1 Introduction


In the field of set theory, Smarandache's neutrosophic set [5, 6] provides a prominent framework. This set encompasses classical sets, fuzzy sets, and different generalizations, like IvFSs, IFSs, and IvIFSs. The application of NSs expands to several fields like control theory, algebra, and topology. Further developing this concept, Wang et al. [7] gave the concept of interval-valued neutrosophic sets, which gives higher adjustability and precision compared to single-valued neutrosophic sets. Jun et al. [8] introduced a novel concept called negative-valued function and created N-structures. These N-structures were further extended by Khan et al. [9], who introduced the idea of neutrosophic N-structure and applied it to a semigroup. Jun et al. [10] applied the idea of neutrosophic N-structure to BCK/BCI-algebras.

In recent years there has been remarkable progress in neutrosophic set theory. In this manuscript, we provide basic definitions essential for our work in Section 2. These basic definitions are d-algebra, d-subalgebra, d-ideal, BCK-ideal, Fuzzy set, Fuzzy d-subalgebra, Fuzzy d-ideal, Fuzzy BCK-ideal, Neutrosophic set,
Neutrosophic N-structure, Interval-valued Neutrosophic N-structure, Interval-valued Neutrosophic N-d-subalgebra, and level sets related to Interval-valued Neutrosophic N-structure. In Section 3, we

- Introduced IvNSN-d-I with an example.
- Introduced IvNSN-BCK-I.
- Proved every IvNSN-d-I is an IvNSN-d-SA (converse not true, which is illustrated through an example).
- Proved every IvNSN-d-I is an IvNSN-BCK-I (converse not true, which is illustrated through an example).
- The proved intersection of an arbitrary family of IvNSN-d-I is also an IvNSN-d-I.
- Proved IvNSN-S is an IvNSN-d-I if its level sets are d-ideal.
- Provided some characteristics of IvNSN-d-I.

Section 4 is the conclusion part, which briefly summarises the key concepts discussed in the manuscript.

2 | Preliminaries

Definition 2.1. [3] Let \( \mathcal{D} (\neq \emptyset) \) be a set with a constant ‘0’ and a binary operation ‘*’. Then \( \mathcal{D} \) is called a d-algebra if it satisfies the following conditions for all \( d_1, d_2 \in \mathcal{D} \):

(d-A 1) \( d_1 * d_1 = 0 \)

(d-A 2) \( 0 * d_1 = 0 \)

(d-A 3) \( d_1 * d_2 = 0 \) and \( d_2 * d_1 = 0 \) implies \( d_1 = d_2 \).

We will refer to \( d_1 \leq d_2 \) if and only if \( d_1 * d_2 = 0 \).

Definition 2.2. [4] Let \( \mathcal{D} \) be a d-algebra with binary operation ‘*’ and \( \mathcal{P} \subseteq \mathcal{D} \). Then, \( \mathcal{P} \) is said to be a d-subalgebra of \( \mathcal{D} \), if \( d_1, d_2 \in \mathcal{P} \) implies \( d_1 * d_2 \in \mathcal{P} \).

Definition 2.3. [4] Let \( \mathcal{D} \) be a d-algebra with binary operation ‘*’ and a constant 0. Then, \( \mathcal{P} \subseteq \mathcal{D} \) is called a d-ideal of \( \mathcal{D} \) if it satisfies the following conditions

(d-I I) \( d_1 * d_2 \in \mathcal{P} \) and \( d_2 \in \mathcal{P} \) implies \( d_1 \in \mathcal{P} \);

(d-I II) \( d_1 \in \mathcal{P} \) and \( d_2 \in \mathcal{D} \) implies \( d_1 * d_2 \in \mathcal{P} \).

Definition 2.4. [4] A subset \( \mathcal{P}(\neq \emptyset) \) of d-algebra \( \mathcal{D} \) is called a BCK-ideal of \( \mathcal{D} \) if satisfies (d-I I) and \( 0 \in \mathcal{P} \).

Definition 2.5. [11] A FS \( \mathcal{A}_T \) in a set \( \mathcal{D} (\neq \emptyset) \) is a function from \( \mathcal{D} \) into a \([0,1]\).

Definition 2.6. [12] A FS \( \mathcal{A}_T \) in a d-algebra \( \mathcal{D} \) is called a fuzzy d-subalgebra of \( \mathcal{D} \) if it satisfies \( \mathcal{A}_T(d_1 * d_2) \geq \min\{\mathcal{A}_T(d_1), \mathcal{A}_T(d_2)\} \), for all \( d_1, d_2 \in \mathcal{D} \).

Definition 2.7. [13] A FS \( \mathcal{A}_T \) in a d-algebra \( \mathcal{D} \) is called a fuzzy d-ideal of \( \mathcal{D} \) if it satisfies \( \mathcal{A}_T(d_1) \geq \min\{\mathcal{A}_T(d_1 * d_2), \mathcal{A}_T(d_2)\} \) and \( \mathcal{A}_T(d_1 * d_2) \geq \mathcal{A}_T(d_1) \) for all \( d_1, d_2 \in \mathcal{D} \).

Definition 2.8. [12] A FS \( \mathcal{A}_T \) in a d-algebra \( \mathcal{D} \) is called a fuzzy BCK-ideal of \( \mathcal{D} \) if it satisfies \( \mathcal{A}_T(0) \geq \mathcal{A}_T(d_1) \) and \( \mathcal{A}_T(d_1) \geq \min\{\mathcal{A}_T(d_1 * d_2), \mathcal{A}_T(d_2)\} \), for all \( d_1, d_2 \in \mathcal{D} \).

Definition 2.9. A mapping \( f: \mathcal{D} \rightarrow \mathcal{Y} \) of d-algebras is called a homomorphism if \( f(d_1 * d_2) = f(d_1) * f(d_2) \), for all \( d_1, d_2 \in \mathcal{D} \).

Note that if \( f: \mathcal{D} \rightarrow \mathcal{Y} \) is a homomorphism of d-algebras, then \( f(0) = 0 \).
Definition 2.10. [5] A neutrosophic set over a universal set \( \mathcal{D} \) is defined as follows
\[
\mathcal{A} = \{(d_1; \mathcal{A}_T(d_1), \mathcal{A}_I(d_1), \mathcal{A}_F(d_1)) \mid d_1 \in \mathcal{D}\}
\]
where \( \mathcal{A}_T(d_1): \mathcal{D} \to [-1,0], \mathcal{A}_I(d_1): \mathcal{D} \to [-1,0], \mathcal{A}_F(d_1): \mathcal{D} \to [-1,0] \) are the truth, indeterminacy, and false degree value of \( d_1 \) and \(-1 \leq \mathcal{A}_T(d_1) + \mathcal{A}_I(d_1) + \mathcal{A}_F(d_1) \leq 1^+ \).

Consider \( \mathcal{F}(\mathcal{D}, [-1,0]) \) to be the set of all functions mapping elements from a set \( \mathcal{D} \) to the interval \([-1,0]\). We define an element of \( \mathcal{F}(\mathcal{D}, [-1,0]) \) as a negative-valued function from \( \mathcal{D} \) to \([-1,0]\), and is abbreviated as an N-function [14].

Definition 2.11. [15] A NSN-S over \( \mathcal{D} \) is defined to be the structure
\[
\mathcal{A} = \{(d_1; \mathcal{A}_T(d_1), \mathcal{A}_I(d_1), \mathcal{A}_F(d_1)) \mid d_1 \in \mathcal{D}\}
\]
where \( \mathcal{A}_T(d_1): \mathcal{D} \to [-1,0], \mathcal{A}_I(d_1): \mathcal{D} \to [-1,0], \mathcal{A}_F(d_1): \mathcal{D} \to [-1,0] \) are N-functions on \( \mathcal{D} \) which are called the negative truth membership function, the negative indeterminacy membership function, and the negative falsity membership function, respectively, on \( \mathcal{D} \) and \(-3 \leq \mathcal{A}_T(d_1) + \mathcal{A}_I(d_1) + \mathcal{A}_F(d_1) \leq 0 \).

An interval number is defined as a closed subinterval \( \bar{\xi} = [\alpha^L, \alpha^U] \) within the interval \([-1,0]\), where \(-1 \leq \alpha^L \leq \alpha^U \leq 0 \). Let \( I \) represent the set of all such interval numbers. We define the refined minimum (denoted by \( rmin \)) and refined maximum (denoted by \( rmax \)) for any two members in \( I \). Further, we define the symbols “\( \ll \)”, “\( \gg \)”, and “\( = \)” for two members in \( I \). For two interval numbers \( \bar{\xi}_1 = [\alpha_1^L, \alpha_1^U] \) and \( \bar{\xi}_2 = [\alpha_2^L, \alpha_2^U] \):

\[
rmin[\bar{\xi}_1, \bar{\xi}_2] = \min\{\alpha_1^L, \alpha_2^L\}, \min\{\alpha_1^U, \alpha_2^U\}
\]

\[
rmax[\bar{\xi}_1, \bar{\xi}_2] = \max\{\alpha_1^L, \alpha_2^L\}, \max\{\alpha_1^U, \alpha_2^U\}
\]

\( \bar{\xi}_1 \ll \bar{\xi}_2 \Leftrightarrow \alpha_1^L \leq \alpha_2^L, \alpha_1^U \leq \alpha_2^U \), and likewise, \( \bar{\xi}_1 \gg \bar{\xi}_2 \) and \( \bar{\xi}_1 = \bar{\xi}_2 \).

Definition 2.12. [16] Consider \( \mathcal{D} \) is a set of objects (or points), where each object in \( \mathcal{D} \) represented by \( d_1 \). An IvNSN-S over \( \mathcal{D} \) is characterized as the set
\[
\mathcal{A} = \left\{(d_1; \mathcal{A}_T(d_1), \mathcal{A}_I(d_1), \mathcal{A}_F(d_1)) \mid d_1 \in \mathcal{D}\right\}
\]

Where \( \mathcal{A}_T(d_1): \mathcal{D} \to I[-1,0], \mathcal{A}_I(d_1): \mathcal{D} \to I[-1,0], \mathcal{A}_F(d_1): \mathcal{D} \to I[-1,0] \) are functions on \( \mathcal{D} \) which are called the negative interval-valued degree of membership, the negative interval-valued degree of indeterminacy, and the negative interval-valued degree of non-membership, respectively, on \( \mathcal{D} \).

Definition 2.13. [16] Let \( \mathcal{D} \) be d-algebra. An IvNSN-S \( \bar{\mathcal{A}} = (\bar{\mathcal{A}}_T, \bar{\mathcal{A}}_I, \bar{\mathcal{A}}_F) \) is called an Interval-valued neutrosophic N-d-subalgebra if it satisfies \( \bar{\mathcal{A}}_T(d_1 * d_2) \leq rmax\{\bar{\mathcal{A}}_T(d_1), \bar{\mathcal{A}}_T(d_2)\}; \bar{\mathcal{A}}_I(d_1 * d_2) \geq rmin\{\bar{\mathcal{A}}_I(d_1), \bar{\mathcal{A}}_I(d_2)\}; \) and \( \bar{\mathcal{A}}_F(d_1 * d_2) \leq rmax\{\bar{\mathcal{A}}_F(d_1), \bar{\mathcal{A}}_F(d_2)\} \) for all \( d_1, d_2 \in \mathcal{D} \).

Definition 2.14. [16] Let \( \bar{\mathcal{A}} = (\bar{\mathcal{A}}_T, \bar{\mathcal{A}}_I, \bar{\mathcal{A}}_F) \) be an interval-valued neutrosophic N-set in \( \mathcal{D} \) and let \( \bar{r} = [r^L, r^U], \bar{s} = [s^L, s^U], \bar{t} = [t^L, t^U] \in I[-1,0] \). Then, we define the following level sets for all \( d_1 \in \mathcal{D} \)
\[
L_1(\bar{\mathcal{A}}_T, \bar{r}) = \{d_1 \in \mathcal{D} : \bar{\mathcal{A}}_T(d_1) \leq [r^L, r^U]\}; U(\bar{\mathcal{A}}_I, \bar{s}) = \{d_1 \in \mathcal{D} : \bar{\mathcal{A}}_I(d_1) \geq [s^L, s^U]\};
\]

\[
L_2(\bar{\mathcal{A}}_F, \bar{t}) = \{d_1 \in \mathcal{D} : \bar{\mathcal{A}}_F(d_1) \leq [t^L, t^U]\}.
\]

### 3 Interval-Valued Neutrosophic N-D-Ideal

Throughout this section, \( \mathcal{D} \) represents a d-algebra unless otherwise specified.
Definition 3.1. An IvNSN-d-I of $\mathcal{D}$ is the IvNSN-S $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ in $\mathcal{D}$ with the following inequalities

(IvNSN-d-I 1) $\tilde{A}_T(d_1) \leq \text{rmax}\{\tilde{A}_T(d_1 \ast d_2), \tilde{A}_T(d_2)\}$

(IvNSN-d-I 2) $\tilde{A}_I(d_1) \geq \text{rmin}\{\tilde{A}_I(d_1 \ast d_2), \tilde{A}_I(d_2)\}$

(IvNSN-d-I 3) $\tilde{A}_F(d_1) \leq \text{rmax}\{\tilde{A}_F(d_1 \ast d_2), \tilde{A}_F(d_2)\}$

(IvNSN-d-I 4) $\tilde{A}_T(d_1 \ast d_2) \leq \tilde{A}_T(d_1)$

(IvNSN-d-I 5) $\tilde{A}_I(d_1 \ast d_2) \geq \tilde{A}_I(d_1)$

(IvNSN-d-I 6) $\tilde{A}_F(d_1 \ast d_2) \leq \tilde{A}_F(d_1)$, for all $d_1, d_2 \in \mathcal{D}$.

Definition 3.2. An IvNSN-S $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ in $\mathcal{D}$ is called an IvNSN-BCK-I if it satisfies the following inequalities

(IvNSN-BCK-I 1) $\tilde{A}_T(0) \leq \tilde{A}_T(d_1)$, $\tilde{A}_I(0) \geq \tilde{A}_I(d_1)$, and $\tilde{A}_F(0) \leq \tilde{A}_F(d_1)$

(IvNSN-BCK-I 2) $\tilde{A}_T(d_1) \leq \text{rmax}\{\tilde{A}_T(d_1 \ast d_2), \tilde{A}_T(d_2)\}$

(IvNSN-BCK-I 3) $\tilde{A}_I(d_1) \geq \text{rmin}\{\tilde{A}_I(d_1 \ast d_2), \tilde{A}_I(d_2)\}$

(IvNSN-BCK-I 4) $\tilde{A}_F(d_1) \leq \text{rmax}\{\tilde{A}_F(d_1 \ast d_2), \tilde{A}_F(d_2)\}$ for all $d_1, d_2 \in \mathcal{D}$.

Example 3.3. Consider a set $\mathcal{D} = \{0, 1, 2\}$ in which the operation “*” is defined as shown in the following Cayley table (see Table 1).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(\mathcal{D}, \ast, 0)$ is a $d$-algebra. Let $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ be an IvNSN-S in $\mathcal{D}$ as defined in the following table (see Table 2).

<table>
<thead>
<tr>
<th>$\mathcal{D}$</th>
<th>$\tilde{A}_T(d_1)$</th>
<th>$\tilde{A}_I(d_1)$</th>
<th>$\tilde{A}_F(d_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$[-0.95, -0.31]$</td>
<td>$[-0.55, -0.27]$</td>
<td>$[-0.85, -0.51]$</td>
</tr>
<tr>
<td>1</td>
<td>$[-0.67, -0.15]$</td>
<td>$[-0.87, -0.42]$</td>
<td>$[-0.79, -0.39]$</td>
</tr>
<tr>
<td>2</td>
<td>$[-0.67, -0.15]$</td>
<td>$[-0.87, -0.42]$</td>
<td>$[-0.79, -0.39]$</td>
</tr>
</tbody>
</table>

Then $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ is an IvNSN-d-I of $d$-algebra $\mathcal{D}$.

Proposition 3.4. If $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ is an interval-valued neutrosophic N-d-ideal of $d$-algebra $\mathcal{D}$, then $\tilde{A}_T(0) \leq \tilde{A}_T(d_1)$, $\tilde{A}_I(0) \geq \tilde{A}_I(d_1)$ and $\tilde{A}_F(0) \leq \tilde{A}_F(d_1)$ for all $d_1 \in \mathcal{D}$.

Proof: Suppose that $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ is an IvNSN-d-I of $\mathcal{D}$, and let $d_1 \in \mathcal{D}$. Now utilizing (IvNSN-d-I 4), (IvNSN-d-I 5), (IvNSN-d-I 6), and $d_2 \ast d_1 = 0$, we get,

$$\begin{align*}
\tilde{A}_T(d_1 \ast d_1) &\leq \tilde{A}_T(d_1) \\
\tilde{A}_I(d_1 \ast d_1) &\geq \tilde{A}_I(d_1) \\
\tilde{A}_F(d_1 \ast d_1) &\leq \tilde{A}_F(d_1)
\end{align*}$$

for all $d_1 \in \mathcal{D}.$

Lemma 3.5. Let an IvNSN-S $\tilde{\mathcal{A}} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F)$ in $\mathcal{D}$ be an IvNSN-d-I of $\mathcal{D}$. If $d_1 \ast d_2 \leq d_3$, then
\[
\begin{align*}
\mathcal{A}_T(d_1) &\leq \text{rmax}\{\mathcal{A}_T(d_2), \mathcal{A}_T(d_3)\} \\
\mathcal{A}_f(d_1) &\geq \text{rmin}\{\mathcal{A}_f(d_2), \mathcal{A}_f(d_3)\} \\
\mathcal{A}_f(d_1) &\leq \text{rmax}\{\mathcal{A}_f(d_2), \mathcal{A}_f(d_3)\}
\end{align*}
\]
for all \(d_1, d_2, d_3 \in \mathcal{D}\).

**Proof:** Suppose that \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_f, \mathcal{A}_f)\) is an IvNSN-d-I of \(\mathcal{D}\). Let \(d_1, d_2,\) and \(d_3\) be any three members of \(\mathcal{D}\) such that \(d_1 \ast d_2 \leq d_3\). Then \((d_1 \ast d_2) \ast d_3 = 0\).

\[
\mathcal{A}_T(d_1) \leq \text{rmax}\{\mathcal{A}_T(d_1 \ast d_2), \mathcal{A}_T(d_2)\}
\leq \text{rmax}\{\text{rmax}\{\mathcal{A}_T((d_1 \ast d_2) \ast d_3), \mathcal{A}_T(d_3)\}, \mathcal{A}_T(d_2)\}
= \text{rmax}\{\text{rmax}\{\mathcal{A}_T(0), \mathcal{A}_T(d_3)\}, \mathcal{A}_T(d_2)\}
= \text{rmax}\{\mathcal{A}_T(d_2), \mathcal{A}_T(d_3)\},\]

\[
\mathcal{A}_f(d_1) \geq \text{rmin}\{\mathcal{A}_f(d_1 \ast d_2), \mathcal{A}_f(d_2)\}
\geq \text{rmin}\{\text{rmin}\{\mathcal{A}_f((d_1 \ast d_2) \ast d_3), \mathcal{A}_f(d_3)\}, \mathcal{A}_f(d_2)\}
= \text{rmin}\{\text{rmin}\{\mathcal{A}_f(0), \mathcal{A}_f(d_3)\}, \mathcal{A}_f(d_2)\}
= \text{rmin}\{\mathcal{A}_f(d_2), \mathcal{A}_f(d_3)\},\]

\[
\mathcal{A}_f(d_1) \leq \text{rmax}\{\mathcal{A}_f(d_1 \ast d_2), \mathcal{A}_f(d_2)\}
\leq \text{rmax}\{\text{rmax}\{\mathcal{A}_f((d_1 \ast d_2) \ast d_3), \mathcal{A}_f(d_3)\}, \mathcal{A}_f(d_2)\}
= \text{rmax}\{\text{rmax}\{\mathcal{A}_f(0), \mathcal{A}_f(d_3)\}, \mathcal{A}_f(d_2)\}
= \text{rmax}\{\mathcal{A}_f(d_2), \mathcal{A}_f(d_3)\}.
\]

This completes the proof.

**Lemma 3.6.** Let \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_f, \mathcal{A}_f)\) be an IvNSN-d-I of \(\mathcal{D}\). If \(d_1 \leq d_2\) in \(\mathcal{D}\), then \(\mathcal{A}_T(d_1) \leq \mathcal{A}_T(d_2)\), \(\mathcal{A}_f(d_1) \geq \mathcal{A}_f(d_2)\), and \(\mathcal{A}_f(d_1) \leq \mathcal{A}_f(d_2)\) for all \(d_1, d_2 \in \mathcal{D}\).

**Proof:** Suppose that \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_f, \mathcal{A}_f)\) is an IvNSN-d-I of \(\mathcal{D}\). Let \(d_1\) and \(d_2\) be any two members of \(\mathcal{D}\) such that \(d_1 \leq d_2\). Then \(d_1 \ast d_2 = 0\).

\[
\mathcal{A}_T(d_1) \leq \text{rmax}\{\mathcal{A}_T(d_1 \ast d_2), \mathcal{A}_T(d_2)\} = \text{rmax}\{\mathcal{A}_T(0), \mathcal{A}_T(d_2)\} = \mathcal{A}_T(d_2)
\Rightarrow \mathcal{A}_T(d_1) \leq \mathcal{A}_T(d_2),
\]

\[
\mathcal{A}_f(d_1) \geq \text{rmin}\{\mathcal{A}_f(d_1 \ast d_2), \mathcal{A}_f(d_2)\} = \text{rmin}\{\mathcal{A}_f(0), \mathcal{A}_f(d_2)\} = \mathcal{A}_f(d_2)
\Rightarrow \mathcal{A}_f(d_1) \geq \mathcal{A}_f(d_2),
\]

\[
\mathcal{A}_f(d_1) \leq \text{rmax}\{\mathcal{A}_f(d_1 \ast d_2), \mathcal{A}_f(d_2)\} = \text{rmax}\{\mathcal{A}_f(0), \mathcal{A}_f(d_2)\} = \mathcal{A}_f(d_2)
\Rightarrow \mathcal{A}_f(d_1) \leq \mathcal{A}_f(d_2).
\]

This completes the proof.

**Theorem 3.7.** If \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_f, \mathcal{A}_f)\) is an IvNSN-d-I of \(\mathcal{D}\), then for any \(d_1, \delta_1, \delta_2, \ldots, \delta_n \in \mathcal{D}\), such that \(((d_1 \ast \delta_1) \ast \delta_2) \ldots \ast \delta_n = 0\) implies
\[
\begin{align*}
\tilde{A}_T(d_1) & \leq \text{rmax}\{\tilde{A}_T(b_1), \tilde{A}_T(b_2), \ldots, \tilde{A}_T(b_n)\} \\
\tilde{A}_i(d_1) & \geq \text{rmin}\{\tilde{A}_i(b_1), \tilde{A}_i(b_2), \ldots, \tilde{A}_i(b_n)\} \\
\tilde{A}_F(d_1) & \leq \text{rmax}\{\tilde{A}_F(b_1), \tilde{A}_F(b_2), \ldots, \tilde{A}_F(b_n)\}
\end{align*}
\]

**Proof:** By applying induction on \( n \) and Lemma 3.5 and Lemma 3.6.

**Theorem 3.8.** Every IvNSN-d-I of \( \mathfrak{D} \) is an interval-valued neutrosophic N-d-subalgebra.

**Proof:** Let \( \tilde{A} = (\tilde{A}_T, \tilde{A}_i, \tilde{A}_F) \) be an IvNSN-d-I of \( \mathfrak{D} \). So

\[
\begin{align*}
\tilde{A}_T(d_1 \ast d_2) & \leq \tilde{A}_T(d_1) \leq \text{rmax}\{\tilde{A}_T(d_1 \ast d_2), \tilde{A}_T(d_2)\} \leq \text{rmax}\{\tilde{A}_T(d_1), \tilde{A}_T(d_2)\}, \\
\tilde{A}_i(d_1 \ast d_2) & \geq \tilde{A}_i(d_1) \geq \text{rmin}\{\tilde{A}_i(d_1 \ast d_2), \tilde{A}_i(d_2)\} \geq \text{rmin}\{\tilde{A}_i(d_1), \tilde{A}_i(d_2)\}, \\
\tilde{A}_F(d_1 \ast d_2) & \leq \tilde{A}_F(d_1) \leq \text{rmax}\{\tilde{A}_F(d_1 \ast d_2), \tilde{A}_F(d_2)\} \leq \text{rmax}\{\tilde{A}_F(d_1), \tilde{A}_F(d_2)\}
\end{align*}
\]

for all \( d_1, d_2 \in \mathfrak{D} \).

Therefore, \( \tilde{A} = (\tilde{A}_T, \tilde{A}_i, \tilde{A}_F) \) is an interval-valued neutrosophic N-d-subalgebra.

The converse of this theorem is not true in general which is shown in the next example.

**Example 3.9.** Consider a set \( \mathfrak{D} = \{0, 1, 2, 3\} \) in which the operation \( \ast \) is defined as shown in the following Cayley table (see Table 3).

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (\mathfrak{D}, \ast, 0) \) is a d-algebra. Let \( \tilde{A} = (\tilde{A}_T, \tilde{A}_i, \tilde{A}_F) \) be an IvNSN-S in \( \mathfrak{D} \) as defined in the following table (see Table 4).

<table>
<thead>
<tr>
<th>( \mathfrak{D} )</th>
<th>( \tilde{A}_T(d_1) )</th>
<th>( \tilde{A}_i(d_1) )</th>
<th>( \tilde{A}_F(d_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[-0.92, -0.53]</td>
<td>[-0.73, -0.35]</td>
<td>[-0.83, -0.63]</td>
</tr>
<tr>
<td>1</td>
<td>[-0.92, -0.53]</td>
<td>[-0.73, -0.35]</td>
<td>[-0.83, -0.63]</td>
</tr>
<tr>
<td>2</td>
<td>[-0.73, -0.42]</td>
<td>[-0.89, -0.74]</td>
<td>[-0.52, -0.31]</td>
</tr>
<tr>
<td>3</td>
<td>[-0.92, -0.53]</td>
<td>[-0.73, -0.35]</td>
<td>[-0.83, -0.63]</td>
</tr>
</tbody>
</table>

It is clear that \( \tilde{A} = (\tilde{A}_T, \tilde{A}_i, \tilde{A}_F) \) is an interval-valued neutrosophic N-d-subalgebra, but \( \tilde{A}_T(2) = [-0.73, -0.42] = \text{rmax}\{\tilde{A}_T(2 \ast 1), \tilde{A}_T(1)\} \geq \tilde{A}_T(1) = [-0.92, -0.53] \), \( \tilde{A}_i(2) = [-0.89, -0.74] = \text{rmin}\{\tilde{A}_i(2 \ast 1), \tilde{A}_i(1)\} \leq \tilde{A}_i(1) = [-0.73, -0.35] \), and \( \tilde{A}_F(2) = [-0.52, -0.31] = \text{rmax}\{\tilde{A}_F(2 \ast 1), \tilde{A}_F(1)\} \geq \tilde{A}_F(1) = [-0.83, -0.63] \).

So \( \tilde{A} = (\tilde{A}_T, \tilde{A}_i, \tilde{A}_F) \) is not an IvNSN-d-I.

**Proposition 3.10.** Every IvNSN-d-I in \( \mathfrak{D} \) is an IvNSN-BCK-I of \( \mathfrak{D} \).

**Proof:** The proof is straightforward.

The converse of this theorem is not true in general which is shown in the next example.
Example 3.11. Consider a set $\mathcal{D} = \{0,1,2,3,4\}$ in which the binary operation “∗” is defined as shown in the following Cayley table (see Table 5).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(\mathcal{D}, \ast, 0)$ is a d-algebra. Let $\mathcal{A} = (\mathcal{A_T}, \mathcal{A_I}, \mathcal{A_F})$ be an IvNSN-S in $\mathcal{D}$ as defined in the following table (see Table 6).

<table>
<thead>
<tr>
<th>$\mathcal{D}$</th>
<th>$\mathcal{A_T}(d_1)$</th>
<th>$\mathcal{A_I}(d_1)$</th>
<th>$\mathcal{A_F}(d_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$[-0.92, -0.54]$</td>
<td>$[-0.58, -0.23]$</td>
<td>$[-0.85, -0.57]$</td>
</tr>
<tr>
<td>1</td>
<td>$[-0.92, -0.54]$</td>
<td>$[-0.58, -0.23]$</td>
<td>$[-0.85, -0.57]$</td>
</tr>
<tr>
<td>2</td>
<td>$[-0.92, -0.54]$</td>
<td>$[-0.58, -0.23]$</td>
<td>$[-0.85, -0.57]$</td>
</tr>
<tr>
<td>3</td>
<td>$[-0.53, -0.21]$</td>
<td>$[-0.97, -0.77]$</td>
<td>$[-0.64, -0.32]$</td>
</tr>
<tr>
<td>4</td>
<td>$[-0.92, -0.54]$</td>
<td>$[-0.58, -0.23]$</td>
<td>$[-0.85, -0.57]$</td>
</tr>
</tbody>
</table>

Then $\mathcal{A} = (\mathcal{A_T}, \mathcal{A_I}, \mathcal{A_F})$ in $\mathcal{D}$ is an IvNSN-BCK-I of $\mathcal{D}$ but it is not an interval-valued neutrosophic N-d-I, because

$\mathcal{A_T}(4 \ast 3) = \mathcal{A_T}(3) = [-0.53, -0.21] \geq [-0.92, -0.54] = \mathcal{A_T}(4)$

$\mathcal{A_I}(4 \ast 3) = \mathcal{A_I}(3) = [-0.97, -0.77] \leq [-0.58, -0.23] = \mathcal{A_I}(4)$

$\mathcal{A_F}(4 \ast 3) = \mathcal{A_F}(3) = [-0.64, -0.32] \geq [-0.85, -0.57] = \mathcal{A_F}(4)$.

Theorem 3.12. If $\{\mathcal{A_i}, i \in \Lambda\}$ is an arbitrary family of IvNSN-d-I of d-algebra, then $\bigcap_{i \in \Lambda} \mathcal{A_i}$ is an IvNSN-d-I of d-algebra, where $\bigcap_{i \in \Lambda} \mathcal{A_i} = \left(\max(\mathcal{A_T}), \min(\mathcal{A_I}), \max(\mathcal{A_F})\right)$.

Proof: Suppose that $\{\mathcal{A_i}, i \in \Lambda\}$ be the collection of an arbitrary family of IvNSN-d-I of d-algebra and

$\bigcap_{i \in \Lambda} \mathcal{A_i} = \left(\max(\mathcal{A_T}), \min(\mathcal{A_I}), \max(\mathcal{A_F})\right)$.

For all $i \in \Lambda$

$\max(\mathcal{A_T})(d_1) \leq \max\{\max(\mathcal{A_T}(d_1 \ast d_2), \mathcal{A_T}(d_2))\}$

$\leq \max\{\max(\mathcal{A_T})(d_1 \ast d_2), \max(\mathcal{A_T})(d_2)\}$,

$\min(\mathcal{A_I})(d_1) \geq \min\{\min(\mathcal{A_I}(d_1 \ast d_2), \mathcal{A_I}(d_2))\}$

$\geq \min\{\min(\mathcal{A_I})(d_1 \ast d_2), \min(\mathcal{A_I})(d_2)\}$,

$\max(\mathcal{A_F})(d_1) \leq \max\{\max(\mathcal{A_F}(d_1 \ast d_2), \mathcal{A_F}(d_2))\}$

$\leq \max\{\max(\mathcal{A_F})(d_1 \ast d_2), \max(\mathcal{A_F})(d_2)\}$.

Since $\mathcal{A_T}(d_1 \ast d_2) \leq \mathcal{A_T}(d_1)$, $\mathcal{A_I}(d_1 \ast d_2) \geq \mathcal{A_I}(d_1)$ and $\mathcal{A_F}(d_1 \ast d_2) \leq \mathcal{A_F}(d_1)$ for all $i \in \Lambda$, we have
\[
\begin{align*}
\text{Proof:} & \text{ Suppose that } \mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_F) \text{ be an IvNSN-d-I of d-algebra } \mathcal{D}. \text{ Then for any } d_1, d_2 \in \mathcal{D} \text{ we have} \\
\Rightarrow & -\mathcal{A}_i(d_1) \leq -r \text{min}\{\mathcal{A}_i(d_1 * d_2), \mathcal{A}_i(d_2)\} \\
& = \text{rmax}\{-\mathcal{A}_i(d_1 * d_2), -\mathcal{A}_i(d_2)\} \\
\Rightarrow & -\mathcal{A}_i(d_1) + [-1, -1] \leq \text{rmax}\{-\mathcal{A}_i(d_1 * d_2) + [-1, -1], -\mathcal{A}_i(d_2) + [-1, -1]\} \\
\Rightarrow & \mathcal{A}_i^c(d_1) \leq \text{rmax}\{\mathcal{A}_i^c(d_1 * d_2), \mathcal{A}_i^c(d_2)\}. \text{ Also} \\
\Rightarrow & \mathcal{A}_i(d_1 * d_2) \geq \mathcal{A}_i(d_1) \\
\Rightarrow & -\mathcal{A}_i(d_1 * d_2) \leq -\mathcal{A}_i(d_1) \\
\Rightarrow & -\mathcal{A}_i(d_1 * d_2) + [-1, -1] \leq -\mathcal{A}_i(d_1) + [-1, -1] \\
\Rightarrow & \mathcal{A}_i^c(d_1 * d_2) \leq \mathcal{A}_i^c(d_1).
\end{align*}
\]

Therefore, the N-interval-valued fuzzy set \(\mathcal{A}_i^c\) is a N-interval-valued fuzzy d-ideal of \(\mathcal{D}\). Hence for an IvNSN-d-I \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_F)\), the corresponding N-interval-valued fuzzy sets \(\mathcal{A}_T, \mathcal{A}_I^c, \text{ and } \mathcal{A}_F\) are N-interval-valued fuzzy d-ideal of \(\mathcal{D}\).

Conversely, suppose that \(\mathcal{A}_T, \mathcal{A}_I^c\), and \(\mathcal{A}_F\) are N-interval-valued fuzzy d-ideal of \(\mathcal{D}\). For any \(d_1, d_2 \in \mathcal{D}\) we get

\[
\begin{align*}
\mathcal{A}_T(d_1) \leq \text{rmax}\{\mathcal{A}_T(d_1 * d_2), \mathcal{A}_T(d_2)\}; \mathcal{A}_T(d_1 * d_2) \leq \mathcal{A}_T(d_1) \\
\mathcal{A}_I^c(d_1) \leq \text{rmax}\{\mathcal{A}_I^c(d_1 * d_2), \mathcal{A}_I^c(d_2)\}; \mathcal{A}_I^c(d_1 * d_2) \leq \mathcal{A}_I^c(d_1) \\
\mathcal{A}_F(d_1) \leq \text{rmax}\{\mathcal{A}_F(d_1 * d_2), \mathcal{A}_F(d_2)\}; \mathcal{A}_F(d_1 * d_2) \leq \mathcal{A}_F(d_1).
\end{align*}
\]

\[
\begin{align*}
[-1, -1] - \mathcal{A}_i(d_1) & \leq \text{rmax}\{[-1, -1] - \mathcal{A}_i(d_1 * d_2), [-1, -1] - \mathcal{A}_i(d_2)\} \\
& = [-1, -1] - \text{rmin}\{\mathcal{A}_i(d_1 * d_2), \mathcal{A}_i(d_2)\} \\
-\mathcal{A}_i(d_1) & \leq -\text{rmin}\{\mathcal{A}_i(d_1 * d_2), \mathcal{A}_i(d_2)\}
\end{align*}
\]
\[
\mathcal{A}_i(d_1) \geq r_{\text{min}}\{\mathcal{A}_i(d_1 \ast d_2), \mathcal{A}_i(d_2)\}. \text{ Also,}
\]
\[
\mathcal{A}_i^c(d_1 \ast d_2) \leqslant \mathcal{A}_i^c(d_1)
\]
\[
\Rightarrow [-1, -1] - \mathcal{A}_i(d_1 \ast d_2) \leq [-1, -1] - \mathcal{A}_i(d_1)
\]
\[
\Rightarrow -\mathcal{A}_i(d_1 \ast d_2) \leq \mathcal{A}_i(d_1)
\]
\[
\Rightarrow \mathcal{A}_i(d_1 \ast d_2) \geq \mathcal{A}_i(d_1).
\]

Hence \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_P)\) is an IvNSN-d-I of \(d\)-algebra \(\mathcal{D}\).

**Theorem 3.14.** Let \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_P)\) be an IvNSN-S over \(d\)-algebra \(\mathcal{D}\). Then \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_P)\) is an IvNSN-d-I of \(\mathcal{D}\) if and only if the following IvNSN-S are an IvNSN-d-I of \(\mathcal{D}\).

\[
\mathcal{A}_1 = (\mathcal{A}_T, \mathcal{A}_T^c, \mathcal{A}_T)
\]
\[
\mathcal{A}_2 = (\mathcal{A}_P, \mathcal{A}_P^c, \mathcal{A}_P)
\]
\[
\mathcal{A}_3 = (\mathcal{A}_I^c, \mathcal{A}_I, \mathcal{A}_I^c)
\]

**Theorem 3.15.** An IvNSN-S \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_P)\) over \(d\)-algebra \(\mathcal{D}\) is an IvNSN-d-I of \(\mathcal{D}\) if and only if for all \(\hat{r}, \hat{s}, \hat{t} \in I[1, 0]\) the sets \(L_1(\mathcal{A}_T, \hat{r})\), \(U(\mathcal{A}_I, \hat{s})\), and \(L_2(\mathcal{A}_P, \hat{t})\) of \(\mathcal{A}\) are either empty or \(d\)-ideal of \(\mathcal{D}\).

**Proof:** Assume that \(\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_P)\) be an IvNSN-d-I of \(\mathcal{D}\). Let \(L_1(\mathcal{A}_T, \hat{r}), U(\mathcal{A}_I, \hat{s}), \text{ and } L_2(\mathcal{A}_P, \hat{t})\) are non-empty sets for any \(\hat{r}, \hat{s}, \hat{t} \in I[1, 0]\). Let \(d_1, d_2 \in D\) such that \(d_1 \ast d_2, d_2 \in L_1(\mathcal{A}_T, \hat{r})\), so \(\mathcal{A}_T(d_1 \ast d_2) \leq \hat{r}\) and \(\mathcal{A}_T(d_2) \leq \hat{r}\). Then, \(\mathcal{A}_T(d_1) \leq \text{rmax}\{\mathcal{A}_T(d_1 \ast d_2), \mathcal{A}_T(d_2)\} \leq \text{rmax}\{\hat{r}, \hat{r}\} \leq \hat{r} \Rightarrow d_1 \in L_1(\mathcal{A}_T, \hat{r})\). And let \(d_1 \in L_1(\mathcal{A}_T, \hat{r}), d_2 \in D\). Then \(\mathcal{A}_T(d_1) \leq \hat{r}\) and \(\mathcal{A}_T(d_1 \ast d_2) \leq \mathcal{A}_T(d_2) \leq \hat{r}\). So, \(d_1 \ast d_2 \in L_1(\mathcal{A}_T, \hat{r})\). Hence, \(L_1(\mathcal{A}_T, \hat{r})\) is a \(d\)-ideal of \(\mathcal{D}\). Also, take \(d_1, d_2 \in D\) such that \(d_1 \ast d_2, d_2 \in U(\mathcal{A}_I, \hat{s})\), so \(\mathcal{A}_I(d_1 \ast d_2) \geq \hat{s}\) and \(\mathcal{A}_I(d_2) \geq \hat{s}\). Then, \(\mathcal{A}_I(d_1) \geq \text{rmin}\{\mathcal{A}_I(d_1 \ast d_2), \mathcal{A}_I(d_2)\} \geq r_{\text{min}}[\hat{s}, \hat{s}] \geq \hat{s} \Rightarrow d_1 \in U(\mathcal{A}_I, \hat{s})\). And let \(d_1 \in U(\mathcal{A}_I, \hat{s}), d_2 \in D\). Then \(\mathcal{A}_I(d_1) \geq \hat{s}\) and \(\mathcal{A}_I(d_1 \ast d_2) \geq \mathcal{A}_I(d_2) \geq \hat{s}\). So, \(d_1 \ast d_2 \in U(\mathcal{A}_I, \hat{s})\). Hence, \(U(\mathcal{A}_I, \hat{s})\) is a \(d\)-ideal of \(\mathcal{D}\). In a similar process, we can show that \(L_2(\mathcal{A}_P, \hat{t})\) is a \(d\)-ideal of \(\mathcal{D}\).

Conversely, suppose that for any \(\hat{r}, \hat{s}, \hat{t} \in I[1, 0]\) the sets \(L_1(\mathcal{A}_T, \hat{r}), U(\mathcal{A}_I, \hat{s}), \text{ and } L_2(\mathcal{A}_P, \hat{t})\) of \(\mathcal{A}\) are \(d\)-ideals of \(\mathcal{D}\). Let us take \(\zeta_1, \zeta_2 \in L_1(\mathcal{A}_T, \hat{r})\) such that \(\mathcal{A}_T(\zeta_1) > \text{rmax}\{\mathcal{A}_T(\zeta_1 \ast \zeta_2), \mathcal{A}_T(\zeta_2)\}\). Suppose that \(\mathcal{A}_T(\zeta_1) = \hat{r}_1 = [r_1^c, r_1^u], \mathcal{A}_T(\zeta_1 \ast \zeta_2) = \hat{r}_2 = [r_2^c, r_2^u]\), and \(\mathcal{A}_T(\zeta_2) = \hat{r}_3 = [r_3^c, r_3^u]\). Then, \(r_1^c > \text{rmax}\{r_2^c, r_3^c\}, [r_2^u, r_3^u], [r_2^c, r_3^u]\) and so,
\[
r_1^u > \text{max}\{r_2^u, r_3^u\} \quad \text{and} \quad r_1^u > \text{max}\{r_2^u, r_3^u\}.
\]

Taking \(r_4 = [r_4^c, r_4^u] = \frac{1}{2}[\mathcal{A}_T(\zeta_1) + \text{rmax}\{\mathcal{A}_T(\zeta_1 \ast \zeta_2), \mathcal{A}_T(\zeta_2)\}]
\]
\[
= \frac{1}{2} [r_1^c, r_1^u] + \text{max}\{r_2^u, r_3^u\}]
\]
\[
= \left[\frac{1}{2} (r_1^c + \text{max}\{r_2^u, r_3^u\}), \frac{1}{2} (r_1^u + \text{max}\{r_2^u, r_3^u\})\right]
\]

It follows that
\[
r_1^c > r_1^c = \frac{1}{2} (r_1^c + \text{max}\{r_2^u, r_3^u\}) > \text{max}\{r_2^u, r_3^u\},
\]
\[
r_1^u > r_1^u = \frac{1}{2} (r_1^u + \text{max}\{r_2^u, r_3^u\}) > \text{max}\{r_2^u, r_3^u\}.
\]

Hence, \(\text{max}\{r_2^u, r_3^u\}, \text{ max}\{r_2^u, r_3^u\} < [r_4^c, r_4^u] < [r_4^c, r_4^u] = \mathcal{A}_T(\zeta_1)\).
Therefore, $\zeta_1 \in L_1(\tilde{A}_T, \tilde{r}_4)$. On the other hand

\[ \mathcal{A}_T(\zeta_1 \ast \zeta_2) = \tilde{r}_2 = [r^f_2, r^u_2] \preceq [\max\{r^f_2, r^f_3\}, \max\{r^u_2, r^u_3\}] < [r^f_4, r^u_4] = \tilde{r}_4, \]

\[ \mathcal{A}_T(\zeta_2) = \tilde{r}_3 = [r^f_3, r^u_3] \preceq [\max\{r^f_2, r^f_3\}, \max\{r^u_2, r^u_3\}] < [r^f_4, r^u_4] = \tilde{r}_4. \]

i.e., $\zeta_1 \ast \zeta_2, \zeta_2 \in L_1(\mathcal{A}_T, \tilde{r}_4)$. This is a contradiction and therefore

\[ \mathcal{A}_T(\zeta_1) \prec \max\{\mathcal{A}_T(\zeta_1 \ast \zeta_2), \mathcal{A}_T(\zeta_2)\} \] for all $\zeta_1, \zeta_2 \in \mathcal{D}.$

Suppose that for any $\zeta_1, \zeta_2 \in \mathcal{D}$, we have $\mathcal{A}_T(\zeta_1 \ast \zeta_2) > \mathcal{A}_T(\zeta_1)$.

Then taking $\tilde{r}_1 = [r^f_1, r^u_1] = \frac{1}{2}[\mathcal{A}_T(\zeta_1 \ast \zeta_2) + \mathcal{A}_T(\zeta_1)]$, we have $\mathcal{A}_T(\zeta_1) \prec [r^f_1, r^u_1] < \mathcal{A}_T(\zeta_1 \ast \zeta_2)$. Hence $\zeta_1 \in L_1(\mathcal{A}_T, \tilde{r}_1)$, $\zeta_2 \in \mathcal{D}$, but $\zeta_1 \ast \zeta_2 \in L_1(\mathcal{A}_T, \tilde{r}_1)$. This is a contradiction and therefore $\mathcal{A}_T(\zeta_1 \ast \zeta_2) \preceq \mathcal{A}_T(\zeta_1)$.

Suppose that $\tilde{A}_T(\zeta_1) < r\min\{\tilde{A}_T(\zeta_1 \ast \zeta_2), \tilde{A}_T(\zeta_2)\}$ for some $\zeta_1, \zeta_2 \in U(\tilde{A}_T, \tilde{s})$. Let us take $\tilde{A}_T(\zeta_1) = \tilde{s}_1 = [s^f_1, s^u_1]$, $\tilde{A}_T(\zeta_1 \ast \zeta_2) = \tilde{s}_2 = [s^f_2, s^u_2]$, and $\tilde{A}_T(\zeta_2) = \tilde{s}_3 = [s^f_3, s^u_3]$. Then,

\[ [s^f_1, s^u_1] < r\min\{[s^f_2, s^u_2], [s^f_3, s^u_3]\} = \min\{s^f_2, s^f_3\}, \min\{s^u_2, s^u_3\} \]

and so,

\[ s^f_1 < \min\{s^f_2, s^f_3\} \text{ and } s^u_1 < \min\{s^u_2, s^u_3\}. \]

Taking $\tilde{s}_4 = [s^f_4, s^u_4] = \frac{1}{2}[\tilde{A}_T(\zeta_1) + r\min\{\tilde{A}_T(\zeta_1 \ast \zeta_2), \tilde{A}_T(\zeta_2)\}]$

\[ = \frac{1}{2}\left[ [s^f_1, s^u_1] + \min\{s^f_2, s^f_3\}, \min\{s^u_2, s^u_3\} \right] \]

\[ = \left[ \frac{1}{2}(s^f_1 + \min\{s^f_2, s^f_3\}), \frac{1}{2}(s^u_1 + \min\{s^u_2, s^u_3\}) \right] \]

It follows that

\[ s^f_1 < \frac{1}{2}(s^f_1 + \min\{s^f_2, s^f_3\}) < \min\{s^f_2, s^f_3\}, \]

\[ s^u_1 < \frac{1}{2}(s^u_1 + \min\{s^u_2, s^u_3\}) < \min\{s^u_2, s^u_3\}. \]

Hence, $\min\{s^f_2, s^f_3\}, \min\{s^u_2, s^u_3\} > [s^f_4, s^u_4] > [s^f_1, s^u_1] = \tilde{A}_T(\zeta_1)$. Therefore, $\zeta_1 \in U(\tilde{A}_T, \tilde{s}_4)$. On the other hand

\[ \tilde{A}_T(\zeta_1 \ast \zeta_2) = \tilde{s}_2 = [s^f_2, s^u_2] \succ [\min\{s^f_2, s^f_3\}, \min\{s^u_2, s^u_3\}] > [s^f_4, s^u_4] = \tilde{s}_4, \]

\[ \tilde{A}_T(\zeta_2) = \tilde{s}_3 = [s^f_3, s^u_3] \succ [\min\{s^f_2, s^f_3\}, \min\{s^u_2, s^u_3\}] > [s^f_4, s^u_4] = \tilde{s}_4. \]

i.e., $\zeta_1 \ast \zeta_2, \zeta_2 \in U(\tilde{A}_T, \tilde{s}_4)$. This is a contradiction and therefore

\[ \tilde{A}_T(\zeta_1) \succ r\min\{\tilde{A}_T(\zeta_1 \ast \zeta_2), \tilde{A}_T(\zeta_2)\} \] for all $\zeta_1, \zeta_2 \in \mathcal{D}$.

Suppose that for any $\zeta_1, \zeta_2 \in \mathcal{D}$, we have $\tilde{A}_T(\zeta_1 \ast \zeta_2) < \tilde{A}_T(\zeta_1)$.

Then taking $\tilde{s}_1 = [s^f_1, s^u_1] = \frac{1}{2}[\tilde{A}_T(\zeta_1 \ast \zeta_2) + \tilde{A}_T(\zeta_1)]$, we have $\tilde{A}_T(\zeta_1) > [s^f_1, s^u_1] > \tilde{A}_T(\zeta_1 \ast \zeta_2)$. Hence $\zeta_1 \in U(\tilde{A}_T, \tilde{s}_1)$, $\zeta_2 \in \mathcal{D}$, but $\zeta_1 \ast \zeta_2 \notin U(\tilde{A}_T, \tilde{s}_1)$. This is a contradiction and therefore $\tilde{A}_T(\zeta_1 \ast \zeta_2) \succ \tilde{A}_T(\zeta_1)$.

Let us take $\zeta_1, \zeta_2 \in L_2(\tilde{A}_F, \tilde{t})$ such that $\tilde{A}_F(\zeta_1) = [t^f_1, t^u_1], \tilde{A}_F(\zeta_1 \ast \zeta_2) = \tilde{t}_2 = [t^f_2, t^u_2]$, and $\tilde{A}_F(\zeta_2) = \tilde{t}_3 = [t^f_3, t^u_3]$. Then,

\[ [t^f_1, t^u_1] > r\max\{[t^f_2, t^u_2], [t^f_3, t^u_3]\} = \max\{t^f_2, t^f_3\}, \max\{t^u_2, t^u_3\} \]

and so,
\[ t^*_f > \max\{t^*_2, t^*_3\} \text{ and } t^*_1 > \max\{t^*_2, t^*_3\}. \]

Taking \( \xi_4 = [t^*_4, t^*_1, t^*_4] = \frac{1}{2}[A_F(\zeta_1) + rmax\{A_F(\zeta_1 * \zeta_2), A_F(\zeta_2)\}] \)

\[ = \frac{1}{2} \left[ \left( t^*_4 + t^*_1 \right) + \max\{t^*_2, t^*_3, \max\{t^*_2, t^*_3\}\} \right] \]

\[ = \frac{1}{2} \left( t^*_4 + \max\{t^*_2, t^*_3\} \right), \]

It follows that

\[ t^*_4 > t^*_4 = \frac{1}{2} \left( t^*_4 + \max\{t^*_2, t^*_3\} \right) > \max\{t^*_2, t^*_3\}, \]

\[ t^*_1 > t^*_1 = \frac{1}{2} \left( t^*_1 + \max\{t^*_2, t^*_3\} \right) > \max\{t^*_2, t^*_3\}. \]

Hence, \( \max\{t^*_2, t^*_3\}, \max\{t^*_2, t^*_3\} \) \(< \left[ t^*_4, t^*_1 \right] < \left[ t^*_4, t^*_1 \right] = A_F(\zeta_1). \)

Therefore, \( \zeta_1 \in L_2(A_F, \xi_4) \). On the other hand

\[ A_F(\zeta_1 * \zeta_2) = \xi_2 = \left[ t^*_2, t^*_3 \right] \leq \left[ \max\{t^*_2, t^*_3\}, \max\{t^*_2, t^*_3\} \right] < \left[ t^*_4, t^*_1 \right] = \xi_4, \]

\[ A_F(\zeta_2) = \xi_3 = \left[ t^*_2, t^*_3 \right] \leq \left[ \max\{t^*_2, t^*_3\}, \max\{t^*_2, t^*_3\} \right] < \left[ t^*_4, t^*_1 \right] = \xi_4. \]

i.e., \( \zeta_1 * \zeta_2, \zeta_2 \in L_2(A_F, \xi_4) \). This is a contradiction and therefore \( A_F(\zeta_1) \leq rmax\{A_F(\zeta_1 * \zeta_2), A_F(\zeta_2)\} \) for all \( \zeta_1, \zeta_2 \in \mathbb{D}. \)

Suppose that for any \( \zeta_1, \zeta_2 \in \mathbb{D} \), we have \( A_F(\zeta_1 * \zeta_2) > A_F(\zeta_1). \)

Then taking \( \xi_1 = [t^*_1, t^*_4] = \frac{1}{2}[A_F(\zeta_1) + A_F(\zeta_1)] \), we have \( A_F(\zeta_1) < \left[ t^*_4, t^*_1 \right] < A_F(\zeta_1 * \zeta_2). \)

Hence \( \zeta_1 \in L_2(\mathbb{F}, \xi_1) \), \( \xi_2 \in \mathbb{D} \), but \( \zeta_1 * \zeta_2 \notin L_2(\mathbb{F}, \xi_1) \). This is a contradiction and therefore \( A_F(\zeta_1 * \zeta_2) \leq A_F(\zeta_1). \)

Therefore \( A = (A_F, A_I, A_P) \) is an IVNSN-d-I of \( \mathbb{D}. \)

**Theorem 3.16.** If an IVNSN-S \( \mathbb{A} = (A_T, A_P, A_P) \) in \( \mathbb{D} \) is an IVNSN-d-I of \( \mathbb{D} \), then the sets \( \mathbb{A} \) is an IVNSN-d-I of \( \mathbb{D}. \)

**Proof:** Assume that \( \mathbb{A} = (A_T, A_I, A_P) \) be an IVNSN-d-I of \( \mathbb{D}. \) Let \( d_1 * d_2, d_2 \in \mathbb{A}_{\mathbb{T}}. \) Therefore \( \mathbb{A}(d_1 * d_2) = A_T(0), \mathbb{A}(d_2) = A_T(0). \) Now by utilising definition 3.1, we obtain \( \mathbb{A}(d_1) \leq rmax\{\mathbb{A}(d_1 * d_2), \mathbb{A}(d_2)\} = rmax\{\mathbb{A}(0), \mathbb{A}(0)\} = A_T(0) \Rightarrow A_T(d_1) \leq A_T(0). \) From proposition 3.4, we have \( A_T(0) \leq A_T(d_1). \) Therefore \( A_T(0) \leq A_T(d_1) \leq A_T(0). \) Again let \( d_1 \in \mathbb{A}_{\mathbb{T}} \) and \( d_2 \in \mathbb{D}. \) Then, \( \mathbb{A}(d_1) = A_T(0) \) and so \( \mathbb{A}(d_1 * d_2) \leq A_T(d_1) = A_T(0) \Rightarrow A_T(d_1 * d_2) \leq A_T(0). \) By using proposition 3.4, we have \( A_T(0) \leq A_T(d_1 * d_2). \) Hence \( A_T(d_1 * d_2) = A_T(0). \) i.e. \( d_1 * d_2 \in \mathbb{A}_{\mathbb{T}}. \) Therefore, \( d_1 \in \mathbb{A}_{\mathbb{T}}, d_2 \in \mathbb{D} \Rightarrow d_1 * d_2 \in \mathbb{A}_{\mathbb{T}}. \) Hence the set \( \mathbb{A}_{\mathbb{T}} \) is a d-ideal of \( \mathbb{D}. \) Similarly, we can show that \( \mathbb{A}_{\mathbb{I}}, \mathbb{A}_{\mathbb{T}} \) are d-ideals of \( \mathbb{D}. \)

**Theorem 3.17.** If \( f \) is a d-homomorphism function from d-algebra \( \mathbb{D}_1 \) into a d-algebra \( \mathbb{D}_2 \) and \( \mathbb{A} = (A_T, A_I, A_P) \) be an IVNSN-d-I of \( \mathbb{D}_2. \) Then, \( f^{-1}(\mathbb{A}) = \left( f^{-1}(A_T), f^{-1}(A_I), f^{-1}(A_P) \right) \) is an IVNSN-d-I of \( \mathbb{D}_1, \) where \( f^{-1}(\mathbb{A})(d_1) = \mathbb{A}(f(d_1)), f^{-1}(\mathbb{A})(d_1) = \mathbb{A}(f(d_1)), \) and \( f^{-1}(\mathbb{A})(d_1) = \mathbb{A}(f(d_1)) \) for all \( d_1 \in \mathbb{D}_1. \)

**Proof:** For any \( d_1, d_2 \in \mathbb{D}_1, \) we have

\[ t^*_f > \max\{t^*_2, t^*_3\} \text{ and } t^*_1 > \max\{t^*_2, t^*_3\}. \]
\[ f^{-1}(\tilde{A}_T(d_1)) = \tilde{A}_T(f(d_1)) \leq r_{\text{max}}\{\tilde{A}_T(f(d_1) \ast f(d_2)), \tilde{A}_T(f(d_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_T(f(d_1 \ast d_2)), \tilde{A}_T(f(d_2))\} \]
\[ = r_{\text{max}}\{f^{-1}(\tilde{A}_T(d_1 \ast d_2)), f^{-1}(\tilde{A}_T(d_2))\} \]
\[ f^{-1}(\tilde{A}_T(d_1 \ast d_2)) = \tilde{A}_T(f(d_1 \ast d_2)) = \tilde{A}_T(f(d_1) \ast f(d_2)) \]
\[ \leq \tilde{A}_T(f(d_1)) = f^{-1}(\tilde{A}_T(d_1)) \]

\[ f^{-1}(\tilde{A}_I(d_1)) = \tilde{A}_I(f(d_1)) \geq r_{\text{min}}\{\tilde{A}_I(f(d_1) \ast f(d_2)), \tilde{A}_I(f(d_2))\} \]
\[ = r_{\text{min}}\{\tilde{A}_I(f(d_1 \ast d_2)), \tilde{A}_I(f(d_2))\} \]
\[ = r_{\text{min}}\{f^{-1}(\tilde{A}_I(d_1 \ast d_2)), f^{-1}(\tilde{A}_I(d_2))\} \]
\[ f^{-1}(\tilde{A}_I(d_1 \ast d_2)) = \tilde{A}_I(f(d_1 \ast d_2)) = \tilde{A}_I(f(d_1) \ast f(d_2)) \]
\[ \geq \tilde{A}_I(f(d_1)) = f^{-1}(\tilde{A}_I(d_1)) \]

\[ f^{-1}(\tilde{A}_F(d_1)) = \tilde{A}_F(f(d_1)) \leq r_{\text{max}}\{\tilde{A}_F(f(d_1) \ast f(d_2)), \tilde{A}_F(f(d_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_F(f(d_1 \ast d_2)), \tilde{A}_F(f(d_2))\} \]
\[ = r_{\text{max}}\{f^{-1}(\tilde{A}_F(d_1 \ast d_2)), f^{-1}(\tilde{A}_F(d_2))\} \]
\[ f^{-1}(\tilde{A}_F(d_1 \ast d_2)) = \tilde{A}_F(f(d_1 \ast d_2)) = \tilde{A}_F(f(d_1) \ast f(d_2)) \]
\[ \leq \tilde{A}_F(f(d_1)) = f^{-1}(\tilde{A}_F(d_1)) \]

Hence, \( f^{-1}(\tilde{A}) \) is an IvNSN-d-I over \( D_1 \).

**Theorem 3.18.** If \( f \) is a d-epimorphism function from d-algebra \( D_1 \) into a d-algebra \( D_2 \) and \( \tilde{A} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F) \) be an IvNSN-S of \( D_2 \). If \( f^{-1}(\tilde{A}) = \left(f^{-1}(\tilde{A}_T), f^{-1}(\tilde{A}_I), f^{-1}(\tilde{A}_F)\right) \) is an IvNSN-d-I of \( D_1 \), then \( \tilde{A} = (\tilde{A}_T, \tilde{A}_I, \tilde{A}_F) \) is an IvNSN-d-I of \( D_2 \).

**Proof:** Let \( d_1, d_2 \in D_2 \). Then their exits \( i_1, i_2 \in D_1 \) such that \( f(i_1) = d_1 \) and \( f(i_2) = d_2 \).

\[ \tilde{A}_T(d_1) = \tilde{A}_T(f(i_1)) = f^{-1}(\tilde{A}_T(i_1)) \leq r_{\text{max}}\{f^{-1}(\tilde{A}_T(i_1 \ast i_2)), f^{-1}(\tilde{A}_T(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_T(f(i_1 \ast i_2)), \tilde{A}_T(f(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_T(f(i_1) \ast f(i_2)), \tilde{A}_T(f(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_T(d_1 \ast d_2), \tilde{A}_T(d_2)\} \]
\[ \tilde{A}_T(d_1 \ast d_2) = \tilde{A}_T(f(i_1) \ast f(i_2)) = \tilde{A}_T(f(i_1 \ast i_2)) = f^{-1}(\tilde{A}_T(i_1 \ast i_2)) \]
\[ \leq f^{-1}(\tilde{A}_T(i_1)) = \tilde{A}_T(f(i_1)) = \tilde{A}_T(d_1) \]

\[ \tilde{A}_I(d_1) = \tilde{A}_I(f(i_1)) = f^{-1}(\tilde{A}_I(i_1)) \geq r_{\text{min}}\{f^{-1}(\tilde{A}_I(i_1 \ast i_2)), f^{-1}(\tilde{A}_I(i_2))\} \]
\[ = r_{\text{min}}\{\tilde{A}_I(f(i_1 \ast i_2)), \tilde{A}_I(f(i_2))\} \]

\[ \tilde{A}_I(d_1 \ast d_2) = \tilde{A}_I(f(i_1) \ast f(i_2)) = \tilde{A}_I(f(i_1 \ast i_2)) = f^{-1}(\tilde{A}_I(i_1 \ast i_2)) \]
\[ \leq f^{-1}(\tilde{A}_I(i_1)) = \tilde{A}_I(f(i_1)) = \tilde{A}_I(d_1) \]

\[ \tilde{A}_F(d_1) = \tilde{A}_F(f(i_1)) = f^{-1}(\tilde{A}_F(i_1)) \leq r_{\text{max}}\{f^{-1}(\tilde{A}_F(i_1 \ast i_2)), f^{-1}(\tilde{A}_F(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_F(f(i_1 \ast i_2)), \tilde{A}_F(f(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_F(f(i_1) \ast f(i_2)), \tilde{A}_F(f(i_2))\} \]
\[ = r_{\text{max}}\{\tilde{A}_F(d_1 \ast d_2), \tilde{A}_F(d_2)\} \]
\[ \tilde{A}_F(d_1 \ast d_2) = \tilde{A}_F(f(i_1) \ast f(i_2)) = \tilde{A}_F(f(i_1 \ast i_2)) = f^{-1}(\tilde{A}_F(i_1 \ast i_2)) \]
\[ \leq f^{-1}(\tilde{A}_F(i_1)) = \tilde{A}_F(f(i_1)) = \tilde{A}_F(d_1) \]
\[ r_{min}(\mathcal{A}_i(f(i_1) \cdot f(i_2)), \mathcal{A}_i(f(i_2))) = r_{max}(\mathcal{A}_i(d_1 \cdot d_2), \mathcal{A}_i(d_2)), \]

\[ \mathcal{A}_i(d_1 \cdot d_2) = \mathcal{A}_i(f(i_1) \cdot f(i_2)) = \mathcal{A}_i(f(i_1 \cdot i_2)) = f^{-1}(\mathcal{A}_i(i_1 \cdot i_2)) \]

\[ \neq f^{-1}(\mathcal{A}_i(i_1)) = \mathcal{A}_i(f(i_1)) = \mathcal{A}_i(d_1), \]

\[ \mathcal{A}_F(d_1) = \mathcal{A}_F(f(i_1)) = f^{-1}(\mathcal{A}_F(i_1)) \leq r_{max}\{r_{max}\{f^{-1}(\mathcal{A}_F(i_1 \cdot i_2)), f^{-1}(\mathcal{A}_F(i_2))\}\} \]

\[ = r_{max}\{\mathcal{A}_F(f(i_1 \cdot i_2)), \mathcal{A}_F(f(i_2))\} \]

\[ = r_{max}\{\mathcal{A}_F(f(i_1) \cdot f(i_2)), \mathcal{A}_F(f(i_2))\} \]

\[ = r_{max}\{\mathcal{A}_F(d_1 \cdot d_2), \mathcal{A}_F(d_2)\}, \]

\[ \mathcal{A}_F(d_1 \cdot d_2) = \mathcal{A}_F(f(i_1) \cdot f(i_2)) = \mathcal{A}_F(f(i_1 \cdot i_2)) = f^{-1}(\mathcal{A}_F(i_1 \cdot i_2)) \]

\[ \leq f^{-1}(\mathcal{A}_F(i_1)) = \mathcal{A}_F(f(i_1)) = \mathcal{A}_F(d_1). \]

Hence the proof is completed.

### 4 | Conclusion

In this research, we presented the innovative idea of IvNSN-d-I by applying IvNSN-S to the d-ideal of d-algebra. We have clearly shown that the pre-image and the image of an IvNSN-d-I under homeomorphism and epimorphism, respectively, remain IvNSN-d-Is. Furthermore, we investigated and characterized several important features of IvNSN-d-Is.

### 4.1 | Limitations and Future Research

In everyday life, some scenarios may involve negative features in expressing opinions about a particular problem. Negative features point out choices that are either prohibited or unattainable. Interval-valued neutrosophic N-structure offers the ability to make decisions based on negative features. The methodology used in this manuscript is also applicable to ideals of other algebraic structures such as semigroups, near-rings, b-algebra, BCH-algebra, etc.

### Acknowledgments

The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

### Author Contributions

Conceptualization, Methodology: Satyanarayana; writing-creating-reviewing and editing: Baji. All authors have read and agreed to the published version of the manuscript.

### Funding

This research received no external funding.
Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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