



Some Special Refined Neutrosophic Ideals in Refined Neutrosophic Rings: A Proof-of-Concept Study

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Abstract: In this research, we created notions of a refined neutrosophic prime (completely prime, semiprime, and completely semiprime) ideal in a refined neutrosophic ring. If $R(I_1, I_2)$ is a refined neutrosophic ring, then each ideal of $R(I_1, I_2)$ has the form $J + KI_1 + LI_2$, where $J \subseteq L \subseteq K$ are ideals of the classical ring R . The objective of this work is to find the necessary and sufficient condition on classical ideals $J, L,$ and K that makes $J + KI_1 + LI_2$ a prime (completely prime, semiprime, and completely semiprime) ideal in $R(I_1, I_2)$. We studied some of the elementary properties of these concepts and the most important properties that link them.

We reached several results, the most important of which are as follows:

- If $J + KI_1 + LI_2 \in RN\mathfrak{Z}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)} \Leftrightarrow J, K,$ and $L \in S\wp_R$.
- $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$, then $J, K, L \in \wp_R$.
- Assuming that $R(I_1, I_2)$ is a finite unity commutation, then $RNM_{R(I_1, I_2)} = RN\wp_{R(I_1, I_2)}$.
- $R(I_1, I_2)$ is a refined neutrosophic field $\Leftrightarrow \{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)$ are only refined neutrosophic ideals in $R(I_1, I_2)$.
- We call $R(I_1, I_2)$ a refined neutrosophic prime ring if $RI_1 + RI_2 \in RN\wp_{R(I_1, I_2)}$ and a fully prime ring if $RN\mathfrak{Z}_{R(I_1, I_2)} \setminus \{0\} = RN\wp_{R(I_1, I_2)}$.

Keywords: Refined Neutrosophic Ring; Refined Neutrosophic Ideal; Completely Semiprime; Fully Prime; Fully Semiprime.

1. Introduction

Neutrosophy is a broad view of intuitionistic fuzzy logic that represents a new development of fuzzy notions. This strategy has a fascinating impact on applied science [1, 2, 3, 4, 5]. Neutrosophy can be applied to algebraic structures as a new branch of philosophy, leading to a better understanding and evolution of these structures. Kandasamy and Smarandache presented the concept of neutrosophic groups, rings, and fields [6], which has been widely investigated [7, 8, 9, 10] and is still being studied. Numerous intriguing discoveries about neutrosophic rings have recently been discussed [11, 12, 13].

Adeleke et al. [14, 15] generalized neutrosophic sets by dividing the degree of indeterminacy I into two degrees of indeterminacy I_1 , and I_2 . This concept has been widely employed in algebra by analyzing refined neutrosophic rings [14, 15] and n -refined neutrosophic rings and modules [16, 17, 18], and many intriguing findings have been established [19]. Abobala [20] characterized the maximal and minimal ideals in a refined neutrosophic ring.

We present a characterization of refined neutrosophic prime (completely prime, semiprime, and completely semiprime) ideals by depending on the properties of classical ideals. This study aims to describe the structure and properties of prime, completely prime, semiprime, and completely semiprime ideals of refined neutrosophic rings.

Our motivation is to close an important research gap by determining all prime, completely prime, semiprime, and completely semiprime ideals and their properties in refined neutrosophic rings. This paper continues the work begun in "On Neutrosophic Prime, Completely Prime, Semiprime, and Completely Semiprime Ideals in Neutrosophic Ring."

2. Definitions and notations

Since most academics interested in the subject are already familiar with classical rings and their ideals, this section will focus on numerous definitions and major results relevant to refined neutrosophic rings and their ideals.

Definition 2.1: [14, 15] Let R be a ring, the collection $R(I_1, I_2) = \{a + bI_1; a, b, c \in R \text{ and } I_1^2 = I_1, I_2^2 = I_2, I_1I_2 = I_2I_1 = I_1\}$ is called a refined neutrosophic ring. $R(I_1, I_2)$ is referred to as a refined neutrosophic field when R is a field.

Properties 2.2: [14, 15]

- (i) R is a unity commutative ring iff $R(I_1, I_2)$ is a unity commutative refined neutrosophic ring.
- (ii) $(I_1)^n = I_1$ and $(I_2)^n = I_2$ for each $n \in \mathbb{Z}^+$.
- (iii) $aI_1 = I_1a$ and $aI_2 = I_2a \forall a \in R$.
- (iv) $0I_1 = 0 = 0I_2$, $\underbrace{I_1 + I_1 + \dots + I_1}_{n \text{ time}} = nI_1$ and $\underbrace{I_2 + I_2 + \dots + I_2}_{n \text{ time}} = nI_2$.

Theorem 2.3: [20] If $R(I_1, I_2)$ is a refined neutrosophic ring, and $J + KI_1 + LI_2 \subseteq R(I_1, I_2)$, then $J + KI_1 + LI_2$ is a neutrosophic ideal iff $J, K,$ and L are ideals of R , where $J \subseteq L \subseteq K$.

Theorem 2.4: [20] If $R(I_1, I_2)$ is a refined neutrosophic ring, and $J + KI_1 + LI_2$ is an ideal of $R(I_1, I_2)$, then $J + KI_1 + LI_2$ is a neutrosophic maximal ideal iff J is a maximal of R , where $L = K = R$ or $J + KI_1 + LI_2 = R(I_1, I_2)$.

3. Results

In a refined neutrosophic ring $R(I_1, I_2)$, we indicate by $RN\mathfrak{I}_{R(I_1, I_2)}$ is the set of refined neutrosophic ideals, $RN\wp_{R(I_1, I_2)}$ the set of refined neutrosophic prime ideals, $RNC\wp_{R(I_1, I_2)}$ the set of refined neutrosophic completely prime ideals, $RNS\wp_{R(I_1, I_2)}$ the set of refined neutrosophic semiprime ideals, $RNCS\wp_{R(I_1, I_2)}$ the collection of refined neutrosophic completely semiprime ideals, and $RNM_{R(I_1, I_2)}$ the collection of refined neutrosophic maximal ideals. In addition, in classical ring R , we indicate by \mathfrak{I}_R the collection of ideals, \wp_R the collection of prime ideals, $C\wp_R$ the collection of completely prime ideals, $S\wp_R$ the collection of semiprime ideals, $SC\wp_R$ the collection of completely semiprime ideals, and \mathcal{M}_R the collection of maximal ideals.

Definition 3.1: If $J + KI_1 + LI_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$; $J \subseteq L \subseteq K$, then

- (i) $J + KI_1 + LI_2$ is a refined neutrosophic semiprime ideal if the following condition is satisfied: $\forall J_1 + K_1I_1 + L_1I_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$; $J_1 \subseteq L_1 \subseteq K_1$; $(J_1 + K_1I_1 + L_1I_2)^2 \subseteq J + KI_1 + LI_2 \Rightarrow J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$.
- (ii) $J + KI_1 + LI_2$ is a refined neutrosophic completely semiprime ideal if the following condition is satisfied: $\forall a + bI_1 + cI_2 \in R(I_1, I_2)$; $(a + bI_1 + cI_2)^2 \in J + KI_1 + LI_2 \Rightarrow a + bI_1 + cI_2 \in J + KI_1 + LI_2$.
- (iii) $J + KI_1 + LI_2$ is a refined neutrosophic prime ideal if the following condition is satisfied: $\forall J_1 + K_1I_1 + L_1I_2, J_2 + K_2I_1 + L_2I_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$; $J_1 \subseteq L_1 \subseteq K_1$ and $J_2 \subseteq L_2 \subseteq K_2$;
 $(J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$
 $\Rightarrow J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$ or $J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$
- (iv) $J + KI_1 + LI_2$ is a refined neutrosophic completely prime ideal if the following condition is satisfied: $\forall a + bI_1 + cI_2$ and $e + fI_1 + gI_2 \in R(I_1, I_2)$; $(a + bI_1 + cI_2)(e + fI_1 + gI_2) \in J + KI_1 + LI_2 \Rightarrow a + bI_1 + cI_2 \in J + KI_1 + LI_2 \vee e + fI_1 + gI_2 \in J + KI_1 + LI_2$

Theorem 3.2: If $J + KI_1 + LI_2 \in RN\mathfrak{X}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)} \Leftrightarrow J, K, \text{ and } L \in S\wp_R$.

Proof.

Firstly, $\forall J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$. Now, suppose that $J_1, K_1, L_1 \in \mathfrak{X}_R$, where $J_1^2 \subseteq J, K_1^2 \subseteq K, \text{ and } L_1^2 \subseteq L$. Subsequently, $J_1^2 \subseteq L \subseteq K$ and $L_1^2 \subseteq K$.

We have $J_1 + J_1I_1 + J_1I_2 \in N\mathfrak{X}_{R(I_1, I_2)}$, and we note

$$(J_1 + J_1I_1 + J_1I_2)^2 = J_1^2 + (J_1^2 + J_1^2 + J_1^2 + J_1^2 + J_1^2)I_1 + (J_1^2 + J_1^2 + J_1^2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, so $J_1 + J_1I_1 + J_1I_2 \subseteq J + KI_1 + LI_2$ and from which $J_1 \subseteq J$.

Therefore, $J \in S\wp_R$.

On the other hand, $\{0\} + L_1I_1 + L_1I_2 \in N\mathfrak{X}_{R(I_1, I_2)}$, and we note $(\{0\} + L_1I_1 + L_1I_2)^2 = \{0\}^2 + (\{0\}.L_1 + L_1.\{0\} + L_1^2 + L_1^2 + L_1^2)I_2 + (\{0\}.L_1 + L_1.\{0\} + L_1^2)I_2 \subseteq J + KI_1 + LI_2$

Since $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, so $\{0\} + L_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$ and from which $L_1 \subseteq L$.

Therefore, $L \in S\wp_R$.

And on the other hand, $\{0\} + K_1I_1 + \{0\}I_2 \in N\mathfrak{X}_{R(I_1, I_2)}$, and we note

$$(\{0\} + K_1I_1 + \{0\}I_2)^2 = \{0\}^2 + (\{0\}.K_1 + K_1.\{0\} + K_1^2 + K_1.\{0\} + \{0\}.K_1)I_2 + (\{0\}^2 + \{0\}^2 + \{0\}^2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, $\{0\} + K_1I_1 + \{0\}I_2 \subseteq J + KI_1 + LI_2$ and from which $K_1 \subseteq K$.

Therefore, $K \in S\wp_R$.

Conversely, suppose that $J, K, L \in S\wp_R$.

Now, if $J_1 + K_1I_1 + L_1I_2 \in N\mathfrak{X}_{R(I_1, I_2)}$; $J_1 \subseteq L_1 \subseteq K_1$, where

$$(J_1 + K_1I_1 + L_1I_2)^2 \subseteq J + KI_1 + LI_2 \Rightarrow J_1^2 + (J_1K_1 + K_1J_1 + K_1^2 + K_1L_1 + L_1K_1)I_1 + (J_1L_1 + L_1J_1 + L_1^2)I_2 \subseteq J + KI_1 + LI_2$$

Therefore, $J_1^2 \subseteq J$ and $J_1K_1 + K_1J_1 + K_1^2 + K_1L_1 + L_1K_1 \subseteq K$ and $J_1L_1 + L_1J_1 + L_1^2 \subseteq L$

Since $J_1^2 \subseteq J \subseteq L$ and $J_1L_1 + L_1J_1 + L_1^2 \subseteq L \subseteq K$, so $J_1^2 + J_1L_1 + L_1J_1 + L_1^2 = (J_1 + L_1)^2 \subseteq L$ and $J_1^2 + J_1K_1 + K_1J_1 + K_1^2 + K_1L_1 + L_1K_1 + J_1L_1 + L_1J_1 + L_1^2 = (J_1 + K_1 + L_1)^2 \subseteq K$.

Since J, K and $L \in S\wp_R$, so $J_1 \subseteq J$ and $J_1 + L_1 \subseteq L \subseteq K$ and $J_1 + K_1 + L_1 \subseteq K$.

Since $J_1 \subseteq J \subseteq L$, so $L_1 \subseteq L \subseteq K$ and $K_1 \subseteq K$.

Therefore, $J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$.

Theorem 3.3: If $J + KI_1 + LI_2 \in RN\mathfrak{X}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)} \Leftrightarrow J, K, L \in CS\wp_R$.

Proof.

Firstly, $\forall J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)}$.

Now, if $j, k, l \in R$, where $j^2 \in J \subseteq L$ and $l^2 \in L \subseteq K$ and $k^2 \in K$.

We have $j + jI_1 + jI_2 \in R(I_1, I_2)$ and we note

$$(j + jI_1 + jI_2)^2 = j^2 + (j^2 + j^2 + j^2 + j^2 + j^2)I_1 + (j^2 + j^2 + j^2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)}$, so $j + jI_1 + jI_2 \in J + KI_1 + LI_2$ and from which $j \in J$.

Therefore, $J \in CS\wp_R$.

On the other hand, we have $0 + lI_1 + lI_2 \in R(I_1, I_2)$, and we note

$$(0 + lI_1 + lI_2)^2 = 0^2 + (0.l + l.0 + l^2 + l^2 + l^2)I_1 + (0.l + l.0 + l^2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNCS_{\wp_{R(I_1, I_2)}}$, so $0 + lI_1 + lI_2 \in J + KI_1 + LI_2$ and from which $l \in L$. Therefore, $L \in CS_{\wp_R}$.

And on the other hand, we have $0 + kI_1 + 0I_2 \in R(I_1, I_2)$, and we note

$$(0 + kI_1 + 0I_2)^2 = 0^2 + (0.k + k.0 + k^2 + k.0 + 0.k)I_1 + (0^2 + 0^2 + 0^2)I_2 \in J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNCS_{\wp_{R(I_1, I_2)}}$, so $0 + kI_1 + 0I_2 \in J + KI_1 + LI_2$ and from which $k \in K$. Therefore, $K \in CS_{\wp_R}$.

Conversely, suppose that $J, K, \text{ and } L \in CS_{\wp_R}$.

Now, if $j_1 + k_1I_1 + l_1I_2 \in R(I_1, I_2)$, where $(j_1 + k_1I_1 + l_1I_2)^2 \in J + KI_1 + LI_2 \Rightarrow j_1^2 + (j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1)I_1 + (j_1l_1 + l_1j_1 + l_1^2)I_2 \in J + KI_1 + LI_2$

Therefore, $j_1^2 \in J$ and $j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1 \in K$ and $j_1l_1 + l_1j_1 + l_1^2 \in L$

Since $j_1^2 \in J \subseteq L$ and $j_1l_1 + l_1j_1 + l_1^2 \in L \subseteq K$, so $j_1^2 + j_1l_1 + l_1j_1 + l_1^2 = (j_1 + l_1)^2 \in L$ and $j_1^2 + j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1 + j_1l_1 + l_1j_1 + l_1^2 = (j_1 + k_1 + l_1)^2 \in K$.

Since $J, L, \text{ and } K \in CS_{\wp_R}$, so $J_1 \in J$ and $J_1 + L_1 \in L \subseteq K$ and $j_1 + k_1 + l_1 \in K$.

Since $j_1 \in J \subseteq L$, so $l_1 \in L \subseteq K$ and $k_1 \in K$. Subsequently, $j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNCS_{\wp_{R(I_1, I_2)}}$.

Theorem 3.4: If $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$, then $J, K, L \in \wp_R$.

Proof.

Suppose that $J_1, J_2, K_1, K_2, L_1, L_2 \in \wp_R$, where $J_1J_2 \subseteq J, K_1K_2 \subseteq K, \text{ and } L_1L_2 \subseteq L$.

Firstly, we have $J_1 + J_1I_1 + J_1I_2$ and $J_2 + J_2I_1 + J_2I_2 \in RN_{\wp_{R(I_1, I_2)}}$, and we note

$$(J_1 + J_1I_1 + J_1I_2)(J_2 + J_2I_1 + J_2I_2) = J_1J_2 + (J_1J_2 + J_1J_2 + J_1J_2 + J_1J_2 + J_1J_2)I_1 + (J_1J_2 + J_1J_2 + J_1J_2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$, so $J_1 + J_1I_1 + J_1I_2 \subseteq J + KI_1 + LI_2$ or $J_2 + J_2I_1 + J_2I_2 \subseteq J + KI_1 + LI_2$. Subsequently, $J_1 \subseteq J$ or $J_2 \subseteq J$. Thus $J \in \wp_R$.

On the other hand, we have $\{0\} + L_1I_1 + L_1I_2$ and $\{0\} + L_2I_1 + L_2I_2 \in N_{\wp_{R(I_1, I_2)}}$, and we note

$$(\{0\} + L_1I_1 + L_1I_2)(\{0\} + L_2I_1 + L_2I_2) = \{0\}^2 + (\{0\}.L_2 + L_1.\{0\} + L_1L_2 + L_1L_2 + L_1L_2)I_1 + (\{0\}.L_2 + L_1.\{0\} + L_1L_2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$, so $\{0\} + L_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$ or $\{0\} + L_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$. Subsequently, $L_1 \subseteq L$ or $L_2 \subseteq L$. Thus $L \in \wp_R$.

Also, we have $\{0\} + K_1I_1 + \{0\}I_2$ and $\{0\} + K_2I_1 + \{0\}I_2 \in N_{\wp_{R(I_1, I_2)}}$, and we note

$$\begin{aligned} & (\{0\} + K_1I_1 + \{0\}I_2)(\{0\} + K_2I_1 + \{0\}I_2) \\ &= \{0\}^2 + (\{0\}.K_2 + K_1.\{0\} + K_1K_2 + K_1.\{0\} + \{0\}.K_2)I_1 + (\{0\}^2 + \{0\}^2 + \{0\}^2)I_2 \\ &\subseteq J + KI_1 + LI_2 \end{aligned}$$

Since $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$, so $\{0\} + K_1I_1 + \{0\}I_2 \subseteq J + KI_1 + LI_2$ or $\{0\} + K_2I_1 + \{0\}I_2 \subseteq J + KI_1 + LI_2$ and from which $K_1 \subseteq K$ or $K_2 \subseteq K$. Thus $K \in \wp_R$.

Corollary 3.5: If $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$, and $J, K, L \in \wp_R$, then not necessarily $J + KI_1 + LI_2 \in RN_{\wp_{R(I_1, I_2)}}$.

Because.

Suppose that J, K , and $L \in \wp_R$. Now, if $J_1 + K_1I_1 + L_1I_2$, and $J_2 + K_2I_1 + L_2I_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$, where,

$$\begin{aligned} (J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) &\subseteq J + KI_1 + LI_2 \\ &\Rightarrow J_1J_2 + (J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2)I_1 + (J_1L_2 + L_1J_2 + L_1L_2)I_2 \\ &\subseteq J + KI_1 + LI_2, \text{ so} \end{aligned}$$

$$J_1J_2 \subseteq J, \quad J_1L_2 + L_1J_2 + L_1L_2 \subseteq L, \quad \text{and } J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2 \subseteq K$$

Since $J_1J_2 \subseteq J \subseteq L$, so $J_1J_2 + J_1L_2 + L_1J_2 + L_1L_2 = (J_1 + L_1)(J_2 + L_2) \subseteq L \subseteq K$

$$\text{and } J_1J_2 + J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2 + J_1L_2 + L_1J_2 + L_1L_2 = (J_1 + K_1 + L_1)(J_2 + K_2 + L_2) \subseteq K$$

Since J, K , and $L \in \wp_R$, so

$(J_1 \subseteq J \text{ or } J_2 \subseteq J), (J_1 + L_1 \subseteq L \text{ or } J_2 + L_2 \subseteq L), \text{ and } (J_1 + K_1 + L_1 \subseteq K \text{ or } J_2 + K_2 + L_2 \subseteq K)$. Thus not necessarily $J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$ or $J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$

Subsequently, not necessarily $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$.

Theorem 3.6: If $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$, then J, K , and $L \in C\wp_R$.

Proof.

If $j_1, j_2, k_1, k_2, l_1, l_2 \in R$, where $j_1j_2 \in J$, $k_1k_2 \in K$, and $l_1l_2 \in L$.

Firstly, $j_1 + j_1I_1 + j_1I_2$ and $j_2 + j_2I_1 + j_2I_2 \in R(I_1, I_2)$ and we note

$$\begin{aligned} (j_1 + j_1I_1 + j_1I_2)(j_2 + j_2I_1 + j_2I_2) &= j_1j_2 + (j_1j_2 + j_1j_2 + j_1j_2 + j_1j_2 + j_1j_2)I_1 + (j_1j_2 + j_1j_2 + j_1j_2)I_2 \\ &\in J + KI_1 + LI_2 \end{aligned}$$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$, so

$$j_1 + j_1I_1 + j_1I_2 \in J + KI_1 + LI_2 \text{ or } j_2 + j_2I_1 + j_2I_2 \in J + KI_1 + LI_2.$$

Therefore, $j_1 \in J$ or $j_2 \in J$. Thus $J \in C\wp_R$.

On the other hand, $0 + l_1I_1 + l_1I_2$ and $0 + l_2I_1 + l_2I_2 \in R(I_1, I_2)$, and we note

$$\begin{aligned} (0 + l_1I_1 + l_1I_2)(0 + l_2I_1 + l_2I_2) &= 0^2 + (0.l_2 + l_1.0 + l_1l_2 + l_1l_2 + l_1l_2)I_1 + (0.l_2 + l_1.0 + l_1l_2)I_2 \\ &\in J + KI_1 + LI_2 \end{aligned}$$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$, so $0 + l_1I_1 + l_1I_2 \in J + KI_1 + LI_2$ or $0 + l_2I_1 + l_2I_2 \in J + KI_1 + LI_2$.

Therefore, $l_1 \in L$ or $l_2 \in L$. Thus $L \in C\wp_R$.

Also, we have $0 + k_1I_1 + 0I_2$ and $0 + k_2I_1 + 0I_2 \in R(I_1, I_2)$, and we note

$$\begin{aligned} (0 + k_1I_1 + 0I_2)(0 + k_2I_1 + 0I_2) &= 0^2 + (0.k_2 + k_1.0 + k_1k_2 + k_1.0 + \{0\}.k_2)I_1 + (0^2 + 0^2 + 0^2)I_2 \\ &\in J + KI_1 + LI_2 \end{aligned}$$

since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$, so $0 + k_1I_1 + 0I_2 \in J + KI_1 + LI_2$ or $0 + k_2I_1 + 0I_2 \in J + KI_1 + LI_2$

and from which $k_1 \in K$ or $k_2 \in K$. Thus $K \in C\wp_R$.

Corollary 3.7: If $J + KI_1 + LI_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$ and J, K , and $L \in C\wp_R$, then not necessarily $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$.

Because.

Suppose that $j_1 + k_1I_1 + l_1I_2$, $j_2 + k_2I_1 + l_2I_2 \in R(I_1, I_2)$, where,

$$\begin{aligned} (j_1 + k_1I_1 + l_1I_2)(j_2 + k_2I_1 + l_2I_2) &\in J + KI_1 + LI_2 \\ &\Rightarrow j_1j_2 + (j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2)I_1 + (j_1l_2 + l_1j_2 + l_1l_2)I_2 \in J + KI_1 + LI_2, \text{ so} \\ j_1j_2 &\in J \text{ and } j_1l_2 + l_1j_2 + l_1l_2 \in L \text{ and } j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2 \in K \end{aligned}$$

Since $j_1j_2 \in J \subseteq L$, so $j_1j_2 + j_1l_2 + l_1j_2 + l_1l_2 = (j_1 + l_1)(j_2 + l_2) \in L \subseteq K$

and $j_1j_2 + j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2 + j_1l_2 + l_1j_2 + l_1l_2 = (j_1 + k_1 + l_1)(j_2 + k_2 + l_2) \in K$
 Since $J, K, L \in C\wp_R$, so $(j_1 \in J \text{ or } j_2 \in J)$, $(j_1 + l_1 \in L \text{ or } j_2 + l_2 \in L)$, and $(j_1 + k_1 + l_1 \in K \text{ or } j_2 + k_2 + l_2 \in K)$. So not necessarily

$j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2$ or $j_2 + k_2I_1 + l_2I_2 \in J + KI_1 + LI_2$. Therefore, not necessarily $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$.

Theorem 3.8: If $J + KI_1 + LI_2 \in RN\wp_{R(I)}$, then

- (i) $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)} \Rightarrow J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$.
- (ii) $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)} \Rightarrow J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$.

Proof.

- (i) Since $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)}$, so J, K , and $L \in CS\wp_R$ according to Theorem.3.3. Therefore, J, K , and $L \in S\wp_R$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$ according to Theorem.3.2.
- (ii) Suppose that $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$. Now, if $J_1 + K_1I_1 + L_1I_2$ and $J_2 + K_2I_1 + L_2I_2 \in RN\wp_{R(I_1, I_2)}$ in which $(J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$.

Firstly, suppose that

$J_1 + K_1I_1 + L_1I_2 \not\subseteq J + KI_1 + LI_2$ and $J_2 + K_2I_1 + L_2I_2 \not\subseteq J + KI_1 + LI_2$. Therefore, $\exists j_1 + k_1I_1 + l_1I_2 \in J_1 + K_1I_1 + L_1I_2$ and $j_2 + k_2I_1 + l_2I_2 \in J_2 + K_2I_1 + L_2I_2$, where $j_1 + k_1I_1 + l_1I_2 \notin J + KI_1 + LI_2$ and $j_2 + k_2I_1 + l_2I_2 \notin J + KI_1 + LI_2$.

On the other hand, we have $(j_1 + k_1I_1 + l_1I_2)(j_2 + k_2I_1 + l_2I_2) \in (J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$, so

$j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2$ or $j_2 + k_2I_1 + l_2I_2 \in J + KI_1 + LI_2$. This is a contradiction. Therefore, $J + KI_1 + LI_2 \subseteq J + KI_1 + LI_2$ or $J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$.

Remark 3.9: Figure 1 shows the resulting relationship between the prime (completely prime, semiprime, and completely semiprime) ideals in any refined neutrosophic and classical ring, as follows:

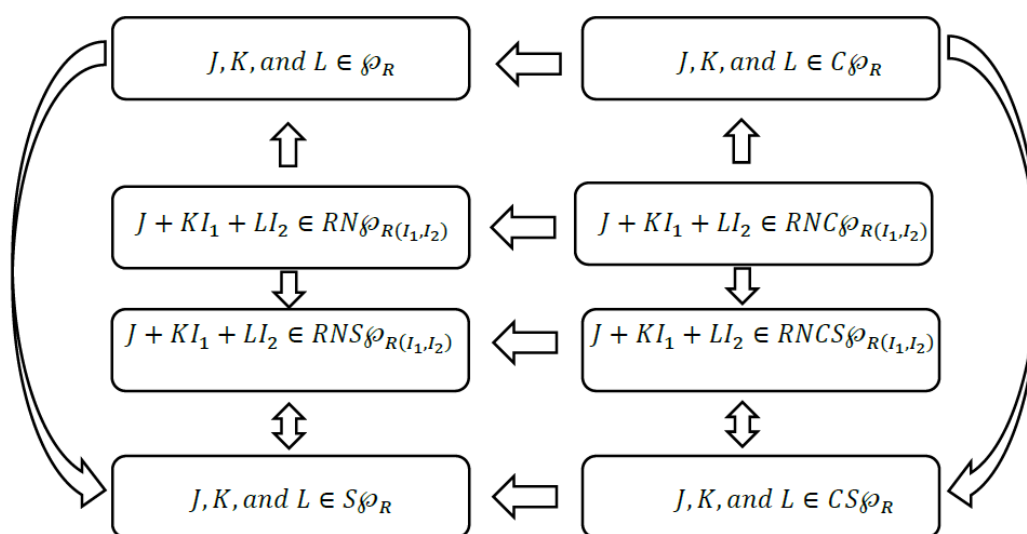


Figure 1. The relationship between the ideals of the refined neutrosophic and classical ring.

Theorem 3.10: If $R(I_1, I_2)$ is a unity, and $J + KI_1 + LI_2 \in RN\mathfrak{X}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)} \Leftrightarrow \forall r_1 + r_2I_1 + r_3I_2 \in R(I_1, I_2); (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) \subseteq J + KI_1 + LI_2 \Rightarrow r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, and we will prove that the condition is satisfied.

$$\begin{aligned} &\forall r_1 + r_2I_1 + r_3I_2 \in R(I_1, I_2); (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) \subseteq J + KI_1 + LI_2 \\ \Rightarrow &(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq (J + KI_1 + LI_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2 \\ &\Rightarrow [(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)]^2 \subseteq J + KI_1 + LI_2 \end{aligned}$$

Since $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, so $(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2$.

On the other hand, we have

$$\begin{aligned} r_1 + r_2I_1 + r_3I_2 &= (r_1 + r_2I_1 + r_3I_2) \cdot 1 \in (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2 \Rightarrow r_1 + r_2I_1 + r_3I_2 \\ &\in J + KI_1 + LI_2 \end{aligned}$$

Conversely, suppose that the condition is true and we will prove that $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$.

Suppose that $J_1 + K_1I_1 + L_1I_2 \in RN\mathfrak{X}_{R(I_1, I_2)}$, where,

$$[J_1 + K_1I_1 + L_1I_2]^2 \subseteq J + KI_1 + LI_2.$$

If we assume the argument $J_1 + K_1I_1 + L_1I_2 \not\subseteq J + KI_1 + LI_2$. Therefore, there is an element $r_1 + r_2I_1 + r_3I_2 \in J_1 + K_1I_1 + L_1I_2$ and $r_1 + r_2I_1 + r_3I_2 \notin J + KI_1 + LI_2$

On the other hand, we have

$$\begin{aligned} r_1 + r_2I_1 + r_3I_2 \in J_1 + K_1I_1 + L_1I_2 &\Rightarrow (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J_1 + K_1I_1 + L_1I_2 \\ \Rightarrow (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) &\subseteq (J_1 + K_1I_1 + L_1I_2)(r_1 + r_2I_1 + r_3I_2) \\ &\subseteq (J_1 + K_1I_1 + L_1I_2)(J_1 + K_1I_1 + L_1I_2) = [J_1 + K_1I_1 + L_1I_2]^2 \subseteq J + KI_1 + LI_2 \end{aligned}$$

Therefore, $r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$, which is a contradiction.

So $J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$.

Theorem.3.11 If $R(I_1, I_2)$ is a unity, and $J + KI_1 + LI_2 \in RN\mathfrak{X}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$ iff the condition is satisfied:

$$\begin{aligned} &\forall r_1 + r_2I_1 + r_3I_2 \text{ and } r'_1 + r'_2I_1 + r'_3I_2 \in R(I_1, I_2); \\ &(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r'_1 + r'_2I_1 + r'_3I_2) \subseteq J + KI_1 + LI_2 \\ \Rightarrow &r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2 \text{ or } r'_1 + r'_2I_1 + r'_3I_2 \in J + KI_1 + LI_2 \end{aligned}$$

Proof. In a similar way to proof of the theorem.3.10.

Corollary 3.12: Let $R(I_1, I_2)$ be a unity commutation.

- (i) If $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$.
- (ii) If $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)}$.

Proof.

1. Suppose that $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$, and $r_1 + r_2I_1 + r_3I_2, r'_1 + r'_2I_1 + r'_3I_2 \in R(I_1, I_2)$, where, $(r_1 + r_2I_1 + r_3I_2)(r'_1 + r'_2I_1 + r'_3I_2) \in J + KI_1 + LI_2$.

$$\Rightarrow (r_1 + r_2I_1 + r_3I_2)(r'_1 + r'_2I_1 + r'_3I_2)R(I_1, I_2) \subseteq (J + KI_1 + LI_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2$$

Since $R(I_1, I_2)$ is a commutative, so

$$(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r'_1 + r'_2I_1 + r'_3I_2) \subseteq J + KI_1 + LI_2$$

And since $J + KI_1 + LI_2 \in N\wp_{R(I_1, I_2)}$, and according to theorem.3.11, so

$r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$ or $r'_1 + r'_2I_1 + r'_3I_2 \in J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNC\wp_{R(I_1, I_2)}$.

2. In a similar way to proof.1. Or in another way, since $R(I_1, I_2)$ is a unity commutative, so R is a unity commutative ring.

We have $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$, therefore, J, K , and $L \in S\wp_R$ according to Theorem.3.2.

Since R is a unity commutative ring, so J, K , and $L \in CS\wp_R$. Using the Theorem.3.3, $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1, I_2)}$.

Theorem 3.13: Assuming that $R(I_1, I_2)$ is a unity. If $J + KI_1 + LI_2 \in RN\mathcal{M}_{R(I_1, I_2)}$, then $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$.

Proof.

Since $J + KI_1 + LI_2 \in RN\mathcal{M}_{R(I_1, I_2)}$, so $(J \in \mathcal{M}_R \text{ and } K = L = R) \text{ or } J + KI_1 + LI_2 = R(I_1, I_2)$ according to theorem.2.4. If $J + KI_1 + LI_2 = R(I_1, I_2)$, then the desired is achieved. Now, suppose that $J + KI_1 + LI_2 \neq R(I_1, I_2)$.

We have $R(I_1, I_2)$ is a unity, therefore, we may apply the condition specified in the theorem.3.11.

$$\forall r_1 + r_2I_1 + r_3I_2 \text{ and } r'_1 + r'_2I_1 + r'_3I_2 \in R(I_1, I_2);$$

$$(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r'_1 + r'_2I_1 + r'_3I_2) \subseteq J + KI_1 + LI_2$$

Now, we will prove that $r_1 + r_2I_1 + r_3I_2 \in J + RI_1 + RI_2$ or $r'_1 + r'_2I_1 + r'_3I_2 \in J + RI_1 + RI_2$.

In fact, it suffices to demonstrate that $r_1 \in J$ or $r'_1 \in J$.

Firstly, $(r_1 + r_2I_1 + r_3I_2)(R + RI_1 + RI_2)(r'_1 + r'_2I_1 + r'_3I_2) \subseteq J + RI_1 + RI_2$

$$\begin{aligned} \Rightarrow r_1Rr'_1 + [r_1Rr'_1 + r_2Rr'_1 + r_2Rr'_1 + r_2Rr'_1 + r_3Rr'_1 + r_1Rr'_2 + r_1Rr'_2 + r_2Rr'_2 + r_2Rr'_2 + r_2Rr'_2 + r_3Rr'_2 \\ + r_1Rr'_2 + r_3Rr'_2 + r_3Rr'_2 + r_1Rr'_3 + r_2Rr'_3 + r_2Rr'_3 + r_2Rr'_3 + r_3Rr'_3]I_1 + [r_1Rr'_1 + r_3Rr'_1 \\ + r_3Rr'_1 + r_1Rr'_3 + r_1Rr'_3 + r_3Rr'_3 + r_3Rr'_3]I_2 \subseteq J + RI_1 + RI_2 \end{aligned}$$

Therefore, $r_1Rr'_1 \subseteq J$.

Suppose that $r_1 \notin J$. Since $J \in \mathcal{M}_R$, so $J + r_1R = R \Rightarrow Jr'_1 + r_1Rr'_1 = Rr'_1$

On the other hand, we have $Jr'_1 \subseteq J$ and $r_1Rr'_1 \subseteq J$. Therefore, $r'_1 = 1 \cdot r'_1 \in Rr'_1 \subseteq J$.

Subsequently, $r'_1 + r'_2I_1 + r'_3I_2 \in J + RI_1 + RI_2$. Thus $J + KI_1 + LI_2 \in N\wp_{R(I_1, I_2)}$.

Remark 3.14: Figure 2 shows the resulting relationship between the prime (completely prime, semiprime, completely semiprime, and maximal) ideals in the unity refined neutrosophic and classical rings, as follows:

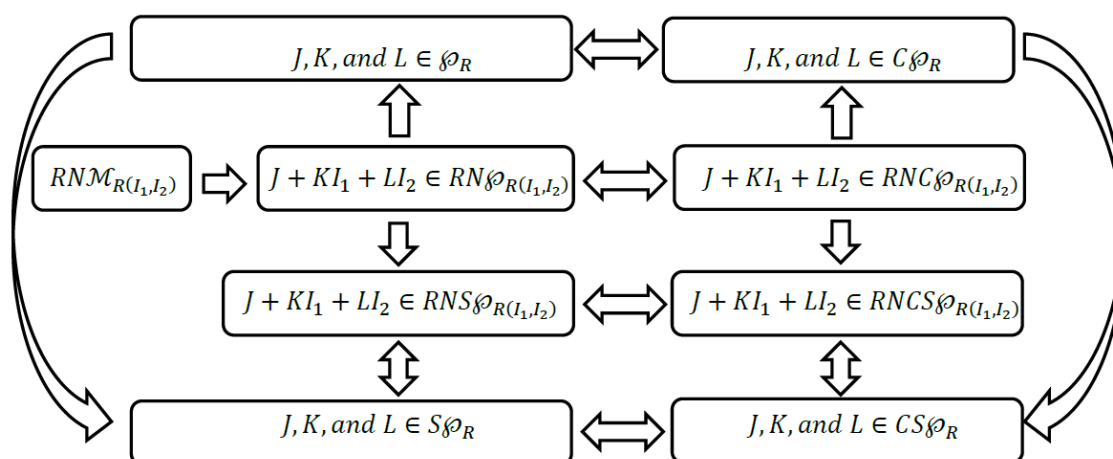


Figure 2. The relationship between the ideals of the unity refined neutrosophic and classical ring.

Theorem 3.15: Assuming that $R(I_1, I_2)$ is a finite unity commutation, then $RNM_{R(I_1, I_2)} = RN\phi_{R(I_1, I_2)}$.

Proof.

Since $R(I_1, I_2)$ is a unity, so $RNM_{R(I_1, I_2)} \subseteq RN\phi_{R(I_1, I_2)}$ according to theorem.3.13.

Now, if $J + KI_1 + LI_2 \in RN\phi_{R(I_1, I_2)}$, then $J, K, \text{ and } L \in \phi_R$ according to theorem.3.4. Since $J \in \phi_R$, and R is a finite unity commutation, so $J \in \mathcal{M}_R$. Since $J \subseteq L \subseteq K$, so $K = L = R$. Thus $J + KI_1 + LI_2 \in RNM_{R(I_1, I_2)}$.

Examples and Notes 3.16:

(1) In $Z(I)$, we have $10Z + 10ZI_1 + 10ZI_2 \in RNS\phi_{Z(I_1, I_2)}$, because $\forall r_1 + r_2I_1 + r_3I_2 \in Z(I_1, I_2)$; $(r_1 + r_2I_1 + r_3I_2)^2 \in 10Z + 10ZI_1 + 10ZI_2$, then $r_1^2 \in 10Z$ and $(r_1 + r_3)^2 \in 10Z$ and $(r_1 + r_2 + r_3)^2 \in 10Z$. Since $10Z \in S\phi_Z$, so $r_1 \in 10Z, r_1 + r_3 \in 10Z, \text{ and } r_1 + r_2 + r_3 \in 10Z$. Therefore $r_1, r_2, r_3 \in 10Z$. Thus $r_1 + r_2I_1 + r_3I_2 \in 10Z + 10ZI_1 + 10ZI_2 \in RNS\phi_Z$.

By the same way we find that $\langle 0 \rangle + \langle 0 \rangle I_1 + \langle 0 \rangle I_2 = \{0\} \in RNS\phi_{Z(I_1, I_2)}$.

(2) By the same way we find that $\langle 2 \rangle + \langle 2 \rangle I_1 + \langle 2 \rangle I_2 \in RNS\phi_{Z_4(I_1, I_2)}$.

(3) In $Z_4(I_1, I_2)$, we have $\langle 2 \rangle = \{0, 2\} \in \phi_{Z_4}$, but $\langle 2 \rangle + \langle 2 \rangle I_1 + \langle 2 \rangle I_2 = \{0, 2I_2, 2I_1, 2I_1 + 2I_2, 2, 2 + 2I_2, 2 + 2I_1, 2 + 2I_1 + 2I_2\} \notin RN\phi_{Z_4(I_1, I_2)}$, because we have $(I_1 + I_2)(2 + I_1) = 2I_2 \in \langle 2 \rangle + \langle 2 \rangle I_1 + \langle 2 \rangle I_2$,

but $I_1 + I_2$ and $2 + I_1 \notin \langle 2 \rangle + \langle 2 \rangle I_1 + \langle 2 \rangle I_2$.

(4) In $Z(I_1, I_2)$, we have $\langle 0 \rangle$ and $\langle 3 \rangle \in \phi_Z$, but $\langle 0 \rangle + \langle 3 \rangle I_1 + \langle 0 \rangle I_2 = \langle 3 \rangle I_1 \notin RN\phi_{Z(I_1, I_2)}$, because we have $(0 + 2I_1)(3 + I_1) = 9I_1 \in \langle 3 \rangle I_1$, but $3 + I_1$ and $2I_1 \notin \langle 3 \rangle I_1$. By the same way, we find $\langle 3 \rangle I_1 + \langle 3 \rangle I_2 \notin RN\phi_{Z(I_1, I_2)}$.

(5) In $Z_6(I_1, I_2)$, we have $\langle 3 \rangle + Z_6I_1 + \langle 3 \rangle I_2 \notin RN\phi_{Z_6(I_1, I_2)}$, because we have $(0 + I_2)(2 + I_2) = 3I_2 \in \langle 3 \rangle + Z_6I_1 + \langle 3 \rangle I_2$, but I_2 and $2 + I_2 \notin \langle 3 \rangle + Z_6I_1 + \langle 3 \rangle I_2$.

(6) We note $\langle 2 \rangle + Z_6I_1 + Z_6I_2 = \{0, 2\} + Z_6I_1 + Z_6I_2 \in N\phi_{Z_6(I_1, I_2)}$, because $\langle 2 \rangle \in \mathcal{M}_{Z_6}$, so $\langle 2 \rangle + Z_6I_1 + Z_6I_2 \in RNM_{Z_6(I_1, I_2)}$ according to theorem.2.7. Therefore, $\langle 2 \rangle + Z_6I_1 + Z_6I_2 \in RN\phi_{Z_6(I_1, I_2)}$ according to Theorem.3.13.

(7) In $Z_7(I_1, I_2)$, we have $\langle 0 \rangle + \langle 0 \rangle I_1 + \langle 0 \rangle I_2 = \{0\} \notin RN\phi_{Z_7(I_1, I_2)}$, because we have

$(6 + I_2)(I_1 + I_2) = 0 \in Z_7(I_1, I_2)$, but $6 + I_2$ and $I_1 + I_2 \notin \langle 0 \rangle + \langle 0 \rangle I_1 + \langle 0 \rangle I_2$.

(8) In any refined neutrosophic field $R(I_1, I_2)$, $RI_1 + RI_2 \in RN\wp_{R(I_1, I_2)}$, because $RI_1 + RI_2 = \langle 0 \rangle + RI_1 + RI_2$, where $\langle 0 \rangle \in NM_R$. Using theorem.3.13, we find $RI_1 + RI_2 \in RN\wp_{R(I_1, I_2)}$.

It can be proven in another way:

If $a + bI_1 + cI_2$ and $d + eI_1 + fI_2 \in R(I_1, I_2)$ where $(a + bI_1 + cI_2)(d + eI_1 + fI_2) \in RI_1 + RI_2 \Rightarrow \exists r, r' \in R$ in which $(a + bI_1 + cI_2)(d + eI_1 + fI_2) = rI_1 + r'I_2$

So $ad + [ae + bd + be + bf + ce]I_1 + [af + cd + cf]I_2 = 0 + rI_1 + r'I_2$

Therefore, $ad = 0$. So $a = 0$ or $d = 0$

if $a = 0$ then $a + bI_1 + cI_2 \in RI_1 + RI_2$

if $d = 0$ then $d + eI_1 + fI_2 \in RI_1 + RI_2$

(9) Generally, in refined neutrosophic rings, $RI_1 + RI_2$ is not necessarily belongs to $RN\wp_{R(I_1, I_2)}$.

(10) $Z_9I_1 + Z_9I_2 \notin RN\wp_{Z_9(I_1, I_2)}$, because we have $(3 + I_1 + 2I_2)^2 = 2I_1 + 4I_2 \in Z_9I_1 + Z_9I_2$, but $3 + I_1 + 2I_2 \notin Z_9I_1 + Z_9I_2$.

(11) In $Z(I_1, I_2)$, we have $\langle 0 \rangle + ZI_1 + ZI_2$ and $\langle p \rangle + ZI_1 + ZI_2 \in RN\wp_{Z(I_1, I_2)}$, where p is prime.

Theorem 3.17: Assuming that $R(I_1, I_2)$ is a unity. Then $R(I_1, I_2)$ is a refined neutrosophic field $\Leftrightarrow \{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)$ are only refined neutrosophic ideals in $R(I_1, I_2)$.

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$. Since $R(I_1, I_2)$ is a refined neutrosophic field, so R is a field. Therefore, R contains only two ideals $\{0\}$ and R . Thus

$$J, K, \text{ and } L = \{0\} \text{ or } R$$

We have $J \subseteq L \subseteq K$ and we note

$$\text{if } J = L = K = \{0\}, \text{ then } J + KI_1 + LI_2 = \{0\}$$

$$\text{if } J = \{0\} \wedge K = L = R, \text{ then } J + KI_1 + LI_2 = RI_1 + RI_2$$

$$\text{if } J = L = \{0\} \wedge K = R, \text{ then } J + KI_1 + LI_2 = RI_1$$

$$\text{if } J = L = K = R, \text{ then } J + KI_1 + LI_2 = R + RI_1 + RI_2$$

Subsequently, $RN\wp_{R(I_1, I_2)} = \{\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)\}$.

Conversely, suppose that $RN\wp_{R(I_1, I_2)} = \{\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)\}$.

Now, If $J + KI_1 + LI_2 \in RN\wp_{R(I_1, I_2)}$, then

$$J + KI_1 + LI_2 = R + RI_1 + RI_2 \vee \{0\} + RI_1 + \{0\}I_2 \vee \{0\} + RI_1 + RI_2 \vee \{0\} + \{0\}I_1 + \{0\}I_2$$

In every case, we see that $J, K, \text{ and } L = \{0\} \vee R$. Therefore, R contains only two ideals $\{0\}$ and R .

Subsequently, R is a field. Thus $R(I_1, I_2)$ is a refined neutrosophic field.

Definition 3.18: Assuming that $R(I_1, I_2)$ is a refined neutrosophic ring.

- (i) We call $R(I_1, I_2)$ a refined neutrosophic semiprime ring if $\{0\} \in RNS\wp_{R(I_1, I_2)}$ and a fully semiprime ring if $RN\wp_{R(I_1, I_2)} = RNS\wp_{R(I_1, I_2)}$.
- (ii) We call $R(I_1, I_2)$ a refined neutrosophic prime ring if $RI_1 + RI_2 \in RN\wp_{R(I_1, I_2)}$ and a fully prime ring if $RN\wp_{R(I_1, I_2)} \setminus \{0\} = RN\wp_{R(I_1, I_2)}$.
- (iii) We call $R(I_1, I_2)$ a refined neutrosophic fully idempotent if all its neutrosophic ideals are idempotent.

Examples 3.19:

- (1) $Z(I_1, I_2)$ is a refined neutrosophic semiprime ring.

(2) $R(I_1, I_2)$ is a refined neutrosophic semiprime (fully semiprime) ring, where R is a field.

(3) $R(I_1, I_2)$ is a refined neutrosophic prime (fully prime) ring, where R is a field.

Theorem 3.20: Assuming that $R(I_1, I_2)$ is a refined neutrosophic ring, $R(I_1, I_2)$ is a refined neutrosophic fully semiprime $\Leftrightarrow R(I_1, I_2)$ is a refined neutrosophic fully idempotent.

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$. Now, we have $(J + KI_1 + LI_2)^2 \in RN\mathfrak{I}_{R(I_1, I_2)}$. Therefore, it belongs to $RNS\wp_{R(I_1, I_2)}$.

Also, we have $(J + KI_1 + LI_2)^2 \subseteq (J + KI_1 + LI_2)^2 \stackrel{\Rightarrow}{(J+KI_1+LI_2)^2 \in RNS\wp_{R(I_1, I_2)}}$

$$J + KI_1 + LI_2 \subseteq (J + KI_1 + LI_2)^2$$

On the other hand, $(J + KI_1 + LI_2)^2 \subseteq J + KI_1 + LI_2$. So $(J + KI_1 + LI_2)^2 = J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2$ is a refined neutrosophic idempotent ideal.

Conversely, suppose that $J + KI_1 + LI_2 \in RN\mathfrak{I}_{R(I_1, I_2)}$.

Now, let's prove that

$\forall P + QI_1 + SI_2 \in N\mathfrak{I}_{R(I_1, I_2)}$, where, $(P + QI_1 + SI_2)^2 \subseteq J + KI_1 + LI_2$,

then $P + QI_1 + SI_2 \subseteq J + KI_1 + LI_2$.

Since $(P + QI_1 + SI_2)^2 = P + QI_1 + SI_2$ (because it is idempotent), then

$P + QI_1 + SI_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1, I_2)}$.

Example.3.21 According to the theorem 3.17, in $\mathbb{Z}_3(I_1, I_2)$, we have $\{0\}$, $\mathbb{Z}_3I_1 + \mathbb{Z}_3I_2$, \mathbb{Z}_3I_1 , and $\mathbb{Z}_3(I_1, I_2)$ are the only neutrosophic ideals. Now we note $\{0\}$, $\mathbb{Z}_3I_1 + \mathbb{Z}_3I_2$, \mathbb{Z}_3I_1 , and $\mathbb{Z}_3(I_1, I_2)$ are refined neutrosophic idempotent ideals. According to definition.3.18, $\mathbb{Z}_3(I_1, I_2)$ is a refined neutrosophic semiprime ideals. Conversely, according to the theorem.3.20, $\mathbb{Z}_3(I_1, I_2)$ is a refined neutrosophic fully semiprime.

Finally, Table 1 depicts the key distinctions between the classical and refined neutrosophic rings.

Table 1. Key distinctions between the classical and refined neutrosophic rings.

R	R(I₁, I₂)
R is a field $\Leftrightarrow \{0\}$, R are only ideals in R.	R(I₁, I₂) is a refined neutrosophic field $\Leftrightarrow \{0\}, \mathbf{R}I_1 + \mathbf{R}I_2, \mathbf{R}I_1, \mathbf{R}(I_1, I_2)$ are only refined neutrosophic ideals.
R is a prime ring if $\{0\} \in \wp_R$	R(I₁, I₂) is a refined neutrosophic prime ring if $\mathbf{R}I_1 + \mathbf{R}I_2 \in \mathbf{RN}\wp_{R(I_1, I_2)}$
$R(I_1, I_2)$ is a fully prime ring if $\mathfrak{I}_R = \wp_R$.	R(I₁, I₂) is a fully prime ring if $\mathbf{RN}\mathfrak{I}_{R(I_1, I_2)} \setminus \{0\} = \mathbf{RN}\wp_{R(I_1, I_2)}$.

4. Conclusion and future works

In this study, the structure and properties of all prime, completely prime, semiprime, and completely semiprime ideals in refined neutrosophic rings were determined. Herein, we present the

concept of fully prime (fully prime) and fully semiprime (fully semiprime) refined neutrosophic rings. In addition, many examples were built to clarify the validity of this work. Certainly, these ideals will find applications in all places where they find their applications, with some indeterminacy. In the future, we plan to generalize the prime (completely prime, semiprime, and completely semiprime) ideals of the n -refined neutrosophic rings.

Acknowledgments

The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Received: 13 Nov 2023, **Revised:** 21 Nov 2023,

Accepted: 04 Dec 2023, **Available online:** 08 Dec 2023.



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