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Review of Rough Turiyam Neutrosophic Directed Graphs and Rough Pentapartitioned Neutrosophic Directed Graphs

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Abstract

Graph theory, a fundamental branch of mathematics, examines relationships between entities through the use of vertices and edges. Within this field, Uncertain Graph Theory has developed as a powerful framework to represent the uncertainties found in real-world networks.
Among the various uncertain graph models, Turiyam Neutrosophic Graphs and Pentapartitioned Neutrosophic Graphs are well-established. However, their extension to Directed Graphs remains relatively unexplored. To address this gap, this paper presents the concepts of the Turiyam Neutrosophic Directed Graph and the Pentapartitioned Neutrosophic Directed Graph, thereby broadening the scope of uncertain graph theory.

Keywords: Neutrosophic graph, Fuzzy graph, Plithogenic graph, Turiyam Neutrosophic Graph, Directed Graph

1 | Introduction

1.1 | Directed Graph Theory

Graph theory is a fundamental branch of mathematics that models networks using vertices (nodes) and edges (connections), representing relationships within various systems [18, 126, 133, 88, 29].

Graphs are generally classified into three main categories: undirected graphs, where edges have no orientation, directed graphs, where edges have specific directions [20, 17, 74, 58, 73, 100], and mixed graphs, which combine both undirected and directed edges [104, 48, 105]. Each of these categories has been widely studied in terms of algorithms and mathematical structures.

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1.2 | Uncertain Graph Theory

This paper delves into various uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs. These models extend classical graph theory by incorporating varying degrees of uncertainty, providing a more nuanced analysis of complex and ambiguous relationships. The versatility of uncertain graph models has broad real-world applications, leading to the development of diverse classes of uncertain graphs [41, 37, 54, 46, 49, 43, 51, 42, 38, 50].

Fundamental concepts such as Fuzzy Sets and Neutrosophic Sets underpin these uncertain graph models and are thoroughly documented in the literature [114, 96, 30, 24, 31, 11, 13, 12].

Within the context of Uncertain Graph Theory, the concept of a Directed Graph has become increasingly significant. Prominent examples include Fuzzy Digraphs [78, 82, 16, 80, 21, 32], Intuitionistic Fuzzy Digraphs [84, 77, 111, 112], and Neutrosophic Digraphs [22]. Additionally, researchers have explored extensions of concepts like Rough Graphs [66, 124, 86], Rough Directed Graphs [129, 3], and Rough Soft Graphs [87, 33] to the uncertain graph framework.

As evident from the extensive research mentioned above, studies on Uncertain Graph Theory are of significant importance. For a comprehensive understanding, readers are encouraged to consult existing survey papers [49, 41, 54].

1.3 | Our Contribution in This Paper

Among the uncertain graph models, the Turiyam Neutrosophic Graph and the Pentapartitioned Neutrosophic Graph are well-known. However, their application to Directed Graphs has received limited attention. Therefore, this paper introduces the concepts of the Turiyam Neutrosophic Directed Graph and the Pentapartitioned Neutrosophic Directed Graphs, expanding the scope of uncertain graph theory.

2 | Preliminaries and Definitions

This section provides an overview of the fundamental definitions and notations used throughout the paper.

2.1 | Basic Graph Concepts

Below are some of the foundational concepts in graph theory. For more comprehensive information on graph theory and its notations, refer to [29, 27, 28, 64, 133].

Definition 1 (Graph)**.** [29] A *graph* G is a mathematical structure that represents relationships between objects. It consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge connects a pair of vertices. Formally, a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges.

Definition 2 (Subgraph)**.** [29] A graph $H = (V_H, E_H)$ is called a *subgraph* of a graph $G = (V, E)$ if:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G,
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G,
- For each edge $e \in E_H$, if $e = \{u, v\}$, then $u, v \in V_H$.

In other words, a subgraph H of G consists of a subset of the vertices and edges of G, with the condition that all edges in E_H connect vertices in V_H.

Definition 3 (Digraph)**.** [29] A *digraph* (directed graph) $\vec{G}$ is a mathematical structure that represents directed relationships between objects. It consists of a set of vertices $V(\vec{G})$ and a set of directed edges $\vec{E}(\vec{G})$, where each directed edge has an orientation from one vertex to another. Formally, a digraph is represented as $\vec{G} = (V, \vec{E})$, where V is the set of vertices and $\vec{E}$ is the set of directed edges.

Definition 4 (Subdigraph)**.** (cf.[109]) A *subdigraph* $Q = (V(Q), E(Q))$ of a directed graph $D = (V(D), E(D))$ is defined as a directed graph where:

(1) $V(Q) \subseteq V(D)$, meaning the vertex set of Q is a subset of the vertex set of D.

(2) $E(Q) \subseteq E(D)$, meaning the edge set of Q is a subset of the edge set of D.

In this case, we write $Q \subseteq D$ to denote that Q is a subdigraph of D.

Induced Subdigraph: If Q includes all edges from D that connect the vertices in $V(Q)$, Q is called an *induced subdigraph* of D.

Definition 5 (Degree)**.** [29] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is defined as the number of edges connected to v. For undirected graphs, the degree is given by:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

For directed graphs, the *in-degree* $\deg^-(v)$ refers to the number of edges directed towards v, while the *out-degree* $\deg^+(v)$ represents the number of edges directed away from v.

2.2 | Uncertain Graph

This paper explores Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic concepts. Additionally, it introduces the definition of the Single-Valued Pentapartitioned Neutrosophic Graph. The definitions are presented as follows.

Definition 6 (Unified Uncertain Graphs Framework)**.** (cf.[41]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

(1) *Fuzzy Graph* [89, 103, 76, 122, 102]*:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$.

(2) *Intuitionistic Fuzzy Graph (IFG)* [71, 127, 134, 2]*:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ and $\nu_B(u, v) \in [0, 1]$, with $\mu_B(u, v) + \nu_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [75, 9, 106, 67, 19, 6]*:*

- Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0, 1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

(4) *Turiyam Neutrosophic Graph* [55, 57, 56]*:*

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where each component is in $[0, 1]$ and $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple.

(5) *Vague Graph* [5, 107, 101, 4]*:*

- Each vertex $v \in V$ is assigned a pair $(\tau(v), \phi(v))$, where $\tau(v) \in [0, 1]$ is the degree of truth-membership and $\phi(v) \in [0, 1]$ is the degree of false-membership, with $\tau(v) + \phi(v) \leq 1$.
- The grade of membership is characterized by the interval $[\tau(v), 1 - \phi(v)]$.
- Each edge $e = (u, v) \in E$ is assigned a pair $(\tau(e), \phi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\}, \quad \phi(e) \geq \max\{\phi(u), \phi(v)\}.$$

(6) *Hesitant Fuzzy Graph* [132, 62, 15, 90, 93]*:*

- Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0, 1]$, denoted $\sigma(v) \subseteq [0, 1]$.
- Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0, 1]$.
- Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.

(7) *Single-Valued Pentapartitioned Neutrosophic Graph* [23, 99, 68, 69]*:*

- Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
  - $T(v) \in [0, 1]$ is the truth-membership degree.
  - $C(v) \in [0, 1]$ is the contradiction-membership degree.
  - $R(v) \in [0, 1]$ is the ignorance-membership degree.
  - $U(v) \in [0, 1]$ is the unknown-membership degree.
  - $F(v) \in [0, 1]$ is the false-membership degree.
  - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
- Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

2.3 | Bipolar Neutrosophic Digraph

A Bipolar Neutrosophic Digraph assigns positive and negative truth, indeterminacy, and falsity memberships to vertices and directed edges, effectively modeling complex uncertainties. It can also be viewed as an extension of a Neutrosophic Digraph with positive and negative values. The definition of the Bipolar Neutrosophic Digraph is provided below [1, 8, 7]. Related concepts include the Bipolar Intuitionistic Fuzzy Digraph [25, 85] and the Bipolar Fuzzy Soft Digraph [108].

Definition 7 (Neutrosophic Digraph)**.** (cf.[113]) A *Neutrosophic Digraph* $G = (V, A, T_V, I_V, F_V, T_A, I_A, F_A)$ is defined as follows:

- V: the set of vertices.
- A: the set of directed edges (arcs), where each arc $(u, v) \in A$ represents a directed connection from vertex u to vertex v.
- $T_V, I_V, F_V : V \to [0, 1]$: the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* functions for vertices, satisfying:
$$0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3, \quad \forall v \in V.$$
- $T_A, I_A, F_A : A \to [0, 1]$: the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* functions for arcs, satisfying:
$$0 \leq T_A(u, v) + I_A(u, v) + F_A(u, v) \leq 3, \quad \forall (u, v) \in A.$$
- The arc membership functions satisfy the following conditions:
$$T_A(u, v) \leq \min(T_V(u), T_V(v)),$$
$$I_A(u, v) \geq \max(I_V(u), I_V(v)),$$

$$F_A(u,v) \geq \max(F_V(u), F_V(v)), \quad \forall (u,v) \in A.$$

Proposition 8. *A Neutrosophic Digraph becomes a Neutrosophic Graph when the directed arcs are replaced with undirected edges.*

Proof: Let $G_D = (V, A, T_V, I_V, F_V, T_A, I_A, F_A)$ be a Neutrosophic Digraph. The key components of G_D are:

- V: the set of vertices.
- A: the set of directed arcs, where $(u,v) \in A$ represents a directed connection from vertex u to vertex v.
- $T_V, I_V, F_V : V \to [0,1]$: the truth-membership, indeterminacy-membership, and falsity-membership functions for vertices.
- $T_A, I_A, F_A : A \to [0,1]$: the truth-membership, indeterminacy-membership, and falsity-membership functions for arcs.

To convert G_D into an undirected Neutrosophic Graph $G_U = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$, perform the following steps:

Replace every directed arc $(u,v) \in A$ with an undirected edge $\{u,v\} \in E$. The new edge set E satisfies $E = \{\{u,v\} \mid (u,v) \in A \text{ or } (v,u) \in A\}$.

Define the membership functions for the undirected edges $T_E, I_E, F_E : E \to [0,1]$ based on the arc membership functions as follows:

$$T_E(\{u,v\}) = \max(T_A(u,v), T_A(v,u)),$$
$$I_E(\{u,v\}) = \min(I_A(u,v), I_A(v,u)),$$
$$F_E(\{u,v\}) = \max(F_A(u,v), F_A(v,u)).$$

The relationships between the edge and vertex membership functions remain consistent:

$$T_E(\{u,v\}) \leq \min(T_V(u), T_V(v)),$$
$$I_E(\{u,v\}) \geq \max(I_V(u), I_V(v)),$$
$$F_E(\{u,v\}) \geq \max(F_V(u), F_V(v)).$$

The resulting graph $G_U = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$ is an undirected Neutrosophic Graph with the same vertex set and updated edge set and membership functions. The directed structure of the digraph has been replaced by an undirected one, and all membership functions are adjusted accordingly.

Thus, the Neutrosophic Digraph G_D is transformed into a Neutrosophic Graph G_U, completing the proof. □

Definition 9 (Bipolar Neutrosophic Digraph)**.** (cf.[1, 8, 7]) A *Bipolar Neutrosophic Digraph* $\vec{G} = (V, \vec{E})$ is defined over a non-empty set of vertices V and a set of directed edges $\vec{E} \subseteq V \times V$. Each vertex $x \in V$ and each directed edge $\vec{e} = (x,y) \in \vec{E}$ are characterized by seven membership functions.

Vertex Membership Functions:

- $t_P(x)$: Positive truth-membership, $t_P : V \to [0,1]$.
- $i_P(x)$: Positive indeterminacy-membership, $i_P : V \to [0,1]$.
- $f_P(x)$: Positive falsity-membership, $f_P : V \to [0,1]$.
- $t_N(x)$: Negative truth-membership, $t_N : V \to [-1,0]$.
- $i_N(x)$: Negative indeterminacy-membership, $i_N : V \to [-1,0]$.
- $f_N(x)$: Negative falsity-membership, $f_N : V \to [-1,0]$.

Edge Membership Functions:

- $\mu_P^t(\vec{e})$: Positive truth-membership of the edge, $\mu_P^t : \vec{E} \to [0,1]$.

- $\mu_P^i(\vec{e})$: Positive indeterminacy-membership of the edge, $\mu_P^i : \vec{E} \to [0,1]$.
- $\mu_P^f(\vec{e})$: Positive falsity-membership of the edge, $\mu_P^f : \vec{E} \to [0,1]$.
- $\mu_N^t(\vec{e})$: Negative truth-membership of the edge, $\mu_N^t : \vec{E} \to [-1,0]$.
- $\mu_N^i(\vec{e})$: Negative indeterminacy-membership of the edge, $\mu_N^i : \vec{E} \to [-1,0]$.
- $\mu_N^f(\vec{e})$: Negative falsity-membership of the edge, $\mu_N^f : \vec{E} \to [-1,0]$.

The edge membership functions must satisfy the following conditions:

$$\mu_P^t(x,y) \leq t_P(x) \wedge t_P(y), \quad \mu_P^i(x,y) \leq i_P(x) \wedge i_P(y), \quad \mu_P^f(x,y) \geq f_P(x) \vee f_P(y),$$

$$\mu_N^t(x,y) \geq t_N(x) \vee t_N(y), \quad \mu_N^i(x,y) \geq i_N(x) \vee i_N(y), \quad \mu_N^f(x,y) \leq f_N(x) \wedge f_N(y),$$

for all $x, y \in V$.

Proposition 10. *A Bipolar Neutrosophic Digraph generalizes both Neutrosophic Digraphs and classical directed graphs (digraphs).*

Proof: Let $\vec{G} = (V, \vec{E})$ be a Bipolar Neutrosophic Digraph with vertex membership functions $t_P, i_P, f_P, t_N, i_N, f_N$ and edge membership functions $\mu_P^t, \mu_P^i, \mu_P^f, \mu_N^t, \mu_N^i, \mu_N^f$. We analyze its relationships with Neutrosophic Digraphs and classical digraphs:

A Neutrosophic Digraph $G = (V, A, T_V, I_V, F_V, T_A, I_A, F_A)$ is obtained by restricting the membership functions of $\vec{G}$ as follows:

$$T_V(v) = t_P(v), \quad I_V(v) = i_P(v), \quad F_V(v) = f_P(v),$$

$$T_A(x,y) = \mu_P^t(x,y), \quad I_A(x,y) = \mu_P^i(x,y), \quad F_A(x,y) = \mu_P^f(x,y).$$

This restriction maps the positive components of the Bipolar Neutrosophic Digraph onto the membership functions of the Neutrosophic Digraph. The negative components t_N, i_N, f_N and $\mu_N^t, \mu_N^i, \mu_N^f$ are omitted, thus reducing the structure to a standard Neutrosophic Digraph.

A classical digraph $\vec{G}_C = (V, A)$ is obtained by further restricting the membership functions:

$$t_P(v) = 1, \quad i_P(v) = 0, \quad f_P(v) = 0, \quad t_N(v) = 0, \quad i_N(v) = 0, \quad f_N(v) = 0,$$

$$\mu_P^t(x,y) = 1, \quad \mu_P^i(x,y) = 0, \quad \mu_P^f(x,y) = 0, \quad \mu_N^t(x,y) = 0, \quad \mu_N^i(x,y) = 0, \quad \mu_N^f(x,y) = 0,$$

for all vertices $v \in V$ and arcs $(x,y) \in A$. This ensures that every arc $(x,y) \in \vec{E}$ has maximum truth-membership and no indeterminacy or falsity, reducing the Bipolar Neutrosophic Digraph to a classical directed graph.

The Bipolar Neutrosophic Digraph includes both positive and negative components, which can independently represent truth, indeterminacy, and falsity across the range $[0,1]$ (positive) and $[-1,0]$ (negative). By allowing both positive and negative evaluations, the Bipolar Neutrosophic Digraph extends the expressive power of the Neutrosophic Digraph and classical digraphs.

Hence, the Bipolar Neutrosophic Digraph generalizes both Neutrosophic Digraphs and classical digraphs. □

2.4 | Rough Neutrosophic Digraph

The definition of the Rough Neutrosophic Digraph is presented below. This graph concept extends the principles of Rough Graphs to Neutrosophic Digraphs [110, 70].

Definition 11. [95] Let X be the universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or an indiscernibility relation) on X, partitioning X into equivalence classes. For any subset $U \subseteq X$, the lower approximation $\underline{U}$ and the upper approximation $\overline{U}$ are defined as follows:

1. *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid R(x) \subseteq U\}$$

This is the set of all elements in X that certainly belong to U based on the equivalence classes defined by R.

2. *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}$$

This set contains all elements in X that possibly belong to U.

The pair $(\underline{U}, \overline{U})$ constitutes a rough set representation of U, where $\underline{U} \subseteq U \subseteq \overline{U}$.

Definition 12 (Rough Neutrosophic Digraph)**.** [110, 70] A *Rough Neutrosophic Digraph* is defined on a non-empty set of vertices V^* as a 4-tuple $G = (R, R_V, S, S_E)$, where:

- R is an equivalence relation on the set V^*.
- S is an equivalence relation on the set of directed edges $E^* \subseteq V^* \times V^*$.
- $R_V = (R_V, \overline{R}_V)$ is a rough neutrosophic set over V^*, where:
  - R_V is the lower approximation of the neutrosophic set,
  - $\overline{R}_V$ is the upper approximation of the neutrosophic set.
- $S_E = (S_E, \overline{S}_E)$ is a rough neutrosophic relation over E^*, where:
  - S_E is the lower approximation of the neutrosophic relation,
  - $\overline{S}_E$ is the upper approximation of the neutrosophic relation.

The rough neutrosophic digraph is represented as two digraphs:

- $G = (R_V, S_E)$: The lower approximate neutrosophic digraph,
- $G = (\overline{R}_V, \overline{S}_E)$: The upper approximate neutrosophic digraph.

Conditions for Membership Functions: For all $x, y \in V^*$, the membership functions must satisfy:

$$\mu_{S_E}(x, y) \leq \min\{\mu_{R_V}(x), \mu_{R_V}(y)\},$$
$$\sigma_{S_E}(x, y) \leq \min\{\sigma_{R_V}(x), \sigma_{R_V}(y)\},$$
$$\lambda_{S_E}(x, y) \leq \max\{\lambda_{R_V}(x), \lambda_{R_V}(y)\},$$

where μ, σ, and λ represent the degree of membership, the degree of indeterminacy, and the degree of falsity, respectively.

Proposition 13. *The Rough Neutrosophic Digraph can be transformed into a Neutrosophic Digraph.*

Proof: Let $G = (R, R_V, S, S_E)$ be a Rough Neutrosophic Digraph, where:

- R is an equivalence relation on the vertex set V^*.
- S is an equivalence relation on the edge set $E^* \subseteq V^* \times V^*$.
- $R_V = (\underline{R}_V, \overline{R}_V)$ is a rough neutrosophic set over V^*, with $\underline{R}_V$ and $\overline{R}_V$ representing the lower and upper approximations, respectively.
- $S_E = (\underline{S}_E, \overline{S}_E)$ is a rough neutrosophic relation over E^*, with $\underline{S}_E$ and $\overline{S}_E$ representing the lower and upper approximations, respectively.

Define a mapping that consolidates the rough neutrosophic structure into a neutrosophic structure. For each vertex $x \in V^*$, let:

$$T_V(x) = \frac{\underline{T}_{R_V}(x) + \overline{T}_{R_V}(x)}{2}, \quad I_V(x) = \frac{\underline{I}_{R_V}(x) + \overline{I}_{R_V}(x)}{2}, \quad F_V(x) = \frac{\underline{F}_{R_V}(x) + \overline{F}_{R_V}(x)}{2}.$$

For each directed edge $(x, y) \in E^*$, let:

$$T_A(x, y) = \frac{\underline{T}_{S_E}(x, y) + \overline{T}_{S_E}(x, y)}{2}, \quad I_A(x, y) = \frac{\underline{I}_{S_E}(x, y) + \overline{I}_{S_E}(x, y)}{2}, \quad F_A(x, y) = \frac{\underline{F}_{S_E}(x, y) + \overline{F}_{S_E}(x, y)}{2}.$$

This mapping ensures that each vertex and edge in the Rough Neutrosophic Digraph is assigned a single membership degree in the neutrosophic framework.

For vertices $x \in V^*$:

$$T_V(x) + I_V(x) + F_V(x) = \frac{\underline{T}_{R_V}(x) + \overline{T}_{R_V}(x)}{2} + \frac{\underline{I}_{R_V}(x) + \overline{I}_{R_V}(x)}{2} + \frac{\underline{F}_{R_V}(x) + \overline{F}_{R_V}(x)}{2}.$$

Since $\underline{T}_{R_V}(x) + \underline{I}_{R_V}(x) + \underline{F}_{R_V}(x) \leq 3$ and $\overline{T}_{R_V}(x) + \overline{I}_{R_V}(x) + \overline{F}_{R_V}(x) \leq 3$, it follows that:

$$T_V(x) + I_V(x) + F_V(x) \leq 3.$$

For edges $(x, y) \in E^*$:

$$T_A(x, y) \leq \min(T_V(x), T_V(y)),$$

$$I_A(x, y) \geq \max(I_V(x), I_V(y)),$$

$$F_A(x, y) \geq \max(F_V(x), F_V(y)).$$

These conditions hold due to the definitions of $\underline{S}_E$ and $\overline{S}_E$ in the rough neutrosophic framework.

Thus, the Rough Neutrosophic Digraph G is transformed into a Neutrosophic Digraph

$$G' = (V, A, T_V, I_V, F_V, T_A, I_A, F_A)$$

. □

3 | Result: Some Graph of Turiyam Neutrosophic Directed Graphs and Pentapartitioned Neutrosophic Directed Graphs

In this section, the results of this paper and the definitions of related concepts are presented.

3.1 | Basic Uncertain Digraph (with Review)

The definitions of the base Turiyam Neutrosophic Directed Graph and Pentapartitioned Neutrosophic Directed Graph are presented below. These graphs will be extended using the concepts of bipolar graphs and rough graphs.

Definition 14. A *Turiyam Neutrosophic Directed Graph* is an extension of the Turiyam Neutrosophic Graph concept to directed graphs, incorporating four membership degrees—truth, indeterminacy, falsity, and liberation—for each vertex and directed edge.

Let $\vec{G} = (V, E)$ be a directed graph, where:

- V is a non-empty finite set of vertices.
- $E \subseteq V \times V$ is the set of directed edges.

Vertex Membership Functions Each vertex $v \in V$ is associated with a quadruple of membership degrees:

- $t(v) \in [0, 1]$: truth-membership degree.
- $iv(v) \in [0, 1]$: indeterminacy-membership degree.
- $fv(v) \in [0, 1]$: falsity-membership degree.
- $lv(v) \in [0, 1]$: liberation-membership degree.

These degrees satisfy:

$$t(v) + iv(v) + fv(v) + lv(v) \leq 4, \quad \forall v \in V.$$

Edge Membership Functions Each directed edge $e = (u, v) \in E$ is associated with a quadruple of membership degrees:

- $t(e) \in [0, 1]$: truth-membership degree.
- $iv(e) \in [0, 1]$: indeterminacy-membership degree.
- $fv(e) \in [0, 1]$: falsity-membership degree.
- $lv(e) \in [0, 1]$: liberation-membership degree.

These degrees satisfy the following conditions:

(1) *Truth-Membership Degree:*

$$t(e) \leq \min\{t(u), t(v)\}.$$

(2) *Indeterminacy-Membership Degree:*

$$iv(e) \geq \max\{iv(u), iv(v)\}.$$

(3) *Falsity-Membership Degree:*

$$fv(e) \geq \max\{fv(u), fv(v)\}.$$

(4) *Liberation-Membership Degree:*

$$lv(e) \geq \max\{lv(u), lv(v)\}.$$

Additionally, the sum of the membership degrees for each edge satisfies:

$$t(e) + iv(e) + fv(e) + lv(e) \leq 4, \quad \forall e \in E.$$

Theorem 15. *The Turiyam Neutrosophic Digraph generalizes the Neutrosophic Digraph.*

Proof: Let $G = (V, A, T_V, I_V, F_V, T_A, I_A, F_A)$ be a Neutrosophic Digraph, where each vertex $v \in V$ and each arc $a \in A$ are characterized by the membership degrees (T, I, F) satisfying:

$$T(v) + I(v) + F(v) \leq 3, \quad T(a) + I(a) + F(a) \leq 3.$$

Now consider a Turiyam Neutrosophic Digraph $G' = (V, A, t_V, iv_V, fv_V, lv_V, t_A, iv_A, fv_A, lv_A)$, where each vertex $v \in V$ and each arc $a \in A$ are characterized by the membership degrees (t, iv, fv, lv) satisfying:

$$t(v) + iv(v) + fv(v) + lv(v) \leq 4, \quad t(a) + iv(a) + fv(a) + lv(a) \leq 4.$$

The Neutrosophic Digraph can be obtained as a special case of the Turiyam Neutrosophic Digraph by setting $lv(v) = lv(a) = 0$, thus reducing the Turiyam membership functions to the Neutrosophic membership functions. Therefore, the Turiyam Neutrosophic Digraph is a generalization of the Neutrosophic Digraph. □

Theorem 16. *A Turiyam Neutrosophic Directed Graph reduces to a Turiyam Neutrosophic Graph when made undirected.*

Proof: Let $\vec{G} = (V, E)$ be a Turiyam Neutrosophic Directed Graph, where:

- Each vertex $v \in V$ has membership degrees $t(v), iv(v), fv(v), lv(v)$ satisfying:

$$t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$

- Each directed edge $e = (u, v) \in E$ has membership degrees $t(e), iv(e), fv(e), lv(e)$ satisfying:

$$t(e) + iv(e) + fv(e) + lv(e) \leq 4.$$

To construct an undirected graph $G = (V, E')$, define E' such that:

$$(u, v) \in E' \iff (u, v) \in E \text{ or } (v, u) \in E.$$

For each edge $\{u, v\} \in E'$, the membership degrees are given by:

$$\begin{aligned} t(\{u, v\}) &= \max\{t((u, v)), t((v, u))\}, \\ iv(\{u, v\}) &= \min\{iv((u, v)), iv((v, u))\}, \\ fv(\{u, v\}) &= \max\{fv((u, v)), fv((v, u))\}, \\ lv(\{u, v\}) &= \max\{lv((u, v)), lv((v, u))\}. \end{aligned}$$

These degrees satisfy:

$$t(\{u, v\}) + iv(\{u, v\}) + fv(\{u, v\}) + lv(\{u, v\}) \leq 4.$$

The resulting graph G is a Turiyam Neutrosophic Graph. Thus, a Turiyam Neutrosophic Directed Graph reduces to a Turiyam Neutrosophic Graph when made undirected. □

Theorem 17. *A Turiyam Neutrosophic Directed Graph generalizes a Directed Graph.*

Proof: Let $\vec{G} = (V, E)$ be a directed graph where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of directed edges.

Now consider a Turiyam Neutrosophic Directed Graph $\vec{G'} = (V, E)$, where:

- Each vertex $v \in V$ is assigned membership degrees $t(v), iv(v), fv(v), lv(v)$ such that:

$$t(v) = 1, \quad iv(v) = fv(v) = lv(v) = 0.$$

- Each edge $e = (u, v) \in E$ is assigned membership degrees $t(e), iv(e), fv(e), lv(e)$ such that:

$$t(e) = 1, \quad iv(e) = fv(e) = lv(e) = 0.$$

Under these assignments, the Turiyam Neutrosophic Directed Graph $\vec{G'}$ becomes equivalent to the original directed graph $\vec{G}$. Thus, the Turiyam Neutrosophic Directed Graph generalizes a Directed Graph. □

Definition 18. A *Pentapartitioned Neutrosophic Directed Graph* extends the neutrosophic graph framework to directed graphs with five membership degrees—truth, contradiction, ignorance, unknown, and falsity—for each vertex and directed edge.

Let $\vec{G} = (V, E)$ be a directed graph, where:

- V is a non-empty finite set of vertices.
- $E \subseteq V \times V$ is the set of directed edges.

Vertex Membership Functions Each vertex $v \in V$ is associated with a quintuple of membership degrees:

- $T(v) \in [0, 1]$: truth-membership degree.
- $C(v) \in [0, 1]$: contradiction-membership degree.
- $R(v) \in [0, 1]$: ignorance-membership degree.
- $U(v) \in [0, 1]$: unknown-membership degree.
- $F(v) \in [0, 1]$: falsity-membership degree.

These degrees satisfy:

$$T(v) + C(v) + R(v) + U(v) + F(v) \leq 5, \quad \forall v \in V.$$

Edge Membership Functions Each directed edge $e = (u, v) \in E$ is associated with a quintuple of membership degrees:

- $T(e) \in [0, 1]$: truth-membership degree.
- $C(e) \in [0, 1]$: contradiction-membership degree.
- $R(e) \in [0, 1]$: ignorance-membership degree.
- $U(e) \in [0, 1]$: unknown-membership degree.
- $F(e) \in [0, 1]$: falsity-membership degree.

These degrees satisfy the following conditions:

(1) *Truth-Membership Degree:*
$$T(e) \leq \min\{T(u), T(v)\}.$$

(2) *Contradiction-Membership Degree:*
$$C(e) \leq \min\{C(u), C(v)\}.$$

(3) *Ignorance-Membership Degree:*
$$R(e) \geq \max\{R(u), R(v)\}.$$

(4) *Unknown-Membership Degree:*
$$U(e) \geq \max\{U(u), U(v)\}.$$

(5) *Falsity-Membership Degree:*
$$F(e) \geq \max\{F(u), F(v)\}.$$

Additionally, the sum of the membership degrees for each edge satisfies:

$$T(e) + C(e) + R(e) + U(e) + F(e) \leq 5, \quad \forall e \in E.$$

Theorem 19. *The Pentapartitioned Neutrosophic Directed Graph generalizes both the Turiyam Neutrosophic Digraph and the Neutrosophic Digraph.*

Proof: Let $G = (V, A, t_V, iv_V, fv_V, lv_V, t_A, iv_A, fv_A, lv_A)$ be a Turiyam Neutrosophic Digraph, where:
$$t(v) + iv(v) + fv(v) + lv(v) \leq 4, \quad t(a) + iv(a) + fv(a) + lv(a) \leq 4.$$

Now consider a Pentapartitioned Neutrosophic Directed Graph
$$G' = (V, A, T_V, C_V, R_V, U_V, F_V, T_A, C_A, R_A, U_A, F_A)$$
, where:
$$T(v) + C(v) + R(v) + U(v) + F(v) \leq 5, \quad T(a) + C(a) + R(a) + U(a) + F(a) \leq 5.$$

By setting $C(v) = R(v) = U(v) = 0$ and $C(a) = R(a) = U(a) = 0$, the membership functions of G' reduce to those of G, thus proving that the Pentapartitioned Neutrosophic Directed Graph generalizes the Turiyam Neutrosophic Digraph.

Additionally, by setting $lv(v) = lv(a) = 0$ and $C(v) = R(v) = U(v) = 0$, the membership functions of G' reduce to those of the Neutrosophic Digraph. Hence, the Pentapartitioned Neutrosophic Directed Graph generalizes both the Turiyam Neutrosophic Digraph and the Neutrosophic Digraph. $\square$

3.2 | Bipolar Turiyam Neutrosophic Digraph

The definitions of the Bipolar Turiyam Neutrosophic Digraph and the Bipolar Pentapartitioned Neutrosophic Directed Graph are presented below.

Definition 20 (Bipolar Turiyam Neutrosophic Digraph)**.** A *Bipolar Turiyam Neutrosophic Digraph* is a directed graph where each vertex and directed edge is characterized by both positive and negative membership degrees across four aspects: truth, indeterminacy, falsity, and liberation.

Let $\vec{G} = (V, E)$ be a directed graph, where:

- V is a non-empty finite set of vertices.
- $E \subseteq V \times V$ is the set of directed edges.

Vertex Membership Functions Each vertex $v \in V$ is associated with eight membership degrees:

- *Positive Membership Degrees:*
  - $t_P(v) \in [0, 1]$: Positive truth-membership degree.
  - $iv_P(v) \in [0, 1]$: Positive indeterminacy-membership degree.
  - $fv_P(v) \in [0, 1]$: Positive falsity-membership degree.
  - $lv_P(v) \in [0, 1]$: Positive liberation-membership degree.
- *Negative Membership Degrees:*
  - $t_N(v) \in [-1, 0]$: Negative truth-membership degree.
  - $iv_N(v) \in [-1, 0]$: Negative indeterminacy-membership degree.
  - $fv_N(v) \in [-1, 0]$: Negative falsity-membership degree.
  - $lv_N(v) \in [-1, 0]$: Negative liberation-membership degree.

These degrees satisfy:

$$t_P(v) + iv_P(v) + fv_P(v) + lv_P(v) \leq 4,$$

$$t_N(v) + iv_N(v) + fv_N(v) + lv_N(v) \geq -4.$$

Edge Membership Functions Each directed edge $e = (u, v) \in E$ is associated with eight membership degrees:

- *Positive Membership Degrees:*
  - $t_P(e) \in [0, 1]$: Positive truth-membership degree.
  - $iv_P(e) \in [0, 1]$: Positive indeterminacy-membership degree.
  - $fv_P(e) \in [0, 1]$: Positive falsity-membership degree.
  - $lv_P(e) \in [0, 1]$: Positive liberation-membership degree.
- *Negative Membership Degrees:*
  - $t_N(e) \in [-1, 0]$: Negative truth-membership degree.
  - $iv_N(e) \in [-1, 0]$: Negative indeterminacy-membership degree.
  - $fv_N(e) \in [-1, 0]$: Negative falsity-membership degree.
  - $lv_N(e) \in [-1, 0]$: Negative liberation-membership degree.

These degrees satisfy the following conditions:

(1) *For Positive Membership Degrees:*

$$t_P(e) \leq \min\{t_P(u), t_P(v)\},$$
$$iv_P(e) \leq \min\{iv_P(u), iv_P(v)\},$$
$$fv_P(e) \geq \max\{fv_P(u), fv_P(v)\},$$
$$lv_P(e) \geq \max\{lv_P(u), lv_P(v)\}.$$

(2) *For Negative Membership Degrees:*

$$t_N(e) \geq \max\{t_N(u), t_N(v)\},$$
$$iv_N(e) \geq \max\{iv_N(u), iv_N(v)\},$$
$$fv_N(e) \leq \min\{fv_N(u), fv_N(v)\},$$
$$lv_N(e) \leq \min\{lv_N(u), lv_N(v)\}.$$

Additionally, the sums of the membership degrees satisfy:

$$t_P(e) + iv_P(e) + fv_P(e) + lv_P(e) \leq 4,$$

$$t_N(e) + iv_N(e) + fv_N(e) + lv_N(e) \geq -4.$$

Theorem 21. *A Bipolar Neutrosophic Turiyam Digraph generalizes a Bipolar Neutrosophic Digraph.*

Proof: Let $\vec{G} = (V, E, t_P, iv_P, fv_P, t_N, iv_N, fv_N)$ be a Bipolar Neutrosophic Digraph, where:

$$t_P(v) + iv_P(v) + fv_P(v) \leq 3, \quad t_N(v) + iv_N(v) + fv_N(v) \geq -3, \quad \forall v \in V,$$

and similar conditions hold for edges.

Consider a Bipolar Neutrosophic Turiyam Digraph $\vec{G}' = (V, E, t_P, iv_P, fv_P, lv_P, t_N, iv_N, fv_N, lv_N)$, where:

$$t_P(v) + iv_P(v) + fv_P(v) + lv_P(v) \leq 4, \quad t_N(v) + iv_N(v) + fv_N(v) + lv_N(v) \geq -4, \quad \forall v \in V.$$

By setting $lv_P(v) = lv_N(v) = 0$, the Bipolar Neutrosophic Turiyam Digraph reduces to a Bipolar Neutrosophic Digraph. Hence, it generalizes the Bipolar Neutrosophic Digraph. □

Theorem 22. *A Bipolar Neutrosophic Turiyam Digraph generalizes a Neutrosophic Turiyam Digraph.*

Proof: Let $\vec{G} = (V, E, t, iv, fv, lv)$ be a Neutrosophic Turiyam Digraph, where:

$$t(v) + iv(v) + fv(v) + lv(v) \leq 4, \quad \forall v \in V,$$

and similar conditions hold for edges.

Consider a Bipolar Neutrosophic Turiyam Digraph $\vec{G}' = (V, E, t_P, iv_P, fv_P, lv_P, t_N, iv_N, fv_N, lv_N)$, where:

$$t_P(v) + iv_P(v) + fv_P(v) + lv_P(v) \leq 4, \quad t_N(v) + iv_N(v) + fv_N(v) + lv_N(v) \geq -4, \quad \forall v \in V.$$

By setting $t_N(v) = iv_N(v) = fv_N(v) = lv_N(v) = 0$, the Bipolar Neutrosophic Turiyam Digraph reduces to a Neutrosophic Turiyam Digraph. Hence, it generalizes the Neutrosophic Turiyam Digraph. □

Theorem 23. *A Bipolar Neutrosophic Turiyam Digraph generalizes a Directed Graph.*

Proof: Let $\vec{G} = (V, E)$ be a directed graph where each vertex and edge is unweighted. Consider a Bipolar Neutrosophic Turiyam Digraph $\vec{G}' = (V, E, t_P, iv_P, fv_P, lv_P, t_N, iv_N, fv_N, lv_N)$, where:

$$t_P(v) = 1, \quad iv_P(v) = fv_P(v) = lv_P(v) = 0, \quad t_N(v) = iv_N(v) = fv_N(v) = lv_N(v) = 0, \quad \forall v \in V,$$

and similar conditions hold for edges.

Under these assignments, $\vec{G}'$ is equivalent to $\vec{G}$. Hence, the Bipolar Neutrosophic Turiyam Digraph generalizes a Directed Graph. □

Definition 24. A *Bipolar Pentapartitioned Neutrosophic Directed Graph* (BPNDG) extends the framework of directed graphs by associating both positive and negative membership degrees across five aspects: truth, contradiction, ignorance, unknown, and falsity.

Let $\vec{G} = (V, E)$ be a directed graph, where:

- V: A non-empty finite set of vertices.
- $E \subseteq V \times V$: The set of directed edges.

Vertex Membership Functions Each vertex $v \in V$ is associated with 10 membership degrees:

- *Positive Membership Degrees:*

  $$T_P(v), C_P(v), R_P(v), U_P(v), F_P(v) \in [0, 1],$$

  where:

  - $T_P(v)$: Positive truth-membership.
  - $C_P(v)$: Positive contradiction-membership.
  - $R_P(v)$: Positive ignorance-membership.
  - $U_P(v)$: Positive unknown-membership.
  - $F_P(v)$: Positive falsity-membership.

- *Negative Membership Degrees:*

  $$T_N(v), C_N(v), R_N(v), U_N(v), F_N(v) \in [-1, 0],$$

  where:

  - $T_N(v)$: Negative truth-membership.
  - $C_N(v)$: Negative contradiction-membership.
  - $R_N(v)$: Negative ignorance-membership.
  - $U_N(v)$: Negative unknown-membership.
  - $F_N(v)$: Negative falsity-membership.

These degrees satisfy:

$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5,$$
$$T_N(v) + C_N(v) + R_N(v) + U_N(v) + F_N(v) \geq -5.$$

Edge Membership Functions Each directed edge $e = (u, v) \in E$ is associated with 10 membership degrees:

- *Positive Membership Degrees:*

  $$T_P(e), C_P(e), R_P(e), U_P(e), F_P(e) \in [0, 1],$$

  where the definitions are analogous to the vertex membership functions.

- *Negative Membership Degrees:*

  $$T_N(e), C_N(e), R_N(e), U_N(e), F_N(e) \in [-1, 0],$$

  where the definitions are analogous to the vertex membership functions.

These degrees satisfy:

$$T_P(e) + C_P(e) + R_P(e) + U_P(e) + F_P(e) \leq 5,$$
$$T_N(e) + C_N(e) + R_N(e) + U_N(e) + F_N(e) \geq -5.$$

Theorem 25. *A Bipolar Pentapartitioned Neutrosophic Directed Graph (BPNDG) can be transformed into:*

(1) *A Bipolar Neutrosophic Directed Graph (BNDG) by setting certain membership degrees to zero and reinterpreting others.*

(2) *A Bipolar Turiyam Neutrosophic Directed Graph (BTDG) by merging specific membership degrees and adjusting the sum conditions accordingly.*

Proof: *1. Transformation from BPNDG to BNDG*

Let $G = (V, E)$ be a Bipolar Pentapartitioned Neutrosophic Directed Graph. Each vertex $v \in V$ has positive and negative membership degrees:

- *Positive Membership Degrees*:
  - $T_P(v) \in [0, 1]$: Positive truth-membership degree.
  - $C_P(v) \in [0, 1]$: Positive contradiction-membership degree.
  - $R_P(v) \in [0, 1]$: Positive ignorance-membership degree.
  - $U_P(v) \in [0, 1]$: Positive unknown-membership degree.
  - $F_P(v) \in [0, 1]$: Positive falsity-membership degree.
- *Negative Membership Degrees*:
  - $T_N(v) \in [-1, 0]$: Negative truth-membership degree.
  - $C_N(v) \in [-1, 0]$: Negative contradiction-membership degree.
  - $R_N(v) \in [-1, 0]$: Negative ignorance-membership degree.
  - $U_N(v) \in [-1, 0]$: Negative unknown-membership degree.
  - $F_N(v) \in [-1, 0]$: Negative falsity-membership degree.

These degrees satisfy:

$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5,$$
$$T_N(v) + C_N(v) + R_N(v) + U_N(v) + F_N(v) \geq -5.$$

Similarly for edges.

To transform G into a Bipolar Neutrosophic Directed Graph, proceed as follows:

- *Set Contradiction and Unknown Membership Degrees to Zero*:
  $$C_P(v) = U_P(v) = 0, \quad C_N(v) = U_N(v) = 0, \quad \forall v \in V.$$
  Similarly for edges:
  $$C_P(e) = U_P(e) = 0, \quad C_N(e) = U_N(e) = 0, \quad \forall e \in E.$$
- *Rename Ignorance Membership Degrees as Indeterminacy Membership Degrees*:
  $$i_P(v) = R_P(v), \quad i_N(v) = R_N(v), \quad \forall v \in V.$$
  Similarly for edges:
  $$i_P(e) = R_P(e), \quad i_N(e) = R_N(e), \quad \forall e \in E.$$
- *Define the Remaining Membership Degrees*:

  - Positive degrees: $t_P(v), i_P(v), f_P(v)$.
  - Negative degrees: $t_N(v), i_N(v), f_N(v)$.
- *Sum Conditions*:

$$t_P(v) + i_P(v) + f_P(v) = T_P(v) + R_P(v) + F_P(v) \leq 5,$$
$$t_N(v) + i_N(v) + f_N(v) = T_N(v) + R_N(v) + F_N(v) \geq -5.$$

- *Normalization*: To align with the standard Bipolar Neutrosophic Directed Graph, where the sum of membership degrees is typically considered within $[0, 3]$ for positive degrees and $[-3, 0]$ for negative degrees, we can normalize the membership degrees:

$$\tilde{t}_P(v) = \frac{t_P(v)}{5} \times 3, \quad \tilde{i}_P(v) = \frac{i_P(v)}{5} \times 3, \quad \tilde{f}_P(v) = \frac{f_P(v)}{5} \times 3.$$

  Similarly for negative degrees:

$$\tilde{t}_N(v) = \frac{t_N(v)}{5} \times 3, \quad \tilde{i}_N(v) = \frac{i_N(v)}{5} \times 3, \quad \tilde{f}_N(v) = \frac{f_N(v)}{5} \times 3.$$

  After normalization, the sum conditions become:

$$\tilde{t}_P(v) + \tilde{i}_P(v) + \tilde{f}_P(v) \leq 3, \quad \tilde{t}_N(v) + \tilde{i}_N(v) + \tilde{f}_N(v) \geq -3.$$

Thus, the Bipolar Pentapartitioned Neutrosophic Directed Graph reduces to a Bipolar Neutrosophic Directed Graph under these transformations.

2. Transformation from BPNDG to BTDG

To transform G into a Bipolar Turiyam Neutrosophic Directed Graph, proceed as follows:

- *Merge Contradiction Membership Degrees with Truth Membership Degrees*:

$$t_P(v) = T_P(v) + C_P(v), \quad t_N(v) = T_N(v) + C_N(v), \quad \forall v \in V.$$

  Similarly for edges:

$$t_P(e) = T_P(e) + C_P(e), \quad t_N(e) = T_N(e) + C_N(e), \quad \forall e \in E.$$

- *Rename Ignorance and Unknown Membership Degrees*:

$$iv_P(v) = R_P(v), \quad iv_N(v) = R_N(v), \quad \forall v \in V.$$
$$lv_P(v) = U_P(v), \quad lv_N(v) = U_N(v), \quad \forall v \in V.$$

  Similarly for edges.

- *Keep Falsity Membership Degrees Unchanged*:

$$fv_P(v) = F_P(v), \quad fv_N(v) = F_N(v), \quad \forall v \in V.$$

  Similarly for edges.

- *Sum Conditions*:

$$t_P(v) + iv_P(v) + fv_P(v) + lv_P(v) = T_P(v) + C_P(v) + R_P(v) + F_P(v) + U_P(v) \leq 5,$$
$$t_N(v) + iv_N(v) + fv_N(v) + lv_N(v) = T_N(v) + C_N(v) + R_N(v) + F_N(v) + U_N(v) \geq -5.$$

- *Normalization*: To conform to the Bipolar Turiyam Neutrosophic Directed Graph, where the sums of membership degrees are within $[0, 4]$ and $[-4, 0]$, normalize as:

$$\tilde{t}_P(v) = \frac{t_P(v)}{5} \times 4, \quad \tilde{iv}_P(v) = \frac{iv_P(v)}{5} \times 4, \quad \tilde{fv}_P(v) = \frac{fv_P(v)}{5} \times 4, \quad \tilde{lv}_P(v) = \frac{lv_P(v)}{5} \times 4.$$

  Similarly for negative degrees.

  After normalization, the sum conditions become:

$$\tilde{t}_P(v) + \tilde{iv}_P(v) + \tilde{fv}_P(v) + \tilde{lv}_P(v) \leq 4, \quad \tilde{t}_N(v) + \tilde{iv}_N(v) + \tilde{fv}_N(v) + \tilde{lv}_N(v) \geq -4.$$

Thus, the Bipolar Pentapartitioned Neutrosophic Directed Graph reduces to a Bipolar Turiyam Neutrosophic Directed Graph under these transformations. □

Theorem 26. *A Bipolar Pentapartitioned Neutrosophic Directed Graph (BPNDG) generalizes a Pentapartitioned Neutrosophic Directed Graph (PNDG).*

Proof: Let $\vec{G} = (V, E, T_P, C_P, R_P, U_P, F_P)$ be a Pentapartitioned Neutrosophic Directed Graph (PNDG), where:

- Each vertex $v \in V$ is characterized by the positive membership degrees:
$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5, \quad T_P(v), C_P(v), R_P(v), U_P(v), F_P(v) \in [0, 1].$$
- Each directed edge $e = (u, v) \in E$ satisfies:
$$T_P(e) + C_P(e) + R_P(e) + U_P(e) + F_P(e) \leq 5, \quad T_P(e), C_P(e), R_P(e), U_P(e), F_P(e) \in [0, 1].$$

Now, consider a Bipolar Pentapartitioned Neutrosophic Directed Graph

$$\vec{G}' = (V, E, T_P, C_P, R_P, U_P, F_P, T_N, C_N, R_N, U_N, F_N)$$

, where:

- Each vertex $v \in V$ is characterized by both positive and negative membership degrees:
$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5, \quad T_P(v), C_P(v), R_P(v), U_P(v), F_P(v) \in [0, 1],$$
$$T_N(v) + C_N(v) + R_N(v) + U_N(v) + F_N(v) \geq -5, \quad T_N(v), C_N(v), R_N(v), U_N(v), F_N(v) \in [-1, 0].$$
- Each directed edge $e = (u, v) \in E$ satisfies:
$$T_P(e) + C_P(e) + R_P(e) + U_P(e) + F_P(e) \leq 5, \quad T_P(e), C_P(e), R_P(e), U_P(e), F_P(e) \in [0, 1],$$
$$T_N(e) + C_N(e) + R_N(e) + U_N(e) + F_N(e) \geq -5, \quad T_N(e), C_N(e), R_N(e), U_N(e), F_N(e) \in [-1, 0].$$

To reduce $\vec{G}'$ to $\vec{G}$, set all negative membership degrees to zero:

$$T_N(v) = C_N(v) = R_N(v) = U_N(v) = F_N(v) = 0, \quad \forall v \in V,$$
$$T_N(e) = C_N(e) = R_N(e) = U_N(e) = F_N(e) = 0, \quad \forall e \in E.$$

Under this transformation, the Bipolar Pentapartitioned Neutrosophic Directed Graph $\vec{G}'$ reduces to the Pentapartitioned Neutrosophic Directed Graph $\vec{G}$, as the negative membership degrees are eliminated, leaving only the positive membership degrees.

Thus, the Bipolar Pentapartitioned Neutrosophic Directed Graph generalizes the Pentapartitioned Neutrosophic Directed Graph. □

3.3 | Rough Digraph

We consider the mathematical structures of the Rough Turiyam Neutrosophic Digraph and the Rough Pentapartitioned Neutrosophic Directed Graph. The definitions are presented below.

Definition 27 (Rough Turiyam Neutrosophic Digraph)**.** A *Rough Turiyam Neutrosophic Digraph* combines the concepts of rough sets and Turiyam Neutrosophic digraphs. It is defined over a universe of discourse V^* with equivalence relations to form lower and upper approximations of Turiyam Neutrosophic sets.

Let V^* be a non-empty finite set of vertices. A Rough Turiyam Neutrosophic Digraph is a quadruple $G = (R, R_V, S, S_E)$, where:

- R is an equivalence relation on V^*.
- S is an equivalence relation on the set of directed edges $E^* \subseteq V^* \times V^*$.
- $R_V = (R_L, R_U)$ is a rough Turiyam Neutrosophic set over V^*, where:
  - R_L is the lower approximation of the Turiyam Neutrosophic set.

  - R_U is the upper approximation of the Turiyam Neutrosophic set.
- $S_E = (S_L, S_U)$ is a rough Turiyam Neutrosophic relation over E^*, where:
  - S_L is the lower approximation of the Turiyam Neutrosophic relation.
  - S_U is the upper approximation of the Turiyam Neutrosophic relation.

Vertex Membership Functions Each vertex $x \in V^*$ has lower and upper approximation membership degrees:

- *Lower Approximation Membership Degrees:*
  - $t_L(x) \in [0,1]$: Lower truth-membership degree.
  - $iv_L(x) \in [0,1]$: Lower indeterminacy-membership degree.
  - $fv_L(x) \in [0,1]$: Lower falsity-membership degree.
  - $lv_L(x) \in [0,1]$: Lower liberation-membership degree.
- *Upper Approximation Membership Degrees:*
  - $t_U(x) \in [0,1]$: Upper truth-membership degree.
  - $iv_U(x) \in [0,1]$: Upper indeterminacy-membership degree.
  - $fv_U(x) \in [0,1]$: Upper falsity-membership degree.
  - $lv_U(x) \in [0,1]$: Upper liberation-membership degree.

These degrees satisfy:

$$\begin{aligned}
&t_L(x) \le t_U(x), \quad iv_L(x) \le iv_U(x),\\
&fv_L(x) \le fv_U(x), \quad lv_L(x) \le lv_U(x),\\
&t_L(x) + iv_L(x) + fv_L(x) + lv_L(x) \le 4,\\
&t_U(x) + iv_U(x) + fv_U(x) + lv_U(x) \le 4.
\end{aligned}$$

Edge Membership Functions Each directed edge $e = (x, y) \in E^*$ has lower and upper approximation membership degrees:

- *Lower Approximation Membership Degrees:*
  - $t_L(e) \in [0,1]$: Lower truth-membership degree.
  - $iv_L(e) \in [0,1]$: Lower indeterminacy-membership degree.
  - $fv_L(e) \in [0,1]$: Lower falsity-membership degree.
  - $lv_L(e) \in [0,1]$: Lower liberation-membership degree.
- *Upper Approximation Membership Degrees:*
  - $t_U(e) \in [0,1]$: Upper truth-membership degree.
  - $iv_U(e) \in [0,1]$: Upper indeterminacy-membership degree.
  - $fv_U(e) \in [0,1]$: Upper falsity-membership degree.
  - $lv_U(e) \in [0,1]$: Upper liberation-membership degree.

These degrees satisfy:

(1) *For Lower Approximations:*

$$\begin{aligned}
t_L(e) &\le \min\{t_L(x), t_L(y)\},\\
iv_L(e) &\ge \max\{iv_L(x), iv_L(y)\},\\
fv_L(e) &\ge \max\{fv_L(x), fv_L(y)\},\\
lv_L(e) &\ge \max\{lv_L(x), lv_L(y)\}.
\end{aligned}$$

(2) *For Upper Approximations:*

$$\begin{aligned} t_U(e) &\leq \min\{t_U(x), t_U(y)\}, \\ iv_U(e) &\geq \max\{iv_U(x), iv_U(y)\}, \\ fv_U(e) &\geq \max\{fv_U(x), fv_U(y)\}, \\ lv_U(e) &\geq \max\{lv_U(x), lv_U(y)\}. \end{aligned}$$

Additionally, the sums of the membership degrees satisfy:

$$t_L(e) + iv_L(e) + fv_L(e) + lv_L(e) \leq 4,$$
$$t_U(e) + iv_U(e) + fv_U(e) + lv_U(e) \leq 4.$$

Theorem 28. *The Rough Turiyam Neutrosophic Digraph generalizes the Rough Neutrosophic Digraph.*

Proof: Let $G = (R, R_V, S, S_E)$ be a Rough Turiyam Neutrosophic Digraph, where:

- $R_V = (R_L, R_U)$ consists of lower and upper approximations of Turiyam Neutrosophic sets.
- $S_E = (S_L, S_U)$ consists of lower and upper approximations of Turiyam Neutrosophic relations.

For any vertex $x \in V^*$, define:

$$\begin{aligned} t_L(x) &= T_L(x), & t_U(x) &= T_U(x), \\ iv_L(x) &= I_L(x), & iv_U(x) &= I_U(x), \\ fv_L(x) &= F_L(x), & fv_U(x) &= F_U(x), \\ lv_L(x) &= 0, & lv_U(x) &= 0. \end{aligned}$$

Similarly, for any edge $e = (x, y) \in E^*$, set:

$$\begin{aligned} t_L(e) &= T_L(e), & t_U(e) &= T_U(e), \\ iv_L(e) &= I_L(e), & iv_U(e) &= I_U(e), \\ fv_L(e) &= F_L(e), & fv_U(e) &= F_U(e), \\ lv_L(e) &= 0, & lv_U(e) &= 0. \end{aligned}$$

Under this mapping, the Rough Turiyam Neutrosophic Digraph reduces to a Rough Neutrosophic Digraph, satisfying the constraints and properties of a Rough Neutrosophic framework. Thus, G generalizes the Rough Neutrosophic Digraph. □

Theorem 29. *The Rough Turiyam Neutrosophic Digraph can be transformed into:*

- *A Turiyam Neutrosophic Digraph.*
- *A Neutrosophic Digraph.*

Proof: Let $G = (R, R_V, S, S_E)$ be a Rough Turiyam Neutrosophic Digraph. For each vertex $x \in V^*$, define:

$$t(x) = \frac{t_L(x) + t_U(x)}{2}, \quad iv(x) = \frac{iv_L(x) + iv_U(x)}{2}, \quad fv(x) = \frac{fv_L(x) + fv_U(x)}{2}, \quad lv(x) = \frac{lv_L(x) + lv_U(x)}{2}.$$

Similarly, for each edge $e = (x, y) \in E^*$, define:

$$t(e) = \frac{t_L(e) + t_U(e)}{2}, \quad iv(e) = \frac{iv_L(e) + iv_U(e)}{2}, \quad fv(e) = \frac{fv_L(e) + fv_U(e)}{2}, \quad lv(e) = \frac{lv_L(e) + lv_U(e)}{2}.$$

The resulting digraph $G' = (V, E)$, with these membership degrees, satisfies the conditions of a Turiyam Neutrosophic Digraph:

$$t(x) + iv(x) + fv(x) + lv(x) \leq 4, \quad t(e) + iv(e) + fv(e) + lv(e) \leq 4.$$

Thus, G transforms into a Turiyam Neutrosophic Digraph.

To further transform G' into a Neutrosophic Digraph, set:

$$lv(x) = 0, \quad lv(e) = 0, \quad \forall x \in V,\, e \in E.$$

The resulting digraph $G'' = (V, E)$ satisfies:

$$t(x) + iv(x) + fv(x) \leq 3, \quad t(e) + iv(e) + fv(e) \leq 3.$$

Thus, G reduces to a Neutrosophic Digraph. □

Definition 30 (Rough Pentapartitioned Neutrosophic Directed Graph)**.** A *Rough Pentapartitioned Neutrosophic Directed Graph* is defined over a non-empty finite set of vertices V and a set of directed edges $E \subseteq V \times V$. Components

- *Equivalence Relations*:
  - $R \subseteq V \times V$: An equivalence relation on V.
  - $S \subseteq E \times E$: An equivalence relation on E.
- *Lower and Upper Approximations*:
  - *Vertices*:
    - Lower approximation: $R_V \subseteq V$.
    - Upper approximation: $\overline{R}_V \subseteq V$.
  - *Edges*:
    - Lower approximation: $S_E \subseteq E$.
    - Upper approximation: $\overline{S}_E \subseteq E$.

Each vertex and edge is characterized by membership degrees in terms of lower and upper approximations. Vertex Membership Functions For each vertex $v \in V$:

- *Lower Approximation Membership Degrees*:
  - $T_L(v) \in [0, 1]$: Truth-membership degree.
  - $C_L(v) \in [0, 1]$: Contradiction-membership degree.
  - $R_L(v) \in [0, 1]$: Ignorance-membership degree.
  - $U_L(v) \in [0, 1]$: Unknown-membership degree.
  - $F_L(v) \in [0, 1]$: Falsity-membership degree.
- *Upper Approximation Membership Degrees*:
  - $T_U(v) \in [0, 1]$.
  - $C_U(v) \in [0, 1]$.
  - $R_U(v) \in [0, 1]$.
  - $U_U(v) \in [0, 1]$.
  - $F_U(v) \in [0, 1]$.

These degrees satisfy:

$$T_L(v) \leq T_U(v),$$
$$C_L(v) \leq C_U(v),$$
$$R_L(v) \leq R_U(v),$$
$$U_L(v) \leq U_U(v),$$
$$F_L(v) \leq F_U(v).$$

And:

$$T_L(v) + C_L(v) + R_L(v) + U_L(v) + F_L(v) \leq 5,$$

$$T_U(v) + C_U(v) + R_U(v) + U_U(v) + F_U(v) \leq 5.$$

Edge Membership Functions For each directed edge $e = (u, v) \in E$:

- *Lower Approximation Membership Degrees*:
  - $T_L(e) \in [0, 1]$.
  - $C_L(e) \in [0, 1]$.
  - $R_L(e) \in [0, 1]$.
  - $U_L(e) \in [0, 1]$.
  - $F_L(e) \in [0, 1]$.
- *Upper Approximation Membership Degrees*:
  - $T_U(e) \in [0, 1]$.
  - $C_U(e) \in [0, 1]$.
  - $R_U(e) \in [0, 1]$.
  - $U_U(e) \in [0, 1]$.
  - $F_U(e) \in [0, 1]$.

These degrees satisfy:

$$T_L(e) \leq T_U(e),$$
$$C_L(e) \leq C_U(e),$$
$$R_L(e) \leq R_U(e),$$
$$U_L(e) \leq U_U(e),$$
$$F_L(e) \leq F_U(e).$$

And:

$$T_L(e) + C_L(e) + R_L(e) + U_L(e) + F_L(e) \leq 5,$$

$$T_U(e) + C_U(e) + R_U(e) + U_U(e) + F_U(e) \leq 5.$$

Conditions For all $v \in V$ and $e = (u, v) \in E$:

$$\begin{aligned} T_L(e) &\leq \min\{T_L(u), T_L(v)\}, & T_U(e) &\leq \min\{T_U(u), T_U(v)\}, \\ C_L(e) &\leq \min\{C_L(u), C_L(v)\}, & C_U(e) &\leq \min\{C_U(u), C_U(v)\}, \\ R_L(e) &\geq \max\{R_L(u), R_L(v)\}, & R_U(e) &\geq \max\{R_U(u), R_U(v)\}, \\ U_L(e) &\geq \max\{U_L(u), U_L(v)\}, & U_U(e) &\geq \max\{U_U(u), U_U(v)\}, \\ F_L(e) &\geq \max\{F_L(u), F_L(v)\}, & F_U(e) &\geq \max\{F_U(u), F_U(v)\}. \end{aligned}$$

Theorem 31. *A Rough Pentapartitioned Neutrosophic Directed Graph (RPNDG) can be transformed into:*

(1) *A Rough Neutrosophic Directed Graph (RNDG) by appropriately setting certain membership degrees to zero and reinterpreting others.*

(2) *A Rough Turiyam Neutrosophic Directed Graph (RTDG) by merging specific membership degrees and adjusting the sum conditions accordingly.*

Proof: Let $G = (V, E)$ be a Rough Pentapartitioned Neutrosophic Directed Graph. Each vertex $v \in V$ has lower and upper approximation membership degrees:

- Lower approximation: $T_L(v), C_L(v), R_L(v), U_L(v), F_L(v)$.
- Upper approximation: $T_U(v), C_U(v), R_U(v), U_U(v), F_U(v)$.

Similarly for each edge $e \in E$.

To transform G into a Rough Neutrosophic Directed Graph, we proceed as follows:

- Set the contradiction-membership degrees and unknown-membership degrees to zero:
$$\begin{aligned} C_L(v) = C_U(v) = 0, \quad U_L(v) = U_U(v) = 0, \quad \forall v \in V, \\ C_L(e) = C_U(e) = 0, \quad U_L(e) = U_U(e) = 0, \quad \forall e \in E. \end{aligned}$$
- Rename the ignorance-membership degrees as indeterminacy-membership degrees:
$$\begin{aligned} I_L(v) = R_L(v), \quad I_U(v) = R_U(v), \quad \forall v \in V, \\ I_L(e) = R_L(e), \quad I_U(e) = R_U(e), \quad \forall e \in E. \end{aligned}$$
- The sum conditions become:
$$T_L(v) + I_L(v) + F_L(v) \leq 5, \quad T_U(v) + I_U(v) + F_U(v) \leq 5.$$
- Normalize the membership degrees to ensure that their sums are less than or equal to 3:
$$\tilde{T}_L(v) = \frac{T_L(v)}{5} \times 3, \quad \tilde{I}_L(v) = \frac{I_L(v)}{5} \times 3, \quad \tilde{F}_L(v) = \frac{F_L(v)}{5} \times 3.$$
Similar definitions apply for upper approximations and edges.
- The sum conditions now satisfy:
$$\tilde{T}_L(v) + \tilde{I}_L(v) + \tilde{F}_L(v) \leq 3.$$

Thus, the RPNDG reduces to an RNDG under these transformations.

Next, to transform G into a Rough Turiyam Neutrosophic Directed Graph, we proceed as follows:

- Merge the contradiction-membership degree with the truth-membership degree, and the unknown-membership degree with the liberation-membership degree:
$$\begin{aligned} t_L(v) = T_L(v) + C_L(v), \quad iv_L(v) = R_L(v), \quad fv_L(v) = F_L(v), \quad lv_L(v) = U_L(v), \quad \forall v \in V, \\ t_U(v) = T_U(v) + C_U(v), \quad iv_U(v) = R_U(v), \quad fv_U(v) = F_U(v), \quad lv_U(v) = U_U(v), \quad \forall v \in V. \end{aligned}$$
Similarly for edges.

- The sum conditions become:

$$t_L(v) + iv_L(v) + fv_L(v) + lv_L(v) = T_L(v) + C_L(v) + R_L(v) + F_L(v) + U_L(v) \leq 5.$$

- Normalize the membership degrees to ensure that their sums are less than or equal to 4:

$$\tilde{t}_L(v) = \frac{t_L(v)}{5} \times 4, \quad \tilde{iv}_L(v) = \frac{iv_L(v)}{5} \times 4, \quad \tilde{fv}_L(v) = \frac{fv_L(v)}{5} \times 4, \quad \tilde{lv}_L(v) = \frac{lv_L(v)}{5} \times 4.$$

- The sum conditions now satisfy:

$$\tilde{t}_L(v) + \tilde{iv}_L(v) + \tilde{fv}_L(v) + \tilde{lv}_L(v) \leq 4.$$

Thus, the RPNDG reduces to an RTDG under these transformations. □

Theorem 32. *A Rough Pentapartitioned Neutrosophic Directed Graph (RPNDG) can be transformed into a Pentapartitioned Neutrosophic Directed Graph (PNDG).*

Proof: Let $\vec{G} = (V, E, T_L, C_L, R_L, U_L, F_L, T_U, C_U, R_U, U_U, F_U)$ be a Rough Pentapartitioned Neutrosophic Directed Graph, where:

- Each vertex $v \in V$ has lower and upper membership degrees satisfying:

$$\begin{aligned} &T_L(v) \leq T_U(v), \quad C_L(v) \leq C_U(v), \quad R_L(v) \leq R_U(v), \\ &U_L(v) \leq U_U(v), \quad F_L(v) \leq F_U(v), \end{aligned}$$

  and:

$$\begin{aligned} T_L(v) + C_L(v) + R_L(v) + U_L(v) + F_L(v) \leq 5, \\ T_U(v) + C_U(v) + R_U(v) + U_U(v) + F_U(v) \leq 5. \end{aligned}$$

- Each directed edge $e = (u, v) \in E$ has lower and upper membership degrees satisfying:

$$\begin{aligned} &T_L(e) \leq T_U(e), \quad C_L(e) \leq C_U(e), \quad R_L(e) \leq R_U(e), \\ &U_L(e) \leq U_U(e), \quad F_L(e) \leq F_U(e), \end{aligned}$$

  and:

$$\begin{aligned} T_L(e) + C_L(e) + R_L(e) + U_L(e) + F_L(e) \leq 5, \\ T_U(e) + C_U(e) + R_U(e) + U_U(e) + F_U(e) \leq 5. \end{aligned}$$

To transform $\vec{G}$ into a Pentapartitioned Neutrosophic Directed Graph, proceed as follows:

For each vertex $v \in V$, set:

$$T_P(v) = T_L(v) = T_U(v), \quad C_P(v) = C_L(v) = C_U(v),$$

$$R_P(v) = R_L(v) = R_U(v), \quad U_P(v) = U_L(v) = U_U(v), \quad F_P(v) = F_L(v) = F_U(v).$$

Similarly, for each directed edge $e \in E$, set:

$$T_P(e) = T_L(e) = T_U(e), \quad C_P(e) = C_L(e) = C_U(e),$$

$$R_P(e) = R_L(e) = R_U(e), \quad U_P(e) = U_L(e) = U_U(e), \quad F_P(e) = F_L(e) = F_U(e).$$

The resulting membership degrees satisfy the following conditions for all $v \in V$ and $e \in E$:

$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5,$$

$$T_P(e) + C_P(e) + R_P(e) + U_P(e) + F_P(e) \leq 5.$$

The transformed graph $\vec{G}' = (V, E, T_P, C_P, R_P, U_P, F_P)$ is a Pentapartitioned Neutrosophic Directed Graph, as it satisfies all the required conditions for vertices and edges with single-valued membership degrees. □

4 | Result: Extension of Plithogenic graphs

We explore the directed graph version of Plithogenic Graphs. The following sections provide the definition and its relationships with other graph classes.

Definition 33. [123, 61, 116, 115, 121] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
- $bCf : Nm \times Nm \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 34. (cf.[49]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

The General Plithogenic Graph is a relax definition of the Plithogenic Graph (cf.[49, 83, 35]).

Definition 35 (General Plithogenic Graph)**.** [49] Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

(1) *General Plithogenic Vertex Set PM*:
$$PM = (M, l, Ml, adf, aCf)$$
where:
- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.

(2) *General Plithogenic Edge Set PN*:
$$PN = (N, m, Nm, bdf, bCf)$$
where:
- $N \subseteq E$: Set of edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following *Reflexivity and Symmetry* properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:
$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Definition 36. A *General Plithogenic Directed Graph* extends the concept of plithogenic graphs to directed graphs, incorporating multiple attributes and degrees of appurtenance and contradiction for both vertices and directed edges.

Let $G = (V, E)$ be a crisp directed graph, where:

- V is a finite set of vertices.
- $E \subseteq V \times V$ is the set of directed edges.

The *General Plithogenic Directed Graph* $G^{GP} = (PM, PN)$ consists of:

(1) *General Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values for l.

- $adf : M \times Ml \to [0,1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$: Degree of Contradiction Function (DCF) for vertices.

(2) *General Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$: Set of directed edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values for m.
- $bdf : N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Directed Graph G^{GP} must satisfy the following properties:

- *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml, \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml, \\ bCf(a,a) &= 0, && \forall a \in Nm, \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm. \end{aligned}$$

Theorem 37. *The General Plithogenic Directed Graph G^{GP} can be transformed into:*

(1) *A Fuzzy Directed Graph when $s = t = 1$.*

(2) *A Neutrosophic Directed Graph when $s = 3$, $t = 1$.*

(3) *A Turiyam Neutrosophic Directed Graph when $s = 4$, $t = 1$.*

(4) *A Pentapartitioned Neutrosophic Directed Graph when $s = 5$, $t = 1$.*

Proof: (1) *Fuzzy Directed Graph*:

- *Parameters*: Set $s = t = 1$.
- *Degree of Appurtenance Function*:

$$adf : M \times Ml \to [0,1], \quad bdf : N \times Nm \to [0,1].$$

- *Interpretation*:
  - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
  - Each directed edge $e = (u,v) \in E$ is assigned a membership degree $\mu(u,v) \in [0,1]$.
- *Conclusion*: The General Plithogenic Directed Graph reduces to a Fuzzy Directed Graph.

(2) *Neutrosophic Directed Graph*:

- *Parameters*: Set $s = 3$, $t = 1$.
- *Degree of Appurtenance Function*:

$$adf : M \times Ml \to [0,1]^3.$$

- *Interpretation*:
  - Each vertex $v \in V$ is assigned a triplet $(\sigma_T(v), \sigma_I(v), \sigma_F(v)) \in [0,1]^3$, representing truth, indeterminacy, and falsity membership degrees.
  - Similarly for edges.
- *Conclusion*: The General Plithogenic Directed Graph reduces to a Neutrosophic Directed Graph.

(3) *Turiyam Neutrosophic Directed Graph*:

- *Parameters*: Set $s = 4$, $t = 1$.
- *Degree of Appurtenance Function*:

$$adf : M \times Ml \to [0, 1]^4.$$

- *Interpretation*:
  - Each vertex $v \in V$ is assigned a quadruple $(t(v), iv(v), fv(v), lv(v)) \in [0, 1]^4$, representing truth, indeterminacy, falsity, and liberation membership degrees.
  - Similarly for edges.
- *Conclusion*: The General Plithogenic Directed Graph reduces to a Turiyam Neutrosophic Directed Graph.

(4) *Pentapartitioned Neutrosophic Directed Graph*:

- *Parameters*: Set $s = 5$, $t = 1$.
- *Degree of Appurtenance Function*:

$$adf : M \times Ml \to [0, 1]^5.$$

- *Interpretation*:
  - Each vertex $v \in V$ is assigned a quintuple $(T(v), C(v), R(v), U(v), F(v)) \in [0, 1]^5$, representing truth, contradiction, ignorance, unknown, and falsity membership degrees.
  - Similarly for edges.
- *Conclusion*: The General Plithogenic Directed Graph reduces to a Pentapartitioned Neutrosophic Directed Graph.

□

Theorem 38. *A General Plithogenic Directed Graph G^{GP} reduces to a General Plithogenic Graph when made undirected.*

Proof: Let $G^{GP} = (PM, PN)$, where:

- $PM = (M, l, Ml, adf, aCf)$: General Plithogenic Vertex Set.
- $PN = (N, m, Nm, bdf, bCf)$: General Plithogenic Edge Set.

To transform G^{GP} into a General Plithogenic Graph G_u^{GP}:

(1) Define the undirected edge set:

$$(u, v) \in N_u \iff (u, v) \in N \text{ or } (v, u) \in N.$$

(2) For each undirected edge $\{u, v\} \in N_u$, combine attributes as:

$$bdf_u(\{u, v\}, m) = \max\{bdf((u, v), m), bdf((v, u), m)\}, \quad \forall m \in Nm.$$

(3) Retain the vertex attributes:

$$adf_u(v, l) = adf(v, l), \quad aCf_u(a, b) = aCf(a, b).$$

(4) The resulting graph $G_u^{GP} = (PM_u, PN_u)$, where $PN_u = (N_u, m, Nm, bdf_u, bCf_u)$, satisfies the properties of a General Plithogenic Graph.

Thus, G_u^{GP} is a valid General Plithogenic Graph obtained by symmetrizing the edge set and combining attributes. □

5 | Future Tasks: Uncertain Directed Hypergraphs and Beyond

This section outlines the future directions of this research. We aim to explore the mathematical structures and applications of Directed Hypergraphs [97, 98, 81, 130, 14] and Mixed Hypergraphs [125, 72, 79] within the framework of uncertain directed graphs, including Fuzzy and Neutrosophic graphs. Additionally, we plan to extend these graph concepts to Superhypergraphs [117, 118, 40, 47, 119, 65, 60, 34, 53] and explore their potential applications.

Furthermore, we intend to investigate extensions involving Bidirected Graphs [39, 63, 131, 26], Multidirected Graph [92, 45, 135, 91], and Treesoft Graphs (or Treesoft sets) [120, 10, 36, 94, 52, 44, 128, 59], as needed.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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