= \begin{Bmatrix} (3 + 3I), (3 + 5I), (5 + 3I), (5 + 5I) \\ (3 + 5I), (3 + 7I), (5 + 5I), (5 + 7I) \\ (5 + 3I), (5 + 5\boldsymbol{I}), (7 + 3\boldsymbol{I}), (7 + 5\boldsymbol{I}) \\ (5 + 4I), (5 + 6I), (7 + 4I), (7 + 6I) \end{Bmatrix}$$

$$= \begin{Bmatrix} (3 + 3I), (3 + 5I), (5 + 3I), (5 + 5I), (3 + 7I), (5 + 7I), (7 + 3\boldsymbol{I}), (7 + 5\boldsymbol{I}), \\ (5 + 4I), (5 + 6I), (7 + 4I), (7 + 6I) \end{Bmatrix}$$

Theorem 4.1. Let $NG = \langle G[I],*\rangle$ be a neutrosophic group, $H_1[I] \lneqq_N G[I])$ and $H_2[I] \lneqq_N G[I])$ then

$H_1[I]H_2[I] \lessapprox_N G[I])$ iff $H_1[I]H_2[I] = H_2[I]H_1[I]$.

Proof. Let $NG = \langle G[I],* \rangle$ be a neutrosophic group, $H_1[I] \lessapprox_N G[I])$ and $H_2[I] \lessapprox_N G[I])$. Suppose that $H_1[I]H_2[I] \lessapprox_N G[I])$. Let $x, y \in H_1[I]H_2[I]$, where $x \in H_1[I]$ and $y \in H_2[I]$.

$\because x \in H_1[I] \Rightarrow \exists\, x_1, x_2 \in H_1[I]$ and indeterminacy I such that $x = x_1 +, x_2 I$, also

$\because y \in H_2[I] \Rightarrow \exists\, y_1, y_2 \in H_2[I]$ and indeterminacy I such that $y = y_1 +, y_2 I$.

$\because H_1[I]H_2[I] \lessapprox_N G[I]) \Rightarrow xy = (x_1 +, x_2 I)(y_1 +, y_2 I) = (x_1 y_1 + (x_1 y_2 + x_2 y_1 + x_2 y_2) I) \in H_1[I]H_2[I]$, therefore $H_1[I]H_2[I] \subset H_2[I]H_1[I]$. On the other hand, let $x, y \in H_2[I]H_1[I]$, implies that,

$$(xy)^{-1} = \left((x_1 +, x_2 I)(y_1 +, y_2 I)\right)^{-1}$$
$$= (y_1 +, y_2 I)^{-1}(x_1 +, x_2 I)^{-1}$$
$$= (y_1^{-1} +, y_2^{-1} I)\,(x_1^{-1} +, x_2^{-1} I),$$ for some $y_1^{-1}, y_2^{-1} \in H_2[I]$ and $x_1^{-1}, x_2^{-1} \in H_1[I]$, thus

$$xy = \left((y_1^{-1} +, y_2^{-1} I)\,(x_1^{-1} +, x_2^{-1} I)\right)^{-1}$$
$$= (x_1^{-1} +, x_2^{-1} I)^{-1}(y_1^{-1} +, y_2^{-1} I)^{-1} \in H_2[I]H_1[I].$$ Hence $H_1[I]H_2[I] = H_2[I]H_1[I]$. Conversely,

Suppose that $H_1[I]H_2[I] = H_2[I]H_1[I]$. To show that $H_1[I]H_2[I] \lessapprox_N G[I])$.

Let $xy = (x_1 + x_2 I)(y_1 + y_2 I),\ zw = (z_1 + zI)(w_1 + y_2 I) \in H_1[I]H_2[I]$, where $x, z \in H_1[I]$, and

$y, w \in H_2[I]$. Now,

$$w^{-1}\ z^{-1} = (w_1 + w_2 I)^{-1}(z_1 + z_2 I)^{-1} \in H_2[I]H_1[I] = H_1[I]H_2[I],$$ this implies that,
$$= (w_1^{-1} + w_2^{-1} I)(z_1^{-1} + z_2^{-1} I)$$
$$= (w'_1 + w'_2 I)(z'_1 + z'_2 I),$$ for some $(w'_1 + w'_2 I) \in H_2[I]$ and $(z'_1 + z'_2 I) \in H_1[I]$. Thus,

$$(xy)(zw)^{-1} = \left((x_1 + x_2 I)(y_1 +, y_2 I)\right)\left((z_1 + zI)(w_1 + y_2 I)\right)^{-1}$$
$$= \left((x_1 + x_2 I)(y_1 +, y_2 I)\right)\left((w_1 + w_2 I)^{-1}(z_1 + z_2 I)^{-1}\right)$$
$$= \left((x_1 + x_2 I)(y_1 +, y_2 I)\right)\left((w_1^{-1} + w_2^{-1} I)(z_1^{-1} + z_2^{-1} I)\right)$$
$$= \left((x_1 + x_2 I)(y_1 +, y_2 I)\right)\left((w'_1 + w'_2 I)(z'_1 + z'_2 I)\right) \in H_1[I]H_2[I].$$ Hence,

$H_1[I]H_2[I] \lessapprox_N G[I])$■.

Example 4.2. Let $N(\mathbb{Z}) = \langle \mathbb{Z}[I], + \rangle$ As usual, neutrosophic addition forms a commutative neutrosophic-integer group. Consider $H_1[I] = \{1 + I, 1 + 3I, 3 + I, 3 + 2I\}$, where $H_1 = \{1,3\}$, and $H_2[I] = \{2 + 2I, 2 + 4I, 4 + 2I, 4 + 4I\}$, where $H_2 = \{2,4\}$ are two non-empty neutrosophic-set of $\mathbb{Z}[I]$. Then:

$$H_1[I]H_2[I] = \left\{\begin{matrix} (1 + I) + (2 + 2I), (1 + I) + (2 + 4I), (1 + I) + (4 + 2I), (1 + I) + (4 + 4I) \\ (1 + 3I) + (2 + 2I), (1 + 3I) + (2 + 4I), (1 + 3I) + (4 + 2I), (1 + 3I) + (4 + 4I) \\ (3 + I) + (2 + 2I), (3 + I) + (2 + 4I), (3 + I) + (4 + 2I), (3 + I) + (4 + 4I) \\ (3 + 2I) + (2 + 2I), (3 + 2I) + (2 + 4I), (3 + 2I) + (4 + 2I), (3 + 2I) + (4 + 4I) \end{matrix}\right\}$$

$$=\begin{Bmatrix} (3+3I),(3+5I),(5+3I),(5+5I) \\ (3+5I),(3+7I),(5+5I),(5+7I) \\ (5+3I),(5+5\boldsymbol{I}),(7+3\boldsymbol{I}),(7+5\boldsymbol{I}) \\ (5+4I),(5+6I),(7+4I),(7+6I) \end{Bmatrix}$$

$$=\begin{Bmatrix} (3+3I),(3+5I),(5+3I),(5+5I),(3+7I),(5+7I),(7+3\boldsymbol{I}),(7+5\boldsymbol{I}), \\ (5+4I),(5+6I),(7+4I),(7+6I) \end{Bmatrix}$$

5 | Conclusions

This paper aims to introduce the characteristics of neutrosophic subgroups. Neutrosophic laws defined by the partial algebra are totally (100%) true. This work completes our previous work on the axiomatic neutrosophic group.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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