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A New Approach for the Statistical Convergence over Non-Archimedean Fields in Neutrosophic Normed Spaces

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Abstract: The goal of the research involves elaborating on the topics of statistical convergence, including statistical Cauchy sequences within non-Archimedean Neutrosophic normed spaces, as well as achieving specific useful conclusions. The present research shows how, within a non-Archimedean field, certain sections of statistically convergent sequences that could not be true often become true. Likewise, we created statistically complete and statistically continuous spaces for such regions that demonstrated certain essential facts. κ indicates a complete field of non-Archimedean and non-trivially valued research.

Keywords: Neutrosophic Normed Spaces; Non-Archimedean Fields; Statistically Cauchy Sequence; Statistically Convergent.

1. Introduction

Zadeh [16] became the initial one person who creates the fuzzy set using a membership function. Many later researchers were adapted this idea to classical set theory. Atanassov [1] introduced an Intuitionistic Fuzzy (IF) set theory. Saadati along with Park proposed the notion of IF normed space. The study of analysis through fields of Non-Archimedean (NA) is referred to as NA analysis. Suja and Srinivasan [15] newly created statistically convergent along with statistically Cauchy sequences within NA fields. Eghbali and Ganji [3] investigated NAL-fuzzy normed spaces for extended statistical convergence. The research shows that statistical convergence exists in Non-Archimedean Neutrosophic Normed Spaces (NA-NNS) and confirms that key properties of statistical convergence from real sequences are still valid in NA fields [2,4-5,8-14]. The research article concentrates primarily upon the analysis of sequences in the field of NA κ.

In 1998, Smarandache [12] developed the ideas of neutrosophic logic in addition to the Neutrosophic Set [NS]. Kirisci and Simsek establish the Neutrosophic Metric Space [NMS] suggestion which is associated with membership, non-membership and neutralness. Jeyaraman, Ramachandran and Shakila [7] established approximate fixed point theorems in 2022 regarding weak contractions on Neutrosophic Normed Spaces (NNS). Statistical Δ^m convergence in NNS was recently presented by Jeyaraman and Jenifer [6].

A sequence $\bar{\bar{x}}= \{v_p\}$ is said to have been statistically convergent towards a limit $\mathfrak{L}$ when for any $\tilde{\omega}>0$, $\lim_{n\to\infty} \frac{1}{n}\{p \le n : |v_p - \mathfrak{L}| \ge \tilde{\omega}\} = 0$.

In that case above, we put $stat - \lim_{p\to\infty} v_p = \mathfrak{L}$.

Example 1.1. Consider to define the $\bar{\bar{x}}= \{v_p\}$ sequence by

$$v_p = \begin{cases} \frac{p-1}{p^2}, & p \text{ is a perfect square.} \\ 0, & \text{otherwise;} \end{cases}$$

Selecting the NA valuation to be 2-adic, the sequence terms become (0,0,0,1,0,0,0,0,1/8,0,0,....).
As a result, it converges to zero statistically.
A sequence of Statistically Cauchy (SC) when for all $\tilde{\omega}>0$, then existing a range $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\{i \le n : n \in \mathbb{N} : |\bar{\bar{x}}_{i+1} - \bar{\bar{x}}_i| \ge \tilde{\omega}\} = 0$$

Consider that κ to be NA fields. A valuation on κ is referred with the NA if it meets these three given axioms: [1]

(i) $|\bar{\bar{x}}| \ge 0$ and $|\bar{\bar{x}}| = 0$ iff $\bar{\bar{x}}= 0$,
(ii) $|\bar{\bar{x}}\breve{y}| = |\bar{\bar{x}}||\breve{y}|$,
(iii) $|\bar{\bar{x}} + \breve{y}| \le \max[|\bar{\bar{x}}|, |\breve{y}|]$ for every $\bar{\bar{x}}, \breve{y} \in \kappa$ (Ultrametric Inequality).

2. Preliminaries

Here, we will go through the notations along with definitions which will be utilized throughout this article in order to ensure a general understanding of the terminology and symbols used.

Definition 2.1. The 7-tuple $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ is said to be a *NA-NNS*, if $*$ acts as a continuous t-norm, $\diamond$ and $\star$ acts as a t-co norms which are continuous, Ξ become a vector space over a field κ and then $\tilde{\varsigma}, \dot{\varphi}, \psi$ are fuzzy sets functions on $\Xi \times \mathbb{R}$ to [0, 1], for all $v, h \in \Xi$ and $\acute{f}$, $\grave{t} \in$ κ.

(cn1) $\tilde{\varsigma}(v, \acute{f}) + \dot{\varphi}(v, \acute{f}) + \psi(v, \acute{f}) \le 3$
(cn2) $0 \le \tilde{\varsigma}(v, \acute{f}) \le 1; 0 \le \dot{\varphi}(v, \acute{f}) \le 1$ *and* $0 \le \psi(v, \acute{f}) \le 1$;
(cn3) $\tilde{\varsigma}(v, \acute{f}) > 0$;
(cn4) $\tilde{\varsigma}(v, \acute{f}) = 1 \Leftrightarrow v = 0$,
(cn5) $\tilde{\varsigma}(\ddot{\gamma}v, \acute{f}) = \tilde{\varsigma}\left(v, \frac{\acute{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \ne 0$;
(cn6) $\tilde{\varsigma}(v + h, \max\{\acute{f} + \grave{t}\}) \ge \tilde{\varsigma}(v, \acute{f}) * \tilde{\varsigma}(h, \grave{t})$,
(cn7) $\tilde{\varsigma}(v, .)\colon (0, \infty) \to [0,1]$ and it is continuous,
(cn8) $\lim_{\acute{f}\to\infty} \tilde{\varsigma}(v, \acute{f}) = 1$ and $\lim_{\acute{f}\to\infty} \tilde{\varsigma}(v, \acute{f}) = 0$;
(cn9) $\dot{\varphi}(v, \acute{f}) < 1$;
(cn10) $\dot{\varphi}(v, \acute{f}) = 0 \Leftrightarrow v = 0$,
(cn11) $\dot{\varphi}(\ddot{\gamma}v, \acute{f}) = \dot{\varphi}\left(v, \frac{\acute{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \ne 0$;

(cn12) $\dot{\varphi}(\mathcal{v} + \hbar, \max\{\not{f} + \mathfrak{t}\}) \leq \dot{\varphi}(\mathcal{v}, \not{f}) \diamond \dot{\varphi}(\hbar, \mathfrak{t})$,
(cn13) $\dot{\varphi}(\mathcal{v}, .): (0, \infty) \to [0,1]$ and it is continuous;
(cn14) $\lim_{\not{f}\to\infty} \dot{\varphi}(\mathcal{v}, \not{f}) = 0$ and $\lim_{\not{f}\to\infty} \dot{\varphi}(\mathcal{v}, \not{f}) = 1$;
(cn15) $\psi(\mathcal{v}, \not{f}) < 1$,
(cn16) $\psi(\mathcal{v}, \not{f}) = 0 \Leftrightarrow \mathcal{v} = 0$,
(cn17) $\psi(\ddot{\gamma}\mathcal{v}, \not{f}) = \psi\left(\mathcal{v}, \frac{\not{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \neq 0$,
(cn18) $\psi(\mathcal{v} + \hbar, \max\{\not{f} + \mathfrak{t}\}) \leq \psi(\mathcal{v}, \not{f}) \star \psi(\hbar, \mathfrak{t})$,
(cn19) $\psi(\mathcal{v}, .): (0, \infty) \to [0,1]$ is continuous and
(cn20) $\lim_{\not{f}\to\infty} \psi(\mathcal{v}, \not{f}) = 0$ and $\lim_{\not{f}\to\infty} \psi(\mathcal{v}, \not{f}) = 1$.

Here, $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ is known as a *NA-NNS.*

A sequence $\{\mathcal{v}_{\wp}\}$ is referred to be convergent in *NA-NNS* $(\mathfrak{V}, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ or simply $(\tilde{\varsigma}, \dot{\varphi}, \psi)$-convergent to $\bar{\bar{\mathfrak{x}}} \in \Xi$ if for all $\not{f}>0$ and $\widetilde{\omega}>0$, then there exist $\wp_0 \in \mathbb{N}$ so that $\wp \geq \wp_0$,

$$\tilde{\varsigma}(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) > 1 - \widetilde{\omega}, \dot{\varphi}(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) < \widetilde{\omega} \text{ and } \psi(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) < \widetilde{\omega}$$

In this case, we write $(\tilde{\varsigma}, \dot{\varphi}, \psi) - \lim \mathcal{v}_{\wp} = \bar{\bar{\mathfrak{x}}}$.

Example 2.2 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA* normed space, $\mathcal{v} * \hbar = \mathcal{v}\hbar$, $\mathcal{v} \diamond \hbar = \min\{\mathcal{v} + \hbar, 1\}$ and $\mathcal{v} \star \hbar = \min\{\mathcal{v} + \hbar, 1\}$ for all $\mathcal{v}, \hbar \in [0,1]$. For every $\bar{\bar{\mathfrak{x}}} \in \Xi$, every $\hat{\xi}>0$ and $\wp = 1, 2, \ldots$. Consider the following form,

$$\tilde{\varsigma}_{\wp}(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \begin{cases} \dfrac{\hat{\xi}}{\hat{\xi} + \wp\|\bar{\bar{\mathfrak{x}}}\|}, & if \quad \hat{\xi} > 0 \\ 0, & \quad \hat{\xi} \leq 0; \end{cases}$$

$$\dot{\varphi}_{\wp}(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \begin{cases} \dfrac{\wp\|\bar{\bar{\mathfrak{x}}}\|}{\hat{\xi} + \wp\|\bar{\bar{\mathfrak{x}}}\|}, & if \quad \hat{\xi} > 0 \\ 0, & \quad \hat{\xi} \leq 0; \end{cases}$$

$$\psi_{\wp}(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \begin{cases} \dfrac{\wp\|\bar{\bar{\mathfrak{x}}}\|}{\hat{\xi}}, & if \quad \hat{\xi} > 0 \\ 0, & \quad \hat{\xi} \leq 0; \end{cases}$$

Then $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ which is a *NA-NNS*.

Definition 2.3 A $\{\mathcal{v}_{\wp}\}$ sequence in a *NA-NNS*$(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ is said to be a statistically convergent towards a limit $\bar{\bar{\mathfrak{x}}} \in \Xi$ relate with the *NA* fuzzy norm $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ when for each $\widetilde{\omega}>0$ and $\not{f}>0$,

$$\lim_{n} \frac{1}{n} \left|\{\wp \leq n : \tilde{\varsigma}(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) \leq 1 - \widetilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) \geq \widetilde{\omega} \text{ and } \psi(\mathcal{v}_{\wp} - \bar{\bar{\mathfrak{x}}}, \not{f}) \geq \widetilde{\omega}\}\right| = 0.$$

In this case, we write $stat_{\tilde{\varsigma}, \dot{\varphi}, \psi} - \lim_{\wp} \mathcal{v}_{\wp} = \bar{\bar{\mathfrak{x}}}$ where $\bar{\bar{\mathfrak{x}}}$ is the $stat_{\tilde{\varsigma}, \dot{\varphi}, \psi}$*− limit*.

Example 2.4 Let $(Q_P, |.|)$ indicate the p-adic numbers space in the standard norm, and consider $\mathcal{v} * \hbar = \mathcal{v}\hbar$, $\mathcal{v} \diamond \hbar = \min\{\mathcal{v} + \hbar, 1\}$ and $\mathcal{v} \star \hbar = \min\{\mathcal{v} + \hbar, 1\}$ for every $\mathcal{v}, \hbar \in [0, 1]$. For every $\bar{\bar{\mathfrak{x}}} \in Q_P$ and all $\hat{\xi}>0$, let $\tilde{\varsigma}_0(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \frac{\hat{\xi}}{\hat{\xi} + |\bar{\bar{\mathfrak{x}}}|}$, $\dot{\varphi}_0(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \frac{|\bar{\bar{\mathfrak{x}}}|}{\hat{\xi} + |\bar{\bar{\mathfrak{x}}}|}$ and $\psi_0(\bar{\bar{\mathfrak{x}}}, \hat{\xi}) = \frac{|\bar{\bar{\mathfrak{x}}}|}{\hat{\xi}}$. In this case observe that $(Q_P, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ is a *NA-NNS*.

Define a sequence $\bar{\bar{x}}=\{v_{p}\}$ the terms of which are provided by

$$v_{p}=\begin{cases}1, & if\ p=m^{2}(m\in\mathbb{N})\\ 0, & otherwise;\end{cases}$$

Then for every $0<\tilde{\omega}<1$ and for any $\hat{\xi}>0$, let $\mathfrak{K}_{n}(\tilde{\omega},\hat{\xi})=p\le n:\tilde{\varsigma}_{0}(v_{p},\hat{\xi})\le 1-\tilde{\omega}$ or $\dot{\varphi}_{0}(v_{p},\hat{\xi})\ge \tilde{\omega}\ and\ \psi_{0}(v_{p},\hat{\xi})\ge\tilde{\omega}$.

Since

$$p_{n}(\tilde{\omega},\hat{\xi})=\{p\le n:\frac{\hat{\xi}}{\hat{\xi}+|v_{p}|}\le 1-\tilde{\omega}\text{ or }\frac{|v_{p}|}{\hat{\xi}+|v_{p}|}\ge\tilde{\omega}\ and\ \frac{|v_{p}|}{\hat{\xi}}\ge\tilde{\omega}\}$$

$$=\left\{p\le n:|v_{p}|\ge\frac{\tilde{\omega}\hat{\xi}}{1-\tilde{\omega}}>0\right\}=\{p\le n:|v_{p}|=1\}=\{p\le n:p=m^{2}\ and\ m\in\mathbb{N}\}.$$

We have,

$$\frac{1}{n}|p_{n}(\tilde{\omega},\hat{\xi})|=\frac{1}{n}\{p\le n:p=m^{2}\ and\ m\in\mathbb{N}\}\le\frac{\sqrt{n}}{n}.$$

This yields that

$$\lim_{n}\frac{1}{n}|p_{n}(\tilde{\omega},\hat{\xi})|=0.$$

Hence by the above definition, $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-lim\,v_{p}=0$.

3. Statistical Convergence on Neutrosophic Normed Spaces

Here, that portion having the goal is to determine theorems concerning convergence and statistical convergence within the context of *NNS* over *NA* fields κ.

Lemma 3.1 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA-NNS*. After that the given statements is equivalent for all $\tilde{\omega}>0$ and $\hat{f}>0$

$(i)\ Stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{n}v_{p}=\bar{\bar{x}}$.

$$(ii)\ \lim_{n}\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_{p}-\bar{\bar{x}},\hat{f})\le 1-\tilde{\omega}\}\right|=\lim_{n}\frac{1}{n}\left|\{p\le n:\dot{\varphi}(v_{p}-\bar{\bar{x}},\hat{f})\ge\tilde{\omega}\}\right|$$

$$=\lim_{n}\frac{1}{n}\left|\{p\le n:\psi(v_{p}-\bar{\bar{x}},\hat{f})\ge\tilde{\omega}\}\right|=0.$$

$$(iii)\ \lim_{n}\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_{p}-\bar{\bar{x}},\hat{f})>1-\tilde{\omega},\dot{\varphi}(v_{p}-\bar{\bar{x}},\hat{f})<\tilde{\omega}\text{ and }\psi(v_{p}-\bar{\bar{x}},\hat{f})<\tilde{\omega}\}\right|=1.$$

$$(iv)\ \lim_{n}\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_{p}-\bar{\bar{x}},\hat{f})>1-\tilde{\omega}\}\right|=\lim_{n}\frac{1}{n}\left|\{p\le n:\dot{\varphi}(v_{p}-\bar{\bar{x}},\hat{f})<\tilde{\omega}\}\right|$$

$$=\lim_{n}\frac{1}{n}\left|\{p\le n:\psi(v_{p}-\bar{\bar{x}},\hat{f})<\tilde{\omega}\}\right|=1.$$

$(v)\ stat-lim\,\tilde{\varsigma}(v_{p}-\bar{\bar{x}},\hat{f})=1, stat-lim\,\dot{\varphi}(v_{p}-\bar{\bar{x}},\hat{f})=0\ and\ \ stat-lim\,\psi(v_{p}-\bar{\bar{x}},\hat{f})=0$

Theorem 3.2 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA-NNS*. If a $\{v_{p}\}$ sequence is statistically convergent with respect to the *NN*$(\tilde{\varsigma},\dot{\varphi},\psi)$, then $stat_{\tilde{\varsigma},\dot{\varphi},\psi}$ −limit is unique.

Proof: Assume that $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{p}v_{p}=\bar{\bar{x}}_{1}$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{p}v_{p}=\bar{\bar{x}}_{2}$. Consider a given $\tilde{\omega}>0$, select $\hat{\xi}>0$ so that we have $(1-\hat{\xi})*(1-\hat{\xi})>1-\tilde{\omega},\hat{\xi}\diamond\hat{\xi}<\tilde{\omega}$ and $\hat{\xi}\star\hat{\xi}<\tilde{\omega}$. After that for any $\hat{f}>0$, define the sets given below:

$\wp_{\tilde{\varsigma},1}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \tilde{\varsigma}(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) \le 1-\hat{\xi}\}$, $\wp_{\tilde{\varsigma},2}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \tilde{\varsigma}(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) \le 1-\hat{\xi}\}$,
$\wp_{\dot{\varphi},1}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \dot{\varphi}(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) \ge \hat{\xi}\}$, $\wp_{\dot{\varphi},2}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \dot{\varphi}(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) \ge \hat{\xi}\}$ and
$\wp_{\psi,1}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \psi(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) \ge \hat{\xi}\}$, $\wp_{\psi,2}(\hat{\xi},\acute{f}) := \{\wp \in \mathbb{N} : \psi(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) \ge \hat{\xi}\}$.

Since $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp} \upsilon_\wp = \bar{\bar{x}}_1$, we have

$$\lim_n \frac{1}{n}\{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f})\} = \lim_n \frac{1}{n}\{\wp_{r,1}(\tilde{\omega},\acute{f})\} = 0 \; for \; all \; \acute{f} > 0.$$

Furthermore, using $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp} \upsilon_\wp = \bar{\bar{x}}_2$, we get

$$\lim_n \frac{1}{n}\{\wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\} = \lim_n \frac{1}{n}\{\wp_{r,2}(\tilde{\omega},\acute{f})\} = 0 \; for \; all \; \acute{f} > 0.$$

Now let,

$$\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{f}) := \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\} \cap \{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{f}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{f})\} \cap \{\wp_{\psi,1}(\tilde{\omega},\acute{f}) \cup \wp_{\psi,2}(\tilde{\omega},\acute{f})\}.$$

If $\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{f}) = \wp_{\tilde{\varsigma},\dot{\varphi},\psi}, \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\} = \wp_{\tilde{\varsigma}}, \{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{f}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{f})\}$and $\{\wp_{\psi,1}(\tilde{\omega},\acute{f}) \cup \wp_{\psi 2}(\tilde{\omega},\acute{f})\} = \wp_{\psi}, then \wp_{\tilde{\varsigma},\dot{\varphi},\psi} = \wp_{\tilde{\varsigma}} \cap \wp_{\dot{\varphi}} \cap \wp_{\psi}$.

Then observe that, $\lim_n \frac{1}{n}\{\wp_{\tilde{\varsigma},\dot{\varphi},\psi}\} = 0$. which implies, $\lim_n \frac{1}{n}\{\wp^C_{\tilde{\varsigma},\dot{\varphi},\psi}\} = 1$.

If $\wp \in \wp^C_{\tilde{\varsigma},\dot{\varphi},\psi}$, then there are three possibilities to consider:
Then we have to select the initial part which is $\wp \in \{\wp^C_{\tilde{\varsigma}}\}$, the second part which is $\wp \in \{\wp^C_{\dot{\varphi}}\}$ and the later is $\wp \in \{\wp^C_{\psi}\}$.

We first consider that $\wp \in \{\wp^C_{\tilde{\varsigma}}\}$, then we have,

$$\begin{aligned}\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) = \tilde{\varsigma}(\bar{\bar{x}}_1 - \upsilon_\wp + \upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) &\ge \tilde{\varsigma}(\bar{\bar{x}}_1 - \upsilon_\wp,\acute{f}) * \tilde{\varsigma}(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) \\ &= \tilde{\varsigma}(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) * \tilde{\varsigma}(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) \\ &> (1-\hat{\xi}) * (1-\hat{\xi}).\end{aligned}$$

Since $(1-\hat{\xi}) * (1-\hat{\xi}) > 1-\tilde{\omega}$, it follows that

$$\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) > 1-\tilde{\omega}.$$

Since $\tilde{\omega} > 0$ was arbitrary, $\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) > 1$ for all $\acute{f} > 0$, which given $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
On the second hand, if $\wp \in \{\wp^C_{\dot{\varphi}}\}$, then we may write that,

$$\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) \le \dot{\varphi}(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) \diamond \dot{\varphi}(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) < \hat{\xi} \diamond \hat{\xi}.$$

Now using the fact$\hat{\xi} \diamond \hat{\xi} < \tilde{\omega}$, we see that$\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) < \tilde{\omega}$.
Again, since $\tilde{\omega} > 0$ was arbitrary, we have $\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) = 0$ for all $\acute{f} > 0$.
This implies $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
And on the other side, if $\wp \in \{\wp^C_{\psi}\}$, then we put

$$\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) \le \psi(\upsilon_\wp - \bar{\bar{x}}_1,\acute{f}) \star \psi(\upsilon_\wp - \bar{\bar{x}}_2,\acute{f}) < \hat{\xi} \star \hat{\xi}.$$

By using$\hat{\xi} \star \hat{\xi} < \tilde{\omega}$, we get $\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) < \tilde{\omega}$.
Hence $\tilde{\omega} > 0$ is arbitrary, $\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2,\acute{f}) = 0$ for every $\acute{f} > 0$, which implies $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
Therefore, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} -$ limit is unique.

Theorem 3.3 If a sequence $\{\upsilon_\wp\}$ in a *NA-NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ is $(\tilde{\varsigma},\dot{\varphi},\psi)$ −convergent to $\bar{\bar{x}} \in \Xi$, then this is $stat_{\tilde{\varsigma},\dot{\varphi},\psi}$ −convergent towards $\bar{\bar{x}} \in \Xi$.

Proof: Since $\{\mathcal{v}_{p}\}$ is $(\tilde{\varsigma},\dot{\varphi},\psi)$ −convergent towards $\bar{\bar{x}} \in \Xi$, for all $\tilde{\omega}>0$ and $\acute{f}>0$, then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})>1-\tilde{\omega}, \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})<\tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})<\tilde{\omega}.$$

This given the set $\{p \in \mathbb{N} : \tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega}\}$ has at the most a finite number of terms.

$$\text{i.e.,} \lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p \leq n : \tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \\ \text{and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

i.e., $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - lim \mathcal{v}_{p} = \bar{\bar{x}}$.

Note: It is interesting to note that the converse of this, which is not true classically, is true in a *NA-NNS* as shown below.

Let $\{\mathcal{v}_{p}\}$ be $stat_{\tilde{\varsigma},\dot{\varphi},\psi} -$ convergent towards $\bar{\bar{x}} \in \Xi$. Then for all $\tilde{\omega}>0$ and $\acute{f}>0$,

$$\lim_{n} \frac{1}{n} \left| \{p \leq n : \tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega}\} \right| = 0.$$

Now to prove that $\{\mathcal{v}_{p}\}$ is $(\tilde{\varsigma},\dot{\varphi},\psi) -$ convergent to $\mathcal{v} \in \Xi$. i.e., to prove that for all $\tilde{\omega} > 0$ and $\acute{f} > 0$ therefore $n_0 \in \mathbb{N}$ exists in that way and for every $n \geq n_0$,

$$\tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})>1-\tilde{\omega} \text{ and } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})<\tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f})<\tilde{\omega}.$$

Let us assume the contrary that,

$$\tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega}.$$

This implies that the set

$$\{p \leq n : \tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega}\}$$

has infinitely many terms.

ie, $\lim_{n} \frac{1}{n} \left| \{p \leq n : \tilde{\varsigma}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \leq 1-\tilde{\omega} \text{ or } \dot{\varphi}(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega} \text{ and } \psi(\mathcal{v}_{p}-\bar{\bar{x}},\acute{f}) \geq \tilde{\omega}\} \right| \neq 0$ which is a contradiction. Therefore, $\{\mathcal{v}_{p}\}$ is $(\tilde{\varsigma},\dot{\varphi},\psi)$ −convergent to $\mathcal{v} \in \Xi$.

Theorem 3.4 Let $\{\mathcal{v}_{p}\}$ and $\{\hbar_{p}\}$ be sequences in a *NA-NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ so that $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} \mathcal{v}_{p} = \mathcal{v}$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} \hbar_{p} = \hbar$, where $\mathcal{v},\hbar \in \Xi$. Then we have $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} (\mathcal{v}_{p} + \hbar_{p}) = \mathcal{v} + \hbar$.

Proof: Let $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} \mathcal{v}_{p} = \mathcal{v}$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} \hbar_{p} = \hbar$, choose $\hat{\xi}>0$ such that $(1-\hat{\xi}) * (1-\hat{\xi}) > 1-\tilde{\omega}, \hat{\xi} \diamond \hat{\xi} < \tilde{\omega}$ and $\hat{\xi} \star \hat{\xi} < \tilde{\omega}$ for a given $\tilde{\omega} > 0$. Then, for $\acute{f}>0$, define

$p_{\tilde{\varsigma},1}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \tilde{\varsigma}(\mathcal{v}_{p}-\mathcal{v},\acute{f}) \leq 1-\hat{\xi}\}$,

$p_{\tilde{\varsigma},2}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \tilde{\varsigma}(\hbar_{p}-\hbar,\acute{f}) \leq 1-\hat{\xi}\}$,

$p_{\dot{\varphi},1}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \dot{\varphi}(\mathcal{v}_{p}-\mathcal{v},\acute{f}) \geq \hat{\xi}\}$,

$p_{\dot{\varphi},2}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \dot{\varphi}(\hbar_{p}-\hbar,\acute{f}) \geq \hat{\xi}\}$ and

$p_{\psi,1}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \psi(\mathcal{v}_{p}-\mathcal{v},\acute{f}) \geq \hat{\xi}\}$,

$p_{\psi,2}(\hat{\xi},\acute{f}) := \{p \in \mathbb{N} : \psi(\hbar_{p}-\hbar,\acute{f}) \geq \hat{\xi}\}$.

Since $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \mathcal{v}_{p} = \mathcal{v}$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \hbar_{p} = \hbar$,

$$\lim_{n} \frac{1}{n} \{p_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f})\} = \lim_{n} \frac{1}{n} \{p_{\dot{\varphi},1}(\tilde{\omega},\acute{f})\} = \lim_{n} \frac{1}{n} \{p_{\psi,1}(\tilde{\omega},\acute{f})\} = 0,$$

$$\lim_{n}\frac{1}{n}\{\wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\} = \lim_{n}\frac{1}{n}\{\wp_{\dot{\varphi},2}(\tilde{\omega},\acute{f})\} = \lim_{n}\frac{1}{n}\{\wp_{\psi,2}(\tilde{\omega},\acute{f})\} = 0.$$

Now let,

$\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{f}) := \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\} \cap \{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{f}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{f})\} \cap \{\wp_{\psi,1}(\tilde{\omega},\acute{f}) \cup \wp_{\psi,2}(\tilde{\omega},\acute{f})\}$

i.e., if $\mathfrak{K} = \mathfrak{K}_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{f}), \mathfrak{K}_1 = \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{f}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{f})\}, \mathfrak{K}_2 = \{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{f}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{f})\}$

$\mathfrak{K}_3 = \{\wp_{\psi,1}(\tilde{\omega},\acute{f}) \cup \wp_{\psi,2}(\tilde{\omega},\acute{f})\}$. Then $\mathfrak{K} = \mathfrak{K}_1 \cap \mathfrak{K}_2 \cap \mathfrak{K}_3$.

Since $\mathfrak{K}^C$ is a non-empty set. Consider $\wp \in \mathfrak{K}^C$, then we have three possible cases. The former is $\wp \in \mathfrak{K}_1^C$, the second is $\wp \in \mathfrak{K}_2^C$ and the later is $\wp \in \mathfrak{K}_3^C$. First consider, $\wp \in \mathfrak{K}_1^C$, then we have,

$$\tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) > 1 - \hat{\xi} \text{ and } \tilde{\varsigma}(\hbar_\wp - \hbar,\acute{f}) > 1 - \hat{\xi}.$$

Now, we have,

$$\tilde{\varsigma}(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) > \tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) * \tilde{\varsigma}(\hbar_\wp - \hbar,\acute{f})$$
$$> (1-\hat{\xi}) * (1-\hat{\xi}).$$

Since $(1-\hat{\xi}) * (1-\hat{\xi}) > 1 - \tilde{\omega}$, it follows that $\tilde{\varsigma}(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) > 1 - \tilde{\omega}$.

Since $\tilde{\omega}$ is arbitrary, $\tilde{\varsigma}(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) = 1$ for all $\acute{f} > 0$,

which yields, $\tilde{\varsigma}(\upsilon_\wp + \hbar_\wp - (\upsilon + \hbar),\acute{f}) = 1$.

Similarly, if $\wp \in \mathfrak{K}_2^C$ then,

$\dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) < \hat{\xi}$ and $\dot{\varphi}(\hbar_\wp - \hbar,\acute{f}) < \hat{\xi}$.

$\Rightarrow \dot{\varphi}(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) \le \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) \diamond \dot{\varphi}(\hbar_\wp - \hbar,\acute{f}) < \hat{\xi} < \hat{\xi} \diamond \hat{\xi} < \tilde{\omega}$.

Since $\tilde{\omega}$ is arbitrary, $\dot{\varphi}(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) = 0$, for all $\acute{f} > 0$

$$\Rightarrow \dot{\varphi}(\upsilon_\wp + \hbar_\wp - (\upsilon + \hbar),\acute{f}) = 0$$

And if $\wp \in \mathfrak{K}_3^C$ then,

$\psi(\upsilon_\wp - \upsilon,\acute{f}) < \hat{\xi}$ and $\psi(\hbar_\wp - \hbar,\acute{f}) < \hat{\xi}$

$\Rightarrow \psi(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) \le \psi(\upsilon_\wp - \upsilon,\acute{f}) \star \psi(\hbar_\wp - \hbar,\acute{f}) < \hat{\xi} < \hat{\xi} \star \hat{\xi} < \tilde{\omega}$.

Since $\tilde{\omega}$ is arbitrary, $\psi(\upsilon_\wp + \hbar_\wp - \upsilon - \hbar,\acute{f}) = 0$, for all $\acute{f} > 0$

$$\Rightarrow \psi(\upsilon_\wp + \hbar_\wp - (\upsilon + \hbar),\acute{f}) = 0.$$

Thus, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp\to\infty}(\upsilon_\wp + \hbar_\wp) = \upsilon + \hbar$.

Theorem 3.5 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be an *NA- NNS* over κ. If $\lim_{\wp\to\infty} \tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) = 1$, $\lim_{\wp\to\infty} \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) =$

1 and $\lim_{\wp\to\infty} \psi(\upsilon_\wp - \upsilon,\acute{f}) = 1$ then $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp\to\infty} \upsilon_\wp = \upsilon$.

Proof: Let $\lim_{\wp\to\infty} \tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) = 1$, $\lim_{\wp\to\infty} \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) = 1$ and $\lim_{\wp\to\infty} \psi(\upsilon_\wp - \upsilon,\acute{f}) = 1$. Then for all $\hat{\xi} > 0$

and $\tilde{\omega} > 0$, that is a number $\wp_0 \in \mathbb{N}$ in that way, $\tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) < \tilde{\omega}$ and $\psi(\upsilon_\wp - \upsilon,\acute{f}) < \tilde{\omega}$ for every $\wp \ge \wp_0$. Hence the set, $\{\wp \in \mathbb{N} : \tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) \le 1 - \tilde{\omega},\ \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) \ge \tilde{\omega} \text{ and } \psi(\upsilon_\wp - \upsilon,\acute{f}) \ge \tilde{\omega}\}$ has a finite number of terms.

So, $\lim_{n}\frac{1}{n}\left|\begin{Bmatrix}\wp \le n : \tilde{\varsigma}(\upsilon_\wp - \upsilon,\acute{f}) \le 1 - \tilde{\omega} \text{ or } \dot{\varphi}(\upsilon_\wp - \upsilon,\acute{f}) \ge \tilde{\omega} \\ \text{and } \psi(\upsilon_\wp - \upsilon,\acute{f}) \ge \tilde{\omega}\end{Bmatrix}\right| = 0.$

Thus, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp\to\infty} \upsilon_\wp = \upsilon$.

4. Statistically Cauchy Sequences on NNS

Definition 4.1 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$be a *NA-NNS* over κ. Then, a $\{v_p\}$ sequence is referred to be SC when for each $\tilde{\omega}>0$ and $\acute{f}>0$ therefore $\mathbb{N}$ exists in which case for every $p,\ m\geq\mathbb{N}$,

$$\lim_n\frac{1}{n}\left|\begin{Bmatrix}p,m\leq n:\tilde{\varsigma}(v_p-v_m,\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_p-v_m,\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-v_m,\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|=0.$$

Definition 4.2 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA- NNS*. A sequence $\{v_p\}$ is refer as a Cauchy sequence when for each$\tilde{\omega}>0$ and $\acute{f}>0$, that is a number$p_0\in\mathbb{N}$ exist that way, for every $p,m\geq p_0$,

$$\tilde{\varsigma}(v_p-v_m,\acute{f})>1-\tilde{\omega},\dot{\varphi}(v_p-v_m,\acute{f})<\tilde{\omega}\ and\ \psi(v_p-v_m,\acute{f})<\tilde{\omega}.$$

Theorem 4.3 Every Cauchy sequence with respect to $(\tilde{\varsigma},\dot{\varphi},\psi)$ in *NA- NNS*$(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ over κ is SC.

Proof: If $\{v_p\}$ is a Cauchy sequence with relate to $(\tilde{\varsigma},\dot{\varphi},\psi)$, then there exists$p_0\in\mathbb{N}$ for all $\tilde{\omega}>0$ and $\acute{f}>0$ and let t be an arbitrary constant, we have

$$\tilde{\varsigma}(v_{p+t}-v_p,\acute{f})>1-\tilde{\omega},\dot{\varphi}\ (v_{p+t}-v_p,\acute{f})<\tilde{\omega}\ and\ \psi(v_{p+t}-v_p,\acute{f})<\tilde{\omega}.$$

The number of terms in the set $\begin{Bmatrix}p\in\mathbb{N}:\tilde{\varsigma}(v_{p+t}-v_p,\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_{p+t}-v_p,\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_{p+t}-v_p,\acute{f})\geq\tilde{\omega}\end{Bmatrix}$ is limited.

So

$$\lim_n\frac{1}{n}\left|\begin{Bmatrix}p+t,p\leq n:\tilde{\varsigma}(v_{p+t}-v_p,\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_{p+t}-v_p,\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_{p+t}-v_p,\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|=0.$$

Theorem 4.4 If a statistically convergent sequence in a *NA- NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ over κ, then it is *SC*.

Proof: If the sequence $\{v_p\}$ is statistically convergent to $\bar{\bar{x}}$then,

$$\lim_n\frac{1}{n}\left|\begin{Bmatrix}p\leq n:\tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|=0.$$

Now, we get

$$\lim_n\frac{1}{n}\left|\begin{Bmatrix}p,m\leq n:\tilde{\varsigma}(v_p-v_m,\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_p-v_m,\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-v_m,\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|$$

$$=\lim_n\frac{1}{n}\left|\begin{Bmatrix}p,m\leq n:\tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})*\tilde{\varsigma}(v_m-\bar{\bar{x}},\acute{f})\ \leq 1-\tilde{\omega}\\ or\ \dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})\diamond\dot{\varphi}(v_m-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-\bar{\bar{x}},\acute{f})\star\psi(v_m-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|=0.$$

5. Statistically complete and statistically continuous on NNS

A *NA- NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ is said to be complete if all$(\tilde{\varsigma},\dot{\varphi},\psi)$-Cauchy is $(\tilde{\varsigma},\dot{\varphi},\psi)$- convergent.

Definition 5.1 A *NA- NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ over κ is said to be statistically complete when all *SC* sequence with respect to $(\tilde{\varsigma},\dot{\varphi},\psi)$ is statistically convergent in relate with the $(\tilde{\varsigma},\dot{\varphi},\psi)$.

Theorem 5.2 Every *NA-NNS* $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ over κ is statistically complete with relate to $(\tilde{\varsigma},\dot{\varphi},\psi)$.

Proof: Let $\{v_p\}$ be *SC*. If it is not statistically convergent to$\bar{\bar{x}}\in\Xi$, then we get,

$$\lim_n\frac{1}{n}\left|\begin{Bmatrix}p,m\leq n:\tilde{\varsigma}(v_p-v_m,\acute{f})\leq 1-\tilde{\omega}\ or\ \dot{\varphi}(v_p-v_m,\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-v_m,\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|$$

$$=\lim_n\frac{1}{n}\left|\begin{Bmatrix}p,m\leq n:\tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})*\tilde{\varsigma}(v_m-\bar{\bar{x}},\acute{f})\leq 1-\tilde{\omega}\\ or\ \dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})\diamond\dot{\varphi}(v_m-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\\ and\ \psi(v_p-\bar{\bar{x}},\acute{f})\star\psi(v_m-\bar{\bar{x}},\acute{f})\geq\tilde{\omega}\end{Bmatrix}\right|=0$$

which is contradiction.

Definition 5.3 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NNS* over κ. A map $\jmath: \Xi \to \Xi$ is called $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ continuousat a point $v \in \Xi$, when the sequence with convergence in the *NA-NNS* implies that the sequence $\jmath(v_p)$ to $\jmath(v)$ convergence in the *NA- NNS*.

Definition 5.4 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NNS* over κ. A map $\jmath: \Xi \to \Xi$ is called statistically continuous at a point $v \in$, when $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$ implies that $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \jmath(v_p) = \jmath(v)$.

Theorem 5.5 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NN* space over κ. If $\jmath: \Xi \to \Xi$ is continuous in relate to the $(\tilde{\varsigma}, \dot{\varphi}, \psi)$, then this is statistically continuous.

Proof: Let $\{v_p\} \in \Xi$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$. Then for every $\tilde{\omega} > 0$ and $\acute{f} > 0$, the inequality,

$\tilde{\varsigma}(v_p - v, \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(v_p - v, \acute{f}) < \tilde{\omega}$ *and* $\psi(v_p - v, \acute{f}) < \tilde{\omega}$ implies that $\tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) < \tilde{\omega}$ *and* $\psi(\jmath(v_p) - \jmath(v), \acute{f}) < \tilde{\omega}$. Since $\jmath$ is continuous in relate to the $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ at $v \in \Xi$. Thus,

$$\left\{\begin{matrix} p \in \mathbb{N} : \tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) \le 1 - \tilde{\omega} \ or \ \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) \ge \tilde{\omega} \\ and \ \psi(\jmath(v_p) - \jmath(v), \acute{f}) \ge \tilde{\omega} \end{matrix}\right\}$$

$$\subset \left\{\begin{matrix} p \in \mathbb{N} : \tilde{\varsigma}(v_p - v, \acute{f}) \le 1 - \tilde{\omega} \ and \ \dot{\varphi}(v_p - v, \acute{f}) \ge \tilde{\omega} \\ and \ \psi(v_p - v, \acute{f}) \ge \tilde{\omega} \end{matrix}\right\}$$

Since, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$.

We have

$$\lim_n \frac{1}{n}\left|\left\{\begin{matrix} p \le n : \tilde{\varsigma}(v_p - v, \acute{f}) \le 1 - \tilde{\omega} \ or \ \dot{\varphi}(v_p - v, \acute{f}) \ge \tilde{\omega} \\ and \ \psi(v_p - v, \acute{f}) \ge \tilde{\omega} \end{matrix}\right\}\right| = 0.$$

This implies that,

$$\lim_n \frac{1}{n}\left|\left\{\begin{matrix} p \le n : \tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) \le 1 - \tilde{\omega} \ or \ \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) \ge \tilde{\omega} \\ and \ \psi(\jmath(v_p) - \jmath(v), \acute{f}) \ge \tilde{\omega} \end{matrix}\right\}\right| = 0.$$

This means that, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \jmath(v_p) = \jmath(v)$.

Hence, $\jmath$ is statistically continuous.

6. Conclusions

The NA fields were extended from Archimedean fields with the established outcomes. In this article, we prove certain including relations involving statistical convergence along with *SC* sequences on the *NNS* regarding NA fields.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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