Neutrosophic Optimization and Intelligent Systems

Journal Homepage: sciencesforce.com/nois

Neutrosophic Opt. Int. Syst. Vol. 5 (2025) 38–47

Paper Type: Original Article

Advanced Analysis of Neutrosophic Functions: A Detailed Examination of Three Types within Neutrosophic Set Theory

Adel Al-Odhari [1,2,*]

[1] Faculty of Education, Humanities and Applied Sciences (Khawlan), Yemen.

[2] Department of Foundations of Sciences, Faculty of Engineering, Sana'a University. Box:13509, Sana'a, Yemen; a.aleidhri@su.edu.ye.

Received: 13 Oct 2024 **Revised:** 28 Dec 2024 **Accepted:** 20 Jan 2025 **Published:** 22 Jan 2025

Abstract

This article aims to present the neutrosophic functions on neutrosophic set theory, that similar to the classical set theory, in this paper, we consider the neutrosophic function of three types, composition of neutrosophic function, one-to-one and onto of neutrosophic function with their properties by theorems and examples.

Keywords: Neutrosophic Functions of Three Types; Neutrosophic Domain and Range; Composition Neutrosophic Functions; Properties of Neutrosophic Functions.

1 | Introduction

Language of mathematics will play a crucial role in explaining(or interpreting) the physical world around us, it begins by describing the many in one by expressing (or interpreting) it by using the concept of set. Namely, that abstract set that describes what it contains, without any relations or operations on it. This procedure represents the first step in the methodology of mathematical thought when that set is born in the mind of someone who uncovers it from the world of nothingness to the world of existence. After that, mathematicians are interested in, how to construct relations and operations, and so they study new mathematical structures similar to those known in old mathematical structures. The Science of Neutrosophy is a modern school of mathematical systems treating a world reality that contains indeterminacy which is the opposite of determinacy; when we encounter physical world problems that include some indeterminacy issues. Indeterminacy can occur in various situations or phenomena related to ontology or epistemology. The concept of indeterminacy, in Neutrosophy, according to Smarandeche" is everything that is in between the opposites. $\langle A\rangle$ and $\langle antA\rangle$, written $\langle antA\rangle$ [14,18]. This paper aims to continue our study in neutrosophic set theory in [4,6,7,8]. It also enhances our work in [3,5,10] with the same approach. In addition, it represents a kind of contribution to the dissemination of neutrosophic knowledge with other works in [1,2,9], and in the field of algebraic neutrosophic. We will refer to the sources of neutrosophic logic and neutrosophic set as

Corresponding Author: a.aleidhri@su.edu.ye

https://doi.org/10.61356/j.nois.2025.5444

generalizations of the Intuitionistic fuzzy set and fuzzy set in [13,15,16,17], and concerning the classical set theory see [11,12,19].

2 | Neutrosophic Functions on Neutrosophic Sets of Three Types

In this section, we will begin our development of the axiomatic neutrosophic set theory that corresponds to the axiomatic set theory. In the literature philosophy of mathematical axiomatic systems, it consists of a set of undefined terms and axioms, axioms mean that a declarative sentence (or proposition) is assumed to be true. We will postulate the basis of neutrosophic functions on a neutrosophic set of three types, and we investigate their properties. This section includes neutrosophic functions on neutrosophic sets of three types with their neutrosophic graph, neutrosophic restriction, extension, identity, and constant functions. In addition, the concepts of one-to-one, onto, and composition functions are addressed with some theorems and examples.

Definition 1.2 Let $X_i^t[I]$ and $Y_i^t[I]$ ($i = 1,2,3$) be two neutrosophic sets of three types generated by X and Y. Assume that $f_n^i(I) = I$, for any $i = 1,2,3$ and $f_n^i(xI) = f_c(x)f_n^i(I)$. Intuitively, we can define the neutrosophic functions. $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ of three types generated by a classical function $f_c: X \longmapsto Y$ as follows:

1. $f_n^1(x) = f_c(x_1) + f_n^1(x_2 I) = f_c(x_1) + f_c(x_2) f_n^1(I)$,
2. $f_n^2(x) = \begin{cases} f_c(x) \\ f_n^2(xI) \end{cases}$, and
3. $f_n^3(x) = \begin{cases} f_c(x_1) \\ f_c(x_1) + f_n^3(x_2 I) \end{cases}$

for all $x \in X_i^t[I\,], x_1\,, x_2 \in X$, and an indeterminacy I. In other words, a correspondence from a neutrosophic set $X_i^t[I]$ to a neutrosophic set $Y_i^t[I]$ is a quadruple $f_n^i = \left(X_i^t[I], Y_i^t[I], f_n^i(I), \Gamma_n[I]\right)$, where $X_i^t[I]$ is a neutrosophic domain of f_n^i, $Y_i^t[I]$ is the neutrosophic co-domain of f_n^i, $f_n^i(I)$ is a neutrosophic image of indeterminacy I, and $\Gamma_n[I]$ is a neutrosophic subset of $X_i^t[I] \times Y_i^t[I]$, and it's called the neutrosophic graph of f_n^i. The neutrosophic set:

$NeuDom(f_n^i) = \{x \in X_i^t[I]: \exists y \in Y_i^t[I] \ni f_n^i(x) = y \Leftrightarrow (x,y) \in \Gamma_n[I]\} \subseteq X_i^t[I]$, is the neutrosophic domain of f_n^i , and the neutrosophic set:

$NeuCod\left(f_n^i\right) = \left\{y \in Y_i^t[I]: \exists x \in X_i^t[I] \ni f_n^i(x) = y \Leftrightarrow (x,y) \in \Gamma_n[I]\right\} \subseteq Y_i^t[I]$, is the The neutrosophic range (or neutrosophic co-domain) of f_n^i.

Example 1.2 Let $X = \{\mathrm{a}, \mathrm{b}\}$ and $Y = \{1,2,3\}$ be two classical sets, with a classical function $f_c: X \longmapsto Y$ such that $f_c(a) = 1$, and $f_c(b) = 2$, the neutrosophic sets of three types which are generated by X, and Y are $X_1^t[I] = \begin{Bmatrix} a + aI, \mathrm{a} + \mathrm{bI}, \\ \mathrm{b} + \mathrm{aI}, \mathrm{b} + \mathrm{bI} \end{Bmatrix}$, $Y_1^t[I\,] = \begin{Bmatrix} 1 + 1I, 1 + 2\mathrm{I}, 1 + 3\mathrm{I}, \\ 2 + 1\mathrm{I}, 2 + 2\mathrm{I}, 2 + 3\mathrm{I}, \\ 3 + 1\mathrm{I}, 3 + 2\mathrm{I}, 3 + 3\mathrm{I} \end{Bmatrix}$, $X_2^t[I] = \begin{Bmatrix} a, aI, \\ \mathrm{b}, \mathrm{bI} \end{Bmatrix}$, $Y_2^t[I] = \begin{Bmatrix} 1, 1I, \\ 2,2\mathrm{I}, \\ 3,3\mathrm{I} \end{Bmatrix}$, $X_3^t[I] = \begin{Bmatrix} a, a + aI, \mathrm{a} + \mathrm{bI}, \\ \mathrm{b}, \mathrm{b} + \mathrm{aI}, \mathrm{b} + \mathrm{bI} \end{Bmatrix}$, and $Y_3^t[I\,] = \begin{Bmatrix} 1,1 + 1I, 1 + 2\mathrm{I}, 1 + 3\mathrm{I}, \\ 2, 2 + 1\mathrm{I}, 2 + 2\mathrm{I}, 2 + 3\mathrm{I}, \\ 3, 3 + 1\mathrm{I}, 3 + 2\mathrm{I}, 3 + 3\mathrm{I} \end{Bmatrix}$ respectively. The neutrosophic function f_n^1 of type-1 is given by:

$f_n^1(a + \mathrm{a}I) = f_c(a) + f_n^1(aI) = f_c(a) + f_c(a)f_n^1(I) = 1 + 1I$,
$f_n^1(a + bI) = f_c(a) + f_n^1(bI) = f_c(a) + f_c(b)f_n^1(I) = 1 + 2I$,
$f_n^1(b + aI) = f_c(b) + f_n^1(aI) = f_c(b) + f_c(a)f_n^1(I) = 2 + 1I$, and
$f_n^1(b + bI) = f_c(b) + f_n^1(bI) = f_c(b) + f_c(b)f_n^1(I) = 2 + 2I$. The neutrosophic graph of the neutrosophic function f_n^1 is shown in Figure 1.

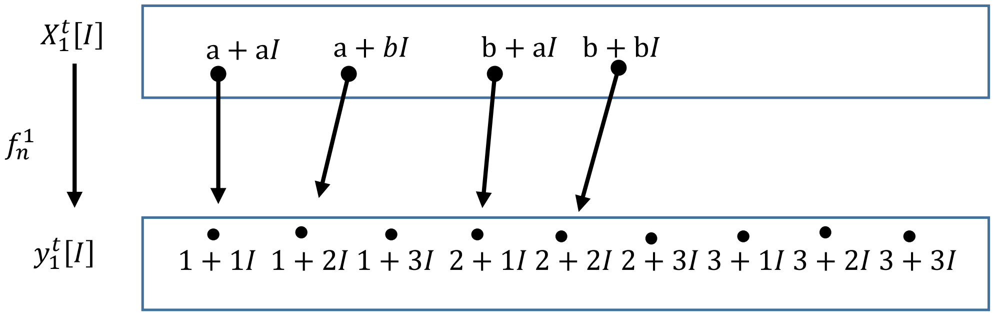

Figure 1. The graph $\Gamma_n[I]$ of f_n^1.

Concerning the neutrosophic function f_n^2 of type-2, we have

$f_n^2(x)=\begin{cases} f_c(x) \\ f_n^2(xI) \end{cases}$, where $f_c(x)$ is a determinacy part and $f_n^2(xI)$ is an indeterminacy part, the values of Neutrosophic function f_n^2 are given by the following:

$f_n^2(a)=f_c(a)=1, f_n^2(aI)=f_c(a)f_n^2(I)=1I, f_n^2(b)=f_c(b)=2, and\ f_n^2(bI)=f_c(b)f_n^2(I)=2I.$

The neutrosophic graph of the neutrosophic function f_n^2 is shown in Figure 2.

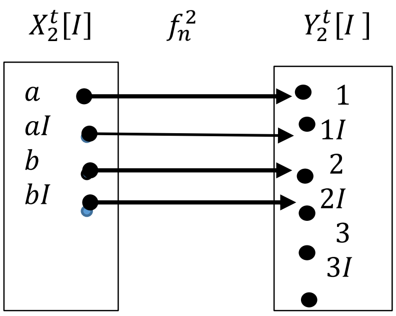

Figure 2. The graph $\Gamma_n[I]$ of f_n^2.

Finally, the neutrosophic function f_n^3 of type3 is given by: $f_n(x)=\begin{cases} f_c(x_1) \\ f_c(x_1)+f_n^3(x_2I) \end{cases}$

In this case, we have,

$f_n^3(a)=f_c(a)=\ 1, f_n^3(a+\mathrm{a}I)=f_c(a)+f_n^3(aI)=f_c(a)+f_c(a)f_n^3(I)=1+1I,$

$f_n^3(a+bI)=f_c(a)+f_n^3(bI)=f_c(a)+f_c(b)f_n^3(I)\ =1+2I,$

$f_n^3(b)=f_c(b)=2, f_n^3(b+aI)=f_c(b)+f_n^3(aI)=f_c(b)+f_c(a)f_n^3(I)\ =2+1I,$ and

$f_n^3(b+bI)=f_c(b)+f_n^3(bI)=f_c(b)+f_c(b)f_n^3(I)\ =2+2I,$ the neutrosophic graph of the neutrosophic function f_n^3 can be represented by a similar previous method.

Definition 2.2 Let $f_n^i: X_i^t[I]\longmapsto Y_i^t[I]$ be the neutrosophic function of three types generated by a classical function $f_c: X\longmapsto Y$ and $A_i^t[I]\subseteq X_i^t[I]$; the function f_n^i considered only on $A_i^t[I]$ is called the neutrosophic restriction of f_n^i to $A_i^t[I]$, written $f_n^i\big|A_i^t[I]$, if $f_n^i\big|A_i^t[I]=f_n^i\cap\left(A_i^t[I]\times Y_i^t[I]\right)$.

Definition 3.2 Consider $A_i^t[I]\subseteq X_i^t[I]$ with $g_n^i: A_i^t[I]\longmapsto Y_i^t[I]$ is a given neutrosophic function, then $f_n^i: X_i^t[I]\longmapsto Y_i^t[I]$ is called the neutrosophic extension function of three types of g_n^i over $X_i^t[I]$, if $f_n^i\big|A_i^t[I]=g_n^i$, for all $x\in A_i^t[I]$.

Definition 4.2 A neutrosophic function of three types, $I_{dn}^{\mathrm{i}}: X_{\mathrm{i}}^{t}[I] \longmapsto X_{\mathrm{i}}^{t}[I]$ is called a neutrosophic identity function, if $I_{dn}^{\mathrm{i}}(x) = x,$ for all $x \in X_1^t[I\,], x_1, x_2 \in X$, and an indeterminacy I.

Definition 5.2 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be the neutrosophic function of three types generated by a classical function $f_c: X \longmapsto Y$, then f_n^i is called a neutrosophic constant function, if there exists a neutrosophic element $y_0 \in Y_i^t[I]$ such that $f_n^i(x) = y_0$, for all $x \in X_i^t[I]$.

Theorem 1.2 Let $f_c: X \longmapsto Y$ be a one-to-one (injective) function, then $f_n^1: X_1^t[I] \longmapsto Y_1^t[I]$ be a one-to-one neutrosophic function.

Proof. Suppose that $f_c: X \longmapsto Y$ is a one-to-one function, andconsider $x, y \in X_1^t[I]$ such that $f_n^1(x) = f_n^1(y)$.

$\Rightarrow f_c(x_1) + f_n^1(x_2 I) = f_c(y_1) + f_n^1(y_2 I)$

$\Rightarrow f_c(x_1) + f_c(x_2) f_n^1(I) = f_c(y_1) + f_c(y_2) f_n^1(I)$

$\Rightarrow f_c(x_1) + f_c(x_2) I = f_c(y_1) + f_c(y_2) I$

$\Rightarrow \big(f_c(x_1) = f_c(y_1)\big) \wedge \big(f_c(x_2) = f_c(y_2)\big)$, because f_c is a one-to-one.

$\Rightarrow (x_1 = y_1) \wedge (x_2 = y_2)$

$\Rightarrow (x_1 + x_2 I) = (y_1 + y_2 I)$

$\Rightarrow x = y$. Hence f_n^1 is a one-to-one neutrosophic function.

Theorem 2.2 Let $f_c: X \longmapsto Y$ be a one-to-one function, then $f_n^3: X_3^t[I] \longmapsto Y_3^t[I]$ be a one-to-one neutrosophic function.

Proof. In a similar manner to theorem 1.2.

Theorem 3.2 Let $f_c: X \longmapsto Y$ be a one-to-one function, then $f_n^2: X_2^t[I] \longmapsto Y_2^t[I]$ be a one-to-one neutrosophic function.

Proof. Consider$f_c: X \longmapsto Y$ is a one-to-one function. Suppose that $x, y \in X_2^t[I]$ such that $f_n^2(x) = f_n^2(y)$. Since,

$$f_n^2(x) = f_n^2(y)$$

$$\Rightarrow \begin{cases} f_c(x_1) = f_c(y_1), determency\ part \\ f_n^2(x_1 I) = f_n^2(y_1 I) \Leftrightarrow f_c(x_1) f_n^2(I) = f_c(y_1) f_n^2(I), indetermency\ part \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = y_1, determency\ part \\ x_1 I = y_1 I, indetermency\ part \end{cases}$$

$$\Rightarrow \begin{cases} x = y, determency\ part \\ xI = yI, indetermency\ part \end{cases}$$

Hence f_n^2 is a one-to-one neutrosophic function.

Theorem 4.2 Let$f_c: X \longmapsto Y$ be an onto (surjective) function, then $f_n^1: X_1^t[I] \longmapsto Y_1^t[I]$ be an onto neutrosophic function.

Proof. Suppose that $f_c: X \longmapsto Y$ is a onto function, and consider $y \in Y_1^t[I] \Rightarrow \exists\, y_1, y_2 \in Y,$ and indeterminacy I such that $y = y_1 + y_2 I \Rightarrow \exists x_1, x_2 \in X,$ and indeterminacy I such that $f_c(x_1) = y_1, f_c(x_2) = y_2,$ and $f_n^1(I) = I$. Therefore, $f_n^1(x) = f_c(x_1) + f_n^1(x_2 I) = f_c(x_1) + f_c(x_2) f_n^1(I) = y_1 + y_2 I = y$. Hence Hence f_n^1 is an onto neutrosophic function.

Theorem 5.2 Let $f_c: X \longmapsto Y$ be an onto (surjective) function, then $f_n^3: X_3^t[I] \longmapsto Y_3^t[I]$ be an onto neutrosophic function.

Proof. By the similar argument of theorem 4.2.

Theorem 6.2 Let $f_c: X \longmapsto Y$ be an onto (surjective) function, then $f_n^2: X_2^t[I] \longmapsto Y_2^t[I]$ be an onto neutrosophic function.

Proof. Assume that $f_c: X \longmapsto Y$ is a onto function. Let $y \in Y_1^t[I] \Longrightarrow \exists\, y_1 \in Y,$ and indeterminacy I such that $y = \begin{cases} y_1, det - \ part \\ y_1 I, ind - \ part \end{cases}$ $\Longrightarrow \exists x_1 \in X$ and indeterminacy I such that $f_c(x_1) = y_1$ or

$$f_n^2(x_1 I) = f_c(x_1) f_n^2(I) = y_1 I \Longrightarrow f_n(x) = \begin{cases} f_c(x_1) = y_1, determency\ part \\ f_n^2(x_1 I) = f_c(x_1) f_n^2(I) = y_1 I, indetermency\ part \end{cases}$$

$\Longrightarrow f_n^2(x) = \begin{cases} y, determency\ part \\ yI, indetermency\ part \end{cases}$ $\Longrightarrow f_n^2$ is an onto neutrosophic function.

Theorem 7.2 Let $f_c: X \longmapsto Y$ be a bijective (injective & surjective) function, then $f_n^i: X_i^t[I] \longmapsto Y_i^t[I], i = 1,2,3$ is a neutrosophic bijective function.

Proof. By theorems 1.2 into 6.2.

Theorem 8.2 Let $I_{dc} : X \longmapsto X$ be a bijective(injective+surjective) identity function, then $I_{dn}^1: X_1^t[I] \longmapsto Y_1^t[I]$ is a bijective neutrosophic identity function.

Proof. Let $I_{dc}: X \longmapsto X$ be a bijective identity function. Assume that $x, y \in X_1^t[I]$ such that

$$I_{dn}^1(x) = I_{dn}^1(y) \Leftrightarrow I_{dc}(x_1) + I_{dn}^1(x_2 I) = I_{dc}(y_1) + I_{dn}^1(y_2 I)$$

$$\Leftrightarrow I_{dc}(x_1) + I_{dc}(x_2) I_{dn}^1(I) = I_{dc}(y_1) + I_{dc}(y_2) I_{dn}^1(I)$$

$$\Leftrightarrow x_1 + x_2 I = y_1 + y_2 I.$$

$\Longrightarrow x = y.$ Hence i_{dn}^1 is a one-to-one neutrosophic identity function. In addition, suppose that $y \in X_1^t[I] \Longrightarrow \exists\, y_1, y_2 \in X,$ and indeterminacy I such that $y = y_1 + x_2 I \Longrightarrow \exists x_1, x_2 \in X,$ and indeterminacy I such that $I_{dc}(x_1) = y_1, I_{dc}(x_2) = y_2,$ and $I_{dn}^1(I) = I.$ Therefore, $x_1 = y_1, x_2 = y_2,$ and $I = I$ Hence, $I_{dn}^1(x) = I_{dc}(x_1) + I_{dc}(x_2) I_{dn}^1(I) = y_1 + y_2 I = y.$ Hence, I_{dn}^1 is an onto neutrosophic identity function, and consequently, I_{dn}^1 is a bijective neutrosophic identity function.

Theorem 9.2 Let $\iota_{dc}: X \longmapsto X$ be a bijective(injective+surjective) identity function, then $I_{dn}^3: X_3^t[I] \longmapsto X_3^t[I]$ is a bijective neutrosophic identity function.

Proof. The same argument is in theorem 8.2.

Theorem 10.2 Let $\iota_{dc}: X \longmapsto X$ be a bijective(injective+surjective) identity function, then $I_{dn}^2: X_2^t[I] \longmapsto X_2^t[I]$ is a bijective neutrosophic identity function.

Proof. Consider $\iota_{dc}: X \longmapsto X$ is a one-to-one function. Suppose that $x, y \in X_2^t[I]$ such that $I_{dn}^2(x) = I_{dn}^2(y)$. Since,

$$I_{dn}^2(x) = I_{dn}^2(y) \Longrightarrow \begin{cases} I_{dc}(x_1) = I_{dc}(y_1), det - part \\ I_{dn}^2(x_1 I) = I_{dn}^2(y_1 I) \Leftrightarrow I_{dc}(x_1) I_{dn}^2(I) = I_{dc}(y_1) I_{dn}^2(I), ind - part \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = y_1, det - part \\ x_1 I = y_1 I,\ ind - part \end{cases}$$

$$\Longrightarrow \begin{cases} x = y, det - part \\ xI = yI, ind - part \end{cases}$$

Hence I_{dn}^2 is a one-to-one neutrosophic identity function, and I_{dn}^2 is obvious is an onto identity function, hence I_{dn}^2 is a bijective neutrosophic identity function.

Definition 6.2 Let $f_n^i, g_n^i: X_i^t[I] \longmapsto Y_i^t[I]\ , i = 1,2,3$ be two neutrosophic functions, where $f_c, g_c: X \longmapsto Y$ be two classical functions, then f_n^i are neutrosophic equal to g_n^i , written $f_n^i = g_n^i$, iff $f_n^i(x) = g_n^i(x)$, for all $x \in X_i^t[I]$ And, $i = 1,2,3$.

Definition 7.2 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ and $g_n^i: Y_i^t[I] \longmapsto Z_i^t[I], i = 1,2,3$ be two neutrosophic functions, where $f_c: X \longmapsto Y$ and $g_c: X \longmapsto Y$ are two classical functions, the composite of f_n^i and g_n^i is defined by: $(g_n^i \circ f_n^i)(x) = g_n^i\left(f_n^i(x)\right)$, for all $x \in X_i^t[I]$and , $i = 1,2,3$.

Example 3.2 Let $\mathbb{R}$ be a set of classical real numbers and $\mathbb{R}_i^t[I]$ be a set of neutrosophic real numbers of three types. Consider two classical functions $f_c, g_c: \mathbb{R} \longmapsto \mathbb{R}$ such that $f_c(x) = x^2$ and $g_c(x) = x + 1, \forall x \in \mathbb{R}$. We can generate two neutrosophic functions $f_n^i, g_n^i: \mathbb{R}_i^t[I] \longmapsto \mathbb{R}_i^t[I]$ induced from f_c and g_c , respectively. Suppose that $x \in \mathbb{R}_i^t[I]$, the neutrosophic composite is given by:

$(g_n^1 \circ f_n^1)(x) = g_n^1(f_n^1(x))$
$= g_n^1(f_c(x_1) + f_c(x_2)I)$
$= g_n^1({x_1}^2 + {x_I}^2 I)$
$= g_c({x_1}^2) + g_c({x_2}^2)I$
$= ({x_1}^2 + 1) + ({x_2}^2 + 1)I$, for instance, the neutrosophic image of the neutrosophic element $(g_n^1 \circ f_n^1)(2 + 3I) = 5 + 10I$. While,

$(f_n^1 \circ g_n^1)(x) = f_n^1(g_n^1(x))$
$= f_n^1(g_c(x_1) + g_c(x_2)I)$
$= f_n^1((x_1 + 1) + (x_2 + 1)I)$
$= f_c((x_1 + 1)) + f_c(x_2 + 1)I$
$= (x_1 + 1)^2 + (x_2 + 1)^2 I$
$= ({x_1}^2 + 2x_1 + 1) + ({x_2}^2 + 2x_2 + 1)I$. So the neutrosophic image of neutrosophic element, $(f_n^1 \circ g_n^1)(2 + 3I) = (4 + 2.2 + 1) + (9 + 2.3 + 1)I = 9 + 16I$. We see that the composition of the neutrosophic function is not commutative. i.e. $(g_n^1 \circ f_n^1)(x) \neq (f_n^1 \circ g_n^1)(x)$.

3 | Properties of Neutrosophic Functions on Neutrosophic Sets of Three Types

This section includes the properties of neutrosophic functions on some neutrosophic subsets of the neutrosophic domain with the operator's union, intersection, difference, and on generalization of union and intersection.

Definition 1.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ Be the neutrosophic function of three types generated from a classical function. $f_c: X \longmapsto Y$ and classical sets X and Y respectively.
Let $C_i^t[I]$ Be a neutrosophic subset of $X_i^t[I]$ Generated by $C \subset X$. Define a neutrosophic direct image of $C_i^t[I]$ under f_n^i , written $f_n^i(C_i^t[I])$, as follows: $f_n^i(C_i^t[I]) = \{y \in Y_i^t[I]: \exists x \in C_i^t[I] \ni f_n^i(x) = y\}$.

Theorem 1.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ Be the neutrosophic function of three types generated from a classical function. $f_c: X \longmapsto Y$ and classical sets X and Y respectively, and Let $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset X_i^t[I]$, **if** $C_i^t[I] = B_i^t[I]$, then $f_n^i(C_i^t[I]) = f_n^i(B_i^t[I])$.

Proof. Suppose that $C_i^t[I] = B_i^t[I]$, and let $y \in f_n^i(C_i^t[I])$, then there exists a neutrosophic element $x \in C_i^t[I]$ such that $f_n^i(x) = y$, for any $i = 1,2,3$. Since $C_i^t[I] = B_i^t[I]$, implies that $x \in B_i^t[I]$, hence $f_n^i(x) \in f_n^i(B_i^t[I])$, therefore $y \in f_n^i(B_i^t[I])$, and consequently,$f_n^i(C_i^t[I]) \subset f_n^i(B_i^t[I])$. By a similar method, we can prove the second part. $f_n^i(B_i^t[I]) \subset f_n^i(C_i^t[I])$, to get the conclusion, $f_n^i(C_i^t[I]) = f_n^i(B_i^t[I])$. The converse of the theorem is not true, by the following example.

Example 1.3 Let $f_n^1: Z_1^t[I] \longmapsto R_1^t[I]$ be a neutrosophic function of type-1 from the neutrosophic set of integers to the neutrosophic set of real numbers defined by $f_n^1(x) = {x_1}^2 + {x_2}^2 I$. Consider $C_1^t[I] = \begin{Bmatrix} -2-2I, -2+3I, \\ 3-2I, 3+3I \end{Bmatrix}$, and $B_1^t[I] = \begin{Bmatrix} 2+2I, 2-3I \\ -3+2I, -3-3I \end{Bmatrix}$, $f_n^1(C_1^t[I]) = \begin{Bmatrix} 4+4I, 4+9I, \\ 9+4I, 9+9I \end{Bmatrix}$ and $f_n^1(B_1^t[I]) = \begin{Bmatrix} 4+4I, 4+9I, \\ 9+4I, 9+9I \end{Bmatrix}$, we say that $\left(f_n^1(B_1^t[I]) = f_n^1(C_1^t[I])\right)$, but $C_1^t[I] \neq B_1^t[I]$.

Theorem 2.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$, and classical sets X and Y respectively, and Let $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset X_i^t[I]$, then:

1. $f_n^i\left(C_i^t[I] \cup B_i^t[I]\right) = f_n^i\left(C_i^t[I]\right) \cup f_n^i\left(B_i^t[I]\right)$,
2. $f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) \subseteq f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)$, and
3. $f_n^i\left(C_i^t[I]\right) - f_n^i\left(B_i^t[I]\right) \subseteq f_n^i\left(C_i^t[I] - B_i^t[I]\right)$.

Proof. (1). Let $y \in f_n^i\left(C_i^t[I] \cup B_i^t[I]\right)$.
Since $y \in f_n^i\left(C_i^t[I] \cup B_i^t[I]\right)$.
$\Rightarrow \exists x \in \left(C_i^t[I] \cup B_i^t[I]\right) \ni f_n^i(x) = y$
$\Rightarrow \exists x \in C_i^t[I] \vee \exists x \in B_i^t[I] \ni f_n^i(x) = y$
$\Rightarrow \left(\exists x \in C_i^t[I] \ni f_n^i(x) = y\right) \vee \left(\exists x \in B_i^t[I] \ni f_n^i(x) = y\right)$
$\Rightarrow \left(f_n^i(x) \in f_n^i(C_i^t[I])\right) \vee \left(f_n^i(x) \in f_n^i(B_i^t[I])\right)$
$\Rightarrow \left(y \in f_n^i(C_i^t[I])\right) \vee \left(y \in f_n^i(B_i^t[I])\right)$
$\Rightarrow \left(y \in (f_n^i(C_i^t[I]) \cup f_n^i(B_i^t[I])\right)$
$\Rightarrow f_n^i\left(C_i^t[I] \cup B_i^t[I]\right) \subseteq f_n^i\left(C_i^t[I]\right) \cup f_n^i\left(B_i^t[I]\right)$ (1). By similar way, let $y \in \left(f_n^i\left(C_i^t[I]\right) \cup f_n^i\left(B_i^t[I]\right)\right)$,
$\Rightarrow y \in f_n^i\left(C_i^t[I]\right) \vee y \in f_n^i\left(B_i^t[I]\right)$
$\Rightarrow \left(\exists x \in C_i^t[I] \ni f_n^i(x) = y\right) \vee \left(\exists z \in B_i^t[I] \ni f_n^i(z) = y\right)$
$\Rightarrow \left(\exists x \in \left(C_i^t[I] \cup B_i^t[I]\right) \ni f_n^i(x) = y\right) \vee \left(\exists z \in \left(C_i^t[I] \cup B_i^t[I]\right) \ni f_n^i(z) = y\right)$
$\Rightarrow \left(y \in f_n^i\left(C_i^t[I] \cup B_i^t[I]\right)\right)$
$\Rightarrow f_n^i\left(C_i^t[I] \cup B_i^t[I]\right) \subseteq f_n^i\left(C_i^t[I] \cup B_i^t[I]\right)$ (2).
$\Rightarrow f_n^i\left(C_i^t[I] \cup B_i^t[I]\right) = f_n^i\left(C_i^t[I]\right) \cup f_n^i\left(B_i^t[I]\right)$.
(2). Let $y \in f_n^i\left(C_i^t[I] \cap B_i^t[I]\right)$
$\Rightarrow \exists x \in \left(C_i^t[I] \cap B_i^t[I]\right) \ni f_n^i(x) = y$
$\Rightarrow \exists x \in C_i^t[I] \wedge \exists x \in B_i^t[I] \ni f_n^i(x) = y$
$\Rightarrow \left(\exists x \in C_i^t[I] \ni f_n^i(x) = y\right) \wedge \left(\exists x \in B_i^t[I] \ni f_n^i(x) = y\right)$
$\Rightarrow \left(f_n^i(x) \in f_n^i(C_i^t[I])\right) \wedge \left(f_n^i(x) \in f_n^i(B_i^t[I])\right)$
$\Rightarrow \left(y \in f_n^i(C_i^t[I])\right) \wedge \left(y \in f_n^i(B_i^t[I])\right)$
$\Rightarrow \left(y \in (f_n^i(C_i^t[I]) \cap f_n^i(B_i^t[I])\right)$
$\Rightarrow f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) \subseteq f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)$.
(3). Let $y \in \left(f_n^i\left(C_i^t[I]\right) - f_n^i\left(B_i^t[I]\right)\right)$,
$\Rightarrow y \in f_n^i\left(C_i^t[I]\right) \wedge y \notin f_n^i\left(B_i^t[I]\right)$
$\because y \in f_n^i\left(C_i^t[I]\right) \Rightarrow \left(\exists x \in C_i^t[I] \ni f_n^i(x) = y\right)$
$\because y \notin f_n^i\left(B_i^t[I]\right) \Rightarrow f_n^i(x) \notin f_n^i\left(B_i^t[I]\right) \Rightarrow x \notin B_i^t[I]$
$\Rightarrow \exists x \in C_i^t[I] \wedge \exists x \notin B_i^t[I] \ni f_n^i(x) = y$
$\Rightarrow \left(\exists x \in \left(C_i^t[I] - B_i^t[I]\right) \ni f_n^i(x) = y\right)$

$\Rightarrow \left(f_n^i(x) \in f_n^i\left(C_i^t[I]\right) - (B_i^t[I]\right)$
$\Rightarrow \left(y \in f_n^i\left(C_i^t[I]\right) - (B_i^t[I]\right)$
$\Rightarrow \left(f_n^i\left(C_i^t[I]\right) - f_n^i\left(B_i^t[I]\right)\right) \subseteq f_n^i\left(C_i^t[I]\right) - (B_i^t[I]$. The following examples illustrate that the equality in part 2 of the previous theorem does not hold.

Example 2.3 Let $f_n^1: X_1^t[I] \longmapsto Y_1^t[I]$ be a constant neutrosophic function of type-1, where $X_1^t[I] = \begin{Bmatrix} 2+2I, 2+7I, \\ 7+2I, 7+7I \end{Bmatrix}$, and $Y_1^t[I] = \{4+4I\}$, take $C_1^t[I] = \{2+2I, 2+7I, 7+2I\}$, and $B_1^t[I] = \{\,7+7I\}$, $C_1^t[I] \cap B_1^t[I] = \emptyset_1^t[I]$, , we hav $f_n^1(C_1^t[I] \cap B_1^t[I]) = f_n^1(\emptyset_1^t[I]) = \emptyset_1^t[I]$,
$f_n^1(C_1^t[I]) = f_n^1(\{2+2I, 2+7I, 7+2I\}) = 4+4I$, and
$f_n^1(B_1^t[I]) = f_n^1(\{\,7+7I\}) = 4+4I$, so $f_n^1(C_1^t[I]) \cap f_n^1(B_1^t[I]) = 4+4I$, we see that
$f_n^1(C_1^t[I]) \cap f_n^1(B_1^t[I]) \nsubseteq f_n^1(C_1^t[I] \cap B_1^t[I])$.

Example 3.3 Let $f_n^1: \mathbb{R}_1^t[I] \longmapsto \mathbb{R}_1^t[I]$ be a neutrosophic function from the neutrosophic set of real numbers to itself such that $f_n^1(x) = f_c(x_1) + f_n^1(x_2 I) = f_c(x_1) + f_c(x_2) f_n^1(I)$, where $f_c(x) = x^2$. Let $C_1^t[I] = \{a+bI: a, b \in C = [-2,0]\}$ and $B_1^t[I] = \{c+dI: c, d \in B = [0,2]\}$be two neutrosophic sets of type-1 generated by C and B, then the intersection of $C_1^t[I] \cap B_1^t[I] = 0+0I$, and
$f_n^1(C_1^t[I] \cap B_1^t[I]) = f_n^1(0+0I) = 0+0I$, while $f_n^1(C_1^t[I]) = f_n^1([-2, -2I]) = 4+4I$, and
$f_n^1(B_1^t[I]) = f_n^1([2, 2I]) = 4+4I$, hence $f_n^1(C_1^t[I]) \cap f_n^1(B_1^t[I]) = 4+4I \neq$
$f_n^1(C_1^t[I] \cap B_1^t[I]) = 0+0I$.

Theorem 3.2. Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic function of three types generated from a classical one-to-one function $f_c: X \longmapsto Y$, and classical sets X and Y respectively, $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset X_i^t[I]$, then:
$f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) = f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)$, if and only if $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a one-to-one neutrosophic function.

Proof. Suppose that $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a one-to-one neutrosophic function, $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset X_i^t[I]$. Let $y \in f_n^i\left(C_i^t[I] \cap B_i^t[I]\right)$
$\Leftrightarrow \exists x \in \left(C_i^t[I] \cap B_i^t[I]\right) \ni f_n^i(x) = y$
$\Leftrightarrow \exists x \in \left(C_i^t[I] \cap B_i^t[I]\right) \ni f_n^i(x) = y$
$\Leftrightarrow \left(\exists x \in C_i^t[I] \ni f_n^i(x) = y\right) \wedge \left(\exists x \in B_i^t[I] \ni f_n^i(x) = y\right)$
$\Leftrightarrow \left(y \in f_n^i\left(C_i^t[I]\right)\right) \wedge \left(y \in f_n^i\left(B_i^t[I]\right)\right)$
$\Leftrightarrow y \in \left(f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)\right)$. Hence $f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) = f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)$. Conversely, Suppose that $f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) = f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right)$, where $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset X_i^t[I]$. To show that $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a one-to-one neutrosophic function, consider $x, z \in X_i^t[I]$, $x \neq z$ such that $f_n^i(x) = f_n^i(z) = y$. Consider $C_i^t[I] = \{x\}$, and $B_i^t[I] = \{z\}$ Are two neutrosophic of type-1, we have $f_n^i\left(C_i^t[I]\right) = f_n^i(x) = y$ and $f_n^i\left(B_i^t[I]\right) = f_n^i(z) = y$. So, $f_n^i\left(C_i^t[I]\right) \cap f_n^i\left(B_i^t[I]\right) = y$, but $C_i^t[I] \cap B_i^t[I] = \emptyset_i^t[I]$, and $f_n^i\left(C_i^t[I] \cap B_i^t[I]\right) = f_n^i\left(\emptyset_i^t[I]\right) = \emptyset_i^t[I]$. Therefore, f_n^i is a one-to-one neutrosophic function.

Theorem 4.2. Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, and Let $\underbrace{H_i^t[I]}_{\alpha}: \alpha \in \mathbb{i} \in \mathbb{Z}^+$ be a family of neutrosophic subsets of $X_i^t[I]$, then:

1. $f_n^i\left(\cup_\alpha \underbrace{H_i^t[I]}_{\alpha}\right) = \cup_\alpha f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right)$, and
2. $f_n^i\left(\cap_\alpha \underbrace{H_i^t[I]}_{\alpha}\right) \subset \cap_\alpha f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right)$.

Proof. (1).Suppose that $y \in \cup_\alpha f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right) \Leftrightarrow \exists \alpha \in \mathbb{i} \ni y \in f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right)$

$$\Leftrightarrow \exists\, x \in \underbrace{H_i^t[I]}_{\alpha} \ni f_n^i(x) = y,$$

$$\Leftrightarrow \exists\, x \in \cup_\alpha \underbrace{H_i^t[I]}_{\alpha} \ni f_n^i(x) = y,$$

$$\Leftrightarrow y = f_n^i(x) \in f_n^i\left(\cup_\alpha \underbrace{H_i^t[I]}_{\alpha}\right).$$

(2). Suppose that $y \in f_n^i\left(\cap_\alpha \underbrace{H_i^t[I]}_{\alpha}\right) \Longrightarrow \exists x \in \cap_\alpha \underbrace{H_i^t[I]}_{\alpha} \ni f_n^i(x) = y$

$$\Longrightarrow \exists x \in \underbrace{H_i^t[I]}_{\alpha} \ni f_n^i(x) = y, \forall \alpha \in \mathbb{i}$$

$$\Longrightarrow y = f_n^i(x) \in f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right), \forall \alpha \in \mathbb{i}$$

$$\Longrightarrow y = f_n^i(x) \in \cap_\alpha f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right)$$

$$\Longrightarrow f_n^i\left(\cap_\alpha \underbrace{H_i^t[I]}_{\alpha}\right) \subset \cap_\alpha f_n^i\left(\underbrace{H_i^t[I]}_{\alpha}\right).$$

Acknowledgments

The researcher thanks the editor-in-chief of the journal and the referees for their guidance throughout the preparation of this paper.

Funding

There is no financial funding for this research.

Data Availability

There is no data available for the research.

Conflicts of Interest

There is no conflict of interest for the research.

References

[1] Al-Odhari, A. M. (2023). A Review Study on Some Properties of The Structure of Neutrosophic Ring. Neutrosophic Sets and Systems, 54, 139-156. doi:10.5281/zenodo.7817736

[2] Al-Odhari, A. M. (2023). Some Algebraic Structure of Neutrosophic Matrices. Prospects for Applied Mathematics and data Analysis (PAMDA), 01(02), 37-44. doi:https://doi.org/10.54216/PAMDA.010204

[3] Al-Odhari, A. M. (2024). Axiomatic of Neutrosophic Groups. Sana'a University Journal of Applied Sciences and Technology, 2(2), 205-214. doi:https://doi.org/10.59628/jast.v2i2.793

[4] Al-Odhari, A. M. (2024). Basic Introduction of Neutrosophic Set Theory. Plithogenic Logic and Computation, 2, 20-28. doi:10.61356/j.plc.2024.2327

[5] Al-Odhari, A. M. (2024). Characteristics Neutrosophic Subgroups of Axiomatic Neutrosophic Groups. Neutrosophic Optimization and Intelligent Systems, 3, 32-40. doi:https://doi.org/10.61356/j.nois.2024.3265

[6] Al-Odhari, A. M. (2024). On the Generalization of Neutrosophic Set Operations: Testing Proofs by Examples. HyperSoft Set Methods in Engineering, 2, 72-82. doi:10.61356/j.hsse.2024.2321

[7] Al-Odhari, A. M. (2024). Some Aspects of Neutrosophic Set Theory. Neutrosophic Optimization and Intelligent Systems, 4, 31-38. doi:https://doi.org/10.61356/j.nois.2024.4359

[8] Al-Odhari, A. M. (2024). Some Results of Neutrosophic Relations for Neutrosophic Set Theory. Neutrosophic Knowledge, 5, 76-86. doi:https://doi.org/10.5281/zenodo.14181753

[9] Al-Odhari, A. M. (2024). The Computations of Algebraic Structure of Neutrosophic Determinants. Sana'a University Journal of Applied Sciences and Technology, 2(1), 41-52. doi:10.59628/jast.v2i1.691

[10] Al-Odhari, A. (n.d.). Revisiting of Axiomatic Neutrosophic Groups According to neutrosophic Set Theory. to appear.

[11] PINTER, C. C. (2014). A Book of SET THEORY. Mineola, New York: DOVER PUBLICATIONS, INC. Retrieved from www.doverpublications.com

[12] Roitman, J. (2011). Introduction to Modern Set Theory. Retrieved from https://www.people.vcu.edu/~clarson/roitman-set-theory.pdf

[13] Smaradache, F. (1999). A UNIFYING FIELD IN LOGICS: NEUTROSOPHIC LOGIC. Rehoboth:American Research Press.

[14] Smarandache, F. (2002). Neutrosophy, A New Branch of Philosophy, in Multiple-Valued Logic. An International Journal, 8(3), 297-384.

[15] Smarandache, F. (2005). Neutrosophic Set, A Generalization of The Intuitionistic Fuzzy Sets. Inter. J. Pure Appl. Math, 24, 287-297.

[16] Smarandache, F. (2016). Degree of Dependence and Independence of the (Sub)Components of Fuzzy Set and Neutrosophic Set. Neutrosophic Sets and Systems, 11, 95-97. doi:org/10.5281/zenodo.571359

[17] Smarandache, F. (2016). Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics. Bruxelles , Belgique: Pons Editions. Retrieved from https://arxiv.org/ftp/arxiv/papers/160

[18] Smarandache, F. (2021). Indeterminacy in Neutrosophic Theories and their Applications. International Journal of Neutrosophic Science (IJNS), 15(2), 89-97. doi:https://doi.org/10.54216/IJNS.150203

[19] Weiss, W. A. (2008). AN INTRODUCTION TO SET THEORY. university of Tornoto. Retrieved from https://www.math.toronto.edu/weiss/set_theory.pdf