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Fixed Point Results in Neutrosophic Rectangular Extended b-Metric Spaces

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Abstract: In this manuscript, we establish the notion of neutrosophic rectangular extended b-metric spaces and derive some fixed point results for contraction mappings. Also, we provide several non-trivial examples. Our results are more generalized with respect to the existing ones in the literature. At the end of the paper, we provide an application to non-linear fractional differential equations to test the validity of the results.

Keywords: Neutrosophic Metric Spaces; Fixed Point; Graphical View; Non-Linear Fractional Differential Equations.

1. Introduction

In 1965, Zadeh [1] developed the "fuzzy notion" to contrast imprecise terms. Fuzzy sets (FSs) presented in [1] and metric spaces presented in [2] are combined to establish the concept of fuzzy metric spaces (FMSs), in which membership function is used. The notion of FMSs first introduced by Kramosil and Michalak [3] in 1975 and then George and Veeramani [4, 5] updated in 1994. Garbiec [6] established the fuzzy version of the Banach fixed point result. The notion of FSs only deals with membership functions, so there is a gap that FSs did not deal with non-membership functions. Atanassov [7] filled this gap to establish the concept of intuitionistic fuzzy sets (IFSs), in which, he used both degrees, the degree of membership and the degree of non-membership. But, there is still a gap that IFSs did not deals with naturalness. Smarandache [30] filled this gap to propose the concept of neutrosophic sets (NSs), as a generalization of IFSs. By combining the concepts of NSs and metric spaces, Kirişci and Simsek [32] presented the notion of neutrosophic metric spaces (NMSs).

Fuzzy rectangular metric spaces and fuzzy rectangular b-metric spaces (FRBMSs) were introduced by Mehmood et al. [9], who also demonstrated the Banach contraction principle in the context of FRBMSs. The concept of orthogonal FMSs was developed by Hezarjaribi [10], who also demonstrated several fixed point theorems. The authors in [11–14, 33-38] established several interesting fixed point results. Park and Jeong [15] established fixed point results for fuzzy mappings. An intuitionistic fuzzy b-metric space was presented by Konwar [16]. The authors in [17–18,] demonstrated a number of fixed point results for in the context of an IFMS. Nice work was done on fractional differential equations by the authors in [19–20]. Several fixed point results were proven by Javed et al. [21] in the setting of fuzzy b-metric-like spaces. Uddin et al. [22] presented a number of fixed point theorems for contraction mappings in the context of orthogonal controlled fuzzy metric spaces. Numerous algebraic structures have been used by mathematicians to apply several novel fuzzy set models [23–27, 32-35]. The idea of pentagonal controlled FMSs was recently given by Aftab et al. [28], who also demonstrated various fixed point theorems. Kattan et al. [29] established some fixed point results in a generalization of an IFMS.

Jeyaraman et al. [39] proved common fixed point theorems in intuitionistic generalized fuzzy cone metric spaces. Ishtiaq et al. [40] derived several a fixed point results in the context of generalized neutrosophic cone metric spaces. Gupta et al. [41] examined the uniqueness of solution by employing CLR-property on V-fuzzy metric spaces. Chauhan et al. [42] examined the existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces. Gupta et al. [43] solved some fixed point theorems for contraction mappings and investigate the xistence of fixed points for J-ψ-fuzzy contractions in fuzzy metric spaces endowed with graph.

In this manuscript, we aim to introduce the concept of neutrosophic rectangular extended b-metric spaces (NREBMSs) and to establish several fixed point results for contraction mappings. Also, we provide some non-trivial examples and an application to non-linear fractional differential equations to show the validity of results herein. An open problem is also raised after the conclusion section.

2. Preliminaries

In this section, we provide some basic notions that are helpful for readers to understand the main results.

Definition 2.1: [8] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm (CTN) if it satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $\hbar * 1 = \hbar$ for all $\hbar \in [0,1]$;

(iv) $\hbar * \ell \leq c * d$ whenever $\hbar \leq c$ and $\ell \leq d$, for all $\hbar, \ell, c, d \in [0,1]$.

Example 2.1: [8] $\hbar * \ell = \hbar\ell$ and $\hbar * \ell = \min\{\hbar, \ell\}$ are CTN.

Definition 2.2: [8] A binary operation $\circ$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-conorm (CTCN) if it meets the below assertions:

T1. $\circ$ is associative and commutative;

T2. $\circ$ is continuous;

T3. $\hbar \circ 0 = 0$, for all $\hbar \in [0, 1]$;

T4. $\hbar \circ \ell \leq c \circ d$ whenever $\hbar \leq c$ and $\ell \leq d$, for all $\hbar, \ell, c, d \in [0,1]$.

Example 2.2: [8] $\hbar \circ \ell = \max\{\hbar, \ell\}$ is CTCN.

Definition 2.3: [4] Let Σ is nonempty set, $\mathbb{K}$ is a FS on $\Sigma \times \Sigma \times (0, +\infty)$, and $*$ is a CTN. Then a triplet $(\Sigma, \mathbb{K}, *)$ is known as FMS, if it verifies the following conditions, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau > 0$:

F1. $\mathbb{K}(\varkappa, \vartheta, \sigma) > 0$;

F2. $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

F3. $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

F4. $\mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, z, \tau) \leq \mathbb{K}(\varkappa, z, \sigma + \tau)$;

F5. $\mathbb{K}(\varkappa, \vartheta, .): (0, + +\infty) \to (0,1]$ is continuous.

Definition 2.4: [9] Let Σ is nonempty set, $\mathbb{K}$ is a FS on $\Sigma \times \Sigma \times [0, +\infty)$, and $*$ is a CTN. Then $(\Sigma, \mathbb{K}, *, \ell)$ is known as FRBMS, if it verifies the following conditions, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w \geq 0$:

S1. $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

S2. $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

S3. $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

S4. $\mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathrm{u}, \tau) * \mathbb{K}(\mathrm{u}, z, w) \leq \mathbb{K}(\varkappa, z, \ell(\sigma + \tau + w))$;

S5. $\mathbb{K}(\varkappa, \vartheta, .): (0, +\infty) \to (0,1]$ is left continuous and $\lim_{\sigma \to +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

Definition 2.5: [32] Let Σ be a non-empty set and $\mathbb{K}, \Pi, \Delta$ are NSs on $\Sigma \times \Sigma \times [0, +\infty)$. Suppose $\psi: \Sigma \times \Sigma \to [1, +\infty)$ be a function, $*$ and $\circ$ are CTN and CTCN respectively. Then, a six tuple $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is known as NMS, if the following conditions are satisfying, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w > 0$,

(N1) $\mathbb{K}(\varkappa, \vartheta, \sigma) + \Pi(\varkappa, \vartheta, \sigma) + \Delta(\varkappa, \vartheta, \sigma) \leq 3$;

(N2) $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

(N3) $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

(N4) $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

(N5) $\mathbb{K}(\varkappa, z, \sigma + \tau) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, z, \tau)$;

(N6) $\mathbb{K}(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

(N7) $\Pi(\varkappa, \vartheta, 0) = 1$;

(N8) $\Pi(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(N9) $\Pi(\varkappa, \vartheta, \sigma) = \Pi(\vartheta, \varkappa, \sigma)$;

(N10) $\Pi(\varkappa, z, \sigma + \tau) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, z, \tau)$;

(N11) $\Pi(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$.

(N12) $\Delta(\varkappa, \vartheta, 0) = 1$;

(N13) $\Delta(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(N14) $\Delta(\varkappa, \vartheta, \sigma) = \Delta(\vartheta, \varkappa, \sigma)$;

(N15) $\Delta(\varkappa, z, \sigma + \tau) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, z, \tau)$;

(N16) $\Delta(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$.

Then $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is called an NMS.

3. Main Section

In this section, we introduce the concept of NREBMS and establish some fixed point results.

Definition 3.1: Let Σ be a non-empty set and $\mathbb{K}, \Pi, \Delta$ are NSs on $\Sigma \times \Sigma \times [0, +\infty)$. Suppose $\psi: \Sigma \times \Sigma \to [1, +\infty)$ be a function, $*$ and $\circ$ are CTN and CTCN respectively. Then, a six tuple $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is known as NREBMS, if the following conditions are satisfying, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w > 0$,

(NRE1) $\mathbb{K}(\varkappa, \vartheta, \sigma) + \Pi(\varkappa, \vartheta, \sigma) + \Delta(\varkappa, \vartheta, \sigma) \leq 3$;

(NRE2) $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

(NRE3) $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

(NRE4) $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

(NRE5) $\mathbb{K}(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathfrak{u}, \tau) * \mathbb{K}(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E} \backslash \{\varkappa, z\}$;

(NRE6) $\mathbb{K}(\varkappa, \vartheta, .): (0,+\infty) \rightarrow [0,1]$ is continuous and $\lim_{\sigma \rightarrow +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

(NRE7) $\Pi(\varkappa, \vartheta, 0) = 1$;

(NRE8) $\Pi(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(NRE9) $\Pi(\varkappa, \vartheta, \sigma) = \Pi(\vartheta, \varkappa, \sigma)$;

(NRE10) $\Pi(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathfrak{u}, \tau) \circ \Pi(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E} \backslash \{\varkappa, z\}$;

(NRE11) $\Pi(\varkappa, \vartheta, .): (0,+\infty) \rightarrow [0,1]$ is continuous and $\lim_{\sigma \rightarrow +\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$.

(NRE12) $\Delta(\varkappa, \vartheta, 0) = 1$;

(NRE13) $\Delta(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(NRE14) $\Delta(\varkappa, \vartheta, \sigma) = \Delta(\vartheta, \varkappa, \sigma)$;

(NRE15) $\Delta(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathfrak{u}, \tau) \circ \Delta(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E} \backslash \{\varkappa, z\}$;

(NRE16) $\Delta(\varkappa, \vartheta, .): (0,+\infty) \rightarrow [0,1]$ is continuous and $\lim_{\sigma \rightarrow +\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$.

Example 3.1: Let $(\mathcal{E}, d)$ be a rectangular metric space, define $\psi: \mathcal{E} \times \mathcal{E} \rightarrow [1,+\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$ and define $\mathbb{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0,+\infty) \rightarrow [0,1]$ by

$$\mathbb{K}(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + d(\varkappa, \vartheta)},$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)} \text{ and } \Delta(\varkappa, \vartheta, \sigma) = \frac{d(\varkappa, \vartheta)}{\sigma} \text{ for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0,$$

with CTN $\hbar * \ell = \min\{\hbar, \ell\}$ and CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$. Then $(\mathcal{E}, \mathbb{K}, \Pi, \Delta, *, \circ)$ is an NREBMS.

Proof: Properties (NRE1)-(NRE4), (NRE6)-(NRE9), (NRE11)-(NRE14) and (NRE16) are easy obvious. Here, we prove (NRE5), (NRE10) and (NRE15).

(NRE5) $\mathbb{K}(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathfrak{u}, \tau) * \mathbb{K}(\mathfrak{u}, z, w)$ for all distinct $\vartheta, \mathfrak{u} \in \mathcal{E} \backslash \{\varkappa, z\}$.

Suppose that

$$\mathbb{K}(\varkappa, \vartheta, \sigma) \leq \mathbb{K}(\vartheta, \mathfrak{u}, \tau)$$

and

$$\mathbb{K}(\varkappa, \vartheta, \sigma) \leq \mathbb{K}(\mathfrak{u}, z, w),$$

which implies that

$$\frac{\sigma}{\sigma + d(\varkappa, \vartheta)} \leq \frac{\tau}{\tau + d(\vartheta, \mathfrak{u})}$$

and

$$\frac{\sigma}{\sigma + d(\varkappa, \vartheta)} \leq \frac{w}{w + d(\mathfrak{u}, z)}.$$

So, we obtain

$$\sigma d(\vartheta, \mathrm{u}) \leq \tau d(\varkappa, \vartheta) \text{ and } \sigma d(\mathrm{u}, z) \leq w d(\varkappa, \vartheta).$$

This implies

$$(\tau + w) d(\varkappa, \vartheta) \geq \sigma[d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)] \qquad (1)$$

Now, observe that

$$\mathrm{K}\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \geq \mathrm{K}(\varkappa, \vartheta, \sigma)$$

$$\Leftrightarrow \frac{\psi(\varkappa, z)(\sigma + \tau + w)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$

$$\Leftrightarrow \frac{\psi(\varkappa, z)(\sigma + \tau + w)}{\psi(\varkappa, z)(\sigma + \tau + w) + \psi(\varkappa, z)[d(\varkappa, \vartheta) + d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)]} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$

$$\Leftrightarrow \frac{\sigma + \tau + w}{\sigma + \tau + w + d(\varkappa, \vartheta) + d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$

$$\Leftrightarrow (\tau + w) d(\varkappa, \vartheta) \geq \sigma[d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)].$$

Hence,

$$\mathrm{K}\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \geq \mathrm{K}(\varkappa, \vartheta, \sigma) * \mathrm{K}(\vartheta, \mathrm{u}, \tau) * \mathrm{K}(\mathrm{u}, z, w).$$

(NRE10) $\Pi\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathrm{u}, \tau) \circ \Pi(\mathrm{u}, z, w)$ for all distinct $\vartheta, \mathrm{u} \in \Sigma\backslash \{\varkappa, z\}$.

Recall that

$$d(\varkappa, z) = d(\varkappa, z) \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Therefore,

$$d(\varkappa, z) \leq [\sigma + \tau + w + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$. Then

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Also observe the fact that

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{\tau + d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{w + d(\mathrm{u}, z)}\right\}.$$

This implies

$$\frac{d(\varkappa, z)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \leq \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{\tau + d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{w + d(\mathrm{u}, z)}\right\}.$$

Then

$$\Pi\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \max\{\Pi(\varkappa, \vartheta, \sigma), \Pi(\vartheta, \mathrm{u}, \tau), \Pi(\mathrm{u}, z, w)\}.$$

Hence,

$$\Pi\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathrm{u}, \tau) \circ \Pi(\mathrm{u}, z, w).$$

(NRE15) $\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathrm{u}, \tau) \circ \Delta(\mathrm{u}, z, w)$ for all distinct $\vartheta, \mathrm{u} \in \Sigma\backslash\{\varkappa, z\}$.

Observe that,

$$d(\varkappa, z) \leq [\sigma + \tau + w + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

Define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$. Then

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

This implies

$$\frac{d(\varkappa, z)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \leq \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

Then

$$\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \max\{\Delta(\varkappa, \vartheta, \sigma), \Delta(\vartheta, \mathrm{u}, \tau), \Delta(\mathrm{u}, z, w)\}.$$

Hence

$$\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathrm{u}, \tau) \circ \Delta(\mathrm{u}, z, w).$$

Therefore, $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a NREBMS.

Remark 3.1: The above example is not a NMS. But, if we let $\psi = 1$, then it is NMS.

Example 3.2: Let $\Sigma = [0,1]$ and define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa^2 + \vartheta^3$ and $\mathrm{K}, \Pi, \Delta: \Sigma \times \Sigma \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ \dfrac{\sigma}{\sigma + \max\{\varkappa, \vartheta\}^p}, & \text{otherwise} \end{cases}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{\max\{\varkappa, \vartheta\}^p}{\sigma + \max\{\varkappa, \vartheta\}^p}, & \text{otherwise} \end{cases}$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{\max\{\varkappa, \vartheta\}^p}{\sigma}, & \text{otherwise} \end{cases} \quad \text{for all } \varkappa, \vartheta \in \Sigma \text{ and } \sigma > 0.$$

Then $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Example 3.3: Let $\Sigma = [0, +\infty)$ and define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \frac{\varkappa}{\vartheta}$ and $\mathrm{K}, \Pi, \Delta: \Sigma \times \Sigma \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ \dfrac{\sigma}{\sigma + (\varkappa + \vartheta)^p}, & \text{otherwise} \end{cases}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{(\varkappa + \vartheta)^p}{\sigma + (\varkappa + \vartheta)^p}, & \text{otherwise} \end{cases}$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{(\varkappa + \vartheta)^p}{\sigma}, & \text{otherwise} \end{cases} \quad \text{for all } \varkappa, \vartheta \in \Sigma \text{ and } \sigma > 0.$$

Then $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Example 3.4: Let $\Sigma = [0, +\infty)$ and define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and $K, \Pi, \Delta: \Sigma \times \Sigma \times [0, +\infty) \to [0,1]$ by

$$K(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \Sigma \text{ and } \sigma > 0.$$

Then $(\Sigma, K, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Remark 3.2: The above examples 3.3 and 3.4 are also NREBMSs if we take $\hbar * \ell = \min\{\hbar, \ell\}$, and $\hbar \circ \ell = \max\{\hbar, \ell\}$.

Definition 3.2: Suppose $(\Sigma, K, \Pi, \Delta, *, \circ)$ is a NREBMS and assume $\{\varkappa_n\}$ be a sequence in Σ. Then

- $\{\varkappa_n\}$ is said to be a convergent sequence if there exists $\varkappa \in \Sigma$ such that

$$\lim_{n\to+\infty} K(\varkappa_n, \varkappa, \sigma) = 1, \text{for all } \sigma > 0$$

$$\lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa, \sigma) = 0, \text{ for all } \sigma > 0.$$

and

$$\lim_{n\to+\infty} \Delta(\varkappa_n, \varkappa, \sigma) = 0, \text{ for all } \sigma > 0.$$

- $\{\varkappa_n\}$ is said to be a Cauchy sequence if

$$\lim_{n\to+\infty} K(\varkappa_n, \varkappa_{n+q}, \sigma) = 1 \text{ for all } \sigma > 0$$

$$\lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa_{n+q}, \sigma) = 0 \text{ for all } \sigma > 0.$$

and

$$\lim_{n\to+\infty} \Delta(\varkappa_n, \varkappa_{n+q}, \sigma) = 0 \text{ for all } \sigma > 0.$$

- The NREBMS $(\Sigma, K, \Pi, \Delta, *, \circ)$ is called complete, if every Cauchy sequence is convergent in Σ.

Example 3.5: Let $\Sigma = [0, +\infty)$ and define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and

$K, \Pi, \Delta: \Sigma \times \Sigma \times [0, +\infty) \to [0,1]$ by

$$K(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa,\vartheta,\sigma)=\frac{|\varkappa-\vartheta|^p}{\sigma+|\varkappa-\vartheta|^p}\ \text{ for all } \varkappa,\vartheta\in\Sigma \text{ and } \sigma>0,$$

then $(\Sigma, K, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar\cdot\ell$, CTCN $\hbar\circ\ell=\max\{\hbar,\ell\}$ and $p\geq 1$.

Let $\{\varkappa_n\}=\frac{1}{n}$ for all $n\in\{1,2,3,\cdots\}$ be a sequence in Σ, then $\{\varkappa_n\}$ converges to 0. Now

$$\lim_{n\to+\infty} K(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\sigma}{\sigma+\left(\frac{1}{n}\right)^p}=1,$$

$$\lim_{n\to+\infty} \Pi(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n}\right)^p}{\sigma+\left(\frac{1}{n}\right)^p}=0,$$

and

$$\lim_{n\to+\infty} \Delta(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n}\right)^p}{\sigma}=0.$$

That is, the sequence $\{\varkappa_n\}$ is convergent.

Example 3.6: Consider the preceding example and a sequence $\varkappa_n=\frac{1}{n}$ for all $n\in\{1,2,3,\cdots,\}$. Then for all $q\in\{1,2,3,\cdots\}$, we get

$$\lim_{n\to+\infty} K(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\sigma}{\sigma+\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}=1,$$

$$\lim_{n\to+\infty} \Pi(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}{\sigma+\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}=0,$$

and

$$\lim_{n\to+\infty} \Delta(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}{\sigma}=0.$$

That is, the sequence $\{\varkappa_n\}$ is Cauchy.

Lemma 3.1: Let $\{\varkappa_n\}$ be a Cauchy sequence in a NREBMS $(\Sigma, K, \Pi, \Delta, *, \circ)$ such that $\varkappa_n\neq\varkappa_m$, whenever $n\neq m$ for all $m,n\in\mathbb{N}$. Then $\{\varkappa_n\}$ converges to at most one point in Σ.

Lemma 3.2: Let $\varkappa$ and ϑ be any two points in a NREBMS $(\Sigma, K, \Pi, \Delta, *, \circ)$. If for any $\eta\in(0,1)$, we have

$$K(\varkappa,\vartheta,\eta\sigma)\geq K(\varkappa,\vartheta,\sigma), \Pi(\varkappa,\vartheta,\eta\sigma)\leq \Pi(\varkappa,\vartheta,\sigma) \text{ and } \Delta(\varkappa,\vartheta,\eta\sigma)\leq\Delta(\varkappa,\vartheta,\sigma),$$

then $\varkappa=\vartheta$.

Theorem 3.1: Let $(\Sigma, K, \Pi, \Delta, *, \circ)$ be a complete NREBMS such that

$$\lim_{\sigma\to+\infty} K(\varkappa,\vartheta,\sigma)=1, \lim_{\sigma\to+\infty}\Pi(\varkappa,\vartheta,\sigma)=0, \text{ and } \lim_{\sigma\to+\infty}\Delta(\varkappa,\vartheta,\sigma)=0, \text{ for all } \varkappa,\vartheta\in\Sigma. \quad (2)$$

Let $\xi:\Sigma\to\Sigma$ be a mapping satisfying

$$K(\xi\varkappa,\xi\vartheta,\eta\sigma)\geq K(\varkappa,\vartheta,\sigma), \qquad \Pi(\xi\varkappa,\xi\vartheta,\eta\sigma)\leq\Pi(\varkappa,\vartheta,\sigma),$$
$$\text{and } \Delta(\xi\varkappa,\xi\vartheta,\eta\sigma)\leq\Delta(\varkappa,\vartheta,\sigma) \qquad (3)$$

for all $\varkappa,\vartheta\in\Sigma,\eta\in(0,1)$. Then ξ has a unique fixed point $\mathfrak{u}\in\Sigma$.

Proof: Let $\varkappa_0 \in \Sigma$ be an arbitrary point and let $n \in \mathbb{N}$ then begin an iterative process such that $\varkappa_{n+1} = \xi\varkappa_n$. Continuously, applying an inequality (3), we deduce that

$$\mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma) \geq \mathbb{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right), \Pi(\varkappa_n, \varkappa_{n+1}, \sigma) \leq \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right) \text{ and } \Delta(\varkappa_n, \varkappa_{n+1}, \sigma)$$

$$\leq \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right). \qquad (4)$$

Since, $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS, then for the sequence $\{\varkappa_n\}$, writing $\sigma = \frac{\sigma}{3} + \frac{\sigma}{3} + \frac{\sigma}{3}$ and using the rectangular inequality given in (NRE5), (NRE10) and (NRE15) on $\mathbb{K}(\varkappa_n, \varkappa_{n+p}, \sigma), \Pi(\varkappa_n, \varkappa_{n+p}, \sigma)$ and $\Delta(\varkappa_n, \varkappa_{n+p}, \sigma)$, we have the following cases.

Case 1: If p is odd, then $p = 2m + 1$ where $m \in \{1,2,3, \dots\}$. So, we have

$$\mathbb{K}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathbb{K}\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Using (4) in the above inequalities, we deduce

$$\mathrm{K}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^m\eta^{n+m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right),$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^m\eta^{n+m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^m\eta^{n+m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Case 2: If p is even, then $p = 2m; m \in \{1,2,3,\cdots\}$. So, the we have

$$\mathbb{K}(\varkappa_n, \varkappa_{n+2m}, \sigma) \geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right) * \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \,\mathbb{K}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathbb{K}\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^{m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

Using (4) in the above inequalities, we deduce

$$\mathbb{K}(\varkappa_n,\varkappa_{n+2m},\sigma)\geq \mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\eta^{n+2m-2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)\eta^{n}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^{n}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

$$\Pi(\varkappa_n,\varkappa_{n+2m},\sigma)\leq \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\cdots\circ$$

$$\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m}\eta^{n+2m-2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_0,1,\frac{\sigma}{(3\eta)\eta^{n}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m}, \sigma) \le \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n}\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m-2}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)$$

$$\le \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n}\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_0, 1, \frac{\sigma}{(3\eta)\eta^{n}\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

Therefore, from $\lim_{\sigma\to+\infty} \mathrm{K}(\varkappa, \vartheta, \sigma) = 1$, $\lim_{\sigma\to+\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$ and $\lim_{\sigma\to+\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$, and cases (1),(2), we get

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \varkappa_{n+p}, \sigma) = 1, \lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa_{n+p}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \varkappa_{n+p}, \sigma) = 0.$$

That is, a sequence $\{\varkappa_n\}$ is Cauchy. Therefore, $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS, so there exists $\mathfrak{u} \in \Sigma$, and we have

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma) = 1, \lim_{n\to+\infty} \Pi(\varkappa_n, \mathfrak{u}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \mathfrak{u}, \sigma) = 0, \text{for all } \sigma > 0 \text{ and } q \geq 1.$$

Now, we show the existence of a fixed point $\mathfrak{u}$.

$$\mathrm{K}(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 1 * 1 * 1 = 1, \text{as } n \to +\infty,$$

$$\Pi(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0, \text{as } n \to +\infty$$

and

$$\Delta(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0 \text{ as } n \to +\infty.$$

Uniqueness: Suppose $v \neq \mathfrak{u}$, be another fixed point, then

$$\mathrm{K}(v, \mathfrak{u}, \sigma) = \mathrm{K}(\xi v, \xi\mathfrak{u}, \sigma) \geq \mathrm{K}\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = \mathrm{K}\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\geq \mathrm{K}\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \geq \cdots \geq \mathrm{K}\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 1 \text{ as } n \to +\infty,$$

$$\Pi(v, \mathfrak{u}, \sigma) = \Pi(\xi v, \xi\mathfrak{u}, \sigma) \leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = \Pi\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \leq \cdots \leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 0 \text{ as } n \to +\infty,$$

and

$$\Delta(v, \mathfrak{u}, \sigma) = \Delta(\xi v, \xi\mathfrak{u}, \sigma) \leq \Delta\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = \Delta\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\leq \Delta\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \leq \cdots \leq \Delta\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 0 \text{ as } n \to +\infty,$$

Hence, $\mathfrak{u} = v$.

Definition 3.3: Suppose $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS. A mapping $\xi: \Sigma \to \Sigma$ is known as neutrosophic rectangular contraction, if

$$\frac{1}{\mathbb{K}(\xi\varkappa, \xi\vartheta, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathbb{K}(\varkappa, \vartheta, \sigma)} - 1\right], \qquad \Pi(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Pi(\varkappa, \vartheta, \sigma)$$

$$\text{and } \Delta(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Delta(\varkappa, \vartheta, \sigma) \qquad (5)$$

for all $\varkappa, \vartheta \in \Sigma, \eta \in (0,1)$ and $\sigma > 0$.

Theorem 3.2: Suppose $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS, such that

$$\lim_{\sigma \to +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1, \lim_{\sigma \to +\infty} \Pi(\varkappa, \vartheta, \sigma) = , \text{and } \lim_{\sigma \to +\infty} \Delta(\varkappa, \vartheta, \sigma) = 0 \quad \text{for all } \varkappa, \vartheta \in \Sigma. \qquad (6)$$

Let $\xi: \Sigma \to \Sigma$ be a Neutrosophic rectangular contraction. Then ξ has a unique fixed point $\mathfrak{u} \in \Sigma$.

Proof: Assume $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS, let an arbitrary point $\varkappa_0 \in \Sigma$, and define a sequence $\{\varkappa_n\}$ in Σ by

$$\varkappa_1 = \xi\varkappa_0 \ , \ \varkappa_2 = \xi^2\varkappa_0 = \xi\varkappa_1, \dots, \varkappa_n = \xi^n\varkappa_0 = \xi\varkappa_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $\varkappa_n = \varkappa_{n-1}$ for some $n \in \mathbb{N}$ then $\varkappa_n$ is a fixed point of ξ. We suppose that $\varkappa_n \neq \varkappa_{n-1}$ for all $n \in \mathbb{N}$. For $\sigma > 0$ and $n \in \mathbb{N}$, utilizing (5), we get

$$\frac{1}{\mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} - 1 = \frac{1}{\mathbb{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathbb{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - 1\right].$$

That is,

$$\frac{1}{\mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} \leq \frac{\eta}{\mathbb{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} + (1-\eta), \forall\, \sigma > 0,$$

$$= \frac{\eta}{\mathbb{K}(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)} + (1-\eta) \leq \frac{\eta^2}{\mathbb{K}(\varkappa_{n-2}, \varkappa_{n-1}, \sigma)} + \eta(1-\eta) + (1-\eta).$$

Continuing this way, we get

$$\frac{1}{\mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} \leq \frac{\eta^n}{\mathbb{K}(\varkappa_0, \varkappa_1, \sigma)} + \eta^{n-1}(1-\eta) + \eta^{n-2}(1-\eta) + \cdots + \eta(1-\eta) + (1-\eta)$$

$$\leq \frac{\eta^n}{\mathbb{K}(\varkappa_0, \varkappa_1, \sigma)} + (\eta^{n-1} + \eta^{n-2} + \cdots + 1)(1-\eta)$$

$$\leq \frac{\eta^n}{\mathbb{K}(\varkappa_0, \varkappa_1, \sigma)} + (1-\eta^n).$$

We have,

$$\frac{1}{\frac{\eta^n}{\mathbb{K}(\varkappa_0, \varkappa_1, \sigma)} + (1-\eta^n)} \leq \mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma), \forall\, \sigma > 0, n \in \mathbb{N}. \qquad (7)$$

$$\Pi(\varkappa_n, \varkappa_{n+1}, \sigma) = \Pi(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma) \leq \eta\Pi(\varkappa_{n-1}, \varkappa_n, \sigma) = \eta\Pi(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)$$
$$\leq \eta^2\Pi(\varkappa_{n-2}, \varkappa_{n-1}, \sigma) \leq \cdots \leq \eta^n\Pi(\varkappa_0, \varkappa_1, \sigma) \qquad (8)$$

and

$$\Delta(\varkappa_n, \varkappa_{n+1}, \sigma) = \Delta(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma) \leq \eta\Delta(\varkappa_{n-1}, \varkappa_n, \sigma) = \eta\Delta(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)$$
$$\leq \eta^2\Delta(\varkappa_{n-2}, \varkappa_{n-1}, \sigma) \leq \cdots \leq \eta^n\Delta(\varkappa_0, \varkappa_1, \sigma). \qquad (9)$$

Since $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS for the sequence $\{\varkappa_n\}$, writing $\sigma = \frac{\sigma}{3} + \frac{\sigma}{3} + \frac{\sigma}{3}$ and using the rectangular inequalities given in (N5), (N10) and (N15) on $\mathrm{K}(\varkappa_n, \varkappa_{n+p}, \sigma), \Pi(\varkappa_n, \varkappa_{n+p}, \sigma)$ and $\Delta(\varkappa_n, \varkappa_{n+p}, \sigma)$, in the following cases.

Case 1: If p is odd, then $p = 2m + 1$ where $m \in \{1,2,3, \ldots\}$. So, we have

$$\mathrm{K}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathrm{K}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathrm{K}\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

By using (7), (8) and (9) in the above inequalities, we have

$$\mathbb{K}(\varkappa_n,\varkappa_{n+2m+1},\sigma)$$
$$\geq \frac{1}{\frac{\eta^n}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^n)} * \frac{1}{\frac{\eta^{n+1}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^{n+1})}$$
$$* \frac{1}{\frac{\eta^{n+2}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-\eta^{n+2})}$$
$$* \frac{1}{\frac{\eta^{n+3}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-\eta^{n+3})}$$
$$*\cdots*$$
$$\frac{1}{\frac{\eta^{n+2m}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)}+(1-\eta^{n+2m})},$$
$$\geq \frac{1}{\frac{\eta^n}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^n)} * \frac{1}{\frac{(\eta)\eta^{n}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-(\eta)\eta^{n})}$$
$$* \frac{1}{\frac{(\eta)^2\eta^{n}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^{n})}$$
$$* \frac{1}{\frac{(\eta)^2\eta^{n+1}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^{n+1})}$$
$$*\cdots*$$
$$\frac{1}{\frac{(\eta)^m\eta^{n+m}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)}+(1-(\eta)^m\eta^{n+m})},$$

$$\Pi(\varkappa_n,\varkappa_{n+2m+1},\sigma) \leq \eta^n\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right) \circ \eta^{n+1}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)$$
$$\circ\, \eta^{n+2}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$
$$\circ\, \eta^{n+3}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$
$$\circ\, \eta^{n+4}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$
$$\circ\, \eta^{n+5}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$
$$\circ\cdots\circ$$
$$\eta^{n+2m}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)$$

$$\leq \eta^n \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta(\eta^n) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^2(\eta^n) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1}) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1}) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2}) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^m(\eta^{n+m}) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \eta^n \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta^{n+1} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^{n+2} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m} \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\leq \eta^n \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta(\eta^n) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^2(\eta^n) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1}) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1}) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2}) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^m(\eta^{n+m}) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Case 2: If p is even, then $p = 2m; m \in \{1,2,3,\cdots\}$. So, we have

$$\mathrm{K}(\varkappa_n,\varkappa_{n+2m},\sigma)\geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+2m},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)*\mathrm{K}\left(\varkappa_{n+4},\varkappa_{n+2m},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+4},\varkappa_{n+5},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+5},\varkappa_{n+6},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathrm{K}\left(\varkappa_{n+2m-2},\varkappa_{n+2m},\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

$$\Pi(\varkappa_n,\varkappa_{n+2m},\sigma)\leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+2},\varkappa_{n+2m},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+4},\varkappa_{n+2m},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

By using (7A), (8A) and (9A) in the above inequalities, we have

$$\mathrm{K}(\varkappa_n,\varkappa_{n+2m},\sigma) \geq \frac{1}{\frac{\eta^n}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)}+(1-\eta^n)} * \frac{1}{\frac{\eta^{n+1}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)}+(1-\eta^{n+1})}$$

$$* \frac{1}{\frac{\eta^{n+2}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-\eta^{n+2})}$$

$$* \frac{1}{\frac{\eta^{n+3}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-\eta^{n+3})}$$

$$* \cdots *$$

$$\frac{1}{\frac{\eta^{n+2m-2}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)}+(1-\eta^{n+2m-2})}$$

$$\geq \frac{1}{\frac{\eta^n}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)}+(1-\eta^n)} * \frac{1}{\frac{(\eta)\eta^n}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)}+(1-(\eta)\eta^n)}$$

$$* \frac{1}{\frac{(\eta)^2\eta^n}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^n)}$$

$$* \frac{1}{\frac{(\eta)^2\eta^{n+1}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^{n+1})}$$

$$* \cdots *$$

$$\frac{1}{\frac{(\eta)^{m-1}\eta^{n+m-1}}{\mathrm{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)}+(1-(\eta)^{m-1}\eta^{n+m-1})}$$

$$\Pi(\varkappa_n,\varkappa_{n+2m},\sigma) \leq \eta^n\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \eta^{n+1}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \eta^{n+2}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m-2}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\leq \eta^n\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ (\eta)\eta^n\Pi\left(\varkappa_0,1,\frac{\sigma}{(3)\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \eta^2(\eta^n)\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$(\eta)^{m-1}\eta^{n+m-1}\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^{m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m}, \sigma) \le \eta^n\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \eta^{n+1}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \eta^{n+2}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m-2}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)$$

$$\le \eta^n\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ (\eta)\eta^n\Delta\left(\varkappa_0, 1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \eta^2(\eta^n)\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$(\eta)^{m-1}\eta^{n+m-1}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^{m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

Therefore, from $\lim_{\sigma\to+\infty} \text{Ҡ}(\varkappa, \vartheta, \sigma) = 1$, $\lim_{\sigma\to+\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$ and $\lim_{\sigma\to+\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$, and Cases (1), (2), we get

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \varkappa_{n+p}, \sigma) = 1, \lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa_{n+p}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \varkappa_{n+p}, \sigma) = 0.$$

Hence $\{\varkappa_n\}$ is a Cauchy sequence. Since, $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS, so there exists $\mathfrak{u} \in \mathcal{E}$ such that

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma) = 1 \quad \lim_{n\to+\infty} \Pi(\varkappa_n, \mathfrak{u}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \mathfrak{u}, \sigma) = 0, \qquad \text{for all } \sigma > 0 \text{ and } q \geq 1.$$

Now, we show the existence of a fixed point $\mathfrak{u}$. Utilizing (5), we have

$$\frac{1}{\mathrm{K}(\xi\varkappa_n, \xi\mathfrak{u}, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} - 1\right] = \frac{\eta}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} - \eta,$$

$$\frac{1}{\frac{\eta}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} + 1 - \eta} \leq \mathrm{K}(\xi\varkappa_n, \xi\mathfrak{u}, \sigma),$$

and

$$\frac{1}{\mathrm{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - 1\right] = \frac{\eta}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - \eta,$$

$$\frac{1}{\frac{\eta}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} + 1 - \eta} \leq \mathrm{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma).$$

Using the above inequalities, we deduce

$$\mathrm{K}(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \frac{1}{\frac{\eta}{\mathrm{K}\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)} + 1 - \eta} * \frac{1}{\frac{\eta}{\mathrm{K}\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)} + 1 - \eta}$$

$$\to 1 * 1 * 1 = 1 \text{ as } n \to +\infty$$

$$\Pi(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Pi\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Pi\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0 \text{ as } n \to +\infty$$

and

$$\Delta(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Delta\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Delta\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\rightarrow 0 \circ 0 \circ 0 = 0 \text{ as } n \rightarrow +\infty.$$

That is, $\xi\mathfrak{u} = \mathfrak{u}$.

Uniqueness: Suppose $v \neq \mathfrak{u}$ be another fixed point of ξ, such that $\mathrm{K}(\mathfrak{u}, v, \mathrm{t}) < 1$ for some $\sigma > 0$, and utilizing (5), we have

$$\frac{1}{\mathrm{K}(\mathfrak{u}, v, \sigma)} - 1 = \frac{1}{\mathrm{K}(\xi\mathfrak{u}, \xi v, \sigma)} - 1$$

$$\leq \eta \left[\frac{1}{\mathrm{K}(\mathfrak{u}, v, \sigma)} - 1\right] < \frac{1}{\mathrm{K}(\mathfrak{u}, v, \sigma)} - 1$$

a contradiction,

$$\Pi(\mathfrak{u}, v, \sigma) = \Pi(\xi\mathfrak{u}, \xi v, \sigma) \leq \eta\Pi(\mathfrak{u}, v, \sigma)$$

and

$$\Delta(\mathfrak{u}, v, \sigma) = \Delta(\xi\mathfrak{u}, \xi v, \sigma) \leq \eta\Delta(\mathfrak{u}, v, \sigma)$$

a contradiction. Therefore, we must have $\mathrm{K}(\mathfrak{u}, v, \sigma) = 1, \Pi(\mathfrak{u}, v, \sigma) = 0$ and $\Delta(\mathfrak{u}, v, \sigma) = 0$ for all $\sigma > 0$, and hence, $\mathfrak{u} = v$.

Example 3.8: Let $\mathcal{E} = [0,1]$. Define $\psi: \mathcal{E} \times \mathcal{E} \rightarrow [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and define $\mathrm{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \rightarrow [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0.$$

Defined by $\hbar * \ell = \hbar \cdot \ell, \hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$, then $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS.
Define $\xi: \mathcal{E} \rightarrow \mathcal{E}$ by $\xi(\varkappa) = \sqrt[p]{\eta}\varkappa$. Then

$$\mathrm{K}(\xi\varkappa, \xi\vartheta, \eta\sigma) = \mathrm{K}\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\eta\sigma}{\eta\sigma + \left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}$$

$$= \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p} = \mathrm{K}(\varkappa, \vartheta, \sigma)$$

$$\Pi(\xi\varkappa, \xi\vartheta, \eta\sigma) = \Pi\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}{\eta\sigma + \left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}$$

$$= \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p} = \Pi(\varkappa, \vartheta, \sigma), \text{ and}$$

$$\Delta(\xi\varkappa, \xi\vartheta, \eta\sigma) = \Delta\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}{\eta\sigma}$$

$$= \frac{|\varkappa - \vartheta|^p}{\sigma} = \Delta(\varkappa, \vartheta, \sigma).$$

Also, contraction conditions of Theorem 3.2,

$\frac{1}{\mathrm{K}(\xi\varkappa, \xi\vartheta, \sigma)} - 1 \leq \eta \left[\frac{1}{\mathrm{K}(\varkappa, \vartheta, \sigma)} - 1\right], \Pi(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Pi(\varkappa, \vartheta, \sigma)$ and $\Delta(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Delta(\varkappa, \vartheta, \sigma)$ are satisfied.

Consequently, all of the assumptions of Theorems 3.1 and 3.2 are satisfied, and 0 is a unique fixed point.

4. Application to Nonlinear Fractional Differential Equation

Theorem 3.1 is used in this section to determine a solution's existence and uniqueness in nonlinear fractional differential equation (see [19]) given by

$$D_c^{\alpha}\varkappa(\varrho) = \psi(\varrho, \varkappa(\varrho)) \qquad (\varrho \in (0,1), \alpha \in (1,2]),$$

with boundary conditions

$$\varkappa(0) = 0, \varkappa'(0) = I\varkappa(\varrho) \qquad \varrho \in (0,1),$$

Where D_c^{α} means caputo fractional derivative of order α, defined by

$$D_c^{\alpha}\psi(\varrho) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{n-\alpha-1}\psi^n(\varpi)d\varpi \qquad (n-1 < \alpha < n,\ \ n = [\alpha]+1),$$

and $\psi: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. We suppose that $\Sigma = C([0,1], \mathbb{R})$, from $[0,1]$ into $\mathbb{R}$ with supremum $|\varkappa| = \operatorname{Sup}_{\theta\in[0,1]} |\varkappa(\varrho)|$.

The Riemann-Liouville fractional integral of order α (see [20]) is given by

$$I^{\alpha}\psi(\varrho) = \frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}\psi(\varpi)d\varpi \qquad (\alpha > 0)$$

We first provide an acceptable form for a nonlinear fractional differential equation before investigating the existence of a solution. Now, we suppose the following fractional differential equation

$$D_c^{\alpha}\varkappa(\varrho) = \psi(\varrho, \varkappa(\varrho)) \qquad (\varrho \in (0,1), \alpha \in (1,2]), \qquad (10)$$

with the boundary conditions

$$\varkappa(0) = 0, \qquad \varkappa'(0) = I\varkappa(\varrho) \qquad (\varrho \in (0,1)),$$

where

i. $\psi: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
ii. $\varkappa(\varrho): [0,1] \to \mathbb{R}$ is continuous,

and satisfying the following condition

$$|\psi(\varrho, \varkappa) - \psi(\varrho, \vartheta)| \leq ЛL|\varkappa - \vartheta|,$$

for all $\varrho \in [0,1]$ and L is a constant with $LЛ < 1$ where

$$Л = \frac{1}{\Gamma(\alpha+1)} + \frac{2\vartheta^{\alpha+1}\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+1)}.$$

Then the equation (10) has a unique solution.

Proof: Suppose that

$$Ꝁ(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p} \text{ , and}$$

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \Sigma \text{ and } \sigma > 0,$$

defined by $\hbar * \ell = \hbar \cdot \ell$, and $\hbar \circ \ell = \max\{\hbar, \ell\}$. Let $|\varkappa - \vartheta| = \sup_{\varrho \in [0,1]} |\varkappa(\varrho) - \vartheta(\varrho)|$, for all $\varkappa, \vartheta \in \Sigma$. Then $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS. We define a mapping $\xi: \Sigma \to \Sigma$ by

$$\xi\varkappa(\varrho) = \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \varpi)^{\alpha-1} \psi\big(\varpi, \varkappa(\varpi)\big) d\varpi$$

$$+ \frac{2\varrho}{(2 - \vartheta^2)\Gamma(\alpha)} \int_0^{\vartheta} \left(\int_0^{\varpi} (\varpi - m)^{\alpha-1} \psi\big(m, \varkappa(m)\big) dm \right) d\varpi \qquad (11)$$

for all $\varrho \in [0,1]$. Equation (10) has a solution $\varkappa \in \Sigma$ iff $\varkappa(\varrho) = \xi\varkappa(\varrho)$ for all $\varrho \in [0,1]$. Now

$$\left.\begin{aligned} \mathrm{K}(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{\sigma}{\sigma + |\varkappa(\varrho) - \vartheta(\varrho)|^p} \\ \Pi(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{|\varkappa(\varrho) - \vartheta(\varrho)|^p}{\sigma + |\varkappa(\varrho) - \vartheta(\varrho)|^p} \\ \Delta(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{|\varkappa(\varrho) - \vartheta(\varrho)|^p}{\sigma} \end{aligned}\right\} \qquad (12)$$

$$|\xi\varkappa(\varrho) - \xi\vartheta(\varrho)| = \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \varpi)^{\alpha-1} \psi\big(\varpi, \varkappa(\varpi)\big) d\varpi$$

$$+ \frac{2\varrho}{(2 - \vartheta^2)\Gamma(\alpha)} \int_0^{\vartheta} \left(\int_0^{\varpi} (\varpi - m)^{\alpha-1} \psi\big(m, \varkappa(m)\big) dm \right) d\varpi| \Bigg|$$

$$- \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \varpi)^{\alpha-1} \psi\big(\varpi, \vartheta(\varpi)\big) d\varpi$$

$$+ \frac{2\varrho}{(2 - \vartheta^2)\Gamma(\alpha)} \int_0^{\vartheta} \left(\int_0^{\varpi} (\varpi - m)^{\alpha-1} \psi\big(m, \vartheta(m)\big) dm \right) d\varpi \Bigg|.$$

That is,

$$|\xi\varkappa(\varrho) - \xi\vartheta(\varrho)| = \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \varpi)^{\alpha-1} \psi\big(\varpi, \varkappa(\varpi)\big) d\varpi$$

$$+ \frac{2\varrho}{(2 - \vartheta^2)\Gamma(\alpha)} \int_0^{\vartheta} \left(\int_0^{\varpi} (\varpi - m)^{\alpha-1} \psi\big(m, \varkappa(m)\big) dm \right) d\varpi$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \varpi)^{\alpha-1} \psi\big(\varpi, \vartheta(\varpi)\big) d\varpi$$

$$- \frac{2\varrho}{(2 - \vartheta^2)\Gamma(\alpha)} \int_0^{\vartheta} \left(\int_0^{\varpi} (\varpi - m)^{\alpha-1} \psi\big(m, \vartheta(m)\big) dm \right) d\varpi \Bigg|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho-\varpi)^{\alpha-1}\left|\psi\big(\varpi,\varkappa(\varpi)\big)-\psi\big(\varpi,\vartheta(\varpi)\big)\right|d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi-m)^{\alpha-1}\left|\psi\big(m,\varkappa(m)\big)-\psi\big(m,\vartheta(m)\big)\right|dm\right)d\varpi$$

$$\leq \frac{L|\varkappa-\vartheta|}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho-\varpi)^{\alpha-1}d\varpi+\frac{2L|\varkappa-\vartheta|}{\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi-m)^{\alpha-1}dm\right)d\varpi$$

$$\leq \frac{L|\varkappa-\vartheta|}{\Gamma(\alpha+1)}+\frac{2\vartheta^{\alpha+1}L|\varkappa-\vartheta|\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+2)}$$

$$\leq L|\varkappa-\vartheta|\left(\frac{1}{\Gamma(\alpha+1)}+\frac{2\vartheta^{\alpha+1}\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+2)}\right)=L\text{Л}|\varkappa-\vartheta|.$$

Utilizing $L\text{Л}<1$ and (12), we have

$$\text{Ƙ}(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{\eta\sigma}{\eta\sigma+|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}\geq\frac{\eta\sigma}{\eta\sigma+L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}$$

$$\geq\frac{\sigma}{\sigma+|\varkappa(\varrho)-\vartheta(\varrho)|^p}=\text{Ƙ}(\varkappa(\varrho),\vartheta(\varrho),\sigma)$$

$$\Pi(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}{\eta\sigma+|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}\leq\frac{L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\eta\sigma+L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}$$

$\leq\frac{|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\sigma+|\varkappa(\varrho)-\vartheta(\varrho)|^p}=\Pi(\varkappa(\varrho),\vartheta(\varrho),\sigma)$ and

$$\Delta(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}{\eta\sigma}\leq\frac{L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\eta\sigma}$$

$$\leq\frac{|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\sigma}=\Delta(\varkappa(\varrho),\vartheta(\varrho),\sigma).$$

As a result, the conditions of Theorem 3.1 are all met. This shows that ξ has unique solution.

5. Conclusion

In this manuscript, we introduced the notion of NREBMS and provided some non-trivial examples of defined space. Several fixed point results for contraction mappings are established with examples. Also, we provided an application to non-linear fractional differential equations to support the validity of main result. This is extendable in several more generalized spaces including neutrosophic rectangular controlled metric spaces, graphical neutrosophic metric spaces, neutrosophic rectangular double controlled metric-like spaces and Hausdorff neutrosophic rectangular metric spaces. Also, this work is extendable by increasing the number of self-mappings.

6. Open Problem

How to prove Theorem 3.1 and Theorem 3.2 (proved in this paper) in the context of graphical neutrosophic extended b-metric spaces?

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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