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A Theoretical Investigation of Quantum n-SuperHypergraph States

Takaaki Fujita[1*]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan; t171d603@gunma-u.ac.jp.,

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Abstract

Hypergraphs generalize ordinary graphs by allowing hyperedges to join any number of vertices rather than just two. Superhypergraphs extend this idea further by iterating the powerset construction, creating hierarchical, nested hyperedge structures. In parallel, quantum graphs model networks of quantum systems, with wavefunctions propagating along edges under specified boundary conditions. Quantum hypergraph states translate hypergraph connectivity into multi-qubit entanglement via generalized controlled-phase gates acting on all vertices of each hyperedge. In this paper, we introduce quantum n-superhypergraph states, which marry the recursive structure of superhypergraphs with the formalism of quantum hypergraph states. We give their precise definition, explore key structural properties, and outline potential directions for their application.

Keywords: Superhypergraph, Hypergraph, Quantum n-superhypergraph state, Quantum hypergraph state, Quantum Graph

1 | Preliminaries and Definitions

This section introduces the fundamental concepts and definitions necessary for the discussions presented in this paper. Throughout the paper, we consider only simple and finite graphs. For foundational operations, concepts, and principles of graph theory, the reader is referred to [1, 2].

1.1 | Graphs and Hypergraphs

In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between elements, thereby enhancing its utility in various fields [3, 4, 5]. In the following, we present rigorous definitions for graphs, subgraphs, and hypergraphs. In this paper, we focus on finite, undirected, and simple graphs.

Corresponding Author: t171d603@gunma-u.ac.jp

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Definition 1 (Hypergraph)**.** [6, 4] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

1.2 | Powerset and n-th Powerset

In what follows, we utilize the concepts of the powerset and the n-th powerset as fundamental building blocks for our subsequent constructions. The n-th powerset represents an iterative application of the powerset operation. Similarly, the superhypergraph, which will be introduced later, is an iterative extension of the hypergraph concept.

Definition 2 (Base Set)**.** [7] A *base set* S is the underlying set from which more elaborate structures, such as powersets and hyperstructures, are constructed. It is defined by

$$S = \{x \mid x \text{ belongs to a specified domain}\}.$$

All elements appearing in constructions like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are drawn from S.

Definition 3 (Powerset)**.** [7, 8] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including both $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 4 (n-th Powerset)**.** (cf. [9, 10, 7, 11, 12]) The n-th powerset of a set H, denoted $P_n(H)$, is defined recursively by:

$$P_1(H) = \mathcal{P}(H), \quad P_{n+1}(H) = \mathcal{P}(P_n(H)) \quad \text{for } n \geq 1.$$

Similarly, the n-th nonempty powerset, denoted $P_n^*(H)$, is given by:

$$P_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(P_n^*(H)),$$

where $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set omitted.

1.3 | SuperHyperGraph

A *SuperHyperGraph* is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model. This concept has been recently introduced and extensively studied in the literature [13, 14, 15, 16, 17, 18, 19].

Definition 5 (n-SuperHyperGraph)**.** [20, 21]

Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An *n-SuperHyperGraph* is then a pair

$$\mathrm{SHT}^{(n)} = (V, E),$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex* and each element of E an *n-superedge*.

Example 6 (A 2-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b\}$. Then

$$\mathcal{P}^1(V_0) = \{\{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}^1(V_0)).$$

Choose three 2-supervertices:

$$v_1 = \{\{a\}\}, \quad v_2 = \{\{b\}\}, \quad v_3 = \{\{a, b\}\},$$

so that

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

Define three 2-superedges:

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3\}, \quad e_3 = \{v_1, v_3\},$$

so that

$$E = \{e_1, e_2, e_3\} \subseteq \mathcal{P}^2(V_0).$$

Then

$$\mathrm{SHT}^{(2)} = (V, E)$$

is a concrete example of a 2-SuperHyperGraph.

1.4 | Quantum hypergraph state

Various quantum-based concepts, such as quantum theory[22, 23, 24, 25] and quantum computing[26, 27, 28, 29], have been extensively studied. This trend is also evident in graph theory. A quantum graph state is an entangled multi-qubit state constructed by applying controlled-Z gates based on a graph's connectivity pattern[30, 31, 32]. As an extension of quantum graphs, concepts like quantum graph neural networks have also been explored [33, 34, 35, 36]. A quantum hypergraph state generalizes this idea by incorporating generalized controlled-phase gates, where hyperedges define interactions among multiple qubits[37, 38, 39].

The formal definition of a quantum hypergraph state is presented below. For details on the operations, refer to [40] and the relevant references.

Definition 7 (Quantum Hypergraph State)**.** [40] Let $G = (V, E)$ be a hypergraph with $|V| = n$. Associate to each vertex $v_i \in V$ a qubit with Hilbert space $\mathcal{H}_i \cong \mathbb{C}^2$, so that the total Hilbert space is

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i.$$

Define the state

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and the product state

$$|+\rangle^{\otimes n} = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle.$$

For each hyperedge $e \in E$, define the *generalized controlled-phase gate* (or *generalized Controlled-Z* gate) acting on the qubits corresponding to the vertices in e by

$$CZ_e = I^{\otimes n} - 2\left(\bigotimes_{i\in e} |1\rangle\langle 1|\right),$$

where $I^{\otimes n}$ denotes the identity on $\mathcal{H}$ and the operator $\bigotimes_{i\in e} |1\rangle\langle 1|$ acts nontrivially only on the qubits indexed by e (and as the identity on the remaining qubits).

Then the *quantum hypergraph state* corresponding to G is defined as

$$|G\rangle = \left(\prod_{e\in E} CZ_e\right) |+\rangle^{\otimes n}.$$

Here the product is taken over all hyperedges in an arbitrary fixed order (the gates commute since they are all diagonal).

Remark 8. *When every hyperedge has cardinality 2 (i.e. when G is a graph), the state $|G\rangle$ reduces to the standard* graph state*, which is a stabilizer state. However, if there exists at least one hyperedge with $|e| > 2$, then the corresponding CZ_e gate is non-Clifford and the resulting state is typically non-stabilizer.*

Even though quantum hypergraph states are not stabilizer states in the conventional sense when hyperedges of size greater than 2 are present, they admit a generalized stabilizer formalism.

Definition 9 (Generalized Stabilizer Generators)**.** [40] For a hypergraph state $|G\rangle$ defined above, define for each vertex $v_i \in V$ the operator

$$S_i = X_i \prod_{\substack{e \in E \\ v_i \in e}} CZ_{e \setminus \{v_i\}},$$

where:

- X_i is the Pauli-X operator acting on the i-th qubit,
- $e \setminus \{v_i\}$ denotes the hyperedge e with the vertex v_i removed, and
- $CZ_{e \setminus \{v_i\}}$ is the corresponding controlled-phase gate acting on the qubits in $e \setminus \{v_i\}$.

These operators satisfy

$$S_i |G\rangle = |G\rangle, \quad \text{for all } i = 1, 2, \dots, n.$$

Example 10 (A 3-Qubit Quantum Hypergraph State)**.** Consider the hypergraph

$$G = (V, E)$$

with vertices

$$V = \{v_1, v_2, v_3\}$$

and hyperedges

$$E = \{e_1, e_2\},$$

where

$$e_1 = \{v_1, v_2\} \quad \text{and} \quad e_2 = \{v_1, v_2, v_3\}.$$

First, assign each vertex a qubit so that the overall Hilbert space is

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3,$$

and prepare the initial product state

$$|+\rangle^{\otimes 3} = |+\rangle \otimes |+\rangle \otimes |+\rangle,$$

with

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Thus,

$$|+\rangle^{\otimes 3} = \frac{1}{\sqrt{8}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big).$$

For each hyperedge, define a generalized controlled-phase gate. For the 2-edge

$$e_1 = \{v_1, v_2\},$$

the gate is

$$CZ_{e_1} = I^{\otimes 3} - 2\Big(|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes I\Big),$$

which flips the sign of any component where qubits 1 and 2 are both in state $|1\rangle$. For the 3-edge

$$e_2 = \{v_1, v_2, v_3\},$$

the generalized controlled-phase gate is

$$CZ_{e_2} = I^{\otimes 3} - 2\Big(|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big),$$

which flips the sign only when all three qubits are in the state $|1\rangle$.

The quantum hypergraph state is then constructed by sequentially applying these gates:

$$|G\rangle = CZ_{e_2}\, CZ_{e_1}\, |+\rangle^{\otimes 3}.$$

Let us detail the action:

(1) Applying CZ_{e_1} on $|+\rangle^{\otimes 3}$ flips the sign of the basis states where qubits 1 and 2 are both $|1\rangle$. In particular,

$$|110\rangle \to -|110\rangle, \quad |111\rangle \to -|111\rangle.$$

(2) Next, applying CZ_{e_2} flips the sign of the state with all qubits in $|1\rangle$, i.e.,

$$-|111\rangle \to (-1) \times (-1)|111\rangle = |111\rangle,$$

while leaving other components unchanged.

Thus, the final state is

$$|G\rangle = \frac{1}{\sqrt{8}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle\Big).$$

This explicit construction shows that the three-qubit state $|G\rangle$ is generated by both a standard controlled-Z gate (on qubits 1 and 2) and a non-Clifford three-qubit controlled-phase gate (on qubits 1, 2, and 3), illustrating the key features that distinguish quantum hypergraph states from conventional graph states.

2 | Result of this paper

2.1 | Quantum n-SuperHypergraphs states

In this subsection we introduce and define the notion of Quantum n-SuperHypergraph States, show that they generalize Quantum Hypergraph States, and prove several of their properties. We now define the associated quantum state.

Definition 11 (Quantum n-SuperHypergraph State)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph with $|V| = m$. Associate to each n-supervertex $v \in V$ a qubit with Hilbert space $\mathcal{H}_v \cong \mathbb{C}^2$, and define the total Hilbert space as

$$\mathcal{H} = \bigotimes_{v\in V} \mathcal{H}_v.$$

Define the single-qubit state

$$|+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big)$$

and the product state

$$|+\rangle^{\otimes m} = \bigotimes_{v\in V} |+\rangle.$$

For each n-superedge $e \in E$, define the *generalized controlled-phase gate* by

$$CZ_e = I^{\otimes m} - 2\left(\bigotimes_{v\in e} |1\rangle\langle 1|\right),$$

where $I^{\otimes m}$ is the identity on $\mathcal{H}$. Then the *Quantum n-SuperHypergraph State* is given by

$$|\mathrm{SHT}^{(n)}\rangle = \left(\prod_{e\in E} CZ_e\right)|+\rangle^{\otimes m}.$$

The product over $e \in E$ is taken in an arbitrary fixed order (the gates commute since they are diagonal).

Theorem 12. *Every Quantum Hypergraph State is a Quantum* 1*-SuperHypergraph State. In particular, if* $n = 1$ *then* $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$*, and the state*

$$|\mathrm{SHT}^{(1)}\rangle = \left(\prod_{e\in E} CZ_e\right)|+\rangle^{\otimes |V|}$$

coincides with the standard Quantum Hypergraph State.

Proof: Let $G = (V, E)$ be any finite hypergraph with vertex set $V \subseteq V_0$ and hyperedge set $E \subseteq \mathcal{P}(V_0)$. By definition, the standard quantum hypergraph state $|G\rangle$ is constructed as follows:

(i) Assign to each $v \in V$ a qubit with Hilbert space $\mathcal{H}_v \cong \mathbb{C}^2$, $\quad \mathcal{H} = \bigotimes_{v\in V} \mathcal{H}_v$,

(ii) $|+\rangle^{\otimes |V|} = \bigotimes_{v\in V} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$,

$$\text{(iii)}\quad CZ_e = I^{\otimes|V|} - 2\,\Big(\bigotimes_{v\in e}|1\rangle\langle 1|\Big)\quad \text{for each } e \in E,$$

$$\text{(iv)}\quad |G\rangle = \Big(\prod_{e\in E} CZ_e\Big)\,|+\rangle^{\otimes|V|}.$$

On the other hand, for $n = 1$ the 1-th powerset satisfies

$$\mathcal{P}^1(V_0) \;=\; \mathcal{P}(V_0),$$

so a 1-SuperHyperGraph $\mathrm{SHT}^{(1)} = (V, E)$ is exactly the same combinatorial data (V, E) that defines G. The quantum 1-SuperHypergraph state $|\mathrm{SHT}^{(1)}\rangle$ is then built by the identical prescription:

$$\text{(i')}\quad \mathcal{H}' \;=\; \bigotimes_{v\in V}\mathcal{H}_v,\quad |+\rangle'^{\otimes|V|} = |+\rangle^{\otimes|V|},$$

$$\text{(ii')}\quad CZ'_e = I^{\otimes|V|} - 2\,\Big(\bigotimes_{v\in e}|1\rangle\langle 1|\Big)\quad \text{for each } e \in E,$$

$$\text{(iii')}\quad |\mathrm{SHT}^{(1)}\rangle = \Big(\prod_{e\in E} CZ'_e\Big)\,|+\rangle'^{\otimes|V|}.$$

Since the assignments in steps (i)–(iv) agree exactly with those in (i')–(iii'), and because all the controlled-phase gates commute (being diagonal in the computational basis), it follows that

$$|\mathrm{SHT}^{(1)}\rangle = \;|G\rangle.$$

Hence every quantum hypergraph state is realized as the quantum 1-SuperHypergraph state. □

Quantum Hypergraph States admit a generalized stabilizer description. The following theorem extends this formalism to Quantum n-SuperHypergraph States.

Theorem 13 (Generalized Stabilizer Property). *Let $|\mathrm{SHT}^{(n)}\rangle$ be a Quantum n-SuperHypergraph State constructed as above. For each n-supervertex $v \in V$, define*

$$S_v = X_v \prod_{\substack{e\in E\\ v\in e}} CZ_{e\setminus\{v\}},$$

where X_v is the Pauli-X operator acting on the qubit corresponding to v and $CZ_{e\setminus\{v\}}$ denotes the generalized controlled-phase gate acting on the qubits associated with $e \setminus \{v\}$. Then,

$$S_v|\mathrm{SHT}^{(n)}\rangle = |\mathrm{SHT}^{(n)}\rangle,\quad \forall\, v \in V.$$

Proof: Let $m = |V|$ and write

$$|\mathrm{SHT}^{(n)}\rangle = \Big(\prod_{e\in E} CZ_e\Big)\,|+\rangle^{\otimes m} =: U\,|+\rangle^{\otimes m},$$

where U denotes the product of all generalized controlled-phase gates. Recall that each $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ is a $+1$ eigenstate of the Pauli-X operator.

Fix any n-supervertex $v \in V$. By definition,

$$S_v = X_v \prod_{\substack{e\in E\\ v\in e}} CZ_{e\setminus\{v\}}.$$

We will show $S_v\,|\mathrm{SHT}^{(n)}\rangle = |\mathrm{SHT}^{(n)}\rangle$.

1. Commutation with non-incident gates. If $e \not\ni v$, then CZ_e acts trivially on qubit v and hence commutes with both X_v and every $CZ_{e'\setminus\{v\}}$.

2. Decomposition of U into incident and non-incident parts. Write $U = U_{\neg v}\,U_v$, where

$$U_v = \prod_{\substack{e\in E\\ v\in e}} CZ_e,\quad U_{\neg v} = \prod_{\substack{e\in E\\ v\notin e}} CZ_e.$$

Since $U_{\neg v}$ commutes with both X_v and $\prod_{e\ni v} CZ_{e\setminus\{v\}}$, we have

$$S_v\, U = U_{\neg v}\; X_v \Big(\prod_{e\ni v} CZ_{e\setminus\{v\}}\Big)\, U_v.$$

3. Cancellation of extra phase factors. For each $e \ni v$, note

$$CZ_e = |0\rangle\langle 0|_v \otimes I + |1\rangle\langle 1|_v \otimes CZ_{e\setminus\{v\}}.$$

It follows that

$$X_v \Big(\prod_{e\ni v} CZ_{e\setminus\{v\}}\Big)\, U_v = U_v\, X_v,$$

because each $CZ_{e\setminus\{v\}}$ cancels the phase inserted by $|1\rangle\langle 1|_v$ when X_v acts.

4. Eigenvalue argument on the product state. Since $|+\rangle^{\otimes m}$ is a $+1$ eigenstate of X_v and is invariant under $U_{\neg v}$,

$$S_v\, |\mathrm{SHT}^{(n)}\rangle = U_{\neg v}\, U_v\, X_v\, |+\rangle^{\otimes m} = U\, |+\rangle^{\otimes m} = |\mathrm{SHT}^{(n)}\rangle.$$

Since v was arbitrary, this holds for every $v \in V$, completing the proof. □

Theorem 14 (Non-Cliffordness)**.** *If there exists an n-superedge $e \in E$ with $|e| > 2$, then the Quantum n-SuperHypergraph State $|\mathrm{SHT}^{(n)}\rangle$ is non-stabilizer in the conventional sense (i.e., it cannot be generated solely by Clifford operations).*

Proof: Recall that the Clifford group on m qubits is the normalizer of the m-qubit Pauli group and is generated by the single-qubit Hadamard and phase gates together with the two-qubit CNOT gate. In particular, every Clifford unitary lies in the second level of the Clifford hierarchy.

Now let $e \in E$ be an n-superedge with $|e| > 2$, and consider the gate

$$CZ_e = I^{\otimes m} \;-\; 2\,\Big(\bigotimes_{v\in e} |1\rangle\langle 1|\Big).$$

This operator acts nontrivially on all qubits indexed by e simultaneously, flipping the phase only on the basis vector $|1\rangle^{\otimes |e|}$. Such a gate belongs to the $|e|$-th level of the Clifford hierarchy: for any Pauli operator P, the conjugation

$$CZ_e\, P CZ_e^{\dagger}$$

yields an operator in the $(|e|-1)$-th level but not necessarily in the Pauli group itself when $|e| > 2$. Since gates in the Clifford group must map Pauli operators back to Pauli operators (remaining within the second level), it follows that CZ_e cannot be decomposed into single-qubit and two-qubit Clifford gates.

Because the circuit preparing $|\mathrm{SHT}^{(n)}\rangle$ includes at least one such non-Clifford gate CZ_e, the resulting state cannot lie entirely within the stabilizer formalism. Hence $|\mathrm{SHT}^{(n)}\rangle$ is a non-stabilizer state. □

Theorem 15 (Commutation of generalized phase gates)**.** *For any two n-superedges $e, f \in E$, the corresponding gates commute:*

$$CZ_e\, CZ_f \;=\; CZ_f\, CZ_e.$$

Proof: Each CZ_e is diagonal in the computational basis, acting by

$$CZ_e|x\rangle = (-1)^{\prod_{v\in e} x_v}|x\rangle,$$

where $x = (x_v)_{v\in V} \in \{0,1\}^m$. Since two diagonal operators always commute,

$$CZ_e\, CZ_f|x\rangle = (-1)^{\sum_{w\in e} x_w + \sum_{w\in f} x_w}|x\rangle = CZ_f\, CZ_e|x\rangle$$

for every basis vector $|x\rangle$. Hence $CZ_e CZ_f = CZ_f CZ_e$. □

Theorem 16 (Explicit amplitude formula)**.** *The quantum n-SuperHypergraph state admits the expansion*

$$|\mathrm{SHT}^{(n)}\rangle = \frac{1}{2^{m/2}} \sum_{x\in\{0,1\}^m} (-1)^{\sum_{e\in E} \prod_{v\in e} x_v}\, |x\rangle.$$

Proof: Starting from

$$|\mathrm{SHT}^{(n)}\rangle = \Big(\prod_{e\in E} CZ_e\Big)\,|+\rangle^{\otimes m}, \quad |+\rangle^{\otimes m} = \frac{1}{2^{m/2}} \sum_{x\in\{0,1\}^m} |x\rangle,$$

apply each CZ_e to $|x\rangle$, which multiplies it by $(-1)^{\prod_{v\in e} x_v}$. Since the gates commute, the total phase is the product over all e, i.e.

$$\prod_{e\in E} (-1)^{\prod_{v\in e} x_v} = (-1)^{\sum_{e\in E}\prod_{v\in e} x_v}.$$

Collecting factors yields the stated sum. □

Theorem 17 (Abelian stabilizer group and uniqueness)**.** *Let G be the subgroup of the m-qubit Pauli group generated by $\{S_v : v \in V\}$. Then G is abelian, has order 2^m, and $|\mathrm{SHT}^{(n)}\rangle$ is its unique common $+1$ eigenstate.*

Proof: Each generator

$$S_v = X_v \prod_{\substack{e\in E\\ v\in e}} CZ_{e\setminus\{v\}}$$

is Hermitian and squares to the identity. To see that $[S_u, S_v] = 0$ for $u \neq v$, note:

- X_u commutes with X_v and with every $CZ_{e\setminus\{w\}}$ whenever $u \neq w$.
- Any two CZ gates commute by the previous theorem.

Hence the m independent involutions S_v generate an abelian group of size 2^m. Since each S_v fixes $|\mathrm{SHT}^{(n)}\rangle$, this state lies in the common $+1$-eigenspace of G. But in an m-qubit system an abelian subgroup of order 2^m has a one-dimensional joint $+1$-eigenspace, so $|\mathrm{SHT}^{(n)}\rangle$ is its unique common eigenvector with eigenvalue $+1$. □

Example 18 (A Quantum 2-SuperHypergraph State)**.** Let the base set be $V_0 = \{a, b\}$. Then

$$\mathcal{P}^1(V_0) = \{\{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}^1(V_0)).$$

We select four 2-supervertices:

$$v_1 = \{\{a\}\}, \quad v_2 = \{\{b\}\}, \quad v_3 = \{\{a, b\}\}, \quad v_4 = \{\{a\}, \{b\}\},$$

so that

$$V = \{v_1, v_2, v_3, v_4\}$$

and define three 2-superedges:

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3, v_4\}, \quad e_3 = \{v_1, v_3\}.$$

Associate to each v_i a qubit with Hilbert space $\mathcal{H}_i \cong \mathbb{C}^2$, and prepare the product state

$$|+\rangle^{\otimes 4} = \big(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\big)^{\otimes 4} = \frac{1}{4} \sum_{x\in\{0,1\}^4} |x_1x_2x_3x_4\rangle.$$

For each superedge $e \in E$, define the generalized controlled-phase gate

$$CZ_e = I^{\otimes 4} - 2\,\Big(\bigotimes_{v_i\in e} |1\rangle\langle 1|\Big),$$

which flips the sign of any basis state for which all qubits indexed by e are 1. Applying these in the order e_1, e_3, e_2 yields

$$|\mathrm{SHT}^{(2)}\rangle = CZ_{e_2}\ CZ_{e_3}\ CZ_{e_1}\ |+\rangle^{\otimes 4} = \frac{1}{4} \sum_{x\in\{0,1\}^4} (-1)^{\,x_1x_2\,+\,x_1x_3\,+\,x_2x_3x_4}\ |x_1x_2x_3x_4\rangle.$$

Here the phase exponent $x_1x_2 + x_1x_3 + x_2x_3x_4$ encodes the combined action of the three gates:

- x_1x_2 from CZ_{e_1},
- x_1x_3 from CZ_{e_3},
- $x_2x_3x_4$ from CZ_{e_2}.

This explicit formula illustrates how the recursion to the second powerset level enriches the phase structure beyond ordinary quantum hypergraph states.

Example 19 (A Quantum 3-SuperHypergraph State)**.** Let the base set be

$$V_0 = \{a\}.$$

Then the iterated powersets are

$$\mathcal{P}^0(V_0) = \{a\}, \quad \mathcal{P}^1(V_0) = \{\{a\}\}, \quad \mathcal{P}^2(V_0) = \{\emptyset, \{\{a\}\}\},$$

$$\mathcal{P}^3(V_0) = \mathcal{P}(\mathcal{P}^2(V_0)) = \{\, \emptyset,\ \{\emptyset\},\ \{\{\{a\}\}\},\ \{\emptyset, \{\{a\}\}\}\,\}.$$

Choose three 3-supervertices

$$v_1 = \{\emptyset\}, \quad v_2 = \{\{\{a\}\}\}, \quad v_3 = \{\emptyset, \{\{a\}\}\},$$

so that

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^3(V_0).$$

Define two 3-superedges

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_1, v_2, v_3\},$$

hence

$$E = \{e_1, e_2\} \subseteq \mathcal{P}^3(V_0).$$

Assign to each v_i a qubit with Hilbert space $\mathcal{H}_i \cong \mathbb{C}^2$, and prepare the product state

$$|+\rangle^{\otimes 3} = \left(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)^{\otimes 3} = \frac{1}{\sqrt{8}} \sum_{x \in \{0,1\}^3} |x_1 x_2 x_3\rangle.$$

For each superedge e, the gate

$$CZ_e = I^{\otimes 3} - 2\left(\bigotimes_{v \in e} |1\rangle\langle 1|\right)$$

flips the sign of any basis vector whose bits at positions in e are all 1. Applying first CZ_{e_1} then CZ_{e_2} yields

$$|\mathrm{SHT}^{(3)}\rangle = CZ_{e_2}\, CZ_{e_1}\, |+\rangle^{\otimes 3}.$$

An explicit expansion in the computational basis is

$$|\mathrm{SHT}^{(3)}\rangle = \frac{1}{\sqrt{8}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle\Big).$$

Here the amplitude $-1/\sqrt{8}$ for $|110\rangle$ arises because CZ_{e_1} flips it (since $x_1 = x_2 = 1$) while CZ_{e_2} does not, and the amplitude for $|111\rangle$ remains $+1/\sqrt{8}$ because it is flipped twice. This fully specifies a concrete three-qubit Quantum 3-SuperHypergraph State.

3 | Conclusion of this paper

In this paper, we introduced the Quantum n-SuperHypergraph State as an extension of the Quantum Hypergraph State and examined its underlying mathematical structure. In future work, we plan to carry out computational studies of these states, develop more refined models, and explore extensions based on directed graphs and bidirected graphs [41].

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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