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An Introduction to Bipolar Pythagorean Refined Sets

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Abstract: The aim of this paper is to introduce the new concept of a Bipolar Pythagorean refined set by combining the two notions called a Bipolar Pythagorean set and a Pythagorean refined set. Also, basic operations and algebraic properties of the Bipolar Pythagorean refined set are discussed with suitable examples.

Keywords: Pythagorean Set; Bipolar Pythagorean Set; Pythagorean Refined Set; Bipolar Pythagorean Refined Set.

1. Introduction

Fuzzy sets were first initiated by Zadeh [24] and he examined the membership function. After introducing some more concepts with fuzzy set theory, Atanassov [1, 2, 3, 4] generalized and introduced the new concept called intuitionistic fuzzy set (IFS) which is a generalized form of FS. Atanassov [5, 6] extended the set to Intuitionistic fuzzy Multi-dimensional sets. Also, Intuitionistic fuzzy topological spaces were introduced by Coker [11].

Yager [22] familiarized the model of Pythagorean fuzzy sets. Peng and Yang [20] presented the basic operators for PFNs. In [21, 23] similarity measures, distance measures, and multiple decision-making problems of Pythagorean fuzzy sets were discussed.

Bosc and Pivert [9] stated "Bipolarity refers to the tendency of the human mind to make decisions on the basis of positive and negative effects. Positive information states what is desired, satisfactory, possible, or considered as being acceptable. At the same time, negative statements express what is rejected, impossible, or forbidden. Negative preferences correspond to constraints while positive preferences correspond to wishes, Later Lee [15] introduced the concept of bipolar fuzzy sets which is a generalization of the fuzzy sets. Recently, bipolar fuzzy models have been studied by many authors on algebraic structures. Chen et. al. [10] studied of m-polar fuzzy set. Then, they examined many results which are related to these concepts and can be generalized to the case of m-polar fuzzy sets. They also proposed numerical examples to show how to apply m-polar fuzzy sets in real-world problems. In [19] Naeem discussed Pythagorean m polar fuzzy sets. In [12, 14] Florentin Smarandache introduced the concept of neutrosophic refined and bipolar neutrosophic sets, as an extension of this [13] Smarandache came with the topic Bipolar Neutrosophic refined sets. R. Jhansi [16] introduced the concept of bipolar Pythagorean fuzzy sets which is an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets. The score function, accuracy function, and some basic operators are also discussed in this paper with real-life applications.

Contrary to ordinary sets, multisets permit us to have multiple occurrences of the members. Blizard [7, 8] introduced multiset theory as a generalization of crisp set theory. As an extension of multiset, Yager introduced the notion of fuzzy multiset (FMS). Muhammad Riaz, Khalid Naeem, Xindong Peng, Deeba Afzal [17] introduced Pythagorean fuzzy multisets that have real-life applications by applying the concept of multiple-valued logic. Pythagorean fuzzy multisets provide a strong mathematical model to deal with multi-attribute group decision-making (MAGDM). While tackling real-world problems, intuitionistic fuzzy multiset cannot deal with the situation if the sum of the membership degree and non-membership degree of the parameter gets larger than 1. It makes decision-making demarcated and affects the optimum decision. PFM sets assist us in handling such situations. [18] Muhammad Saeed explained the properties, Set-Theoretic Operations, and Axiomatic results of Refined Pythagorean fuzzy sets.

In this paper, we introduce the concept of a bipolar Pythagorean refined set which is the combination of the bipolar Pythagorean fuzzy sets and Pythagorean refined sets. Also, we give some basic operators and algebraic properties of bipolar Pythagorean refined set operations with desirable examples.

2. Preliminaries

In this section, we recall the basic definitions and related results for developing the desired set.

Definition 2.1. (Fuzzy set) [23] Let M be a fixed set, then a fuzzy sets Q in M can be define as: Q = {(m, μ_Q (m)) / m ϵ M } Where μ_Q : M → [0,1] is called the membership degree of m ϵ M.

Definition 2.2: (Pythagorean Fuzzy set) [21] Let X be a non-empty set and I the unit interval [0, 1]. A PF set S is an object having the form $P=\{\langle x,\mu_p(x),v_p(x)\rangle : x\in X\}$ where the functions $\mu_p(x): X\to[0,1]$ and $v_p(x): X\to[0,1]$ denote respectively the degree of membership and degree of non-membership of each element x ϵ X to the set P, and $0\le(\mu_p(x))^2+(v_p(x))^2\le 1$ for each x ϵ X.

Definition 2.3: (Bipolar Pythagorean Fuzzy set) [16] Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A=\{\langle x,(\mu_A^P,\eta_A^P),(\mu_A^N,\eta_A^N)\rangle : x\in X\}$ where $\mu_A^P : X\to[0,1]$, $\eta_A^P : X\to[0,1]$ $\mu_A^N : X\to[-1,0]$, $\eta_A^N : X\to[-1,0]$ are the mappings such that

$0\le(\mu_p(x))^2+(v_p(x))^2\le 1$,

$-1\le-((\mu_p(x))^2+(v_p(x))^2)\le 0$ and

μ_A^P denote the positive membership degree, η_A^P denote the positive non-membership degree, μ_A^N denote the negative membership degree and η_A^N denote the negative non membership degree.

Definition 2.4. [16] Let $A=\{\langle x,(\mu_A^P,\eta_A^P),(\mu_A^N,\eta_A^N)\rangle : x\in X\}$ and $B=\{\langle x,(\mu_B^P,\eta_B^P),(\mu_B^N,\eta_B^N)\rangle : x\in X\}$ be two BPFSs, then their operations are defined as follows:

(i) $A\cup B=\{x,\max(\mu_A^P,\mu_B^P),\min(\eta_A^P,\eta_B^P),\min(\mu_A^N,\mu_B^N),\max(\eta_A^N,\eta_B^N) : x\in X\}$

(ii) $A\cap B=\{x,\min(\mu_A^P,\mu_B^P),\max(\eta_A^P,\eta_B^P),\max(\mu_A^N,\mu_B^N),\min(\eta_A^N,\eta_B^N) : x\in X\}$

(iii) $A^C=\{x,(\eta_A^P,\mu_A^P),(\eta_A^N,\mu_A^N) : x\in X\}$

Definition 2.5. (Refined Pythagorean Fuzzy Set) [18] A Refined Pythagorean fuzzy set (rpfs) A_{RP} in U is given by $A_{RP}=\{\langle x,\mu_A{}^{\alpha}(x),v_A{}^{\beta}(x)\rangle : \omega\in N_1^{\gamma},\lambda\in N_1^{\delta},\alpha+\beta\ge 3,u\in U\}$ where $\alpha,\beta\in N$ such that

$\mu_A{}^{\alpha}(x),v_A{}^{\beta}(x): U\to I$ with the condition that, $0\le\sum_{\alpha=1}^{\mu}(\mu_A^{\alpha})^2+\sum_{\beta=1}^{\delta}(v_A^{\beta})^2\le 1$.

3. Bipolar Pythagorean Refined Sets

Despite the fact that electric cars have the potential to greatly decrease GHG emissions and enhance air quality, there are still obstacles that must be overcome before their widespread adoption can be achieved.

Definition 3.1. (Bipolar Pythagorean refined set) Let X be the non - empty set in U. A Bipolar Pythagorean refined set (in short BPRs) A_{BPR} on X can be defined by the form

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),......,\varsigma^{P+}_{A_{BPR}}(x), \vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),......,\vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),......,\varsigma^{P-}_{A_{BPR}}(x), \vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),......,\vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

Where ,

$$\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),....,\varsigma^{P+}_{A_{BPR}}(x), \vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),...,\vartheta^{P+}_{A_{BPR}}(x) : X \to [0,1]$$

$$\varsigma^{1-}_{A_{BPR}}(x),\ \varsigma^{2-}_{A_{BPR}}(x),...,\varsigma^{P-}_{A_{BPR}}(x)\ ,\ \vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),...,\vartheta^{P-}_{A_{BPR}}(x) : \quad X \to [-1,0]$$

Such that $$0 \le (\varsigma^{i+}_{A_{BPR}}(x))^2 + (\vartheta^{i+}_{A_{BPR}}(x))^2 \le 1$$

$$-1 \le -(\varsigma^{i-}_{A_{BPR}}(x))^2 + (\vartheta^{i-}_{A_{BPR}}(x))^2 \le 0$$ for i=1,2,….,p for any element x ϵ X

$\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),......,\varsigma^{P+}_{A_{BPR}}(x)$ denote the positive membership degree.

$\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),......,\vartheta^{P+}_{A_{BPR}}(x)$ denote the positive non membership degree.

$\varsigma^{1-}_{A_{BPR}}(x),\ \varsigma^{2-}_{A_{BPR}}(x),......,\varsigma^{P-}_{A_{BPR}}(x)$ denote the negative membership degree.

$\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),......,\vartheta^{P-}_{A_{BPR}}(x)$ denote the negative non membership degree.

***Definition** 3.2.* **(Subset)** Let $A_{\mathrm{BPR}}, B_{\mathrm{BPR}}$ ϵ BPRS(X) ,where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),......,\varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),......,\vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),......,\varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),......,\vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x),\ \varsigma^{2+}_{B_{BPR}}(x),......,\varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x),\ \vartheta^{2+}_{B_{BPR}}(x),......,\vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x),......,\varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x),\ \vartheta^{2-}_{B_{BPR}}(x),......,\vartheta^{P-}_{B_{BPR}}(x)) : x \in X \}$$

Then A_{BPR} is said to be BPR Subset of B_{BPR} and is denoted by $A_{\mathrm{BPR}} \subseteq B_{\mathrm{BPR}}$ if

$$\varsigma^{i+}_{A_{BPR}}(x) \le \varsigma^{i+}_{B_{BPR}}(x),\ \vartheta^{i+}_{A_{BPR}}(x) \ge \vartheta^{i+}_{B_{BPR}}(x)$$

$$\varsigma^{i-}_{A_{BPR}}(x) \geq \varsigma^{i-}_{B_{BPR}}(x), \quad \vartheta^{i-}_{A_{BPR}}(x) \leq \vartheta^{i-}_{B_{BPR}}(x)$$

for every x ϵ X and i=1,2,…..,p

Example 3.3:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x, ([0.3,.0.4,0.6],[0.5,0.2,0.8])([-0.6,-0.5,-0.4],[-0.2,-0.4,-0.3])\rangle : x \in X\}$$

We can say that $A_{\text{BPR}} \subseteq B_{\text{BPR}}$

Definition 3.4: (Equality) Let A_{BPR} , B_{BPR} ϵ BPRS(X) , where

$$A_{\text{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$B_{\text{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X \}$$

Then A_{BPR} is said to be BPR set equal of B_{BPR} and is denoted by $A_{\text{BPR}} = B_{\text{BPR}}$ if

$$\varsigma^{i+}_{A_{BPR}}(x) = \varsigma^{i+}_{B_{BPR}}(x), \quad \vartheta^{i+}_{A_{BPR}}(x) = \vartheta^{i+}_{B_{BPR}}(x)$$

$$\varsigma^{i-}_{A_{BPR}}(x) = \varsigma^{i-}_{B_{BPR}}(x), \quad \vartheta^{i-}_{A_{BPR}}(x) = \vartheta^{i-}_{B_{BPR}}(x)$$

for every x ϵ X and i=1,2,…..,p

Example 3.5.

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x, ([0.2,.0.3,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

We can say that $A_{\text{BPR}} = B_{\text{BPR}}$

Definition 3.6. (Complement)

Let A_{BPR} ϵ BPRS(X) . where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X$$

The complement of A_{BPR} denoted by $A^{\mathrm{C}}_{\mathrm{BPR}}$ and is defined by

$$A_{BPR}{}^{C} = \{(x, (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x), \varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x)), (\vartheta^{1-}_{A_{BPR}}(x),$$

$$\vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x), \varsigma^{1-}_{A_{BPR}}(x), \varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x)) : x \in X \}$$

for every x ϵ X and i=1,2,…..,p

Example 3.7:

Let X be a non empty set in U. If A_{BPR} is bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

Then the complement of A_{BPR}

$$A^{C}_{BPR} = \{\langle x, ([0.7,0.6,0.9],[0.2,.0.3,0.5])([-0.4,-0.6,-0.7],[-0.5,-0.4,-0.3])\rangle : x \in X\}$$

Definition 3.8. (Union) Let A_{BPR} , B_{BPR} ϵ BPRS(X) . , where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X \}$$

The union of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \cup B_{\mathrm{BPR}} = C_{\mathrm{BPR}}$ and is defined by

$$C_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{C_{\mathrm{BPR}}}(x), \varsigma^{2+}_{C_{\mathrm{BPR}}}(x), \ldots\ldots, \varsigma^{P+}_{C_{\mathrm{BPR}}}(x), \vartheta^{1+}_{C_{\mathrm{BPR}}}(x), \vartheta^{2+}_{C_{\mathrm{BPR}}}(x), \ldots\ldots, \vartheta^{P+}_{C_{\mathrm{BPR}}}(x)), (\varsigma^{1-}_{C_{\mathrm{BPR}}}(x),$$

$$\varsigma^{2-}_{C_{\mathrm{BPR}}}(x), \ldots\ldots, \varsigma^{P-}_{C_{\mathrm{BPR}}}(x), (\vartheta^{1-}_{C_{\mathrm{BPR}}}(x), \vartheta^{2-}_{C_{\mathrm{BPR}}}(x), \ldots\ldots, \vartheta^{P-}_{C_{\mathrm{BPR}}}(x)) : x \in X \}$$

Where,

$$\varsigma^{i+}_{C_{\text{BPR}}}(x) = \max\{\varsigma^{i+}_{A_{\text{BPR}}}(x), \varsigma^{i+}_{B_{\text{BPR}}}(x)\}$$
$$\vartheta^{i+}_{C_{\text{BPR}}}(x) = \min\{\vartheta^{i+}_{A_{\text{BPR}}}(x), \vartheta^{i+}_{B_{\text{BPR}}}(x)\}$$

$$\varsigma^{i-}_{C_{\text{BPR}}}(x) = \min\{\varsigma^{i-}_{A_{\text{BPR}}}(x), \varsigma^{i-}_{B_{\text{BPR}}}(x)\}$$
$$\vartheta^{i-}_{C_{\text{BPR}}}(x) = \max\{\vartheta^{i-}_{A_{\text{BPR}}}(x), \vartheta^{i-}_{B_{\text{BPR}}}(x)\}$$ for every x ϵ X and i=1,2,…..,p

Example 3.9:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x,([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x,([0.2,.0.3,0.6],[0.4,0.8,0.3])([-0.3,-0.2,-0.7],[-0.7,-0.8,-0.5])\rangle : x \in X\}$$

then the union of two sets is

$$C_{\text{BPR}} = \{\langle x,([0.3,.0.5,0.7],[0.4,0.8,0.3])([-0.3,-0.5,-0.7],[-0.5,-0.6,-0.5])\rangle : x \in X\}$$

Definition 3.10. (Intersection)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\text{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x),\ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x),\ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$
$$\varsigma^{2-}_{A_{BPR}}(x),\ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x),\ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

$$B_{\text{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x),\ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x),\ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$
$$\varsigma^{2-}_{B_{BPR}}(x),\ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x),\ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X\}$$

The intersection of A_{BPR} and B_{BPR} is denoted by $A_{\text{BPR}} \cap B_{\text{BPR}} = D_{\text{BPR}}$

$$D_{\text{BPR}} = \{(x, (\varsigma^{1+}_{D_{\text{BPR}}}(x), \varsigma^{2+}_{D_{\text{BPR}}}(x),\ldots\ldots, \varsigma^{P+}_{D_{\text{BPR}}}(x), \vartheta^{1+}_{D_{\text{BPR}}}(x), \vartheta^{2+}_{D_{\text{BPR}}}(x),\ldots\ldots, \vartheta^{P+}_{D_{\text{BPR}}}(x)),$$

$$(\varsigma^{1-}_{D_{\text{BPR}}}(x)$$

$$, \varsigma^{2-}_{D_{\text{BPR}}}(x),\ldots\ldots, \varsigma^{P-}_{D_{\text{BPR}}}(x), (\vartheta^{1-}_{D_{\text{BPR}}}(x), \vartheta^{2-}_{D_{\text{BPR}}}(x),\ldots\ldots, \vartheta^{P-}_{D_{\text{BPR}}}(x)) : x \in X\}$$

Where,

$$\varsigma^{i+}_{D_{\text{BPR}}}(x) = \min\{\varsigma^{i+}_{A_{\text{BPR}}}(x), \varsigma^{i+}_{B_{\text{BPR}}}(x)\}$$
$$\vartheta^{i+}_{D_{\text{BPR}}}(x) = \max\{\vartheta^{i+}_{A_{\text{BPR}}}(x), \vartheta^{i+}_{B_{\text{BPR}}}(x)\}$$

$$\varsigma_{D_{\text{BPR}}}^{i-}(x) = \max\{\varsigma_{A_{\text{BPR}}}^{i-}(x), \varsigma_{B_{\text{BPR}}}^{i-}(x)\}$$
$$\vartheta_{D_{\text{BPR}}}^{i-}(x) = \min\{\vartheta_{A_{\text{BPR}}}^{i-}(x), \vartheta_{B_{\text{BPR}}}^{i-}(x)\}$$ for every x ϵ X and i=1,2,…..,p

Example 3.11:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x, ([0.2,.0.3,0.6],[0.4,0.8,0.3])([-0.3,-0.4,-0.7],[-0.7,-0.8,-0.5])\rangle : x \in X\}$$

then the intersection of two sets is

$$D_{\text{BPR}} = \{\langle x, ([0.2,.0.3,0.6,[0.6,0.8,0.9])([-0.2,-0.4,-0.6],[-0.7,-0.8,-0.9])\rangle : x \in X\}$$

Definition 3.12: (Addition)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\text{BPR}} = \{(x, (\varsigma_{A_{BPR}}^{1+}(x), \varsigma_{A_{BPR}}^{2+}(x), \ldots\ldots, \varsigma_{A_{BPR}}^{P+}(x), (\vartheta_{A_{BPR}}^{1+}(x), \vartheta_{A_{BPR}}^{2+}(x), \ldots\ldots, \vartheta_{A_{BPR}}^{P+}(x)), (\varsigma_{A_{BPR}}^{1-}(x),$$
$$\varsigma_{A_{BPR}}^{2-}(x), \ldots\ldots, \varsigma_{A_{BPR}}^{P-}(x), (\vartheta_{A_{BPR}}^{1-}(x), \vartheta_{A_{BPR}}^{2-}(x), \ldots\ldots, \vartheta_{A_{BPR}}^{P-}(x)) : x \in X\}$$

$$B_{\text{BPR}} = \{(x, (\varsigma_{B_{BPR}}^{1+}(x), \varsigma_{B_{BPR}}^{2+}(x), \ldots\ldots, \varsigma_{B_{BPR}}^{P+}(x), (\vartheta_{B_{BPR}}^{1+}(x), \vartheta_{B_{BPR}}^{2+}(x), \ldots\ldots, \vartheta_{B_{BPR}}^{P+}(x)), (\varsigma_{B_{BPR}}^{1-}(x),$$
$$\varsigma_{B_{BPR}}^{2-}(x), \ldots\ldots, \varsigma_{B_{BPR}}^{P-}(x), (\vartheta_{B_{BPR}}^{1-}(x), \vartheta_{B_{BPR}}^{2-}(x), \ldots\ldots, \vartheta_{B_{BPR}}^{P-}(x)) : x \in X\}$$

Then the addition of A_{BPR} and B_{BPR} is denoted by $A_{\text{BPR}} \oplus B_{\text{BPR}}$

$$A_{\text{BPR}} \oplus B_{\text{BPR}} = \begin{pmatrix} \left(\varsigma_{A_{BPR}}^{i+}(x) + \varsigma_{B_{BPR}}^{i+}(x) - \varsigma_{A_{BPR}}^{i+}(x)\varsigma_{B_{BPR}}^{i+}(x), \vartheta_{A_{BPR}}^{i+}(x)\vartheta_{B_{BPR}}^{i+}(x)\right), \\ \left(-\varsigma_{A_{BPR}}^{i-}(x)\varsigma_{B_{BPR}}^{i-}(x), -(\vartheta_{A_{BPR}}^{i-}(x) + \vartheta_{B_{BPR}}^{i-}(x) - \vartheta_{A_{BPR}}^{i-}(x)\vartheta_{B_{BPR}}^{i-}(x))\right) \end{pmatrix}$$

For x ϵ X , i=1,2,…,p

Definition 3.13: (Multiplication)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\text{BPR}} = \{(x, (\varsigma_{A_{BPR}}^{1+}(x), \varsigma_{A_{BPR}}^{2+}(x), \ldots\ldots, \varsigma_{A_{BPR}}^{P+}(x), (\vartheta_{A_{BPR}}^{1+}(x), \vartheta_{A_{BPR}}^{2+}(x), \ldots\ldots, \vartheta_{A_{BPR}}^{P+}(x)), (\varsigma_{A_{BPR}}^{1-}(x),$$
$$\varsigma_{A_{BPR}}^{2-}(x), \ldots\ldots, \varsigma_{A_{BPR}}^{P-}(x), (\vartheta_{A_{BPR}}^{1-}(x), \vartheta_{A_{BPR}}^{2-}(x), \ldots\ldots, \vartheta_{A_{BPR}}^{P-}(x)) : x \in X\}$$

B_{BPR} = {(x, ($\varsigma^{1+}_{B_{BPR}}(x)$, $\varsigma^{2+}_{B_{BPR}}(x)$,……, $\varsigma^{P+}_{B_{BPR}}(x)$,($\vartheta^{1+}_{B_{BPR}}(x)$, $\vartheta^{2+}_{B_{BPR}}(x)$,……, $\vartheta^{P+}_{B_{BPR}}(x)$) , ($\varsigma^{1-}_{B_{BPR}}(x)$,

$\varsigma^{2-}_{B_{BPR}}(x)$,……, $\varsigma^{P-}_{B_{BPR}}(x)$,($\vartheta^{1-}_{B_{BPR}}(x)$, $\vartheta^{2-}_{B_{BPR}}(x)$,……, $\vartheta^{P-}_{B_{BPR}}(x)$) : x ϵ X }

Then the multiplication of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \otimes B_{\mathrm{BPR}}$

$$A_{\mathrm{BPR}} \otimes B_{\mathrm{BPR}} = \begin{pmatrix} \left(\varsigma^{i+}_{A_{BPR}}(x)\varsigma^{i+}_{B_{BPR}}(x), \vartheta^{i+}_{A_{BPR}}(x) + \vartheta^{i+}_{B_{BPR}}(x) - \vartheta^{i+}_{A_{BPR}}(x)\vartheta^{i+}_{B_{BPR}}(x)\right), \\ \left(-(\varsigma^{i-}_{A_{BPR}}(x) + \varsigma^{i-}_{B_{BPR}}(x), -\varsigma^{i-}_{A_{BPR}}(x)\varsigma^{i-}_{B_{BPR}}(x)), -\vartheta^{i-}_{A_{BPR}}(x)\vartheta^{i-}_{B_{BPR}}(x))\right) \end{pmatrix}$$

For x ϵ X, i=1, 2, …, p.

4. Algebraic Properties of Bipolar Pythagorean Refined Set Operations

Proposition 4.1: (Commutative Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .Then

(a) $A_{BPR} \cup B_{BPR} = B_{BPR} \cup A_{BPR}$

(b) $A_{BPR} \cap B_{BPR} = B_{BPR} \cap A_{BPR}$

Proof: The proofs can be easily made.

Proposition 4.2 : (Associative Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .Then

(a) $A_{BPR} \cup (B_{BPR} \cup C_{BPR}) = (A_{BPR} \cup B_{BPR}) \cup C_{BPR}$

(b) $A_{BPR} \cap (B_{BPR} \cap C_{BPR}) = (A_{BPR} \cap B_{BPR}) \cap C_{BPR}$

Proof : Let A_{BPR}, B_{BPR} and C_{BPR} be three bipolar Pythagorean refined sets defined as follows.

A_{BPR} = {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$) , ($\varsigma^{1-}_{A_{BPR}}(x)$,

$\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }

B_{BPR} = {(x, ($\varsigma^{1+}_{B_{BPR}}(x)$, $\varsigma^{2+}_{B_{BPR}}(x)$,……, $\varsigma^{P+}_{B_{BPR}}(x)$,($\vartheta^{1+}_{B_{BPR}}(x)$, $\vartheta^{2+}_{B_{BPR}}(x)$,……, $\vartheta^{P+}_{B_{BPR}}(x)$) , ($\varsigma^{1-}_{B_{BPR}}(x)$,

$\varsigma^{2-}_{B_{BPR}}(x)$,……, $\varsigma^{P-}_{B_{BPR}}(x)$,($\vartheta^{1-}_{B_{BPR}}(x)$, $\vartheta^{2-}_{B_{BPR}}(x)$,……, $\vartheta^{P-}_{B_{BPR}}(x)$) : x ϵ X }

C_{BPR} = {(x, ($\varsigma^{1+}_{C_{BPR}}(x)$, $\varsigma^{2+}_{C_{BPR}}(x)$,……, $\varsigma^{P+}_{C_{BPR}}(x)$,($\vartheta^{1+}_{C_{BPR}}(x)$, $\vartheta^{2+}_{C_{BPR}}(x)$,……, $\vartheta^{P+}_{C_{BPR}}(x)$) , ($\varsigma^{1-}_{C_{BPR}}(x)$,

$\varsigma^{2-}_{C_{BPR}}(x)$,……, $\varsigma^{P-}_{C_{BPR}}(x)$,($\vartheta^{1-}_{C_{BPR}}(x)$, $\vartheta^{2-}_{C_{BPR}}(x)$,……, $\vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

Then $(A_{BPR} \cup B_{BPR}) \cup C_{BPR} =$

{(x, ($\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x))$, ($\vartheta^{1+}_{A_{BPR}}(x) \vee$ $\vartheta^{1+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x))$, ($\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x)),($ $\vartheta^{1-}_{A_{BPR}}(x) \vee$ $\vartheta^{1-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x))$: x ϵ X }∪ {(x, ($\varsigma^{1+}_{C_{BPR}}(x), \ldots\ldots\ldots, \varsigma^{P+}_{C_{BPR}}(x), (\vartheta^{1+}_{C_{BPR}}(x), \ldots\ldots,$ $\vartheta^{P+}_{C_{BPR}}(x)), (\varsigma^{1-}_{C_{BPR}}(x), \ldots\ldots\ldots\ldots, \varsigma^{P-}_{C_{BPR}}(x), (\vartheta^{1-}_{C_{BPR}}(x), \ldots\ldots\ldots\ldots, \vartheta^{P-}_{C_{BPR}}(x)$): x ϵ X }

= {(x, ($\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x)), \ldots\ldots, (\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x) \vee$ $\varsigma^{P+}_{C_{BPR}}(x))$, ($\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{B_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x)), \ldots\ldots, (\vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x) \vee \vartheta^{P+}_{C_{BPR}}(x))$, ($\varsigma^{1-}_{A_{BPR}}(x) \vee$ $\varsigma^{1-}_{B_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x)), \ldots\ldots\ldots\ldots,$ ($\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x)),($ $\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{B_{BPR}}(x) \vee, \vartheta^{1-}_{C_{BPR}}(x))$,………..,($\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x) \vee, \vartheta^{P-}_{C_{BPR}}(x))$: x ϵ X }.

= {(x, $\varsigma^{1+}_{A_{BPR}}(x) \vee (\varsigma^{1+}_{B_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x)), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x) \vee (\varsigma^{P+}_{B_{BPR}}(x) \vee$ $\varsigma^{P+}_{C_{BPR}}(x))$, $\vartheta^{1+}_{A_{BPR}}(x) \vee (\vartheta^{1+}_{B_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x)), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x) \vee (\vartheta^{P+}_{B_{BPR}}(x) \vee F^{P+}_{C_{BPR}}(x))$, $\varsigma^{1-}_{A_{BPR}}(x) \vee$ $(\varsigma^{1-}_{B_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x)), \ldots\ldots\ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x) \vee (\varsigma^{P-}_{B_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x)),$ $\vartheta^{1-}_{A_{BPR}}(x) \vee (\vartheta^{1-}_{B_{BPR}}(x) \vee, \vartheta^{1-}_{C_{BPR}}(x))$,……….., $\vartheta^{P-}_{A_{BPR}}(x) \vee (\vartheta^{P-}_{B_{BPR}}(x) \vee, \vartheta^{P-}_{C_{BPR}}(x))$: x ϵ X }.

$= A_{BPR} \cup (B_{BPR} \cup C_{BPR})$

(b) The proof is obvious.

Proposition 4.3: (Idempotent Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .Then

(a) $A_{BPR} \cup A_{BPR} = A_{BPR}$

(b) $A_{BPR} \cap A_{BPR} = A_{BPR}$

Proof: The proofs can be easily made.

Example 4.4:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

Then ,

$$A_{BPR} \cup A_{BPR} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

Hence , $A_{BPR} \cup A_{BPR} = A_{BPR}$

(b) The proof is obvious.

Proposition 4.5: (Demorgan's Law)

Let A_{BPR} , $B_{\mathrm{BPR}} \in$ BPRS(X) .

(a) $(A_{BPR} \cup B_{BPR})^C = {B_{BPR}}^C \cap {A_{BPR}}^C$

(b) $(A_{BPR} \cap B_{BPR})^C = {B_{BPR}}^C \cup {A_{BPR}}^C$

Proof: The proofs can be easily made.

Proposition 4.6: (Distributive Law)

Let A_{BPR} , $B_{\mathrm{BPR}} \in$ BPRS(X) .

(a) $A_{BPR} \cup (B_{BPR} \cap C_{BPR}) = (A_{BPR} \cup B_{BPR}) \cap (A_{BPR} \cup C_{BPR})$

(b) $A_{BPR} \cap (B_{BPR} \cup C_{BPR}) = (A_{BPR} \cap B_{BPR}) \cup (A_{BPR} \cap C_{BPR})$

Proof : Let A_{BPR}, B_{BPR} and C_{BPR} be three bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{(\text{ x}, (\ \varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x), \varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : \text{x} \in \text{X}\}$$

$$B_{\mathrm{BPR}} = \{(\text{ x}, (\ \varsigma^{1+}_{B_{BPR}}(x),\ \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x),\ \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x), \varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x),\ \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : \text{x} \in \text{X}\}$$

$$C_{\mathrm{BPR}} = \{(\text{ x}, (\ \varsigma^{1+}_{C_{BPR}}(x),\ \varsigma^{2+}_{C_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{C_{BPR}}(x), (\vartheta^{1+}_{C_{BPR}}(x),\ \vartheta^{2+}_{C_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{C_{BPR}}(x)), (\varsigma^{1-}_{C_{BPR}}(x), \varsigma^{2-}_{C_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{C_{BPR}}(x), (\vartheta^{1-}_{C_{BPR}}(x),\ \vartheta^{2-}_{C_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{C_{BPR}}(x)) : \text{x} \in \text{X}\}$$

Then $A_{BPR} \cup (B_{BPR} \cap C_{BPR})$

= {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$), ($\varsigma^{1-}_{A_{BPR}}(x)$, $\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }U {(x, ($\varsigma^{1+}_{B_{BPR}}(x) \wedge \varsigma^{1+}_{C_{BPR}}(x)$) ,……, ($\varsigma^{P+}_{B_{BPR}}(x) \wedge \varsigma^{P+}_{C_{BPR}}(x)$) ,($\vartheta^{1+}_{B_{BPR}}(x) \wedge \vartheta^{1+}_{C_{BPR}}(x)$) ,……, ($\vartheta^{P+}_{B_{BPR}}(x) \wedge \vartheta^{P+}_{C_{BPR}}(x)$), ($\varsigma^{1-}_{B_{BPR}}(x) \wedge \varsigma^{1-}_{C_{BPR}}(x)$) ,……, ($\varsigma^{P-}_{B_{BPR}}(x) \wedge \varsigma^{P-}_{C_{BPR}}(x)$),($\vartheta^{1-}_{B_{BPR}}(x) \wedge \vartheta^{1-}_{C_{BPR}}(x)$) ,……, ($\vartheta^{P-}_{B_{BPR}}(x) \wedge \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

= {(x, $\varsigma^{1+}_{A_{BPR}}(x) \vee$ ($\varsigma^{1+}_{B_{BPR}}(x) \wedge \varsigma^{1+}_{C_{BPR}}(x)$) ,……, $\varsigma^{P+}_{A_{BPR}}(x) \vee (\varsigma^{P+}_{B_{BPR}}(x) \wedge \varsigma^{P+}_{C_{BPR}}(x))$, $\vartheta^{1+}_{A_{BPR}}(x) \vee$ ($\vartheta^{1+}_{B_{BPR}}(x) \wedge \vartheta^{1+}_{C_{BPR}}(x)$) ,……, $\vartheta^{P+}_{A_{BPR}}(x)) \vee (\vartheta^{P+}_{B_{BPR}}(x) \wedge \vartheta^{P+}_{C_{BPR}}(x))$, $\varsigma^{1-}_{A_{BPR}}(x) \vee$ ($\varsigma^{1-}_{B_{BPR}}(x) \wedge \varsigma^{1-}_{C_{BPR}}(x)$) ,……, $\varsigma^{P-}_{A_{BPR}}(x) \vee (\varsigma^{P-}_{B_{BPR}}(x) \wedge \varsigma^{P-}_{C_{BPR}}(x))$, $\vartheta^{1-}_{A_{BPR}}(x) \vee$ ($\vartheta^{1-}_{B_{BPR}}(x) \wedge \vartheta^{1-}_{C_{BPR}}(x)$) ,……, $\vartheta^{P-}_{A_{BPR}}(x) \vee (\vartheta^{P-}_{B_{BPR}}(x) \wedge \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

= {(x, ($\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x)) \wedge (\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x))$,……, $(\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x)) \wedge (\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{C_{BPR}}(x))$, $(\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{B_{BPR}}(x)) \wedge (\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x))$,……, $(\vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x)) \wedge (\vartheta^{P+}_{A_{BPR}}(x)) \vee \vartheta^{P+}_{C_{BPR}}(x))$, ($\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{B_{BPR}}(x)) \wedge (\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x))$,……, $(\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x)) \wedge (\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x))$, $(\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{B_{BPR}}(x)) \wedge (\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{C_{BPR}}(x))$,……, $(\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x)) \wedge (\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

$= (A_{BPR} \cup B_{BPR}) \cap (A_{BPR} \cup C_{BPR})$

(b) The proof is obvious.

Proposition 4.7: (Double complement Law)

Let A_{BPR} ϵ BPRS(X) , then

$(A^{c}_{BPR})^{c} = A_{\mathrm{BPR}}$

Proof: Let A_{BPR} be the bipolar Pythagorean refined set defined as follows.

A_{BPR} = {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$), ($\varsigma^{1-}_{A_{BPR}}(x)$,

$\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }

(A^{c}_{BPR})= {(x, ($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$, $\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$), ($\vartheta^{1-}_{A_{BPR}}(x)$,

$\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$, $\varsigma^{1-}_{A_{BPR}}(x)$, $\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$) : x ϵ X }

$(A^{c}_{BPR})^{c}$ = {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$), (

$\varsigma^{1-}_{A_{BPR}}(x)$, $\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }

Hence $(A^{c}_{BPR})^{c}$ = A_{BPR}

Proposition 4.8: (Absorption Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .

(a) $A_{BPR} \cup (A_{BPR} \cap B_{BPR}) = A_{BPR}$

(b) $A_{BPR} \cap (A_{BPR} \cup B_{BPR}) = A_{BPR}$

Proof: The proofs can be easily made.

5. Conclusion

This paper ensures the work of introducing the new set namely the Bipolar Pythagorean refined set by using the theory of the Bipolar Pythagorean set and Pythagorean refined(multi) set. Several operations and laws have been discussed along with some examples. In the future, Bipolar Pythagorean refined topological spaces can be introduced. And also, decision-making problems on bipolar Pythagorean refined sets can be introduced.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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