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Results and Discussion of the Food Chain Model with Different Fear Effects under Uncertain Environment

Pramodh Bharati [1,2], Balaram Manna [1], Ashish Acharya [3], Subrata Paul [4], Animesh Mahata [5,*], Subhabrata Mondal [1], Supriya Mukherjee [6], and Banamali Roy [7]

[1] Department of Mathematics, Swami Vivekananda university, Barrackpore-700121, India.
Emails: pramodbharatinilu@gmail.com; balarammanna.1986@gmail.com; subhabratab@svu.ac.in.
[2] Department of Mathematics, Ramnagar College, Depal, PurbaMedinipur, West Bengal-721453, India; pramodbharatinilu@gmail.com.
[3] Department of Mathematics, Swami Vivekananda Institute of Modern Science, Karbala More, West Bengal –700103, India; ashishacharya314@gmail.com.
[4] Department of Mathematics, Arambagh Govt. Polytechnic, Arambagh, West Bengal, India; paulsubrata564@gmail.com.
[5] Department of Mathematics, Sri RamkrishnaSaradaVidyaMahapitha, Kamarpukur, Hooghly -712612, West Bengal, India; animeshmahata8@gmail.com.
[6] Department of Mathematics, Gurudas College,1/1 Suren Sarkar Road,Kolkata-700054, West Bengal, India; supriyasbku2013@gmail.com.
[7] Department of Mathematics, Bangabasi Evening College, Kolkata-700009, West Bengal, India; banamaliroy@yahoo.co.in.

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Abstract

The parameter involve in ecological model always uncertain due to environmental and demographic fuzziness. In this paper, we consider a food chain model with different fear effects in which all ecological parameters are taken into account in a parametric functional form of an interval number. In the suggested model,the middle predator is the specialist. Using the interval differential equation approach, we point out the importance of modeling on uncertainity by the proposed model. Existence and uniqueness along with non-negativity and boundedness of the model system have been investigated. Equilibrium points and their feasibility and nature of critical points of the proposed system are discussed in uncertain environment with the help of MATLAB. We were able to conduct graphical demonstrations and numerical simulations.

Keywords: Tri-tropic Food Chain Model, Different Level Fear Effect, Stability Analysis, Numerical Simulation.

1 | Introduction

Ecology, a field within the realm of biological science, focuses on the interactions among living organisms and their physical surroundings, encompassing both living and non-living elements. As technological advancements and environmental impacts have rapidly expanded, ecological research has flourished across

Corresponding Author: animeshmahata8@gmail.com
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various scales. Ecosystem ecologists often emphasize the flow of energy and the recycling of nutrients. In ecological studies, one of the central elements is the interaction between predators and prey, which dictates the transfer of energy and biomass from one trophic level to another, influencing population sizes. The responses of prey to predators have a profound impact on the dynamics of these interactions, either directly or indirectly affecting the predator and prey populations. Following the pioneering work of Lotka and Volterra [1-2], numerous intricate predator-prey models have been developed to elucidate the dynamics of these systems in diverse real-world settings [3-9]. Researchers have proposed several hypotheses to explain the coexistence of interacting populations in different environments. Much attention has been devoted to two-species or three-species predator-prey models. Hastings and Powell [10] and Alidoust and Ghahfarokhi [11] explored a three-species food chain model, revealing chaotic behavior. Roy and Alam [12] analyzed versions of a food chain model, considering intra-species competition in the top predator. Freedman and Waltman [13] identified conditions for the persistence of a three-species predator-prey model. Moura and Silva [14] elucidated the food web and ecological models applicable to aquatic ecosystems. Abrams [15] demonstrated the relationship between food availability and the foraging efforts of species in their ecological environment. Nath et al. [16] illustrated how refuge and Allee effects in prey species can stabilize chaos in tri-trophic food chain models. Barbier et al. [17] discussed pyramids, cascades, and the synthesis of functionality and stability in food chains. Banerjee and Das [18] investigated the impulsive effects on a tri-trophic food chain model, considering mixed functional responses under seasonal perturbations. Castellonas et al. [19] observed both Hopf and Bautin bifurcations in a tritrophic food chain model. Huang et al. [20] delved into the dynamic behavior of a food chain model with stage structure and time delays.

1.1 | Motivation and Novelty

Numerous studies of mathematical models of biology concentrate on the deterministic model, which is primarily based on the law of large numbers and the presumption that, if there are enough biological individuals, the system's behaviors will exhibit a reasonably stable statistical regularity. Numerous scholars previously used the interval approach and stochastic approach to address these uncertainty concerns [21-28]. The unknown parameters are described by interval-valued functions in the interval method. We use fuzzy set theory to represent erroneous parameters to get around these problems. For the first time, Professor Zadeh established the fuzzy set theory [29] and he also suggested using fuzzy differential equations as a natural technique to model dynamic systems with probabilistic uncertainty [30]. It is important to note that based on the fuzzy instantaneous annual rate of discount, the optimal harvesting strategy is explored by Sadhukhan et al. [31] when they investigated the optimal harvesting of a food chain model in a fuzzy environment. The majority of research studies in mathematical modeling, particularly bio-mathematical modeling, take place in a crisp (precise) environment. Therefore, in these circumstances, the majority of a biological model's parameters are not exact. A biological model becomes inaccurate if some ambiguous parameters are included. We have considered a tri-tropic food chain model where the middle predator is a generalized predator and the top predator is a specialist. In addition, two fear effects are suggested in this paper.

1.2 | Structure of the Paper

The rest of this article is organized as follows: Some basic definitions such as interval number, parametric representation of a number in the interval, and some properties of interval number are discussed in detail in Section 2. We develop the mathematical model as well as the system's positivity, and boundedness in Section 3. In Section 4, we study the local stability analysis and global stability analysis. In Section 5, we use numerical simulations to validate the outcomes of our analysis. The article concludes with a few conclusions in Section 6.

2 | Pre-requisite Concept

Definition. For an interval $[T_{m_1}, T_{n_1}]$ The interval-valued function can be created as$k_1(\eta) = (T_{m_1})^{1-\eta}(T_{n_1})^{\eta}for\eta \in [0,1]$, which is also called parametric form in interval figure.

Properties. Let, two intervals (parametric form) as $k_1(\eta) = (T_{m_1})^{1-\eta}(T_{n_1})^{\eta}$ and $h_1(\eta) = (R_{m_1})^{1-\eta}(R_{n_1})^{\eta}$ for $\in [0,1]$, then the following operation was obtained:

1. $g_1(\eta) = k_1(\eta) + h_1(\eta) = (T_{m_1} + R_{m_1})^{1-\eta}(T_{n_1} + R_{n_1})^{\eta}$.
2. $s_1(\eta) = k_1(\eta) - h_1(\eta) = (T_{m_1} - R_{m_1})^{1-\eta}(T_{n_1} - R_{n_1})^{\eta}$.
3. $r_1(\eta) = k_1(\eta)h_1(\eta) = (\min\{T_{m_1}R_{m_1}, T_{n_1}R_{n_1}, T_{m_1}T_{n_1}, R_{m_1}R_{n_1}\})^{1-\eta}(\max\{\{T_{m_1}R_{m_1}, T_{n_1}R_{n_1}, T_{m_1}T_{n_1}, R_{m_1}R_{n_1}\}\})^{\eta}$
4. $y\varsigma_1(\eta) = e(\eta) = y(T_{m_1})^{1-\eta}(T_{n_1})^{\eta} if y > 0,$

   $= y(T_{m_1})^{1-\eta}(T_{n_1})^{\eta} if y < 0.$
5. $p_1(\alpha) = \frac{k_1(\eta)}{h_1(\eta)} = (\min\{\frac{T_{m_1}}{R_{m_1}}, \frac{R_{n_1}}{T_{n_1}}, \frac{R_{m_1}}{T_{n_1}}, \frac{T_{n_1}}{R_{m_1}}\})^{1-\eta}(\max\{\{\frac{R_{m_1}}{T_{m_1}}, \frac{R_{n_1}}{T_{n_1}}, \frac{T_{m_1}}{R_{n_1}}, \frac{T_{n_1}}{R_{m_1}}\}\})^{\eta}$.

Where the parametric interval-valued function is $g_1(\eta), s_1(\eta), r_1(\eta), yk_1(\eta), p_1(\alpha), e(\eta)$ for constant ς_1 and $\eta \in [0,1]$.

3 | Proposed Model

Let $u(t), v(t)$, and $w(t)$ represent the density of the prey population, intermediate predator, and top predator, respectively, at any time t.To formulate the model system, the following assumption is made: $u\ (t)$ and $v\ (t)$ increase at the growth rate of r_1 and r_2, with two carrying capacity K_1 and K_2, p_1 and p_2 represents the predation rate, according to Holling Type-II functional response q_1 and q_2 represents the rate of conversion, d_1 represent natural data rate, m_1 and m_2 represents the fear levels, c_1 represent a half-saturation constant. Let E_1 and E_2 be the harvesting effort and d_1, d_2 be the catchability coefficients of intermediate predator and top predator respectively.

Our proposed bio-mathematical model is

$$\frac{du}{dt} = \frac{r_1 u}{1+m_1 w}\left(1 - \frac{u}{K_1}\right) - p_1 uv,$$

$$\frac{dv}{dt} = \frac{r_2 v}{1+m_2 w}\left(1 - \frac{v}{K_2}\right) + p_2 uv - \frac{q_1 vw}{c_1+v} - E_1 d_1 v, \quad (1)$$

$$\frac{dw}{dt} = \frac{q_2 vw}{c_1 + v} - d_1 w - E_2 d_2 w.$$

3.1 | Model in Imprecise Environment

The proposed model (1) in the imprecise environment can be changed with the coefficients taken as interval numbers as follows:

$$\frac{du}{dt} = \frac{\hat{r}_1 u}{1+\widehat{m}_1 w}\left(1 - \frac{u}{K_1}\right) - \hat{p}_1 uv,$$

$$\frac{dv}{dt} = \frac{\hat{r}_2 v}{1+\widehat{m}_2 w}\left(1 - \frac{v}{K_2}\right) + \hat{p}_2 uv - \frac{\hat{q}_1 vw}{\hat{c}_1+v} - E_1 d_1 v, \quad (2)$$

$$\frac{dw}{dt} = \frac{\hat{q}_2 vw}{\hat{c}_1+v} - \hat{d}_1 w - E_2 d_2 w.$$

Now, we take $I_l(\rho) = I_{1L}^{1-\rho} I_{1R}^{\rho}$ for $\rho \in [0,1]$ for an interval $[I_{1L}, I_{1R}]$. Then the above system (2) can be written as follows:

$$\frac{du}{dt}=\frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+wm_{1L}^{1-\rho}m_{1R}^{\rho}}u\left(1-\frac{u}{K_1}\right)-p_{1R}^{1-\rho}p_{1L}^{\rho}uv,$$

$$\frac{dv}{dt}=\frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{2L}^{1-\rho}m_{2R}^{\rho}w}v\left(1-\frac{v}{K_2}\right)+p_{2L}^{1-\rho}p_{2R}^{\rho}uv-\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{c_{1R}^{1-\rho}c_{1L}^{\rho}+v}vw-E_1d_1v, \quad (3)$$

$$\frac{dw}{dt}=\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v}vw-d_{1R}^{1-\rho}d_{1L}^{\rho}w-E_2d_2w.$$

where, $\rho\in[0,1]$.

3.2 | Positivity and Boundedness of the Proposed Model

Theorem 1. All the solutions of the model system (3) are positive.

Proof. From the model system (3), we have $du=u\psi(u,v,w)dt$, with

$$\psi(u,v,w)=\left[\frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w}\left(1-\frac{u}{K_1}\right)-p_{1R}^{1-\rho}p_{1L}^{\rho}v\right].$$

Taking integration, then $u(t)=u(0)e^{\int_0^t\psi(u,v,w)dt}>0\ \forall t.$

Similarly, $v(t)=v(0)e^{\int_0^t\varphi(u,v,w)dt}>0\forall t.$

Where, $\varphi(u,v,w)=\frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{2L}^{1-\rho}m_{2R}^{\rho}w}\left(1-\frac{v}{K_2}\right)+p_{2L}^{1-\rho}p_{2R}^{\rho}u-\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{c_{1R}^{1-\rho}c_{1L}^{\rho}+v}w$

And $w(t)=w(0)e^{\int_0^t\theta(u,v,w)dt}>0\ \forall t.$

where, $\theta(u,v,w)=\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v}v-d_{1R}^{1-\rho}d_{1L}^{\rho}$

As a result, all of the solutions are positive.

Theorem 2. The compact set S $=\left\{(u,v,w)\in R_+^3:\ 0<z(t)<\frac{\beta_L^{1-\rho}\beta_R^{\rho}}{\Omega_L^{1-\rho}\Omega_R^{\rho}}\right\}$. All the solution of the model system (3) is ultimately bounded solution.

Proof. Let the function $Z=u+\frac{p_{1R}^{1-\rho}P_{1L}^{\rho}}{p_{2L}^{1-\rho}P_{2R}^{\rho}}v+\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}}w.$

Differentiating with respect to time of the above function, we have

$$\frac{dZ}{dt}=\frac{du}{dt}+\frac{p_{1R}^{1-\rho}P_{1L}^{\rho}}{p_{2L}^{1-\rho}P_{2R}^{\rho}}\frac{dv}{dt}+\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}}\frac{dw}{dt},$$

$$=\frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w}u\left(1-\frac{u}{K_1}\right)-p_{1R}^{1-\rho}p_{1L}^{\rho}uv+\frac{p_{1R}^{1-\rho}P_{1L}^{\rho}}{p_{2L}^{1-\rho}P_{2R}^{\rho}}\left(\frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{2L}^{1-\rho}m_{2R}^{\rho}w}v\left(1-\frac{v}{K_2}\right)+p_{2L}^{1-\rho}p_{2R}^{\rho}uv-\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{c_{1R}^{1-\rho}c_{1L}^{\rho}+v}vw\right)+\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}}\left(\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v}vw-d_{1R}^{1-\rho}d_{1L}^{\rho}w\right)$$

$$\leq-\Omega\left(u+\frac{p_{1R}^{1-\rho}p_{1L}^{\rho}}{p_{2L}^{1-\rho}p_{2R}^{\rho}}v+\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}}w\right)-\frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{K_1(1+m_{1L}^{1-\rho}m_{1R}^{\rho}w)}(u-K_1)^2-\frac{p_{1R}^{1-\rho}p_{1L}^{\rho}r_{2L}^{1-\rho}r_{2R}^{\rho}}{K_2p_{2L}^{1-\rho}p_{2R}^{\rho}\left(1+m_{2L}^{1-\rho}m_{2R}^{\rho}w\right)}v^2+\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v}vw.$$

Where, $\Omega=\min\{r_{1L}^{1-\rho}r_{1R}^{\rho},\ p_{1R}^{1-\rho}p_{1L}^{\rho},d_{1R}^{1-\rho}d_{1L}^{\rho},r_{2L}^{1-\rho}r_{2R}^{\rho}\}$

$$\frac{dZ}{dt}\leq-\Omega_L^{1-\rho}\Omega_R^{\rho}z+\beta_L^{1-\rho}\beta_R^{\rho}. \quad (4)$$

where, $\beta_L^{1-\rho}\beta_R^{\rho}=\frac{q_{1R}^{1-\rho}q_{1L}^{\rho}}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v}vw$.

Solving the above inequality (4), using the theory of differential inequality, we get

$$0< z(t) < \frac{\beta_L^{1-\rho}\beta_R^{\rho}}{\Omega_L^{1-\rho}\Omega_R^{\rho}}\left(1-e^{-\Omega_L^{1-\rho}\Omega_R^{\rho}t}\right)+z(0)e^{-\Omega_L^{1-\rho}\Omega_R^{\rho}t}.$$

Now taking the limit of the above inequality as $t \to \infty$, we have

$$0< z(t) < \frac{\beta_L^{1-\rho}\beta_R^{\rho}}{\Omega_L^{1-\rho}\Omega_R^{\rho}}. \tag{5}$$

From Eq. (5) all the solutions are bounded in $\mathbb{R}_+^3$.

4 | Stability Analysis

In this section, we obtained trivial and non-trivial equilibrium points of the system (3) and stated their existence condition, also including the stability analysis of these equilibrium points.

4.1 | Existence of Equilibrium Point

The model system (3) has five equilibrium points, considered as follows:

1. Trivial equilibrium point $E_{00}(0,0,0)$, which always exist.
2. Axial equilibrium point $E_{11}(K_1,0,0)$, which always exists.
3. Another axial point $E_{22}(0,K_2,0)$, which always exists.
4. Planer equilibrium point $E_P(u_2,v_2,0)$ on the uv −plane where, $u_2=\frac{K_1 r_{2L}^{1-\rho}r_{2R}^{\rho}(r_{1L}^{1-\rho}r_{1R}^{\rho}-p_{1R}^{1-\rho}p_{1L}^{\rho}\mathrm{K}_2)}{r_{1L}^{1-\rho}r_{1R}^{\rho}r_{2L}^{1-\rho}r_{2R}^{\rho}+p_{1R}^{1-\rho}p_{1L}^{\rho}p_{2L}^{1-\rho}p_{2R}^{\rho}\mathrm{K}_1\mathrm{K}_2}$, $v_2=\frac{r_{1L}^{1-\rho}r_{1R}^{\rho}p_{1R}^{1-\rho}p_{1L}^{\rho}\mathrm{K}_2(\mathrm{K}_1+r_{2L}^{1-\rho}r_{2R}^{\rho})}{r_{1L}^{1-\rho}r_{1R}^{\rho}r_{2L}^{1-\rho}r_{2R}^{\rho}+p_{1R}^{1-\rho}p_{1L}^{\rho}p_{2L}^{1-\rho}p_{2R}^{\rho}\mathrm{K}_1\mathrm{K}_2}$. Therefore, the planer equilibrium point exists if $r_{1L}^{1-\rho}r_{1R}^{\rho}>p_{1R}^{1-\rho}p_{1L}^{\rho}\mathrm{K}_2$.
5. The interior equilibrium point is $E_I^*(u^*,v^*,w^*)$, where u^*,v^*,w^* obtained as follows,

$u^*$is the positive root of the following quadratic equation,${u^*}^2+B_1u^*+B_2=0$,

where, $B_1=\frac{K_1^2}{K_2 r_{1L}^{1-\rho}r_{1R}^{\rho}m_{2L}^{1-\rho}m_{2R}^{\rho}\left(q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}\right)q_{1R}^{1-\rho}q_{1L}^{\rho}-\mathrm{K}_1p_{2L}^{1-\rho}p_{2R}^{\rho}(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*)}\Big[p_{2L}^{1-\rho}p_{2R}^{\rho}\mathrm{K}_2\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)+\frac{(m_{2L}^{1-\rho}m_{2R}^{\rho}+d_{1R}^{1-\rho}d_{1L}^{\rho})K_2\left(q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}\right)r_{1L}^{1-\rho}r_{1R}^{\rho}}{p_{1R}^{1-\rho}p_{1L}^{\rho}m_{1L}^{1-\rho}m_{1R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}\mathrm{K}_1}-\frac{m_{2L}^{1-\rho}m_{2R}^{\rho}K_2}{m_{1L}^{1-\rho}m_{1R}^{\rho}}+q_{1R}^{1-\rho}q_{1L}^{\rho}m_{2L}^{1-\rho}m_{2R}^{\rho}k_2\left(\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}{p_{1R}^{1-\rho}p_{1L}^{\rho}m_{1L}^{1-\rho}m_{1R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}\right)^2\left(\frac{2(r_{1L}^{1-\rho}r_{1R}^{\rho})^2}{K_1}-\frac{2r_{1L}^{1-\rho}r_{1R}^{\rho}p_{1R}^{1-\rho}p_{1L}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}{K_1\left(q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}\right)}\right)\Big]$,

$B_2=\frac{K_1^2}{K_2 r_{1L}^{1-\rho}r_{1R}^{\rho}m_{2L}^{1-\rho}m_{2R}^{\rho}\left(q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}\right)q_{1R}^{1-\rho}q_{1L}^{\rho}-\mathrm{K}_1p_{2L}^{1-\rho}p_{2R}^{\rho}(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*)}\Big[r_{2L}^{1-\rho}r_{2R}^{\rho}(\mathrm{K}_2-v^*)\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)-q_{1R}^{1-\rho}q_{1L}^{\rho}\mathrm{K}_2\left(\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}{p_{1R}^{1-\rho}p_{1L}^{\rho}m_{1L}^{1-\rho}m_{1R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}\right)\left(r_{1L}^{1-\rho}r_{1R}^{\rho}-\frac{p_{1R}^{1-\rho}p_{1L}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}\right)-q_{1R}^{1-\rho}q_{1L}^{\rho}m_{2L}^{1-\rho}m_{2R}^{\rho}\mathrm{K}_2\left(\frac{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}{p_{1R}^{1-\rho}p_{1L}^{\rho}m_{1L}^{1-\rho}m_{1R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}\right)^2\left\{\left(r_{1L}^{1-\rho}r_{1R}^{\rho}\right)^2-\frac{2r_{1L}^{1-\rho}r_{1R}^{\rho}p_{1R}^{1-\rho}p_{1L}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}+\left(\frac{p_{1R}^{1-\rho}p_{1L}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}d_{1R}^{1-\rho}d_{1L}^{\rho}}{q_{2L}^{1-\rho}q_{2R}^{\rho}-d_{1R}^{1-\rho}d_{1L}^{\rho}}\right)^2\right\}\Big]$.

Therefore, the roots of the above quadratic equation given by,

$u^* = \frac{-B_1 \pm \sqrt{B_1^2 - 4B_1B_2}}{2}.$

So, for a positive equilibrium point E_I^*, u^*attains at least one positive root if the following condition is satisfied as follows

i). $B_1 > 0, B_2 > 0$,

ii). $B_1 < 0, B_2 < 0$,

iii). $B_1 > 0, B_2 > 0$ and $B_1^2 - 4B_1B_2 > 0$.

The positiveness of v^*, w^* holds as follows:

$v^* = \frac{c_{1R}^{1-\rho} c_{1L}^{\rho} d_{1R}^{1-\rho} d_{1L}^{\rho}}{q_{2L}^{1-\rho} q_{2R}^{\rho} - d_{1R}^{1-\rho} d_{1L}^{\rho}},$

$w^* = \frac{q_{2L}^{1-\rho} q_{2R}^{\rho} - d_{1R}^{1-\rho} d_{1L}^{\rho}}{p_{1R}^{1-\rho} p_{1L}^{\rho} m_{1L}^{1-\rho} m_{1R}^{\rho} c_{1R}^{1-\rho} c_{1L}^{\rho} d_{1R}^{1-\rho} d_{1L}^{\rho}} \left[r_{1L}^{1-\rho} r_{1R}^{\rho} \left(1 - \frac{u^*}{K_1}\right) - \frac{p_{1R}^{1-\rho} p_{1L}^{\rho} m_{1L}^{1-\rho} c_{1R}^{1-\rho} c_{1L}^{\rho} d_{1R}^{1-\rho} d_{1L}^{\rho}}{q_{2L}^{1-\rho} q_{2R}^{\rho} - d_{1R}^{1-\rho} d_{1L}^{\rho}} \right],$

which exists if, $r_{1L}^{1-\rho} r_{1R}^{\rho} \left(1 - \frac{u^*}{K_1}\right) > \frac{p_{1R}^{1-\rho} p_{1L}^{\rho} m_{1L}^{1-\rho} c_{1R}^{1-\rho} c_{1L}^{\rho} d_{1R}^{1-\rho} d_{1L}^{\rho}}{q_{2L}^{1-\rho} q_{2R}^{\rho} - d_{1R}^{1-\rho} d_{1L}^{\rho}}.$

Now, we analyze the stability condition of the model system (3) at all equilibrium points as follows theorem.

Theorem 3. The trivial equilibrium point $E_{00}(0,0,0)$ of the system (3) always unstable.

Proof. The variational matrix $V_{E_{00}}$at E_{00} is given by,

$$V_{E_{00}} = \begin{pmatrix} r_{1L}^{1-\rho} r_{1R}^{\rho} & 0 & 0 \\ 0 & r_{2L}^{1-\rho} r_{2R}^{\rho} & 0 \\ 0 & 0 & -d_{1R}^{1-\rho} d_{1L}^{\rho} \end{pmatrix}.$$

Consider η is the eigenvalue of $V_{E_{00}}$, then the characteristics equationdet $(V_{E_{00}} - \eta I) = 0$.

Therefore, the eigenvalues of $V_{E_{00}}$are $r_{1L}^{1-\rho} r_{1R}^{\rho}$,$r_{2L}^{1-\rho} r_{2R}^{\rho}$, $-d_{1R}^{1-\rho} d_{1L}^{\rho}$. Here one of two eigenvalues is positive, E_{00} is the saddle point. The system (3) is unstable at E_{00}.

Theorem 4. The axial equilibrium point $E_1(k_1, 0,0)$ of the system (3) is unstable.

Proof. Thevariational matrix V_{E_1}at E_1 is given by,

$$V_{E_1} = \begin{pmatrix} -r_{1L}^{1-\rho} r_{1R}^{\rho} & -p_{1R}^{1-\rho} p_{1L}^{\rho} K_1 & 0 \\ 0 & r_{2L}^{1-\rho} r_{2R}^{\rho} + p_{2L}^{1-\rho} p_{2R}^{\rho} K_1 & 0 \\ 0 & 0 & -d_{1R}^{1-\rho} d_{1L}^{\rho} \end{pmatrix}.$$

Consider η_1 be the eigenvalue of V_{E_1}, then the characteristic equation is det$(V_{E_1} - \eta_1 I) = 0$.

The eigenvalues are $-r_{1L}^{1-\rho} r_{1R}^{\rho}, (r_{2L}^{1-\rho} r_{2R}^{\rho} + p_{2L}^{1-\rho} p_{2R}^{\rho} K_1), -- d_{1R}^{1-\rho} d_{1L}^{\rho}$.Since one eigenvalue $(r_{2L}^{1-\rho} r_{2R}^{\rho} + p_{2L}^{1-\rho} p_{2R}^{\rho} K_1) > 0, E_1$ is a saddle point. Therefore, the system (3) is unstable at E_1.

Theorem 5. Another axial equilibrium point $E_2(0, K_2, 0)$ of the system (3) is LAS if $p_{1R}^{1-\rho} p_{1L}^{\rho} K_2 > r_{1L}^{1-\rho} r_{1R}^{\rho}$, $d_{1R}^{1-\rho} d_{1L}^{\rho} (c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2) > q_{2L}^{1-\rho} q_{2R}^{\rho} K_2$.

Proof. The variational matrix V_{E_2}at E_2 is given by,

$$V_{E_2} = \begin{pmatrix} r_{1L}^{1-\rho} r_{1R}^{\rho} - p_{1R}^{1-\rho} p_{1L}^{\rho} \mathrm{K}_2 & 0 & 0 \\ p_{2L}^{1-\rho} p_{2R}^{\rho} \mathrm{K}_2 & -r_{2L}^{1-\rho} r_{2R}^{\rho} & -\frac{q_{1R}^{1-\rho} q_{1L}^{\rho} \mathrm{K}_2}{c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2} \\ 0 & 0 & \frac{q_{2L}^{1-\rho} q_{2R}^{\rho} K_2}{c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2} - d_{1R}^{1-\rho} d_{1L}^{\rho} \end{pmatrix}.$$

Consider η_2 be the eigenvalue of V_{E_2}, then the characteristic equation is $\det(V_{E_2} - \eta_2 I) = 0$

The eigenvalues of V_{E_2} are $(r_{1L}^{1-\rho} r_{1R}^{\rho} - p_{1R}^{1-\rho} p_{1L}^{\rho} \mathrm{K}_2), -r_{2L}^{1-\rho} r_{2R}^{\rho}, \left(\frac{q_{2L}^{1-\rho} q_{2R}^{\rho} K_2}{c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2} - d_{1R}^{1-\rho} d_{1L}^{\rho}\right)$.

Since the 1st, and 3rd eigenvalues are less than zero when $p_{1R}^{1-\rho} p_{1L}^{\rho} \mathrm{K}_2 > r_{1L}^{1-\rho} r_{1R}^{\rho}$, $d_{1R}^{1-\rho} d_{1L}^{\rho}(c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2) > q_{2L}^{1-\rho} q_{2R}^{\rho} K_2$. Therefore the system (3) is locally asymptotically stable when $p_{1R}^{1-\rho} p_{1L}^{\rho} \mathrm{K}_2 > r_{1L}^{1-\rho} r_{1R}^{\rho}$, $d_{1R}^{1-\rho} d_{1L}^{\rho}(c_{1R}^{1-\rho} c_{1L}^{\rho} + K_2) > q_{2L}^{1-\rho} q_{2R}^{\rho} K_2$.

Theorem 6. The planer equilibrium point $E_P(u_2, v_2, 0)$ of the system (3) is LAS if

$q_{2L}^{1-\rho} q_{2R}^{\rho} v_2 < d_{1R}^{1-\rho} d_{1L}^{\rho}(c_{1R}^{1-\rho} c_{1L}^{\rho} + v_2), r_{1L}^{1-\rho} r_{1R}^{\rho} + r_{2L}^{1-\rho} r_{2R}^{\rho} + p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 < 2\left(\frac{r_{1L}^{1-\rho} r_{1R}^{\rho} u_2}{K_1} + \frac{r_{2L}^{1-\rho} r_{2R}^{\rho} v_2}{K_2}\right) + p_{2L}^{1-\rho} p_{2R}^{\rho} v_2$, $\left(r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} + r_{2L}^{1-\rho} r_{2R}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 + \frac{4 r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} u_2 v_2}{K_1 K_2} + \frac{2 r_{2L}^{1-\rho} r_{2R}^{\rho} v_2^2 p_{2L}^{1-\rho} p_{2R}^{\rho}}{K_2} + p_{1R}^{1-\rho} p_{1L}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 v_2\right) > 2 r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho}\left(\frac{v_2}{K_2} + \frac{u_2}{K_1}\right) + \frac{2 r_{1L}^{1-\rho} r_{1R}^{\rho} u_2^2 p_{2L}^{1-\rho} p_{2R}^{\rho}}{K_1} + p_{2L}^{1-\rho} p_{2R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} v_2 + (p_{2L}^{1-\rho} p_{2R}^{\rho})^2 u_2 v_2, r_{1L}^{1-\rho} r_{1R}^{\rho} > p_{1R}^{1-\rho} p_{1L}^{\rho} K_2$.

Proof. The variational matrix V_{E_p} at E_p is given by,

$$V_{E_p} = \begin{pmatrix} A_1 & -p_{1R}^{1-\rho} p_{1L}^{\rho} u_2 & -\left(1 - \frac{u_2}{K_1}\right) r_{1L}^{1-\rho} r_{1R}^{\rho} m_{1L}^{1-\rho} m_{1R}^{\rho} u_2 \\ p_{2L}^{1-\rho} p_{2R}^{\rho} v_2 & A_2 & A_3 \\ 0 & 0 & A_4 \end{pmatrix},$$

Where $A_1 = r_{1L}^{1-\rho} r_{1R}^{\rho} - \frac{2 r_{1L}^{1-\rho} r_{1R}^{\rho} u_2}{K_1} - p_{1R}^{1-\rho} p_{1L}^{\rho} v_2$, $A_2 = r_{2L}^{1-\rho} r_{2R}^{\rho} - \frac{2 r_{2L}^{1-\rho} r_{2R}^{\rho} v_2}{K_2} + p_{2L}^{1-\rho} p_{2R}^{\rho} u_2$, $A_3 = -r_{2L}^{1-\rho} r_{2R}^{\rho} v_2 m_{2L}^{1-\rho} m_{2R}^{\rho}\left(1 - \frac{v_2}{K_2}\right) - \frac{q_{1R}^{1-\rho} q_{1L}^{\rho} v_2}{c_{1R}^{1-\rho} c_{1L}^{\rho} + v_2}$, $A_4 = \frac{q_{2L}^{1-\rho} q_{2R}^{\rho} v_2}{c_{1R}^{1-\rho} c_{1L}^{\rho} + v_2} - d_{1R}^{1-\rho} d_{1L}^{\rho}$.

If η_3 be the eigenvalue of V_{E_p}, then the characteristic equation becomes,

$$\{(A_1 - \eta_3)(A_2 - \eta_3) - p_{1R}^{1-\rho} p_{1L}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 v_2\}(A_4 - \eta_3) = 0$$

Here, the one eigenvalue is A_4 which is negative when $q_{2L}^{1-\rho} q_{2R}^{\rho} v_2 < d_{1R}^{1-\rho} d_{1L}^{\rho}(c_{1R}^{1-\rho} c_{1L}^{\rho} + v_2)$

And, another eigen equation given as,

$$\eta_3^2 - (A_1 + A_2)\eta_3 + (A_1 A_2 + p_{1R}^{1-\rho} p_{1L}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 v_2) = 0$$

The roots of the above quadratic equation give negative values if,

$A_1 + A_2 < 0, r_{1L}^{1-\rho} r_{1R}^{\rho} + r_{2L}^{1-\rho} r_{2R}^{\rho} + p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 < 2\left(\frac{r_{1L}^{1-\rho} r_{1R}^{\rho} u_2}{K_1} + \frac{r_{2L}^{1-\rho} r_{2R}^{\rho} v_2}{K_2}\right) + p_{2L}^{1-\rho} p_{2R}^{\rho} v_2$,

And $(A_1 A_2 + p_{1R}^{1-\rho} p_{1L}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 v_2) > 0, \left(r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} + r_{2L}^{1-\rho} r_{2R}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 + \frac{4 r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} u_2 v_2}{K_1 K_2} + \frac{2 r_{2L}^{1-\rho} r_{2R}^{\rho} v_2^2 p_{2L}^{1-\rho} p_{2R}^{\rho}}{K_2} + p_{1R}^{1-\rho} p_{1L}^{\rho} p_{2L}^{1-\rho} p_{2R}^{\rho} u_2 v_2\right) > 2 r_{1L}^{1-\rho} r_{1R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho}\left(\frac{v_2}{K_2} + \frac{u_2}{K_1}\right) + \frac{2 r_{1L}^{1-\rho} r_{1R}^{\rho} u_2^2 p_{2L}^{1-\rho} p_{2R}^{\rho}}{K_1} + p_{2L}^{1-\rho} p_{2R}^{\rho} r_{2L}^{1-\rho} r_{2R}^{\rho} v_2 + (p_{2L}^{1-\rho} p_{2R}^{\rho})^2 u_2 v_2, r_{1L}^{1-\rho} r_{1R}^{\rho} > p_{1R}^{1-\rho} p_{1L}^{\rho} K_2$.

From the above mathematical calculation, we obtain that the planer equilibrium point E_p of the system (3) is locally asymptotically stable.

Theorem 7. The interior equilibrium point $E_I^*(u^*, v^*, w^*)$ of the system (3) is LAS if $\frac{2r_{1L}^{1-\rho}r_{1R}^{\rho}u^*}{K_1(1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*)} + p_{1R}^{1-\rho}p_{1L}^{\rho}v^* + \frac{2r_{2L}^{1-\rho}r_{2R}^{\rho}v^*}{K_2(1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*)} + d_{1R}^{1-\rho}d_{1L}^{\rho}w^*\left(\frac{c_{1R}^{1-\rho}c_{1L}^{\rho}}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}\right) + d_{1R}^{1-\rho}d_{1L}^{\rho} > \frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*} + \frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*} + p_{2L}^{1-\rho}p_{2R}^{\rho}u^* + \frac{q_{2L}^{1-\rho}q_{2R}^{\rho}v^*}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v^*}$, $\frac{Bp_{2L}^{1-\rho}p_{2R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}v^*w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} > ACE + \frac{Dc_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$,$(A + C + E)\left(AC + CE + p_{1R}^{1-\rho}p_{1L}^{\rho}p_{2L}^{1-\rho}p_{2R}^{\rho}u^*v^*\right) + ACE + \frac{c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{D}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} > \frac{p_{2L}^{1-\rho}p_{2R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{B}v^*w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$.

Proof. The variational matrix V_{E_I}at E_I^* is given by,

$$V_{E_I} = \begin{pmatrix} A & -p_{1R}^{1-\rho}p_{1L}^{\rho}u^* & -B \\ p_{2L}^{1-\rho}p_{2R}^{\rho}v^* & C & -D \\ 0 & \frac{c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} & E \end{pmatrix}.$$

where, $A = \frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*} - \frac{2r_{1L}^{1-\rho}r_{1R}^{\rho}u^*}{K_1(1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*)} - p_{1R}^{1-\rho}p_{1L}^{\rho}v^*, B = (1-\frac{u^*}{K_1})\frac{m_{1L}^{1-\rho}m_{1R}^{\rho}r_{1L}^{1-\rho}r_{1R}^{\rho}u^*}{(1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*)^2}$, $C = \frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*} - \frac{2r_{2L}^{1-\rho}r_{2R}^{\rho}v^*}{K_2(1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*)} + p_{2L}^{1-\rho}p_{2R}^{\rho}u^* - \frac{c_{1R}^{1-\rho}c_{1L}^{\rho}q_{1R}^{1-\rho}q_{1L}^{\rho}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$, $D = r_{2L}^{1-\rho}r_{2R}^{\rho}v^*\left(1-\frac{v^*}{K_2}\right)\frac{m_{2L}^{1-\rho}m_{2R}^{\rho}}{(1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*)^2} + \frac{q_{1R}^{1-\rho}q_{1L}^{\rho}v^*}{c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*}, E = \frac{q_2v^*}{c_1+v^*} - d_1$.

Consider η_4 be the eigenvalue of V_{E_I}, then the characteristic equation becomes,

$$\eta_4^3 + f_{11}\eta_4^2 + f_{22}\eta_4 + f_{33} = 0, \tag{6}$$

where, $f_{11} = -(A+C+E), f_{22} = AC + CE + EC + p_{1R}^{1-\rho}p_{1L}^{\rho}p_{2L}^{1-\rho}p_{2R}^{\rho}u^*v^*$, $f_{33} = \frac{p_{2L}^{1-\rho}p_{2R}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{B}v^*w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} - ACE - \frac{c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{D}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$.

Using Routh-Hurwitz criteria the system (3) is locally asymptotically stable at the interior equilibrium point E_I^*if $f_{11} > 0,\ f_{22} > 0, f_{33} > 0, f_{11}f_{22} > f_{33}$ and $\frac{2r_{1L}^{1-\rho}r_{1R}^{\rho}u^*}{K_1(1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*)} + p_{1R}^{1-\rho}p_{1L}^{\rho}v^* + \frac{2r_{2L}^{1-\rho}r_{2R}^{\rho}v^*}{K_2(1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*)} + d_{1R}^{1-\rho}d_{1L}^{\rho}w^*\left(\frac{c_{1R}^{1-\rho}c_{1L}^{\rho}}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}\right) + d_{1R}^{1-\rho}d_{1L}^{\rho} > \frac{r_{1L}^{1-\rho}r_{1R}^{\rho}}{1+m_{1L}^{1-\rho}m_{1R}^{\rho}w^*} + \frac{r_{2L}^{1-\rho}r_{2R}^{\rho}}{1+m_{2L}^{1-\rho}m_{2R}^{\rho}w^*} + p_{2L}^{1-\rho}p_{2R}^{\rho}u^* + \frac{q_{2L}^{1-\rho}q_{2R}^{\rho}v^*}{c_{1L}^{1-\rho}c_{1R}^{\rho}+v^*}$, $\frac{Bp_{2L}^{1-\rho}p_{2R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}v^*w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} > ACE + \frac{Dc_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$,$(A + C + E)\left(AC + CE + p_{1R}^{1-\rho}p_{1L}^{\rho}p_{2L}^{1-\rho}p_{2R}^{\rho}u^*v^*\right) + ACE + \frac{c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{D}w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2} > \frac{p_{2L}^{1-\rho}p_{2R}^{\rho}c_{1R}^{1-\rho}c_{1L}^{\rho}q_{2L}^{1-\rho}q_{2R}^{\rho}\mathrm{B}v^*w^*}{\left(c_{1R}^{1-\rho}c_{1L}^{\rho}+v^*\right)^2}$.

5 | Numerical Simulation

In this part, we run careful numerical simulations to test and confirm our model system's analytical conclusions. To numerically estimate the solution of our model system, we utilized the mathematical tools Matlab (2018) and Matcont. It is legitimate to state that determining numerical values for the model system's range of constraints based on real-world data is tough enough. In several scenarios, we simulated the model system with various model parameters. The stability of our proposed model is discussed at $E_p$and $E_1{}^*$in Part I and Part II.

5.1 | Part I: Analyze the Notion of Planer Equilibrium E_p

In this scenario, we simulate the system (3) using the model parameter values shown in Table 1 and set the value of parameter 'ρ' into three different levels (ρ = 0, 0.5, 1), which satisfies the condition as given in Theorem 6. Figure 1 shows the time series plot of the model system (3) in the time range [0, 200], which indicates the stability of the planer equilibrium point E_p for different values of parameter 'ρ'.

Table 1. Shows the values of different parameters that have been used to simulate the system in Part I.

Parameters	Values (For Planer)	Values (For Interior)
r_1	[1.51, 1.71]	**[2.051, 2.071]**
r_2	[1.15, 1.35]	**[3.15, 3.35]**
K_1	12.4	**0.5**
K_2	0.5	**2.4**
p_1	[0.64, 0.75]	**[0.64, 0.75]**
p_2	[0.15, 0.25]	**[0.15, 0.25]**
m_1	[0.035, 0.055]	**[0.035, 0.055]**
m_2	[0.51, 0.71]	**[0.51, 0.71]**
q_1	[0.43, 0.55]	**[0.43, 0.55]**
q_2	[0.041, 0.051	**[0.041, 0.051]**
c_1	[0.9, 1.3]	**[9, 13]**
d_1	**[0.15, 0.35]**	**[0.0015, 0.0035]**

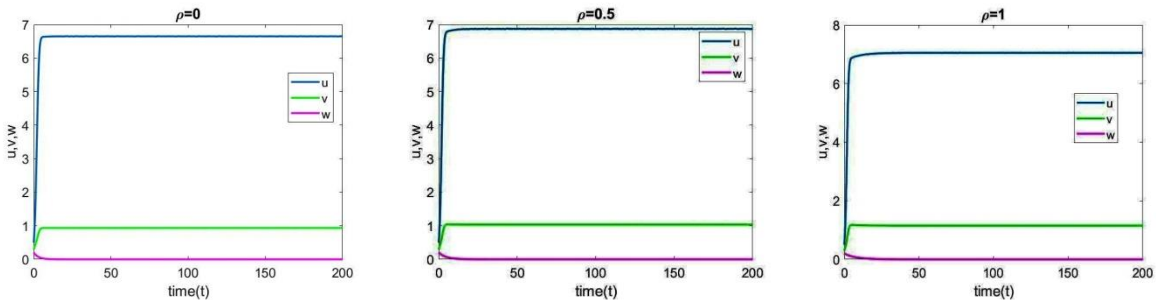

Figure 1. For various values of the parameter ρ, depicts the stable nature of the planer equilibrium point.

5.2 | Part II: Explore the Effects of the Interior Equilibrium Point $E_1{}^*$

In this scenario, we simulate the system (3) using the model parameter values shown in Table 1 and set the value of parameter 'ρ' into three different levels (ρ = 0, 0.5, 1), which satisfies the condition as given in Theorem 7. Figure 2 shows the time series plot of the model system (3) in the time range [0, 200], which indicates the stability of the planer equilibrium point $E_1{}^*$ for different values of parameter 'p'.

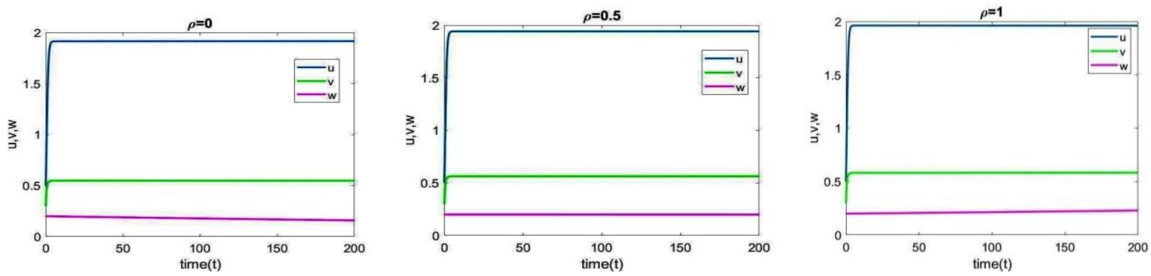

Figure 2. Indicates the time series plot of the system (3) for various values of parameter 'ρ' in the time range [0, 200] at the interior equilibrium point ${E_1}^*$.

6 | Conclusion

We have considered a tri-tropic food chain model where the middle predator is a generalized predator and the top predator is a specialist. In addition, two fear effects are suggested in this paper. It discusses the dynamic characteristics of the model within the context of an uncertain environment. The model incorporates interval uncertainty and takes into account the fear's effects on prey caused by the predator population. The study discusses a straightforward predator-prey model where ecological parameters are represented as parametric-functional interval values for biological parameters, except for the environmental carrying capacity and the fear factor, which are introduced in the field of mathematical biology. These ecological parameters, including prey growth rate, prey consumption, prey-to-predator conversion, transition rate from immature to mature predator, and death rate of immature and mature predators, are treated as parametric-functional intervals. The article establishes the positivity and boundedness of the solutions starting from any non-negative initial points. We have studied the local stability at all equilibrium points. In future studies, the authors plan to analyze further ecological modeling using neutrosophic environments and apply mathematical modeling techniques to suggest medical research models.

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Author Contributions

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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