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Interval-Valued Neutrosophic N-d-ideal in d-algebra

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Abstract

In this article we present the concept of interval-valued neutrosophic N-d-ideal by applying interval-valued neutrosophic N-structure to d-ideal of algebraic structure d-algebra. We proved that an interval-valued neutrosophic N-d-ideal's pre-image and image under homeomorphism and epimorphism, respectively, are also interval-valued neutrosophic N-d-ideals. We provided some characteristics of interval-valued neutrosophic N-d-ideal.

Keywords: Neutrosophic N-Structure (NSN-S), Interval-Valued Neutrosophic N-structure (IvNSN-S), d-algebra (d-A), d-ideal (d-I), Interval-Valued Neutrosophic N-d-ideal (IvNSN-d-I).

1 | Introduction

In the field of abstract algebra, K. Iseki and Y. Imai [1, 2] proposed two extraordinary structures, called BCK-algebra and BCI-algebra. In particular, BCK-algebras are a special type of BCI-algebra. Building on these foundational concepts, J. Negger and H. S. Kim [3] proposed d-algebras, a broader approach that generalizes the scope of BCK-algebras. The theory of ideals in d-algebras is discussed by Y. B. Jun, J. Negger, and H. S. Kim [4].

In the field of set theory, Smarandache's neutrosophic set [5, 6] provides a prominent framework. This set encompasses classical sets, fuzzy sets, and different generalizations, like IvFSs, IFSs, and IvIFSs. The application of NSs expands to several fields like control theory, algebra, and topology. Further developing this concept, Wang et al. [7] gave the concept of interval-valued neutrosophic sets, which gives higher adjustability and precision compared to single-valued neutrosophic sets. Jun et al. [8] introduced a novel concept called negative-valued function and created N-structures. These N-structures were further extended by Khan et al. [9], who introduced the idea of neutrosophic N-structure and applied it to a semigroup. Jun et al. [10] applied the idea of neutrosophic N-structure to BCK/BCI-algebras.

In recent years there has been remarkable progress in neutrosophic set theory. In this manuscript, we provide basic definitions essential for our work in Section 2. These basic definitions are d-algebra, d-subalgebra, d-ideal, BCK-ideal, Fuzzy set, Fuzzy d-subalgebra, Fuzzy d-ideal, Fuzzy BCK-ideal, Neutrosophic set,

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Neutrosophic N-structure, Interval-valued Neutrosophic N-structure, Interval-valued Neutrosophic N-d-subalgebra, and level sets related to Interval-valued Neutrosophic N-structure. In Section 3, we

- Introduced IvNSN-d-I with an example.
- Introduced IvNSN-BCK-I.
- Proved every IvNSN-d-I is an IvNSN-d-SA (converse not true, which is illustrated through an example).
- Proved every IvNSN-d-I is an IvNSN-BCK-I (converse not true, which is illustrated through an example).
- The proved intersection of an arbitrary family of IvNSN-d-I is also an IvNSN-d-I.
- Proved IvNSN-S is an IvNSN-d-I if its level sets are d-ideal.
- Provided some characteristics of IvNSN-d-I.

Section 4 is the conclusion part, which briefly summarises the key concepts discussed in the manuscript.

2 | Preliminaries

Definition 2.1. [3] Let $\mathfrak{D}\ (\neq \emptyset)$ be a set with a constant '0' and a binary operation '$*$'. Then $\mathfrak{D}$ is called a d-algebra if it satisfies the following conditions for all $d_1, d_2 \in \mathfrak{D}$.

(d-A 1) $d_1 * d_1 = 0$

(d-A 2) $0 * d_1 = 0$

(d-A 3) $d_1 * d_2 = 0$ and $d_2 * d_1 = 0$ implies $d_1 = d_2$.

We will refer to $d_1 \leq d_2$ if and only if $d_1 * d_2 = 0$.

Definition 2.2. [4] Let $\mathfrak{D}$ be a d-algebra with binary operation '$*$' and $\mathcal{P} \subseteq \mathfrak{D}$. Then, $\mathcal{P}$ is said to be a d-subalgebra of $\mathfrak{D}$, if $d_1, d_2 \in \mathcal{P}$ implies $d_1 * d_2 \in \mathcal{P}$.

Definition 2.3. [4] Let $\mathfrak{D}$ be a d-algebra with binary operation '$*$' and a constant 0. Then, $\mathcal{P} \subseteq \mathfrak{D}$ is called a d-ideal of $\mathfrak{D}$ if it satisfies the following conditions

(d-I I) $d_1 * d_2 \in \mathcal{P}$ and $d_2 \in \mathcal{P}$ implies $d_1 \in \mathcal{P}$;

(d-I II) $d_1 \in \mathcal{P}$ and $d_2 \in \mathfrak{D}$ implies $d_1 * d_2 \in \mathcal{P}$.

Definition 2.4. [4] A subset $\mathcal{P}(\neq \emptyset)$ of d-algebra $\mathfrak{D}$ is called a BCK-ideal of $\mathfrak{D}$ if satisfies (d-I I) and $0 \in \mathcal{P}$.

Definition 2.5. [11] A FS $\mathcal{A}_T$ in a set $\mathfrak{D}\ (\neq \emptyset)$ is a function from $\mathfrak{D}$ into a $[0,1]$.

Definition 2.6. [12] A FS $\mathcal{A}_T$ in a d-algebra $\mathfrak{D}$ is called a fuzzy d-subalgebra of $\mathfrak{D}$ if it satisfies $\mathcal{A}_T(d_1 * d_2) \geq min\{\mathcal{A}_T(d_1), \mathcal{A}_T(d_2)\}$, for all $d_1, d_2 \in \mathfrak{D}$.

Definition 2.7. [13] A FS $\mathcal{A}_T$ in a d-algebra $\mathfrak{D}$ is called a fuzzy d-ideal of $\mathfrak{D}$ if it satisfies $\mathcal{A}_T(d_1) \geq min\{\mathcal{A}_T(d_1 * d_2), \mathcal{A}_T(d_2)\}$ and $\mathcal{A}_T(d_1 * d_2) \geq \mathcal{A}_T(d_1)$ for all $d_1, d_2 \in \mathfrak{D}$.

Definition 2.8. [12] A FS $\check{\mathcal{A}}_T$ in a d-algebra $\mathfrak{D}$ is called a fuzzy BCK-ideal of $\mathfrak{D}$ if it satisfies $\mathcal{A}_T(0) \geq \mathcal{A}_T(d_1)$ and $\mathcal{A}_T(d_1) \geq min\{\mathcal{A}_T(d_1 * d_2), \mathcal{A}_T(d_2)\}$, for all $d_1, d_2 \in \mathfrak{D}$.

Definition 2.9. A mapping $f: \mathfrak{D} \to \mathcal{Y}$ of d-algebras is called a homomorphism if $f(d_1 * d_2) = f(d_1) * f(d_2)$, for all $d_1, d_2 \in \mathfrak{D}$.

Note that if $f: \mathfrak{D} \to \mathcal{Y}$ is a homomorphism of d-algebras, then $f(0) = 0$.

Definition 2.10. [5] A neutrosophic set over a universal set $\mathfrak{D}$ is defined as follows

$$\mathcal{A} = \{\langle d_1; \mathcal{A}_T(d_1), \mathcal{A}_I(d_1), \mathcal{A}_F(d_1)\rangle \mid d_1 \in \mathfrak{D}\}$$

where $\mathcal{A}_T(d_1): \mathfrak{D} \to]^-0, 1^+[$, $\mathcal{A}_I(d_1): \mathfrak{D} \to]^-0, 1^+[$, and $\mathcal{A}_F(d_1): \mathfrak{D} \to]^-0, 1^+[$ are the truth, indeterminacy, and false degree value of d_1 and $^-0 \leq \mathcal{A}_T(d_1) + \mathcal{A}_I(d_1) + \mathcal{A}_F(d_1) \leq 1^+$.

Consider $\mathcal{F}(\mathfrak{D}, [-1,0])$ to be the set of all functions mapping elements from a set $\mathfrak{D}$ to the interval $[-1,0]$. We define an element of $\mathcal{F}(\mathfrak{D}, [-1,0])$ as a negative-valued function from $\mathfrak{D}$ to $[-1,0]$, and is abbreviated as an N-function [14].

Definition 2.11. [15] A NSN-S over $\mathfrak{D}$ is defined to be the structure

$$\check{\mathcal{A}} = \{\langle d_1; \mathcal{A}_T(d_1), \mathcal{A}_I(d_1), \mathcal{A}_F(d_1)\rangle \mid d_1 \in \mathfrak{D}\}$$

where $\mathcal{A}_T(d_1): \mathfrak{D} \to [-1,0]$, $\mathcal{A}_I(d_1): \mathfrak{D} \to [-1,0]$, and $\mathcal{A}_F(d_1): \mathfrak{D} \to [-1,0]$ are N-functions on $\mathfrak{D}$ which are called the negative truth membership function, the negative indeterminacy membership function, and the negative falsity membership function, respectively, on $\mathfrak{D}$ and $-3 \leq \mathcal{A}_T(d_1) + \mathcal{A}_I(d_1) + \mathcal{A}_F(d_1) \leq 0$.

An interval number is defined as a closed subinterval $\breve{\propto} = [\propto^{\mathcal{L}}, \propto^{\mathcal{U}}]$ within the interval $[-1,0]$, where $-1 \leq \propto^{\mathcal{L}} \leq \propto^{\mathcal{U}} \leq 0$. Let I represent the set of all such interval numbers. We define the refined minimum (denoted by rmin) and refined maximum (denoted by rmax) for any two members in I. Further, we define the symbols "$\preccurlyeq$", "$\succcurlyeq$", and "=" for two members in I. For two interval numbers $\breve{\propto}_1 = [\propto_1{}^{\mathcal{L}}, \propto_1{}^{\mathcal{U}}]$ and $\breve{\propto}_2 = [\propto_2{}^{\mathcal{L}}, \propto_2{}^{\mathcal{U}}]$:

$$rmin\{\breve{\propto}_1, \breve{\propto}_2\} = [min\{\propto_1{}^{\mathcal{L}}, \propto_2{}^{\mathcal{L}}\}, min\{\propto_1{}^{\mathcal{U}}, \propto_2{}^{\mathcal{U}}\}]$$

$$rmax\{\breve{\propto}_1, \breve{\propto}_2\} = [max\{\propto_1{}^{\mathcal{L}}, \propto_2{}^{\mathcal{L}}\}, max\{\propto_1{}^{\mathcal{U}}, \propto_2{}^{\mathcal{U}}\}]$$

$\breve{\propto}_1 \preccurlyeq \breve{\propto}_2 \Leftrightarrow \propto_1{}^{\mathcal{L}} \leq \propto_2{}^{\mathcal{L}}, \propto_1{}^{\mathcal{U}} \leq \propto_2{}^{\mathcal{U}}$, and likewise, $\breve{\propto}_1 \succcurlyeq \breve{\propto}_2$ and $\breve{\propto}_1 = \breve{\propto}_2$.

Definition 2.12. [16] Consider $\mathfrak{D}$ is a set of objects (or points), where each object in $\mathfrak{D}$ represented by d_1. An IvNSN-S over $\mathfrak{D}$ is characterized as the set

$$\check{\mathcal{A}} = \left\{ \frac{\left(\check{\mathcal{A}}_T = [\check{\mathcal{A}}_T{}^{\mathcal{L}}(d_1), \check{\mathcal{A}}_T{}^{\mathcal{U}}(d_1)], \check{\mathcal{A}}_I = [\check{\mathcal{A}}_I{}^{\mathcal{L}}(d_1), \check{\mathcal{A}}_I{}^{\mathcal{U}}(d_1)], \check{\mathcal{A}}_F = [\check{\mathcal{A}}_F{}^{\mathcal{L}}(d_1), \check{\mathcal{A}}_F{}^{\mathcal{U}}(d_1)]\right)}{d_1} \mid d_1 \in \mathfrak{D} \right\}$$

Where $\check{\mathcal{A}}_T(d_1): \mathfrak{D} \to I[-1,0]$, $\check{\mathcal{A}}_I(d_1): \mathfrak{D} \to I[-1,0]$, and $\check{\mathcal{A}}_F(d_1): \mathfrak{D} \to I[-1,0]$ are functions on $\mathfrak{D}$ which are called the negative interval-valued degree of membership, the negative interval-valued degree of indeterminacy, and the negative interval-valued degree of non-membership, respectively, on $\mathfrak{D}$.

Definition 2.13. [16] Let $\mathfrak{D}$ be d-algebra. An IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is called an Interval-valued neutrosophic N-d-subalgebra if it satisfies $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1), \check{\mathcal{A}}_T(d_2)\}$; $\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1), \check{\mathcal{A}}_I(d_2)\}$; and $\check{\mathcal{A}}_F(d_1 * d_2) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1), \check{\mathcal{A}}_F(d_2)\}$ for all $d_1, d_2 \in \mathfrak{D}$.

Definition 2.14. [16] Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an interval-valued neutrosophic N-set in $\mathfrak{D}$ and let $\check{r} = [r^{\mathcal{L}}, r^{\mathcal{U}}]$, $\check{s} = [s^{\mathcal{L}}, s^{\mathcal{U}}]$, $\check{t} = [t^{\mathcal{L}}, t^{\mathcal{U}}] \in I[-1,0]$. Then, we define the following level sets for all $d_1 \in \mathfrak{D}$

$L_1(\check{\mathcal{A}}_T, \check{r}) = \{d_1 \in \mathfrak{D}: \check{\mathcal{A}}_T(d_1) \preccurlyeq [r^{\mathcal{L}}, r^{\mathcal{U}}]\}$; $U(\check{\mathcal{A}}_I, \check{s}) = \{d_1 \in \mathfrak{D}: \check{\mathcal{A}}_I(d_1) \succcurlyeq [s^{\mathcal{L}}, s^{\mathcal{U}}]\}$; and

$L_2(\check{\mathcal{A}}_F, \check{t}) = \{d_1 \in \mathfrak{D}: \check{\mathcal{A}}_F(d_1) \preccurlyeq [t^{\mathcal{L}}, t^{\mathcal{U}}]\}$.

3 | Interval-Valued Neutrosophic N-D-Ideal

Throughout this section, $\mathfrak{D}$ represents a d-algebra unless otherwise specified.

Definition 3.1. An IvNSN-d-I of $\mathfrak{D}$ is the IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ in $\mathfrak{D}$ with the following inequalities

(IvNSN-d-I 1) $\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\}$

(IvNSN-d-I 2) $\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$

(IvNSN-d-I 3) $\check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\}$

(IvNSN-d-I 4) $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_T(d_1)$

(IvNSN-d-I 5) $\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1)$

(IvNSN-d-I 6) $\check{\mathcal{A}}_F(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_F(d_1)$, for all $d_1, d_2 \in \mathfrak{D}$.

Definition 3.2. An IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ in $\mathfrak{D}$ is called an IvNSN-BCK-I if it satisfies the following inequalities

(IvNSN-BCK-I 1) $\check{\mathcal{A}}_T(0) \preccurlyeq \check{\mathcal{A}}_T(d_1)$, $\check{\mathcal{A}}_I(0) \succcurlyeq \check{\mathcal{A}}_I(d_1)$, and $\check{\mathcal{A}}_F(0) \preccurlyeq \check{\mathcal{A}}_F(d_1)$

(IvNSN-BCK-I 2) $\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\}$

(IvNSN-BCK-I 3) $\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$

(IvNSN-BCK-I 4) $\check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\}$ for all $d_1, d_2 \in \mathfrak{D}$.

Example 3.3. Consider a set $\mathfrak{D} = \{0,1,2\}$ in which the operation "$*$" is defined as shown in the following Cayley table (see Table 1).

Table 1. d-algebra.

$*$	**0**	**1**	**2**
0	0	0	0
1	2	0	2
2	1	1	0

Then $(\mathfrak{D}, *, 0)$ is a d-algebra. Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-S in $\mathfrak{D}$ as defined in the following table (see Table 2).

Table 2. Interval-valued neutrosophic N-d-ideal.

$\mathfrak{D}$	$\check{\mathcal{A}}_T(d_1)$	$\check{\mathcal{A}}_I(d_1)$	$\check{\mathcal{A}}_F(d_1)$
0	$[-0.95, -0.31]$	$[-0.55, -0.27]$	$[-0.85, -0.51,]$
1	$[-0.67, -0.15]$	$[-0.87, -0.42]$	$[-0.79, -0.39]$
2	$[-0.67, -0.15]$	$[-0.87, -0.42]$	$[-0.79, -0.39]$

Then $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of d-algebra $\mathfrak{D}$.

Proposition 3.4. If $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an interval-valued neutrosophic N-d-ideal of d-algebra $\mathfrak{D}$, then $\check{\mathcal{A}}_T(0) \preccurlyeq \check{\mathcal{A}}_T(d_1)$, $\check{\mathcal{A}}_I(0) \succcurlyeq \check{\mathcal{A}}_I(d_1)$ and $\check{\mathcal{A}}_F(0) \preccurlyeq \check{\mathcal{A}}_F(d_1)$ for all $d_1 \in \mathfrak{D}$.

Proof: Suppose that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$, and let $d_1 \in \mathfrak{D}$. Now utilizing (IvNSN-d-I 4), (IvNSN-d-I 5), (IvNSN-d-I 6), and $d_1 * d_1 = 0$, we get,

$$\begin{pmatrix} \check{\mathcal{A}}_T(d_1 * d_1) \preccurlyeq \check{\mathcal{A}}_T(d_1) \Rightarrow \check{\mathcal{A}}_T(0) \preccurlyeq \check{\mathcal{A}}_T(d_1) \\ \check{\mathcal{A}}_I(d_1 * d_1) \succcurlyeq \check{\mathcal{A}}_I(d_1) \Rightarrow \check{\mathcal{A}}_I(0) \succcurlyeq \check{\mathcal{A}}_I(d_1) \\ \check{\mathcal{A}}_F(d_1 * d_1) \preccurlyeq \check{\mathcal{A}}_F(d_1) \Rightarrow \check{\mathcal{A}}_F(0) \preccurlyeq \check{\mathcal{A}}_F(d_1) \end{pmatrix} \text{for all } d_1 \in \mathfrak{D}.$$

Lemma 3.5. Let an IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ in $\mathfrak{D}$ be an IvNSN-d-I of $\mathfrak{D}$. If $d_1 * d_2 \leq d_3$, then

$$\begin{pmatrix} \check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_2), \check{\mathcal{A}}_T(d_3)\} \\ \check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_2), \check{\mathcal{A}}_I(d_3)\} \\ \check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_2), \check{\mathcal{A}}_F(d_3)\} \end{pmatrix} \text{ for all } d_1, d_2, d_3 \in \mathfrak{D}.$$

Proof: Suppose that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$. Let $d_1, d_2,$ and d_3 be any three members of $\mathfrak{D}$ such that $d_1 * d_2 \leq d_3$. Then $(d_1 * d_2) * d_3 = 0$.

$$\begin{aligned} \check{\mathcal{A}}_T(d_1) &\preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\} \\ &\preccurlyeq rmax\{rmax\{\check{\mathcal{A}}_T((d_1 * d_2) * d_3), \check{\mathcal{A}}_T(d_3)\}, \check{\mathcal{A}}_T(d_2)\} \\ &= rmax\{rmax\{\check{\mathcal{A}}_T(0), \check{\mathcal{A}}_T(d_3)\}, \check{\mathcal{A}}_T(d_2)\} \\ &= rmax\{\check{\mathcal{A}}_T(d_2), \check{\mathcal{A}}_T(d_3)\}, \end{aligned}$$

$$\begin{aligned} \check{\mathcal{A}}_I(d_1) &\succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\} \\ &\succcurlyeq rmin\{rmin\{\check{\mathcal{A}}_I((d_1 * d_2) * d_3), \check{\mathcal{A}}_I(d_3)\}, \check{\mathcal{A}}_I(d_2)\} \\ &= rmin\{rmin\{\check{\mathcal{A}}_I(0), \check{\mathcal{A}}_I(d_3)\}, \check{\mathcal{A}}_I(d_2)\} \\ &= rmin\{\check{\mathcal{A}}_I(d_2), \check{\mathcal{A}}_I(d_3)\}, \end{aligned}$$

$$\begin{aligned} \check{\mathcal{A}}_F(d_1) &\preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\} \\ &\preccurlyeq rmax\{rmax\{\check{\mathcal{A}}_F((d_1 * d_2) * d_3), \check{\mathcal{A}}_F(d_3)\}, \check{\mathcal{A}}_F(d_2)\} \\ &= rmax\{rmax\{\check{\mathcal{A}}_F(0), \check{\mathcal{A}}_F(d_3)\}, \check{\mathcal{A}}_F(d_2)\} \\ &= rmax\{\check{\mathcal{A}}_F(d_2), \check{\mathcal{A}}_F(d_3)\}. \end{aligned}$$

This completes the proof.

Lemma 3.6. Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-d-I of $\mathfrak{D}$. If $d_1 \leq d_2$ in $\mathfrak{D}$, then $\check{\mathcal{A}}_T(d_1) \preccurlyeq \check{\mathcal{A}}_T(d_2)$, $\check{\mathcal{A}}_I(d_1) \succcurlyeq \check{\mathcal{A}}_I(d_2)$, and $\check{\mathcal{A}}_F(d_1) \preccurlyeq \check{\mathcal{A}}_F(d_2)$ for all $d_1, d_2 \in \mathfrak{D}$.

Proof: Suppose that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$. Let d_1 and d_2 be any two members of $\mathfrak{D}$ such that $d_1 \leq d_2$. Then $d_1 * d_2 = 0$.

$$\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\} = rmax\{\check{\mathcal{A}}_T(0), \check{\mathcal{A}}_T(d_2)\} = \check{\mathcal{A}}_T(d_2)$$

$$\Rightarrow \check{\mathcal{A}}_T(d_1) \preccurlyeq \check{\mathcal{A}}_T(d_2),$$

$$\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\} = rmin\{\check{\mathcal{A}}_I(0), \check{\mathcal{A}}_I(d_2)\} = \check{\mathcal{A}}_I(d_2)$$

$$\Rightarrow \check{\mathcal{A}}_I(d_1) \succcurlyeq \check{\mathcal{A}}_I(d_2),$$

$$\check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\} = rmax\{\check{\mathcal{A}}_F(0), \check{\mathcal{A}}_F(d_2)\} = \check{\mathcal{A}}_F(d_2)$$

$$\Rightarrow \check{\mathcal{A}}_F(d_1) \preccurlyeq \check{\mathcal{A}}_F(d_2).$$

This completes the proof.

Theorem 3.7. If $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$, then for any $d_1, \mathscr{b}_1, \mathscr{b}_2, \dots\dots\dots, \mathscr{b}_n \in \mathfrak{D}$, such that $\left(((d_1 * \mathscr{b}_1) * \mathscr{b}_2) \dots\dots\dots\right) * \mathscr{b}_n = 0$ implies

$$\begin{pmatrix} \check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(\mathscr{b}_1), \check{\mathcal{A}}_T(\mathscr{b}_2), \ldots \ldots \ldots . \check{\mathcal{A}}_T(\mathscr{b}_n)\} \\ \check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(\mathscr{b}_1), \check{\mathcal{A}}_I(\mathscr{b}_2), \ldots \ldots \ldots , \check{\mathcal{A}}_I(\mathscr{b}_n)\} \\ \check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(\mathscr{b}_1), \check{\mathcal{A}}_F(\mathscr{b}_2), \ldots \ldots \ldots , \check{\mathcal{A}}_F(\mathscr{b}_n)\} \end{pmatrix}$$

Proof: By applying induction on n and Lemma 3.5 and Lemma 3.6.

Theorem 3.8. Every IvNSN-d-I of $\mathfrak{D}$ is an interval-valued neutrosophic N-d-subalgebra.

Proof: Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-d-I of $\mathfrak{D}$. So

$\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\} \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1), \check{\mathcal{A}}_T(d_2)\}$,

$\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\} \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1), \check{\mathcal{A}}_I(d_2)\}$,

$\check{\mathcal{A}}_F(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\} \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1), \check{\mathcal{A}}_F(d_2)\}$ for all $d_1, d_2 \in \mathfrak{D}$.

Therefore, $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an interval-valued neutrosophic N-d-subalgebra.

The converse of this theorem is not true in general which is shown in the next example.

Example 3.9. Consider a set $\mathfrak{D} = \{0,1,2,3\}$ in which the operation "$*$" is defined as shown in the following Cayley table (see Table 3).

Table 3. d-algebra.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(\mathfrak{D}, *, 0)$ is a d-algebra. Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-S in $\mathfrak{D}$ as defined in the following table (see Table 4).

Table 4. Interval-valued neutrosophic N-d-subalgebra.

$\mathfrak{D}$	$\check{\mathcal{A}}_T(d_1)$	$\check{\mathcal{A}}_I(d_1)$	$\check{\mathcal{A}}_F(d_1)$
0	$[-0.92, -0.53]$	$[-0.73, -0.35]$	$[-0.83, -0.63]$
1	$[-0.92, -0.53]$	$[-0.73, -0.35]$	$[-0.83, -0.63]$
2	$[-0.73, -0.42]$	$[-0.89, -0.74]$	$[-0.52, -0.31]$
3	$[-0.92, -0.53]$	$[-0.73, -0.35]$	$[-0.83, -0.63]$

It is clear that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an interval-valued neutrosophic N-d-subalgebra, but

$\check{\mathcal{A}}_T(2) = [-0.73, -0.42] = rmax\{\check{\mathcal{A}}_T(2 * 1), \check{\mathcal{A}}_T(1)\} \succcurlyeq \check{\mathcal{A}}_T(1) = [-0.92, -0.53]$,

$\check{\mathcal{A}}_I(2) = [-0.89, -0.74] = rmin\{\check{\mathcal{A}}_I(2 * 1), \check{\mathcal{A}}_I(1)\} \preccurlyeq \check{\mathcal{A}}_I(1) = [-0.73, -0.35]$, and

$\check{\mathcal{A}}_F(2) = [-0.52, -0.31] = rmax\{\check{\mathcal{A}}_F(2 * 1), \check{\mathcal{A}}_F(1)\} \succcurlyeq \check{\mathcal{A}}_F(1) = [-0.83, -0.63]$.

So $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is not an IvNSN-d-I.

Proposition 3.10. Every IvNSN-d-I in $\mathfrak{D}$ is an IvNSN-BCK-I of $\mathfrak{D}$.

Proof: The proof is straightforward.

The converse of this theorem is not true in general which is shown in the next example.

Example 3.11. Consider a set $\mathfrak{D} = \{0,1,2,3,4\}$ in which the binary operation "$*$" is defined as shown in the following Cayley table (see Table 5).

Table 5. d-algebra.

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	3	3	3	0

Then $(\mathfrak{D},*,0)$ is a d-algebra. Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-S in $\mathfrak{D}$ as defined in the following table (see Table 6).

Table 6. Interval-valued Neutrosophic N-BCK-ideal.

$\mathfrak{D}$	$\check{\mathcal{A}}_T(d_1)$	$\check{\mathcal{A}}_I(d_1)$	$\check{\mathcal{A}}_F(d_1)$
0	$[-0.92, -0.54]$	$[-0.58, -0.23]$	$[-0.85, -0.57]$
1	$[-0.92, -0.54]$	$[-0.58, -0.23]$	$[-0.85, -0.57]$
2	$[-0.92, -0.54]$	$[-0.58, -0.23]$	$[-0.85, -0.57]$
3	$[-0.53, -0.21]$	$[-0.97, -0.77]$	$[-0.64, -0.32]$
4	$[-0.92, -0.54]$	$[-0.58, -0.23]$	$[-0.85, -0.57]$

Then $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ in $\mathfrak{D}$ is an IvNSN-BCK-I of $\mathfrak{D}$ but it is not an interval-valued neutrosophic N-d-I, because

$$\check{\mathcal{A}}_T(4*3) = \check{\mathcal{A}}_T(3) = [-0.53, -0.21] \succcurlyeq [-0.92, -0.54] = \check{\mathcal{A}}_T(4)$$

$$\check{\mathcal{A}}_I(4*3) = \check{\mathcal{A}}_I(3) = [-0.97, -0.77] \preccurlyeq [-0.58, -0.23] = \check{\mathcal{A}}_I(4)$$

$$\check{\mathcal{A}}_F(4*3) = \check{\mathcal{A}}_F(3) = [-0.64, -0.32] \succcurlyeq [-0.85, -0.57] = \check{\mathcal{A}}_F(4).$$

Theorem 3.12. If $\{\check{\mathcal{A}}_i, i \in \Lambda\}$ is an arbitrary family of IvNSN-d-I of d-algebra, then $\underset{i \in \Lambda}{\cap} \check{\mathcal{A}}_i$ is an IvNSN-d-I of d-algebra, where $\underset{i \in \Lambda}{\cap} \check{\mathcal{A}}_i = \left(rmax(\check{\mathcal{A}}_{T_i}), rmin(\check{\mathcal{A}}_{I_i}), rmax(\check{\mathcal{A}}_{F_i})\right)$

Proof: Suppose that $\{\check{\mathcal{A}}_i, i \in \Lambda\}$ be the collection of an arbitrary family of IvNSN-d-I of d-algebra and

$$\underset{i \in \Lambda}{\cap} \check{\mathcal{A}}_i = \left(rmax(\check{\mathcal{A}}_{T_i}), rmin(\check{\mathcal{A}}_{I_i}), rmax(\check{\mathcal{A}}_{F_i})\right).$$

For all $i \in \Lambda$

$$rmax(\check{\mathcal{A}}_{T_i})(d_1) \preccurlyeq rmax\left\{rmax\{\check{\mathcal{A}}_{T_i}(d_1 * d_2), \check{\mathcal{A}}_{T_i}(d_2)\}\right\}$$
$$\preccurlyeq rmax\{rmax(\check{\mathcal{A}}_{T_i})(d_1 * d_2), rmax(\check{\mathcal{A}}_{T_i})(d_2)\},$$

$$rmin(\check{\mathcal{A}}_{I_i})(d_1) \succcurlyeq rmin\left\{rmin\{\check{\mathcal{A}}_{I_i}(d_1 * d_2), \check{\mathcal{A}}_{I_i}(d_2)\}\right\}$$
$$\succcurlyeq rmin\{rmin(\check{\mathcal{A}}_{I_i})(d_1 * d_2), rmin(\check{\mathcal{A}}_{I_i})(d_2)\},$$

$$rmax(\check{\mathcal{A}}_{F_i})(d_1) \preccurlyeq rmax\left\{rmax\{\check{\mathcal{A}}_{F_i}(d_1 * d_2), \check{\mathcal{A}}_{F_i}(d_2)\}\right\}$$
$$\preccurlyeq rmax\{rmax(\check{\mathcal{A}}_{F_i})(d_1 * d_2), rmax(\check{\mathcal{A}}_{F_i})(d_2)\}.$$

Since $\check{\mathcal{A}}_{T_i}(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_{T_i}(d_1)$, $\check{\mathcal{A}}_{I_i}(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_{I_i}(d_1)$ and $\check{\mathcal{A}}_{F_i}(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_{F_i}(d_1)$ for all $i \in \Lambda$, we have

$$rmax(\check{\mathcal{A}}_{T_i})(d_1 * d_2) \preccurlyeq rmax(\check{\mathcal{A}}_{T_i})(d_1)$$
$$rmin(\check{\mathcal{A}}_{I_i})(d_1 * d_2) \succcurlyeq rmin(\check{\mathcal{A}}_{I_i})(d_1)$$
$$rmax(\check{\mathcal{A}}_{F_i})(d_1 * d_2) \preccurlyeq rmax(\check{\mathcal{A}}_{F_i})(d_1)$$

Hence $\underset{i \in \Lambda}{\cap} \check{\mathcal{A}}_i = \left(rmax(\check{\mathcal{A}}_{T_i}), rmin(\check{\mathcal{A}}_{I_i}), rmax(\check{\mathcal{A}}_{F_i})\right)$is an IvNSN-d-I of d-algebra.

Theorem 3.13 An IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of d-algebra $\mathfrak{D}$ if and only if the corresponding N-interval-valued fuzzy sets $\check{\mathcal{A}}_T$, ${\check{\mathcal{A}}_I}^c$, and $\check{\mathcal{A}}_F$ are N-interval-valued fuzzy d-ideal of $\mathfrak{D}$.

Proof: Suppose that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-d-I of d-algebra $\mathfrak{D}$. Then for any $d_1, d_2 \in \mathfrak{D}$ we have

$\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\}$; $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_T(d_1)$

$\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$; $\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1)$

$\check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\}$; $\check{\mathcal{A}}_F(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_F(d_1)$, for all $d_1, d_2 \in \mathfrak{D}$. Clearly, $\check{\mathcal{A}}_T$ and $\check{\mathcal{A}}_F$ are N-interval-valued fuzzy d-ideal of $\mathfrak{D}$. Now for all $d_1, d_2 \in \mathfrak{D}$, we obtain

$$\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$$
$$\Rightarrow -\check{\mathcal{A}}_I(d_1) \preccurlyeq -rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$$
$$= rmax\{-\check{\mathcal{A}}_I(d_1 * d_2), -\check{\mathcal{A}}_I(d_2)\}$$
$$\Rightarrow -\check{\mathcal{A}}_I(d_1) + [-1,-1] \preccurlyeq rmax\{-\check{\mathcal{A}}_I(d_1 * d_2) + [-1,-1], -\check{\mathcal{A}}_I(d_2) + [-1,-1]\}$$

$\Rightarrow {\check{\mathcal{A}}_I}^c(d_1) \preccurlyeq rmax\{{\check{\mathcal{A}}_I}^c(d_1 * d_2), {\check{\mathcal{A}}_I}^c(d_2)\}$. Also

$$\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1)$$
$$\Rightarrow -\check{\mathcal{A}}_I(d_1 * d_2) \preccurlyeq -\check{\mathcal{A}}_I(d_1)$$
$$\Rightarrow -\check{\mathcal{A}}_I(d_1 * d_2) + [-1,-1] \preccurlyeq -\check{\mathcal{A}}_I(d_1) + [-1,-1]$$
$$\Rightarrow {\check{\mathcal{A}}_I}^c(d_1 * d_2) \preccurlyeq {\check{\mathcal{A}}_I}^c(d_1).$$

Therefore, the N-interval-valued fuzzy set ${\check{\mathcal{A}}_I}^c$ is a N-interval-valued fuzzy d-ideal of $\mathfrak{D}$. Hence for an IvNSN-d-I $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$, the corresponding N-interval-valued fuzzy sets $\check{\mathcal{A}}_T$, ${\check{\mathcal{A}}_I}^c$, and $\check{\mathcal{A}}_F$ are N-interval-valued fuzzy d-ideal of $\mathfrak{D}$.

Conversely, suppose that $\check{\mathcal{A}}_T$, ${\check{\mathcal{A}}_I}^c$, and $\check{\mathcal{A}}_F$ are N-interval-valued fuzzy d-ideal of $\mathfrak{D}$. For any $d_1, d_2 \in \mathfrak{D}$ we get

$\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\}$; $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_T(d_1)$

${\check{\mathcal{A}}_I}^c(d_1) \preccurlyeq rmax\{{\check{\mathcal{A}}_I}^c(d_1 * d_2), {\check{\mathcal{A}}_I}^c(d_2)\}$; ${\check{\mathcal{A}}_I}^c(d_1 * d_2) \preccurlyeq {\check{\mathcal{A}}_I}^c(d_1)$

$\check{\mathcal{A}}_F(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\}$; $\check{\mathcal{A}}_F(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_F(d_1)$. Now,

$$ {\check{\mathcal{A}}_I}^c(d_1) \preccurlyeq rmax\{{\check{\mathcal{A}}_I}^c(d_1 * d_2), {\check{\mathcal{A}}_I}^c(d_2)\}$$
$$[-1,-1] - \check{\mathcal{A}}_I(d_1) \preccurlyeq rmax\{[-1,-1] - \check{\mathcal{A}}_I(d_1 * d_2), [-1,-1] - \check{\mathcal{A}}_I(d_2)\}$$
$$= [-1,-1] - rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$$
$$-\check{\mathcal{A}}_I(d_1) \preccurlyeq -rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$$

$\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\}$. Also,

$$\check{\mathcal{A}}_I{}^c(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_I{}^c(d_1)$$

$$\Rightarrow [-1,-1] - \check{\mathcal{A}}_I(d_1 * d_2) \preccurlyeq [-1,-1] - \check{\mathcal{A}}_I(d_1)$$

$$\Rightarrow -\check{\mathcal{A}}_I(d_1 * d_2) \preccurlyeq -\check{\mathcal{A}}_I(d_1)$$

$$\Rightarrow \check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1).$$

Hence $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of d-algebra $\mathfrak{D}$.

Theorem 3.14. Let $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-S over d-algebra $\mathfrak{D}$. Then $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$ if and only if the following IvNSN-S are an IvNSN-d-I of $\mathfrak{D}$.

$$\check{\mathcal{A}}_1 = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_T{}^c, \check{\mathcal{A}}_T)$$

$$\check{\mathcal{A}}_2 = (\check{\mathcal{A}}_F, \check{\mathcal{A}}_F{}^c, \check{\mathcal{A}}_F)$$

$$\check{\mathcal{A}}_3 = (\check{\mathcal{A}}_I{}^c, \check{\mathcal{A}}_I, \check{\mathcal{A}}_I{}^c)$$

Theorem 3.15. An IvNSN-S $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ over d-algebra $\mathfrak{D}$ is an IvNSN-d-I of $\mathfrak{D}$ if and only if for all $\check{r}, \check{s}, \check{t} \in I[-1,0]$ the sets $L_1(\check{\mathcal{A}}_T, \check{r})$, $U(\check{\mathcal{A}}_I, \check{s})$, and $L_2(\check{\mathcal{A}}_F, \check{t})$ of $\check{\mathcal{A}}$ are either empty are d-ideal of $\mathfrak{D}$.

Proof: Assume that $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-d-I of $\mathfrak{D}$. Let $L_1(\check{\mathcal{A}}_T, \check{r})$, $U(\check{\mathcal{A}}_I, \check{s})$, and $L_2(\check{\mathcal{A}}_F, \check{t})$ are non-empty sets for any $\check{r}, \check{s}, \check{t} \in I[-1,0]$. Let $d_1, d_2 \in \mathfrak{D}$ such that $d_1 * d_2,\ d_2 \in L_1(\check{\mathcal{A}}_T, \check{r})$, so $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{r}$ and $\check{\mathcal{A}}_T(d_2) \preccurlyeq \check{r}$. Then, $\check{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\} \preccurlyeq rmax\{\check{r}, \check{r}\} \preccurlyeq \check{r} \Rightarrow d_1 \in L_1(\check{\mathcal{A}}_T, \check{r})$. And let $d_1 \in L_1(\check{\mathcal{A}}_T, \check{r})$, $d_2 \in \mathfrak{D}$. Then $\check{\mathcal{A}}_T(d_1) \preccurlyeq \check{r}$ and $\check{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \check{\mathcal{A}}_T(d_1) \preccurlyeq \check{r}$. So, $d_1 * d_2 \in L_1(\check{\mathcal{A}}_T, \check{r})$. Hence, $L_1(\check{\mathcal{A}}_T, \check{r})$ is a d-ideal of $\mathfrak{D}$. Also, take $d_1, d_2 \in \mathfrak{D}$ such that $d_1 * d_2,\ d_2 \in U(\check{\mathcal{A}}_I, \check{s})$, so $\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{s}$ and $\check{\mathcal{A}}_I(d_2) \succcurlyeq \check{s}$. Then, $\check{\mathcal{A}}_I(d_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\} \succcurlyeq rmin\{\check{s}, \check{s}\} \succcurlyeq \check{s} \Rightarrow d_1 \in U(\check{\mathcal{A}}_I, \check{s})$. And let $d_1 \in U(\check{\mathcal{A}}_I, \check{s})$, $d_2 \in \mathfrak{D}$. Then $\check{\mathcal{A}}_I(d_1) \succcurlyeq \check{s}$ and $\check{\mathcal{A}}_I(d_1 * d_2) \succcurlyeq \check{\mathcal{A}}_I(d_1) \succcurlyeq \check{s}$. So, $d_1 * d_2 \in U(\check{\mathcal{A}}_I, \check{s})$. Hence, $U(\check{\mathcal{A}}_I, \check{s})$ is a d-ideal of $\mathfrak{D}$. In a similar process, we can show that $L_2(\check{\mathcal{A}}_F, \check{t})$ is a d-ideal of $\mathfrak{D}$.

Conversely, suppose that for any $\check{r}, \check{s}, \check{t} \in I[-1,0]$ the sets $L_1(\check{\mathcal{A}}_T, \check{r})$, $U(\check{\mathcal{A}}_I, \check{s})$, and $L_2(\check{\mathcal{A}}_F, \check{t})$ of $\check{\mathcal{A}}$ are d-ideals of $\mathfrak{D}$. Let us take $\zeta_1, \zeta_2 \in L_1(\check{\mathcal{A}}_T, \check{r})$ such that $\check{\mathcal{A}}_T(\zeta_1) \succ rmax\{\check{\mathcal{A}}_T(\zeta_1 * \zeta_2), \check{\mathcal{A}}_T(\zeta_2)\}$. Suppose that $\check{\mathcal{A}}_T(\zeta_1) = \check{r}_1 = [r_1^{\mathcal{L}}, r_1^{\mathcal{U}}]$, $\check{\mathcal{A}}_T(\zeta_1 * \zeta_2) = \check{r}_2 = [r_2^{\mathcal{L}}, r_2^{\mathcal{U}}]$, and $\check{\mathcal{A}}_T(\zeta_2) = \check{r}_3 = [r_3^{\mathcal{L}}, r_3^{\mathcal{U}}]$. Then, $[r_1^{\mathcal{L}}, r_1^{\mathcal{U}}] \succ rmax\{[r_2^{\mathcal{L}}, r_2^{\mathcal{U}}], [r_3^{\mathcal{L}}, r_3^{\mathcal{U}}]\} = [max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}, max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}]$ and so,

$r_1^{\mathcal{L}} > max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}$ and $r_1^{\mathcal{U}} > max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}$.

Taking $\check{r}_4 = [r_4^{\mathcal{L}}, r_4^{\mathcal{U}}] = \frac{1}{2}[\check{\mathcal{A}}_T(\zeta_1) + rmax\{\check{\mathcal{A}}_T(\zeta_1 * \zeta_2), \check{\mathcal{A}}_T(\zeta_2)\}]$

$$= \frac{1}{2}\left[[r_1^{\mathcal{L}}, r_1^{\mathcal{U}}] + [max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}, max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}]\right]$$

$$= \left[\frac{1}{2}(r_1^{\mathcal{L}} + max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}), \frac{1}{2}(r_1^{\mathcal{U}} + max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\})\right]$$

It follows that

$$r_1^{\mathcal{L}} > r_4^{\mathcal{L}} = \frac{1}{2}(r_1^{\mathcal{L}} + max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}) > max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\},$$

$$r_1^{\mathcal{U}} > r_4^{\mathcal{U}} = \frac{1}{2}(r_1^{\mathcal{U}} + max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}) > max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}.$$

Hence, $[max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\},\ max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}] \prec [r_4^{\mathcal{L}}, r_4^{\mathcal{U}}] \prec [r_1^{\mathcal{L}}, r_1^{\mathcal{U}}] = \check{\mathcal{A}}_T(\zeta_1)$.

Therefore, $\zeta_1 \notin L_1(\check{\mathcal{A}}_T, \breve{r}_4)$. On the other hand

$\check{\mathcal{A}}_T(\zeta_1 * \zeta_2) = \breve{r}_2 = [r_2^{\mathcal{L}}, r_2^{\mathcal{U}}] \preccurlyeq [max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}, max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}] \prec [r_4^{\mathcal{L}}, r_4^{\mathcal{U}}] = \breve{r}_4$,

$\check{\mathcal{A}}_T(\zeta_2) = \breve{r}_3 = [r_3^{\mathcal{L}}, r_3^{\mathcal{U}}] \preccurlyeq [max\{r_2^{\mathcal{L}}, r_3^{\mathcal{L}}\}, max\{r_2^{\mathcal{U}}, r_3^{\mathcal{U}}\}] \prec [r_4^{\mathcal{L}}, r_4^{\mathcal{U}}] = \breve{r}_4$.

i.e., $\zeta_1 * \zeta_2, \zeta_2 \in L_1(\check{\mathcal{A}}_T, \breve{r}_4)$. This is a contradiction and therefore

$\check{\mathcal{A}}_T(\zeta_1) \preccurlyeq rmax\{\check{\mathcal{A}}_T(\zeta_1 * \zeta_2), \check{\mathcal{A}}_T(\zeta_2)\}$ for all $\zeta_1, \zeta_2 \in \mathfrak{D}$.

Suppose that for any $\zeta_1, \zeta_2 \in \mathfrak{D}$, we have $\check{\mathcal{A}}_T(\zeta_1 * \zeta_2) \succ \check{\mathcal{A}}_T(\zeta_1)$.

Then taking $\breve{r}_1 = [r_1^{\mathcal{L}}, r_1^{\mathcal{U}}] = \frac{1}{2}[\check{\mathcal{A}}_T(\zeta_1 * \zeta_2) + \check{\mathcal{A}}_T(\zeta_1)]$, we have $\check{\mathcal{A}}_T(\zeta_1) \prec [r_1^{\mathcal{L}}, r_1^{\mathcal{U}}] \prec \check{\mathcal{A}}_T(\zeta_1 * \zeta_2)$. Hence $\zeta_1 \in L_1(\check{\mathcal{A}}_T, \breve{r}_1)$, $\zeta_2 \in \mathfrak{D}$, but $\zeta_1 * \zeta_2 \notin L_1(\check{\mathcal{A}}_T, \breve{r}_1)$. This is a contradiction and therefore $\check{\mathcal{A}}_T(\zeta_1 * \zeta_2) \preccurlyeq \check{\mathcal{A}}_T(\zeta_1)$.

Suppose that $\check{\mathcal{A}}_I(\zeta_1) \prec rmin\{\check{\mathcal{A}}_I(\zeta_1 * \zeta_2), \check{\mathcal{A}}_I(\zeta_2)\}$ for some $\zeta_1, \zeta_2 \in U(\check{\mathcal{A}}_I, \breve{s})$. Let us take $\check{\mathcal{A}}_I(\zeta_1) = \breve{s}_1 = [s_1^{\mathcal{L}}, s_1^{\mathcal{U}}]$, $\check{\mathcal{A}}_I(\zeta_1 * \zeta_2) = \breve{s}_2 = [s_2^{\mathcal{L}}, s_2^{\mathcal{U}}]$, and $\check{\mathcal{A}}_I(\zeta_2) = \breve{s}_3 = [s_3^{\mathcal{L}}, s_3^{\mathcal{U}}]$. Then,

$[s_1^{\mathcal{L}}, s_1^{\mathcal{U}}] \prec rmin\{[s_2^{\mathcal{L}}, s_2^{\mathcal{U}}], [s_3^{\mathcal{L}}, s_3^{\mathcal{U}}]\} = [min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}, min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}]$ and so,

$s_1^{\mathcal{L}} < min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}$ and $s_1^{\mathcal{U}} < min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}$.

Taking $\breve{s}_4 = [s_4^{\mathcal{L}}, s_4^{\mathcal{U}}] = \frac{1}{2}[\check{\mathcal{A}}_I(\zeta_1) + rmin\{\check{\mathcal{A}}_I(\zeta_1 * \zeta_2), \check{\mathcal{A}}_I(\zeta_2)\}]$

$$= \frac{1}{2}\left[[s_1^{\mathcal{L}}, s_1^{\mathcal{U}}] + [min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}, min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}]\right]$$

$$= \left[\frac{1}{2}(s_1^{\mathcal{L}} + min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}), \frac{1}{2}(s_1^{\mathcal{U}} + min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\})\right]$$

It follows that

$s_1^{\mathcal{L}} < s_4^{\mathcal{L}} = \frac{1}{2}(s_1^{\mathcal{L}} + min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}) < min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}$,

$s_1^{\mathcal{U}} < s_4^{\mathcal{U}} = \frac{1}{2}(s_1^{\mathcal{U}} + min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}) < min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}$.

Hence, $[min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\},\ min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}] \succ [s_4^{\mathcal{L}}, s_4^{\mathcal{U}}] \succ [s_1^{\mathcal{L}}, s_1^{\mathcal{U}}] = \check{\mathcal{A}}_I(\zeta_1)$.

Therefore, $\zeta_1 \notin U(\check{\mathcal{A}}_I, \breve{s}_4)$. On the other hand

$\check{\mathcal{A}}_I(\zeta_1 * \zeta_2) = \breve{s}_2 = [s_2^{\mathcal{L}}, s_2^{\mathcal{U}}] \succcurlyeq [min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}, min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}] \succ [s_4^{\mathcal{L}}, s_4^{\mathcal{U}}] = \breve{s}_4$,

$\check{\mathcal{A}}_I(\zeta_2) = \breve{s}_3 = [s_3^{\mathcal{L}}, s_3^{\mathcal{U}}] \succcurlyeq [min\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\}, min\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\}] \succ [s_4^{\mathcal{L}}, s_4^{\mathcal{U}}] = \breve{s}_4$.

i.e., $\zeta_1 * \zeta_2, \zeta_2 \in U(\check{\mathcal{A}}_I, \breve{s}_4)$. This is a contradiction and therefore

$\check{\mathcal{A}}_I(\zeta_1) \succcurlyeq rmin\{\check{\mathcal{A}}_I(\zeta_1 * \zeta_2), \check{\mathcal{A}}_I(\zeta_2)\}$ for all $\zeta_1, \zeta_2 \in \mathfrak{D}$.

Suppose that for any $\zeta_1, \zeta_2 \in \mathfrak{D}$, we have $\check{\mathcal{A}}_I(\zeta_1 * \zeta_2) \prec \check{\mathcal{A}}_I(\zeta_1)$.

Then taking $\breve{s}_1 = [s_1^{\mathcal{L}}, s_1^{\mathcal{U}}] = \frac{1}{2}[\check{\mathcal{A}}_I(\zeta_1 * \zeta_2) + \check{\mathcal{A}}_I(\zeta_1)]$, we have $\check{\mathcal{A}}_I(\zeta_1) \succ [s_1^{\mathcal{L}}, s_1^{\mathcal{U}}] \succ \check{\mathcal{A}}_I(\zeta_1 * \zeta_2)$. Hence $\zeta_1 \in U(\check{\mathcal{A}}_I, \breve{s}_1)$, $\zeta_2 \in \mathfrak{D}$, but $\zeta_1 * \zeta_2 \notin U(\check{\mathcal{A}}_I, \breve{s}_1)$. This is a contradiction and therefore $\check{\mathcal{A}}_I(\zeta_1 * \zeta_2) \succcurlyeq \check{\mathcal{A}}_I(\zeta_1)$.

Let us take $\zeta_1, \zeta_2 \in L_2(\check{\mathcal{A}}_F, \breve{t})$ such that $\check{\mathcal{A}}_F(\zeta_1) \succ rmax\{\check{\mathcal{A}}_F(\zeta_1 * \zeta_2), \check{\mathcal{A}}_F(\zeta_2)\}$. Suppose that $\check{\mathcal{A}}_F(\zeta_1) = \breve{t}_1 = [t_1^{\mathcal{L}}, t_1^{\mathcal{U}}]$, $\check{\mathcal{A}}_F(\zeta_1 * \zeta_2) = \breve{t}_2 = [t_2^{\mathcal{L}}, t_2^{\mathcal{U}}]$, and $\check{\mathcal{A}}_F(\zeta_2) = \breve{t}_3 = [t_3^{\mathcal{L}}, t_3^{\mathcal{U}}]$. Then,

$[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}] \succ rmax\{[t_2^{\mathcal{L}}, t_2^{\mathcal{U}}], [t_3^{\mathcal{L}}, t_3^{\mathcal{U}}]\} = [max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}, max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}]$ and so,

$t_1^{\mathcal{L}} > max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}$ and $t_1^{\mathcal{U}} > max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}$.

Taking $\breve{t_4} = [t_4^{\mathcal{L}}, t_4^{\mathcal{U}}] = \frac{1}{2}[\breve{\mathcal{A}}_F(\zeta_1) + rmax\{\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2), \breve{\mathcal{A}}_F(\zeta_2)\}]$

$$= \frac{1}{2}\left[[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}] + [max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}, max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}]\right]$$

$$= \left[\frac{1}{2}(t_1^{\mathcal{L}} + max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}), \frac{1}{2}(t_1^{\mathcal{U}} + max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\})\right]$$

It follows that

$t_1^{\mathcal{L}} > t_4^{\mathcal{L}} = \frac{1}{2}(t_1^{\mathcal{L}} + max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}) > max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}$,

$t_1^{\mathcal{U}} > t_4^{\mathcal{U}} = \frac{1}{2}(t_1^{\mathcal{U}} + max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}) > max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}$.

Hence, $[max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\},\ max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}] \prec [t_4^{\mathcal{L}}, t_4^{\mathcal{U}}] \prec [t_1^{\mathcal{L}}, t_1^{\mathcal{U}}] = \breve{\mathcal{A}}_F(\zeta_1)$.

Therefore, $\zeta_1 \notin L_2(\breve{\mathcal{A}}_F, \breve{t_4})$. On the other hand

$\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2) = \breve{t_2} = [t_2^{\mathcal{L}}, t_2^{\mathcal{U}}] \preccurlyeq [max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}, max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}] \prec [t_4^{\mathcal{L}}, t_4^{\mathcal{U}}] = \breve{t_4}$,

$\breve{\mathcal{A}}_F(\zeta_2) = \breve{t_3} = [t_3^{\mathcal{L}}, t_3^{\mathcal{U}}] \preccurlyeq [max\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\}, max\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\}] \prec [t_4^{\mathcal{L}}, t_4^{\mathcal{U}}] = \breve{t_4}$.

i.e., $\zeta_1 * \zeta_2, \zeta_2 \in L_2(\breve{\mathcal{A}}_F, \breve{t_4})$. This is a contradiction and therefore

$\breve{\mathcal{A}}_F(\zeta_1) \preccurlyeq rmax\{\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2), \breve{\mathcal{A}}_F(\zeta_2)\}$ for all $\zeta_1, \zeta_2 \in \mathfrak{D}$.

Suppose that for any $\zeta_1, \zeta_2 \in \mathfrak{D}$, we have $\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2) \succ \breve{\mathcal{A}}_F(\zeta_1)$.

Then taking $\breve{t_1} = [t_1^{\mathcal{L}}, t_1^{\mathcal{U}}] = \frac{1}{2}[\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2) + \breve{\mathcal{A}}_F(\zeta_1)]$, we have $\breve{\mathcal{A}}_F(\zeta_1) \prec [t_1^{\mathcal{L}}, t_1^{\mathcal{U}}] \prec \breve{\mathcal{A}}_F(\zeta_1 * \zeta_2)$. Hence $\zeta_1 \in L_2(\breve{\mathcal{A}}_F, \breve{t_1})$, $\zeta_2 \in \mathfrak{D}$, but $\zeta_1 * \zeta_2 \notin L_2(\breve{\mathcal{A}}_F, \breve{t_1})$. This is a contradiction and therefore $\breve{\mathcal{A}}_F(\zeta_1 * \zeta_2) \preccurlyeq \breve{\mathcal{A}}_F(\zeta_1)$. Therefore $\breve{\mathcal{A}} = (\breve{\mathcal{A}}_T, \breve{\mathcal{A}}_I, \breve{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}$.

Theorem 3.16. If an IvNSN-S $\breve{\mathcal{A}} = (\breve{\mathcal{A}}_T, \breve{\mathcal{A}}_I, \breve{\mathcal{A}}_F)$ in $\mathfrak{D}$ is an IvNSN-d-I of $\mathfrak{D}$, then the sets $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_T} = \{d_1 \in \mathfrak{D} \mid \breve{\mathcal{A}}_T(d_1) = \breve{\mathcal{A}}_T(0)\}$, $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_I} = \{d_1 \in \mathfrak{D} \mid \breve{\mathcal{A}}_I(d_1) = \breve{\mathcal{A}}_I(0)\}$, and $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_F} = \{d_1 \in \mathfrak{D} \mid \breve{\mathcal{A}}_F(d_1) = \breve{\mathcal{A}}_F(0)\}$ are d-ideals of $\mathfrak{D}$.

Proof: Assume that $\breve{\mathcal{A}} = (\breve{\mathcal{A}}_T, \breve{\mathcal{A}}_I, \breve{\mathcal{A}}_F)$ be an IvNSN-d-I of $\mathfrak{D}$. Let $d_1 * d_2, d_2 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$. Therefore $\breve{\mathcal{A}}_T(d_1 * d_2) = \breve{\mathcal{A}}_T(0)$, $\breve{\mathcal{A}}_T(d_2) = \breve{\mathcal{A}}_T(0)$. Now by utilising definition 3.1, we obtain $\breve{\mathcal{A}}_T(d_1) \preccurlyeq rmax\{\breve{\mathcal{A}}_T(d_1 * d_2), \breve{\mathcal{A}}_T(d_2)\} = rmax\{\breve{\mathcal{A}}_T(0), \breve{\mathcal{A}}_T(0)\} = \breve{\mathcal{A}}_T(0) \Rightarrow \breve{\mathcal{A}}_T(d_1) \preccurlyeq \breve{\mathcal{A}}_T(0)$. From proposition 3.4, we have $\breve{\mathcal{A}}_T(0) \preccurlyeq \breve{\mathcal{A}}_T(d_1)$. Therefore $\breve{\mathcal{A}}_T(d_1) = \breve{\mathcal{A}}_T(0) \Rightarrow d_1 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$. Again let $d_1 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$ and $d_2 \in \mathfrak{D}$. Then, $\breve{\mathcal{A}}_T(d_1) = \breve{\mathcal{A}}_T(0)$ and so $\breve{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \breve{\mathcal{A}}_T(d_1) = \breve{\mathcal{A}}_T(0) \Rightarrow \breve{\mathcal{A}}_T(d_1 * d_2) \preccurlyeq \breve{\mathcal{A}}_T(0)$. By using proposition 3.4, we have $\breve{\mathcal{A}}_T(0) \preccurlyeq \breve{\mathcal{A}}_T(d_1 * d_2)$. Hence $\breve{\mathcal{A}}_T(d_1 * d_2) = \breve{\mathcal{A}}_T(0)$. i.e. $d_1 * d_2 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$. Therefore, $d_1 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}, d_2 \in \mathfrak{D} \Rightarrow d_1 * d_2 \in \breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$. Hence the set $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_T}$ is a d-ideal of $\mathfrak{D}$. Similarly, we can show that $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_I}$ and $\breve{\mathcal{A}}_{\breve{\mathcal{A}}_F}$ are d- ideals of $\mathfrak{D}$.

Theorem 3.17. If f is a d-homomorphism function from d-algebra $\mathfrak{D}_1$ into a d-algebra $\mathfrak{D}_2$ and $\breve{\mathcal{A}} = (\breve{\mathcal{A}}_T, \breve{\mathcal{A}}_I, \breve{\mathcal{A}}_F)$ be an IvNSN-d-I of $\mathfrak{D}_2$. Then, $f^{-1}(\breve{\mathcal{A}}) = \left(f^{-1}(\breve{\mathcal{A}}_T), f^{-1}(\breve{\mathcal{A}}_I), f^{-1}(\breve{\mathcal{A}}_F)\right)$ is an IvNSN-d-I of $\mathfrak{D}_1$, where $f^{-1}(\breve{\mathcal{A}}_T)(d_1) = \breve{\mathcal{A}}_T(f(d_1))$, $f^{-1}(\breve{\mathcal{A}}_I)(d_1) = \breve{\mathcal{A}}_I(f(d_1))$, and $f^{-1}(\breve{\mathcal{A}}_F)(d_1) = \breve{\mathcal{A}}_F(f(d_1))$ for all $d_1 \in \mathfrak{D}_1$.

Proof: For any $d_1, d_2 \in \mathfrak{D}_1$, we have

$$f^{-1}\left(\check{\mathcal{A}}_T(d_1)\right) = \check{\mathcal{A}}_T(f(d_1)) \preccurlyeq rmax\{\check{\mathcal{A}}_T(f(d_1) * f(d_2)), \check{\mathcal{A}}_T(f(d_2))\}$$
$$= rmax\{\check{\mathcal{A}}_T(f(d_1 * d_2)), \check{\mathcal{A}}_T(f(d_2))\}$$
$$= rmax\left\{f^{-1}\left(\check{\mathcal{A}}_T(d_1 * d_2)\right), f^{-1}\left(\check{\mathcal{A}}_T(d_2)\right)\right\},$$

$$f^{-1}\left(\check{\mathcal{A}}_T(d_1 * d_2)\right) = \check{\mathcal{A}}_T(f(d_1 * d_2)) = \check{\mathcal{A}}_T(f(d_1) * f(d_2))$$
$$\preccurlyeq \check{\mathcal{A}}_T(f(d_1)) = f^{-1}\left(\check{\mathcal{A}}_T(d_1)\right),$$

$$f^{-1}\left(\check{\mathcal{A}}_I(d_1)\right) = \check{\mathcal{A}}_I(f(d_1)) \succcurlyeq rmin\{\check{\mathcal{A}}_I(f(d_1) * f(d_2)), \check{\mathcal{A}}_I(f(d_2))\}$$
$$= rmin\{\check{\mathcal{A}}_I(f(d_1 * d_2)), \check{\mathcal{A}}_I(f(d_2))\}$$
$$= rmin\left\{f^{-1}\left(\check{\mathcal{A}}_I(d_1 * d_2)\right), f^{-1}\left(\check{\mathcal{A}}_I(d_2)\right)\right\},$$

$$f^{-1}\left(\check{\mathcal{A}}_I(d_1 * d_2)\right) = \check{\mathcal{A}}_I(f(d_1 * d_2)) = \check{\mathcal{A}}_I(f(d_1) * f(d_2))$$
$$\succcurlyeq \check{\mathcal{A}}_I(f(d_1)) = f^{-1}\left(\check{\mathcal{A}}_I(d_1)\right),$$

$$f^{-1}\left(\check{\mathcal{A}}_F(d_1)\right) = \check{\mathcal{A}}_F(f(d_1)) \preccurlyeq rmax\{\check{\mathcal{A}}_F(f(d_1) * f(d_2)), \check{\mathcal{A}}_F(f(d_2))\}$$
$$= rmax\{\check{\mathcal{A}}_F(f(d_1 * d_2)), \check{\mathcal{A}}_F(f(d_2))\}$$
$$= rmax\left\{f^{-1}\left(\check{\mathcal{A}}_F(d_1 * d_2)\right), f^{-1}\left(\check{\mathcal{A}}_F(d_2)\right)\right\},$$

$$f^{-1}\left(\check{\mathcal{A}}_F(d_1 * d_2)\right) = \check{\mathcal{A}}_F(f(d_1 * d_2)) = \check{\mathcal{A}}_F(f(d_1) * f(d_2))$$
$$\preccurlyeq \check{\mathcal{A}}_F(f(d_1)) = f^{-1}\left(\check{\mathcal{A}}_F(d_1)\right).$$

Hence, $f^{-1}(\check{\mathcal{A}})$ is an IvNSN-d-I over $\mathfrak{D}_1$.

Theorem 3.18. If f is a d-epimorphism function from d-algebra $\mathfrak{D}_1$ into a d-algebra $\mathfrak{D}_2$ and $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ be an IvNSN-S of $\mathfrak{D}_2$. If $f^{-1}(\check{\mathcal{A}}) = \left(f^{-1}(\check{\mathcal{A}}_T), f^{-1}(\check{\mathcal{A}}_I), f^{-1}(\check{\mathcal{A}}_F)\right)$ is an IvNSN-d-I of $\mathfrak{D}_1$, then $\check{\mathcal{A}} = (\check{\mathcal{A}}_T, \check{\mathcal{A}}_I, \check{\mathcal{A}}_F)$ is an IvNSN-d-I of $\mathfrak{D}_2$.

Proof: Let $d_1, d_2 \in \mathfrak{D}_2$. Then their exits $\mathrm{i}_1, \mathrm{i}_2 \in \mathfrak{D}_1$ such that $f(\mathrm{i}_1) = d_1$ and $f(\mathrm{i}_2) = d_2$.

$$\check{\mathcal{A}}_T(d_1) = \check{\mathcal{A}}_T(f(\mathrm{i}_1)) = f^{-1}\left(\check{\mathcal{A}}_T(\mathrm{i}_1)\right) \preccurlyeq rmax\left\{f^{-1}\left(\check{\mathcal{A}}_T(\mathrm{i}_1 * \mathrm{i}_2)\right), f^{-1}\left(\check{\mathcal{A}}_T(\mathrm{i}_2)\right)\right\}$$
$$= rmax\{\check{\mathcal{A}}_T(f(\mathrm{i}_1 * \mathrm{i}_2)), \check{\mathcal{A}}_T(f(\mathrm{i}_2))\}$$
$$= rmax\{\check{\mathcal{A}}_T(f(\mathrm{i}_1) * f(\mathrm{i}_2)), \check{\mathcal{A}}_T(f(\mathrm{i}_2))\}$$
$$= rmax\{\check{\mathcal{A}}_T(d_1 * d_2), \check{\mathcal{A}}_T(d_2)\},$$

$$\check{\mathcal{A}}_T(d_1 * d_2) = \check{\mathcal{A}}_T(f(\mathrm{i}_1) * f(\mathrm{i}_2)) = \check{\mathcal{A}}_T(f(\mathrm{i}_1 * \mathrm{i}_2)) = f^{-1}\left(\check{\mathcal{A}}_T(\mathrm{i}_1 * \mathrm{i}_2)\right)$$
$$\preccurlyeq f^{-1}\left(\check{\mathcal{A}}_T(\mathrm{i}_1)\right) = \check{\mathcal{A}}_T(f(\mathrm{i}_1)) = \check{\mathcal{A}}_T(d_1),$$

$$\check{\mathcal{A}}_I(d_1) = \check{\mathcal{A}}_I(f(\mathrm{i}_1)) = f^{-1}\left(\check{\mathcal{A}}_I(\mathrm{i}_1)\right) \succcurlyeq rmin\left\{f^{-1}\left(\check{\mathcal{A}}_I(\mathrm{i}_1 * \mathrm{i}_2)\right), f^{-1}\left(\check{\mathcal{A}}_I(\mathrm{i}_2)\right)\right\}$$
$$= rmin\{\check{\mathcal{A}}_I(f(\mathrm{i}_1 * \mathrm{i}_2)), \check{\mathcal{A}}_I(f(\mathrm{i}_2))\}$$

$$= rmin\{\check{\mathcal{A}}_I(f(\mathrm{i}_1) * f(\mathrm{i}_2)), \check{\mathcal{A}}_I(f(\mathrm{i}_2))\}$$

$$= rmin\{\check{\mathcal{A}}_I(d_1 * d_2), \check{\mathcal{A}}_I(d_2)\},$$

$$\check{\mathcal{A}}_I(d_1 * d_2) = \check{\mathcal{A}}_I(f(\mathrm{i}_1) * f(\mathrm{i}_2)) = \check{\mathcal{A}}_I(f(\mathrm{i}_1 * \mathrm{i}_2)) = f^{-1}\left(\check{\mathcal{A}}_I(\mathrm{i}_1 * \mathrm{i}_2)\right)$$

$$\succcurlyeq f^{-1}\left(\check{\mathcal{A}}_I(\mathrm{i}_1)\right) = \check{\mathcal{A}}_I(f(\mathrm{i}_1)) = \check{\mathcal{A}}_I(d_1),$$

$$\check{\mathcal{A}}_F(d_1) = \check{\mathcal{A}}_F(f(\mathrm{i}_1)) = f^{-1}\left(\check{\mathcal{A}}_F(\mathrm{i}_1)\right) \preccurlyeq rmax\left\{f^{-1}\left(\check{\mathcal{A}}_F(\mathrm{i}_1 * \mathrm{i}_2)\right), f^{-1}\left(\check{\mathcal{A}}_F(\mathrm{i}_2)\right)\right\}$$

$$= rmax\{\check{\mathcal{A}}_F(f(\mathrm{i}_1 * \mathrm{i}_2)), \check{\mathcal{A}}_F(f(\mathrm{i}_2))\}$$

$$= rmax\{\check{\mathcal{A}}_F(f(\mathrm{i}_1) * f(\mathrm{i}_2)), \check{\mathcal{A}}_F(f(\mathrm{i}_2))\}$$

$$= rmax\{\check{\mathcal{A}}_F(d_1 * d_2), \check{\mathcal{A}}_F(d_2)\},$$

$$\check{\mathcal{A}}_F(d_1 * d_2) = \check{\mathcal{A}}_F(f(\mathrm{i}_1) * f(\mathrm{i}_2)) = \check{\mathcal{A}}_F(f(\mathrm{i}_1 * \mathrm{i}_2)) = f^{-1}\left(\check{\mathcal{A}}_F(\mathrm{i}_1 * \mathrm{i}_2)\right)$$

$$\preccurlyeq f^{-1}\left(\check{\mathcal{A}}_F(\mathrm{i}_1)\right) = \check{\mathcal{A}}_F(f(\mathrm{i}_1)) = \check{\mathcal{A}}_F(d_1).$$

Hence the proof is completed.

4 | Conclusion

In this research, we presented the innovative idea of IvNSN-d-I by applying IvNSN-S to the d-ideal of d-algebra. We have clearly shown that the pre-image and the image of an IvNSN-d-I under homeomorphism and epimorphism, respectively, remain IvNSN-d-Is. Furthermore, we investigated and characterized several important features of IvNSN-d-Is.

4.1 | Limitations and Future Research

In everyday life, some scenarios may involve negative features in expressing opinions about a particular problem. Negative features point out choices that are either prohibited or unattainable. Interval-valued neutrosophic N-structure offers the ability to make decisions based on negative features. The methodology used in this manuscript is also applicable to ideals of other algebraic structures such as semigroups, near-rings, b-algebra, BCH-algebra, etc.

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Author Contributions

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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