



Compactness and Neutrosophic Topological Space via Grills

Runu Dhar ^{1,*} 

¹ Department of Mathematics, Maharaja BirBikram University, Agartala-799004, Tripura, India; gourpal74@gmail.com.

* Correspondence: runu.dhar@gmail.com.

Abstract: The aim of this paper is to introduce the concept of various types of compactness in neutrosophic topological space via grills. We shall generalize neutrosophic C - compact space and neutrosophic G - compact space and introduce $C(G)$ - compact space in neutrosophic topological space with respect to grills. We shall call it as neutrosophic C - compact with respect to grills and term it as neutrosophic $C(G)$ - compact space. We shall also investigate some of its basic properties and characterization theorems. We shall also study the neutrosophic quasi - H - closed space with respect to a grill.

Keywords: Neutrosophic Space; Grill; Neutrosophic G - compact; Neutrosophic C - compact; Neutrosophic quasi - H - closed.

1. Introduction

The notion of a grill was initiated by Choquet [1]. Subsequently it turned out to be a very convenient tool for various topological and neutrosophic topological investigations. From the literature it finds that in many situations, grills are more effective than certain similar concepts like nets and filters. According to Choquet, a grill G on a topological space X is a non - null collection of nonempty subsets of X satisfying two conditions: (i) $A \in G$ and $A \subseteq B \subseteq X \Rightarrow B \in G$ and (ii) $A, B \subseteq X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$.

Zadeh [2] introduced the notion of fuzzy set. As it was not sufficient to control uncertainty, Atanasov [3] introduced the notion of intuitionistic fuzzy set with membership and non - membership values. Thereafter, Smarandache [4] considered the elements with membership, non - membership and indeterministic values and introduced the notion of neutrosophic set in order to overcome all sorts of difficulty to handle all types of problems under uncertainty. The notion of neutrosophic topological space was first introduced by Salama and Alblowi [5], followed by Salama and Alblowi [6]. Alimohammady and Roohi [7] introduced fuzzy minimal structure and fuzzy minimal vector spaces. Alimohammady and Roohi [8] introduced the notion of compactness in fuzzy minimal spaces. Pal et al. [9] introduced the notion of grill in neutrosophic topological space. Pal and Dhar [10] introduced the notion of compactness in neutrosophic minimal space. Roy and Mukherjee [11] introduced the notion of compactness in topological space. Gupta and Gaur [12] introduced the notion of C - compactness in topological space by grills. Besides them, many researchers [13, 14, 15, 16, 17, 18, 19] contributed compactness in neutrosophic topological space. Following their works we would introduce and study C - cocompactness via grills in neutrosophic topological space. We would also introduce neutrosophic quasi - H - closed space with respect to a grill.

2. Preliminaries

In this section, we recall some basic concepts and results which are relevant for this article.

Definition 2.1. [11] Let G be a grill on a topological space (X, τ) . A cover $\{U_\alpha : \alpha \in \Lambda\}$ of X is said to be a G - cover if there exists a finite subset Λ_0 of Λ such that $X - \cup_{\alpha \in \Lambda_0} U_\alpha \notin G$.

Definition 2.2. [4] Let X be an universal set. A neutrosophic set A in X is a set contains triplet having truthness, falseness and indeterminacy membership values that can be characterized independently, denoted by T_A, F_A, I_A in $[0,1]$. The neutrosophic set is denoted as follows:

$$A = \{(x, T_A(x), F_A(x), I_A(x)): x \in X, \text{ and } T_A(x), F_A(x), I_A(x) \in [0,1]\}.$$

There is no restriction on the sum of $T_A(x), F_A(x)$ and $I_A(x)$, so $0 \leq T_A(x) + F_A(x) + I_A(x) \leq 3$.

Definition 2.3. [5] Let X be a non - empty set and T be the collection of neutrosophic subsets of X . Then T is said to be a neutrosophic topology (in short NT) on X if the following properties hold:

(i) $0_N, 1_N \in T$.

(ii) $U_1, U_2 \in T \Rightarrow U_1 \cap U_2 \in T$.

(iii) $\cup_{i \in \Delta} U_i \in T$, for every $\{u_i: i \in \Delta\} \subseteq T$.

Then (X, T) is called a neutrosophic topological space (in short NTS) over X . The members of T are called neutrosophic open sets (in short NOS). A neutrosophic set D is called neutrosophic closed set (in short NCS) if and only if D^c is a neutrosophic open set.

Definition 2.4. [9] Let X be a set and $P(X)$ denotes the power set of X . A family M of neutrosophic subsets of X where $M \subset P(X)$ is said to be a minimal structure on X if 0_N and 1_N belong to M . By (X, M) , we denote the neutrosophic minimal space.

We consider the elements of M as neutrosophic m - open subset of X . The complement of neutrosophic m - open set A is called a neutrosophic m -closed set.

Definition 2.5. [9] Let X be a set and $P(X)$ denotes the power set of X . A sub - collection of neutrosophic sets G (not containing 0_N) of $P(X)$ is called a grill on X if G satisfies the following conditions:

(i) $A \in G$ and $A \subseteq B$ implies $B \in G$.

(ii) $A, B \subseteq X$ and $A \cup B \in G$ implies that $A \in G$ or $B \in G$.

Remark 2.6. [9] Since $0_N \notin G$, so G is not a minimal structure on X . A minimal structure with a grill is called as a grill minimal space, denoted by (X, M, G) .

3. Neutrosophic C - Compactness with Respect to a Grill

In this section, our main focus is to propose the concept of neutrosophic C - compactness with respect to a grill and to investigate various properties of this notion.

Definition 3.1. A neutrosophic space (X, T) is said to be neutrosophic C - compact if for each neutrosophic closed set A and each neutrosophic T - open covering U of A , there exists a finite subfamily $\{U_1, U_2, U_3 \dots \dots, U_n\}$ such that $A \subset \cup_{i=1}^n Cl(U_i)$.

Definition 3.2. Let (X, T) be a neutrosophic topological space and G be a grill on X . (X, T) is said to be neutrosophic C - compact with respect to grill or just $NC(G)$ if for every neutrosophic T -open covering of U of A , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of U such that $A - \sum_{i=1}^n cl(U_i) \notin G$.

Every neutrosophic C - compact space (X, T) is $NC(G)$ - compact for any grill G on X .

Theorem 3.3. For a neutrosophic topological space (X, T) , the following statements are true:

- (a) (X, T) is neutrosophic C - compact.
- (b) (X, T) is neutrosophic $C(0_N)$ - compact.
- (c) (X, T) is neutrosophic $C(1_N)$ - compact.

Proof. Obvious.

Theorem 3.4. If a space is neutrosophic G - compact then it is neutrosophic $C(G)$ - compact.

Proof. Let X be a neutrosophic G - compact space, A a neutrosophic closed subset of X and $\{V_\alpha\}_{\alpha \in \Lambda}$ an open cover of A . Then $(X - A) \cup_{\alpha \in \Lambda} (V_\alpha)$ is a neutrosophic open cover of X . Since X is neutrosophic G - compact, therefore there exists finite $\Lambda_0 \subseteq \Lambda$ such that $X - \{(X - A) \cup_{\alpha \in \Lambda_0} (V_\alpha)\} \notin G$. This implies $A - \{\cup_{\alpha \in \Lambda_0} (V_\alpha)\} \notin G$. Since $V_\alpha \subset Cl(V_\alpha)$, therefore $A - \cup_{\alpha \in \Lambda_0} Cl(V_\alpha) \notin G$, implies that X is neutrosophic $C(G)$ - compact.

Theorem 3.5. Let (X, T) be a neutrosophic space and G be a grill on X . Then the following are equivalent:

- (a) (X, T) is neutrosophic $C(G)$ - compact.
- (b) For each neutrosophic closed subset A of X and each family of neutrosophic closed subsets of X such that $\cap\{T \cap A : T \in F\} = 0_N$, there is a finite subfamily $\{T_1, T_2, T_3, \dots, T_n\}$ such that $\cap_{i=1}^n (int(T_i)) \cap A \in G$.
- (c) For each neutrosophic closed set A and each family F of neutrosophic closed subsets of X such that $\{int(T) \cap A : T \in F\}$ (G) FIP, one has $\cap\{T \cap A : T \in F\} \neq 0_N$.
- (d) For each neutrosophic closed set A and each neutrosophic regular open cover U of A , there exists a finite sub collection $\{U_1, U_2, U_3, \dots, U_n\}$ such that $A - \cup_{i=1}^n Cl(U_i) \notin G$.
- (e) For each neutrosophic closed set A and each family F of neutrosophic regular closed sets such that $\cap\{T \cap A : T \in F\} = 0_N$, there is a finite subfamily $\{T_1, T_2, T_3, \dots, T_n\}$ such that $\cap_{i=1}^n (int(T_i)) \cap A \notin G$.
- (f) For each neutrosophic closed set A and each family F of neutrosophic regular closed sets such that $\{int(T) \cap A : T \in F\}$ has grill neutrosophic finite intersection property, one has $\cap\{T \cap A : T \in F\} \neq 0_N$.
- (g) For each neutrosophic closed set A , each neutrosophic open cover U of $X - A$ and each neutrosophic open neighbourhood U of A , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of U such that $X - (U \cup (\cup_{i=1}^n Cl(U_i))) \notin G$.
- (h) For each neutrosophic closed set A and each neutrosophic open filter base B on X such that $\{B \cap A : B \in B\} \subset G$, one has $\cap\{Cl(B) : B \in B\} \cap A \neq 0_N$.

Proof. (a) \Rightarrow (b). Let (X, T) be neutrosophic $C(G)$ - compact, A a neutrosophic closed subset and F family of neutrosophic closed subsets with $\cap\{T \cap A : T \in F\} = 0_N$. Then $\{X - T : T \in F\}$ is a neutrosophic open cover of A and hence admits a finite subfamily $\{X \setminus T_i : i = 1, 2, \dots, n\}$ such that $A - \cup_{i=1}^n Cl(X - T_i) \notin G$ is easily seen to be $\{\cap_{i=1}^n (int(T_i)) \cap A\}$.

(b) \Rightarrow (c). It is obvious.

(c) \Rightarrow (a). Let A be a neutrosophic closed subset. Let U be a neutrosophic open cover of A with the property that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of U , one has $A - \cup_{i=1}^n Cl(U_i) \notin G$. Then $\{X - U_i : U_i \in U, i = 1, 2, \dots, n\}$ is a family of closed sets. Since $\cap_{i=1}^n \{(X \setminus Cl(U_i))\} \cap A = \cap_{i=1}^n \{A - Cl(U_i)\} = A - \cup_{i=1}^n Cl(U_i)$, the family $\{int(X \setminus U_i) \cap A : U_i \in U, i = 1, 2, \dots, n\}$ has neutrosophic finite intersection property with a grill G . By the hypothesis $\cap\{(X - U_i) \cap A : U_i \in U\} \neq 0_N \Rightarrow \cap\{A - U_i : U_i \in U, i = 1, 2, \dots, n\} \neq 0_N \Rightarrow A - \cup\{U_i : U_i \in U\} \neq 0_N \Rightarrow U$ is not a cover of A , a contradiction.

(d) \Rightarrow (a). Let A be a neutrosophic closed subset of X and U be a neutrosophic open cover of A . Then $\{int(Cl(U_i)) : U_i \in U\}$ is a neutrosophic regular open cover of A . Let $\{int(Cl(U_i)), i =$

$1, 2, \dots, n\}$ be a finite subfamily such that $A - \bigcup_{i=1}^n Cl(int(Cl(U_i))) \notin G$. Since U_i is neutrosophic open and for each open set U_i , we have, $Cl(int(Cl(U_i))) = Cl(U_i)$. We have $A - \bigcup_{i=1}^n Cl(U_i) \notin G$.

Hence X is neutrosophic $C(G)$ - compact.

(a) \Rightarrow (d). It is obvious.

(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (d) are same as (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) respectively.

(a) \Rightarrow (g). Let A be a neutrosophic closed set, V a neutrosophic open neighborhood of A and U an open cover of $X - A$. Since $X - V \subset X - A$, U is a neutrosophic open cover of $X - V$. Let $\{U_1, U_2, U_3, \dots, U_n\}$ be a finite collection of U , such that $(X - V) - \bigcup_{i=1}^n Cl(U_i) \notin G$. Since $(X - V) - \bigcup_{i=1}^n Cl(U_i) = X - (V \cup (\bigcup_{i=1}^n Cl(U_i)))$. This shows $X - (V \cup (\bigcup_{i=1}^n Cl(U_i))) \notin G$.

(g) \Rightarrow (a). Let A be a neutrosophic closed set and U a neutrosophic open covering of A . If H denotes the union of members of U , then $T = X - H$ is neutrosophic closed set and $X - A$ is a neutrosophic open neighborhood of T . Also U is a neutrosophic open cover of $X - T$. By hypothesis, there is a finite sub - collection $\{U_1, U_2, U_3 \dots, U_n\}$ of U , such that $X - ((X - A) \cup (\bigcup_{i=1}^n Cl(U_i))) \notin G$. However, this set not in G is nothing but $A - \bigcup_{i=1}^n Cl(U_i)$.

(a) \Rightarrow (h). Suppose A is a neutrosophic closed set and B is neutrosophic open filter base on X with $\{D \cap A : D \in B\} \subset G$. Suppose, if possible, $\bigcap \{Cl(D) : D \in B\} \cap A = 0_N$. Then $\{X - Cl(D) : D \in B\}$ is a neutrosophic open cover of A . By hypothesis, there exists a finite subfamily $\{X - Cl(D_i) : i = 1, 2, \dots, n\}$ such that $A - \bigcup_{i=1}^n Cl(X - Cl(D_i)) \notin G$. However, this set is $A \cap (\bigcap_{i=1}^n int(Cl(D_i)))$ and $A \cap (\bigcap_{i=1}^n (D_i))$ is a subset of it. Therefore, $A \cap (\bigcap_{i=1}^n (D_i)) \notin G$. Since B is a filter base, we have a $D \in B$ such that $D \subset \bigcap_{i=1}^n D_i$. But then $A \cap D \notin G$ which contradicts the fact that $\{D \cap A : D \in B\} \subset G$.

(h) \Rightarrow (a). Suppose that (X, T) is not neutrosophic $C(G)$ - compact. Then there exists a neutrosophic closed subset A of X and a neutrosophic open cover U of A such that for any finite subfamily $\{U_1, U_2, U_3 \dots, U_n\}$ of U , we have $A - \bigcup_{i=1}^n Cl(A_i) \in G$. We may assume that U is neutrosophic closed under finite unions. Then the family $B = \{X - Cl(U_i) : U_i \in U, i = 1, 2, \dots, n\}$ is a neutrosophic open filter base on X such that $\{D \cap A : D \in B\} \subset G$. So, by the hypothesis, $\bigcap \{Cl(X - Cl(U_i)) : U_i \in U\} \cap A \neq 0_N$. Let x be a neutrosophic point in the intersection. Then $x \in A$ and $x \in Cl(X - Cl(U)) \subset X - U$ for each $U_i \in U$. But this contradicts the fact that U is a cover of A . Hence, (X, T) is a neutrosophic $C(G)$ - compact.

Definition 3.6. A neutrosophic filter base B is said to be neutrosophic G adherent if for every neutrosophic neighborhood N of the adherent set of B , there exists an element $D \in B$ such that $(X - N) \cap D \notin G$.

Theorem 3.7. A space (X, T) is neutrosophic $C(G)$ - compact if and only if every neutrosophic open filter base on G is G - adherent convergent.

Proof. Let (X, T) be neutrosophic $C(G)$ - compact and let B be an open filter base on G with A as its adherent set. Let G be an open neighborhood of A . Then $A = \bigcap \{Cl(D) : D \in B\}$, $A \subset G$ and $X - G$ is neutrosophic closed. Now $\{X - cl(D) : D \in B\}$ is a neutrosophic open cover of $X - G$ and so by the hypothesis, it admits a finite subfamily $\{X - cl(D_i) : i = 1, 2, \dots, n\}$ such that $(X - G) - \bigcup_{i=1}^n cl(X - cl(D_i)) \notin G$. But this implies $(X - G) \cap (\bigcap_{i=1}^n int(cl(D_i))) \notin G$. However, $D_i \subset int(cl(D_i))$ implies

$(X - G) \cap (\bigcap_{i=1}^n (D_i)) \notin G$. Since B is a filter base and $D_i \in B$, there is a $E \in B$ such that $E \subset \bigcap_{i=1}^n (B_i)$. But $(X - G) \cap E \notin G$ is required.

Conversely, let (X, T) be not neutrosophic $C(G)$ - compact, and A be a neutrosophic closed set and U be a neutrosophic open cover of A such that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of U , one has $A - \bigcup_{i=1}^n cl(U_i) \in G$. Without loss of generality, we may assume that U is closed for finite unions. Therefore, $B = \{X - cl(U_i) : U_i \in U\}$ becomes a neutrosophic filter base on G . If x is a neutrosophic adherent point of B , that is, if $x \in \{cl(X - cl(U_i)) : U_i \in U\} = X - \bigcup\{int(cl(U_i)) : U_i \in U\}$, then $x \notin A$, because U is a neutrosophic open cover of A and for $U_i \in U$, $U_i \subset int(cl(U_i))$. Therefore, the neutrosophic adherent set of B is contained in $X - A$, which is a neutrosophic open set. By hypothesis, there exists an element $D \in B$ such that $(X - (X - A)) \cap D \notin G$, that is, $A \cap D \notin G$, that is $A \cap (X - cl(U_i)) \notin G$ for some $U \in V$. This however contradicts our assumption. This completes the proof.

4. Neutrosophic Quasi - H - Closed with Respect to a Grill

In this section, our aim is to introduce the concept of neutrosophic quasi - H - closed and study various properties of this notion with respect to a grill.

Definition 4.1. A neutrosophic topological space (X, T) is said to be neutrosophic quasi - H-closed or simply NQHC, if for every open cover U of X , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ such that $X = \bigcup_{i=1}^n cl(U_i)$.

Definition 4.2. Let (X, T) be a neutrosophic topological space and G be a grill on X . X is neutrosophic quasi - H - closed with respect to G or just NQHC(G) if for every open cover U of X , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of U such that $X - \bigcup_{i=1}^n cl(U_i) \notin G$.

Definition 4.3. A grill G of subsets of a neutrosophic topological space (X, T) is said to be co non - dense if the complement of each of its members is non - dense.

Definition 4.4. Let (X, T) be a neutrosophic topological space. A family F of subsets of X is said to have the finite intersection property with respect to a grill G on X or just FIP(G) if the intersection of finite subfamily of F is a member of G .

Theorem 4.5. For a neutrosophic topological space (X, T) and a grill G on X , the following are equivalent:

- (a) (X, T) is NQHC(G).
- (b) For each family F of closed sets having empty intersection, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^n int(F_i) \notin G$.
- (c) For each family K of neutrosophic closed sets such that $\{int(F) : F \in K\}$ has FIP(G), one has $\bigcap\{F : F \in K\} \neq 0_N$.
- (d) Every neutrosophic regular open cover has a proximate G cover.
- (e) For each family F of non empty neutrosophic regular closed sets having empty intersection, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^n int(F_i) \notin G$.
- (f) For each collection K of non empty neutrosophic regular closed sets such that $\{int(F) : F \in K\}$ has FIP(G), one has $\bigcap\{F : F \in K\} \neq 0_N$.
- (g) For each neutrosophic open filter base $Con G$, $\bigcap\{cl(B) : B \in C\} \neq 0_N$.
- (h) Every neutrosophic open ultra filter on G converges.

Proof. Obvious.

5. Conclusion

In this article, we have defined neutrosophic C - compactness with respect to a grill. We have investigated some properties of this newly defined compactness. Some characterization theorems have also been established. We have also defined neutrosophic quasi - H - closed with respect to a grill. Some characterization theorems on this newly concept have been investigated. It is expected that the work done will help in further investigation of the compactness in neutrosophic topological space.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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