Neutrosophic Systems with Applications, Vol. 13, 2024
https://doi.org/10.61356/j.nswa.2024.106

The Energy of Interval-Valued Complex Neutrosophic Graph Structures: Framework, Application and Future Research Directions

S.N. Suber Bathusha [1,*], Sowndharya Jayakumar [2], and S. Angelin Kavitha Raj [3]

[1] Department of Mathematics, Sadakathullah Appa College, Tirunelveli, Tamil Nadu, India.
Emails: mohamed.suber.96@gmail.com; mohamed.suber.96@sadakath.ac.in.
[2] Department of Bachelor of Computer Applications, Cauvery College for Women, Trichy, Tamil Nadu, India; sowndharyamail03@gmail.com.
[3] Department of Mathematics, Sadakathullah Appa College, Tirunelveli, Tamil Nadu, India.
Emails: angelinkavitha.s@gmail.com; angelinkavitha.s@sadakath.ac.in.

* Correspondence: mohamed.suber.96@gmail.com; mohamed.suber.96@sadakath.ac.in.

Abstract: Graph structure is a developing field with many real-world applications and advancements, particularly effective frameworks for integrative problem-solving in computer networks and artificial intelligence systems. To define the idea of an Interval-Valued Complex Neutrosophic Graph Structure (IVCNGS), the concept of an Interval-Valued Complex Neutrosophic Set (IVCNS) is applied to the graph structure. Using the adjacency matrix to calculate the degree of vertex, we have defined some findings about the IVCNGS. Further, we compute the energy and Laplacian energy of IVCNGS. Moreover, we derive the lower and upper bounds for the energy and Laplacian energy of IVCNGS, and we have discussed their application in IVCNGS. Finally, we develop an algorithm that clarifies the fundamental processes of the application.

Keywords: Graph Structure; Interval-Valued Complex Neutrosophic Graph Structure; Energy and Laplacian Energy; Applications.

1. Introduction

Real-world problems with uncertainty and ambiguity are not always amenable to the standard techniques of classical mathematics. The concept of a fuzzy set (FS) was first proposed by Zadeh [1] in 1965 as an extension of the conventional notion of sets. A gradual determination of an element's membership in a set is allowed by the fuzzy set theory, as represented by a membership function with a value in the real unit interval [0, 1]. Since then, numerous scholars have investigated the concept of fuzzy logic and fuzzy sets to resolve a range of ambiguous and uncertain real-world problems. Interval-valued fuzzy sets are the development that the author initiated in Turksen [2] in 1986. As a result of using numbers as the membership function, it also takes into account the values of number intervals to account for uncertainty. Usually, it is indicated by the symbol $[\mu^{-}_{AL}(x), \mu^{+}_{AU}(x)]$. Use the equation $0 \leq \mu^{-}_{AL}(x) + \mu^{+}_{AU}(x) \leq 1$ to represent the degree of membership of the fuzzy set A.

Likewise, the membership function is single-valued and it is not always possible to use it to capture both support and objection evidence. The intuitionistic fuzzy set (IFS) was developed by Atanassov [3] as a generalization of Zadeh's fuzzy set. IFS, which has both a membership and a non-membership function, can be created by deriving a new component, the degree of membership and non-membership, from the fuzzy set's properties. When defining intuitionistic fuzzy sets, he also included interval-valued intuitionistic fuzzy sets [4] for representing uncertainty, interval-valued intuitionistic fuzzy sets instead of traditional fuzzy sets are preferred. Defuzzification, a technique employed in fuzzy control in many ways, is the phase of the process that needs the most processing.

To interpret the degree of true and false membership functions, it is defined as a pair of intervals $[\mu^-, \mu^+], 0 \leq \mu^- + \mu^+ \leq 1$ and $[\lambda^-, \lambda^+], 0 \leq \lambda^- + \lambda^+ \leq 1$ with $0 \leq \mu^+ + \lambda^+ \leq 1$.

On the other hand, erroneous, inconsistent, and incomplete periodic information cannot be handled by FSs, IFSs, or IVIFSs. Although these theories have applications in many different scientific domains, they are all hampered by the inability to accurately describe two-dimensional events. Ramot [5] proposed the concept of a complex fuzzy set (CFS) in 2012 to address this problem. A helpful generalization of FS is the membership grade of this concept, which is expressed as $re^{i\theta}$, where r stands for the amplitude term and θ for the phase term. Values are restricted to only derived from the complex plane's unit circle. The phase term of CFS is significant since it is better equipped to control cyclical difficulties or recurrent troublesome phenomena. There will undoubtedly be circumstances where the second dimension is required because the phase term is present in CFS. This phrase distinguishes CFS from every other kind of information that is currently available. This use best exemplifies the original notion with a CF representation of solar activity. The concepts of complex intuitionistic fuzzy sets (CIFSs), which they translated to complex intuitionistic fuzzy sets using the degree of complex-valued non-membership functions, were initially described by Alkouri and Salleh [6] in 2012. Complex Interval-Valued Intuitionistic Fuzzy Sets (CIVIFSs) and its associated Aggregation Operator are novel concepts introduced by Harish Garg and Dimple Rani [9]. It is defined as a pair of intervals $\left[\mu^- e^{i\alpha^-}, \mu^+ e^{i\alpha^+}\right], 0 \leq \mu^- + \mu^+ \leq 1, 0 \leq \alpha^- + \alpha^+ \leq 2\pi$ and $\left[\lambda^- e^{i\beta^-}, \lambda^+ e^{i\beta^-}\right], 0 \leq \lambda^- + \lambda^+ \leq 1, 0 \leq \beta^- + \beta^+ \leq 2\pi$ with $0 \leq \mu^+ + \lambda^+ \leq 1$ and $0 \leq \alpha^+ + \beta^+ \leq 1$ to interpret the complex degree of true and false membership functions.

Unfortunately, it is limited to processing incomplete and ambiguous data; it is unable to process inconsistent and ambiguous data, which is common in situations in the real world. It cannot handle the kind of ambiguous and indeterminate information that frequently arises in real-life situations; it can only handle partial and ambiguous information. Thus, Florentin Smarandache introduces the terms neutrosophic set, a unifying field in logics, and A Generalization of the intuitionistic fuzzy sets [7-11] and they are used in many domains to handle contradictory and ambiguous data. Truth membership, indeterminacy membership, and false membership are defined completely independently if the sum of these values in the neutrosophic set lies between 0 and 3. This is known as the indeterminacy value. Neutrosophy: Neutral Logic, Neutral Set, and Neutral Probability Give a more thorough explanation of the ideas of neutrosophy, set, logic, and neutrosophic probability. The neutrosophic set has quickly attracted the attention of many scholars because of the wide range of descriptive situations it covers. Additionally, this new set aids in controlling the ambiguity resulting from the neutrosophic scope. A comprehensive bibliometric examination of the neutrosophic collection is showcased, encompassing the years from 1998 to 2017. Mumtaz Ali and Florentin Smarandache developed the idea of a Complex neutrosophic set in 2016 [12]. When a set of real-valued amplitude terms for truth, indeterminacy, and falsehood are combined with their corresponding phase terms, we have a complex neutrosophic set. This set has a complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsehood membership function. The complex neutrosophic set extends the neutrosophic set. Moreover, Atiqe U. R., Muhammad.S, Florentin Smarandache, and Muhammad R. A. [13] present the development of hybrids of hypersoft sets with complex fuzzy sets, complex intuitionistic fuzzy sets, and complex neutrosophic sets in 2020.

Figure 1 presents the development of IVCNS, including the CS Crisp Set, FS Fuzzy Set, IFS Intuitionistic Fuzzy Set, IVFS Interval-Valued Fuzzy Set, CFS Complex Fuzzy Set, NS Neutrosophic Set, CIFS Complex Intuitionistic Fuzzy Set, CIVFS Complex Interval-Valued Fuzzy Set, CNS Complex Neutrosophic Set, CIVIFS Complex Interval-Valued Intuitionistic Fuzzy Set, and IVCNS Interval-Valued Complex Neutrosophic Set.

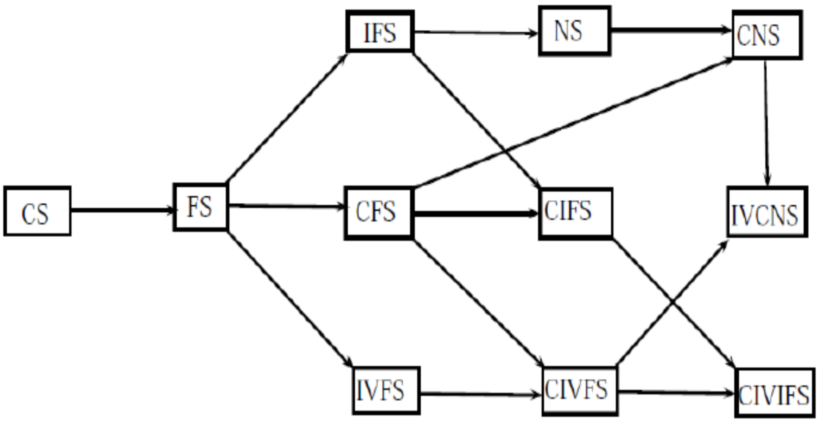

Figure 1. The development of IVNCNS.

Ivan Gutman and Bo Zhou [14] introduced the idea of a graph's Laplacian energy in 2006. Its definition is the sum of the absolute values of the adjacency matrix's eigenvalues for the graph. The energy of a graph is used in quantum theory and many other applications in the context of energy, and it is defined as the sum of the absolute values of the differences of the average vertex degree of the graph to the Laplacian eigenvalues of the graph. This is done by connecting the edge of a graph to the electron energy of a particular type of molecule. Rosenfeld [15] created fuzzy graph theory in 1975 and studied the fuzzy graphs that Kauffmann used to develop the basic idea in 1973. He explored some basic concepts in graph theory and established some of their characteristics. Bhattacharya [16] showed that the inferences from (crisp) graph theory are not always relevant to FGs in his remarks on FGs. Intuitionistic fuzzy relations and intuitionistic fuzzy graphs were introduced by Shannon and Atanassov in 1994. Fuzzy graphs with irregular interval values were examined by Rashmanlou [17]. Additionally, they defined fuzzy graphs [18] and various features of very irregular interval-valued fuzzy graphs. M.G. Karunambigai and K. Palanivel [19] first proposed the Edge Regular Intuitionistic Fuzzy Graph in 2015.

Thirunavukarasu et al. [20] created complex fuzzy graphs (CFGs) to handle uncertain and ambiguous relationships that have a periodic nature. According to Yaqoob et al. [21], complex intuitionistic fuzzy graphs (CIFGs) were defined. They looked into the homomorphisms of CIFG and demonstrated a CIFG application in cellular network provider companies to test their proposed approach. To broaden the concept of neutrosophic graphs and CIFGs, Yaqoob and Akram introduced complex neutrosophic graphs (CNGs) [22]. They covered several basic CNG functions and provided examples to illustrate them. They also presented the energy of CNGs. The concept of Complex Neutrosophic Hypergraphs: New Social Network Models was expounded upon in 2019 by Anam Luqman, Muhammad Akram, and Florentin Smarandache [23]. The best examples and motivation for CNS derive from two voting procedures, and they use this example to support the applicability of their proposed model in their introduction. Laplacian energy of fuzzy graphs is a concept introduced by Sharbaf and Fayazi [24], and some results on Laplacian energy bounds extend to fuzzy graphs. For more details, see the research papers by Soumitra Poulik and Ganesh Ghorai [25–28] on detour g-interior nodes and Detour g-boundary nodes in bipolar fuzzy graphs with applications, pragmatic results in Taiwan education system-based IVFG & IVNG, and empirical results on operations of Bipolar fuzzy graphs with their degree. Further, a note on "Bipolar fuzzy graphs with applications" was proposed in 2020. A graph structure can be produced by enlarging an undirected graph; this structure can then be used to investigate other sorts of structures, such as graphs and signed graphs. The concept of graph structures was first proposed by Sampath Kumar in his essay from 2006 [29]. The concept of a fuzzy graph structure was first proposed by T. Dinesh and T. V. Ramakrishnan in 2011 [30]. To use this model in IVCNGS, it can be rewritten in an abstract form. Muhammad Akram recently proposed the idea of Operations on Intuitionistic Fuzzy Graph Structures [31].

1.1 The framework of this research

This idea can be applied in IVCNGS after being restated abstractly. This work is structured as shown in Figure 2 and as follows:

- The concept of Interval-Valued Complex Neutrosophic Graph Structures (IVCNGS) is introduced in this work. Some results that we can share are that the IVCNGS adjacency matrix and the degree of vertex presence are being further examined.
- Further, the energy and Laplacian energy of IVCNGS are calculated. Also, we determine IVCNGS's energy and Laplacian energy upper and lower bounds.
- Moreover, IVCNGS applications and algorithm explanations were provided. Finally, an explanation of all these studies is provided in conclusion and future works.

In order for researchers to further investigate this theory using analysis of the energy and Laplacian energies of IVCNGS, we recommended readers to read this article.

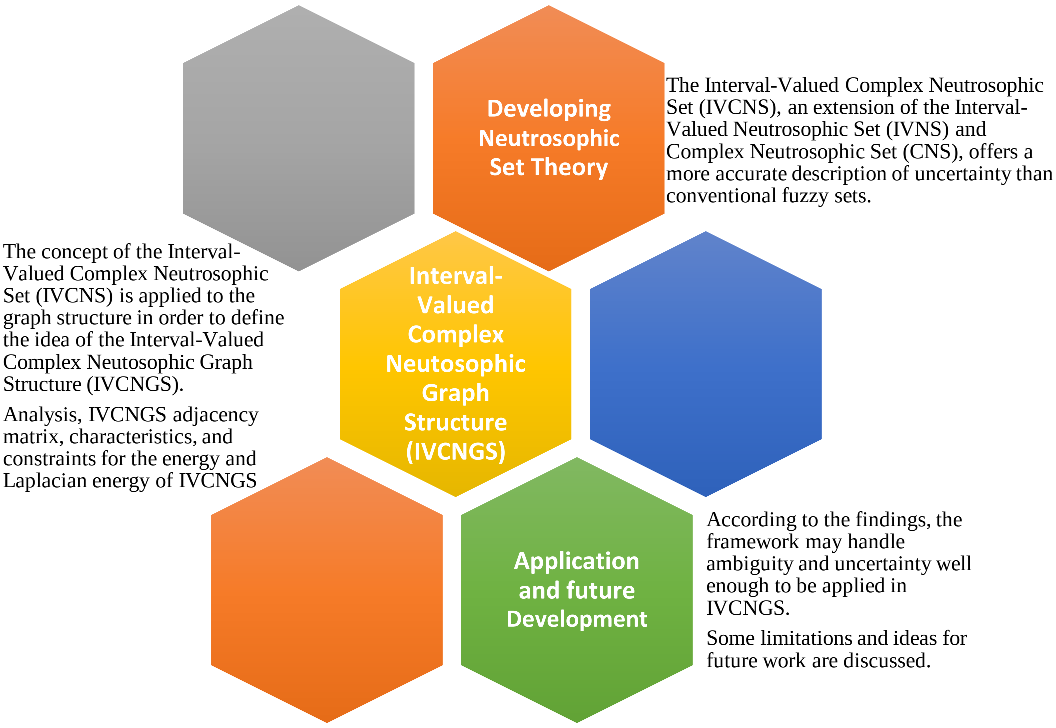

Figure 2. The development of IVNCNS.

2. Preliminaries

The development of the research work will be helped by the deliberation of some fundamental concepts and attributes in this field.

Definition 1. Let's say that conversation is the universe Y. Interval-Valued Complex Neutrosophic Set (IVCNS) A defined on Y is the object of the form.

$$A = \{(a, [\mu^-_{A_1}(a)e^{i\alpha^-_{A_1}(a)}, \mu^+_{A_1}(a)e^{i\alpha^+_{A_1}(a)}], [\mu^-_{A_2}(a)e^{i\alpha^-_{A_2}(a)}, \mu^+_{A_2}(a)e^{i\alpha^+_{A_2}(a)}], [\mu^-_{A_3}(a)e^{i\alpha^-_{A_3}(a)}, \mu^+_{A_3}(a)e^{i\alpha^+_{A_3}(a)}] : a \in Y\}, \textit{ where}$$

$i=\sqrt{-1},\ \mu_{A_1}^{-}(a),\mu_{A_1}^{+}(a),\mu_{A_2}^{-}(a),\mu_{A_2}^{+}(a),\mu_{A_3}^{-}(a),\mu_{A_3}^{+}(a)\in[0,1]$,
$\alpha_{A_1}^{-}(a),\alpha_{A_1}^{+}(a),\alpha_{A_2}^{-}(a),\alpha_{A_2}^{+}(a),\alpha_{A_3}^{-}(a),\alpha_{A_3}^{+}(a)\in[0,2\pi], 0\le\left(\left(\mu_{A_1}^{+}(a)\right)+\left(\mu_{A_2}^{+}(a)\right)+\left(\mu_{A_3}^{+}(a)\right)\right)\le 3.$

Definition 2. Let $A=\{(a,\left[\mu_{A_1}^{-}(a)e^{i\alpha_{A_1}^{-}(a)},\mu_{A_1}^{+}(a)e^{i\alpha_{A_1}^{+}(a)}\right],[\mu_{A_2}^{-}(a)e^{i\alpha_{A_2}^{-}(a)},\mu_{A_2}^{+}(a)e^{i\alpha_{A_2}^{+}(a)}],$ $\left[\mu_{A_3}^{-}(a)e^{i\alpha_{A_3}^{-}(a)},\mu_{A_3}^{+}(a)e^{i\alpha_{A_3}^{+}(a)}\right]:a\in Y\}$ and $B=\{(a,\left[\mu_{B_1}^{-}(a)e^{i\alpha_{B_1}^{-}(a)},\mu_{B_1}^{+}(a)e^{i\alpha_{B_1}^{+}(a)}\right]$ $,\left[\mu_{B_2}^{-}(a)e^{i\alpha_{B_2}^{-}(a)},\mu_{B_2}^{+}(a)e^{i\alpha_{B_2}^{+}(a)}\right],\left[\mu_{B_3}^{-}(a)e^{i\alpha_{B_3}^{-}(a)},\mu_{B_3}^{+}(a)e^{i\alpha_{B_3}^{+}(a)}\right]:a\in Y\}$ be the two IVCNSs in Y, then

- $A\subseteq B$ if and only if $\mu_{A_1}^{-}(a)\le\mu_{B_1}^{-}(a)$, $\mu_{A_1}^{+}(a)\le\mu_{B_1}^{+}(a)$, $\mu_{A_2}^{-}(a)\le\mu_{B_2}^{-}(a)$, $\mu_{A_2}^{+}(a)\le\mu_{B_2}^{+}(a)$ and $\mu_{A_3}^{-}(a)\le\mu_{B_3}^{-}(a)$, $\mu_{A_3}^{+}(a)\le\mu_{B_3}^{+}(a)$ for amplitude terms and $\alpha_{A_1}^{-}(a)\le\alpha_{B_1}^{-}(a)$, $\alpha_{A_1}^{+}(a)\le\alpha_{A_1}^{+}(a)$, $\alpha_{A_2}^{-}(a)\le\alpha_{B_2}^{-}(a)$, $\alpha_{A_2}^{+}(a)\le\alpha_{A_2}^{+}(a)$ and $\alpha_{A_3}^{-}(a)\le\alpha_{B_3}^{-}(a)$, $\alpha_{A_3}^{+}(a)\le\alpha_{A_3}^{+}(a)$ for phase terms, for all $a\in Y$;
- $A=B$ if and only if $\mu_{A_1}^{-}(a)=\mu_{B_1}^{-}(a)$, $\mu_{A_1}^{+}(a)=\mu_{B_1}^{+}(a)$, $\mu_{A_2}^{-}(a)=\mu_{B_2}^{-}(a)$, $\mu_{A_2}^{+}(a)=\mu_{B_2}^{+}(a)$ and $\mu_{A_3}^{-}(a)=\mu_{B_3}^{-}(a)$, $\mu_{A_3}^{+}(a)=\mu_{B_3}^{+}(a)$ for amplitude terms and $\alpha_{A_1}^{-}(a)=\alpha_{B_1}^{-}(a)$, $\alpha_{A_1}^{+}(a)=\alpha_{A_1}^{+}(a)$, $\alpha_{A_2}^{-}(a)=\alpha_{B_2}^{-}(a)$, $\alpha_{A_2}^{+}(a)=\alpha_{A_2}^{+}(a)$ and $\alpha_{A_3}^{-}(a)=\alpha_{B_3}^{-}(a)$, $\alpha_{A_3}^{+}(a)=\alpha_{A_3}^{+}(a)$ for phase terms, for all $a\in Y$;

For simplicity, the
$(\left[\mu_{A_1}^{-}(a)e^{i\alpha_{A_1}^{-}(a)},\mu_{A_1}^{+}(a)e^{i\alpha_{A_1}^{+}(a)}\right],\left[\mu_{A_2}^{-}(a)e^{i\alpha_{A_2}^{-}(a)},\mu_{A_2}^{+}(a)e^{i\alpha_{A_2}^{+}(a)}\right],\left[\mu_{A_3}^{-}(a)e^{i\alpha_{A_3}^{-}(a)},\mu_{A_3}^{+}(a)e^{i\alpha_{A_3}^{+}(a)}\right])$ *is called the IVCNS, where,* $\mu_{A_1}^{+},\mu_{A_2}^{+},\mu_{A_3}^{+}\in[0,1]$ *such that* $\mu_{A_1}^{+}+\mu_{A_2}^{+}+\mu_{A_3}^{+}\le 3$.

Definition 3. A Interval-valued complex Neutrosophic relation in Y is described as a IVCNS X in $Y\times Y$ and is characterised by:
$X=\{(ab,\left[\mu_{X_1}^{-}(ab)e^{i\alpha_{X_1}^{-}(ab)},\mu_{X_1}^{+}(ab)e^{i\alpha_{X_1}^{+}(ab)}\right],\left[\mu_{X_2}^{-}(ab)e^{i\alpha_{X_2}^{-}(ab)},\mu_{X_2}^{+}(ab)e^{i\alpha_{X_2}^{+}(ab)}\right],$ $\left[\mu_{X_3}^{-}(ab)e^{i\alpha_{X_3}^{-}(ab)},\mu_{X_3}^{+}(ab)e^{i\alpha_{X_3}^{+}(ab)}\right])/ab\in Y\times Y\}$ *where the Inter-valued complex Neutrosophic truth-membership, complex indeterminate-membership and complex false-membership functions of X are mapping to [0,1], such that* $0\le\mu_{X_1}^{+}(rs)+\mu_{X_2}^{+}(rs)+\mu_{X_3}^{+}(rs)\le 3$ *for all* $rs\in Y\times Y$.

Definition 4. On a non-empty set X, a Interval-valued complex Neutrosophic graph is a pair $G=(A,B)$, where A and B are complex Neutrosophic sets on X and a Interval-valued complex Neutrosophic relation on X, respectively, such that:

$$\begin{aligned}
(i)\mu_{B_1}^{-}(rs)e^{i\alpha_{B_1}^{-}(rs)} &\le min\{\mu_{A_1}^{-}(r),\mu_{A_1}^{-}(s)\}e^{imin\{\alpha_{A_1}^{-}(r),\alpha_{A_1}^{-}(s)\}}\\
(ii)\mu_{B_1}^{+}(rs)e^{i\alpha_{B_1}^{+}(rs)} &\le min\{\mu_{A_1}^{+}(r),\mu_{A_1}^{+}(s)\}e^{imin\{\alpha_{A_1}^{+}(r),\alpha_{A_1}^{+}(s)\}}\\
(iii)\mu_{B_2}^{-}(rs)e^{i\alpha_{B_2}^{-}(rs)} &\le max\{\mu_{A_2}^{-}(r),\mu_{A_2}^{-}(s)\}e^{imax\{\alpha_{A_2}^{-}(r),\alpha_{A_2}^{-}(s)\}}\\
(iv)\mu_{B_2}^{+}(rs)e^{i\alpha_{B_2}^{+}(rs)} &\le max\{\mu_{A_2}^{+}(r),\mu_{A_2}^{+}(s)\}e^{imax\{\alpha_{A_2}^{+}(r),\alpha_{A_2}^{+}(s)\}}\\
(v)\mu_{B_3}^{-}(rs)e^{i\alpha_{B_3}^{-}(rs)} &\le max\{\mu_{A_3}^{-}(r),\mu_{A_3}^{-}(s)\}e^{imax\{\alpha_{A_3}^{-}(r),\alpha_{A_3}^{-}(s)\}}\\
(vi)\mu_{B_3}^{+}(rs)e^{i\alpha_{B_3}^{+}(rs)} &\le max\{\mu_{A_3}^{+}(r),\mu_{A_3}^{+}(s)\}e^{imax\{\alpha_{A_3}^{+}(r),\alpha_{A_3}^{+}(s)\}}
\end{aligned}$$

$0\le\mu_{B_1}^{+}(rs)+\mu_{B_2}^{+}(rs)+\mu_{B_3}^{+}(rs)\le 3$ *for all* $rs\in Y\times Y$.

3. Energy of IVCNGS

In this part, the concept of routine IVCNGS is introduced. To further explain some of the fundamental IVCNGS features, examples are also provided.

Definition 5. Let $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is referred to as an IVCNGS of graph structure (GS) $\zeta^* = \{Q, R_1, R_2, \dots, R_k\}$ if $\eta = (\eta_1, \eta_2, \eta_3) = ([\eta_1^- e^{i\alpha_1^-}, \eta_1^+ e^{i\alpha_1^+}], [\eta_2^- e^{i\alpha_2^-}, \eta_2^+ e^{i\alpha_2^+}]\ [\eta_3^- e^{i\alpha_3^-}, \eta_3^+ e^{i\alpha_3^+}])$ is an IVCNS on Q and$\delta_J = (\delta_{1J}, \delta_{2J}, \delta_{3J}) = ([\delta_{1J}^- e^{i\beta_{1J}^-}, \delta_{1J}^+ e^{i\beta_{1J}^+}], [\delta_{2J}^- e^{i\beta_{2J}^-}, \delta_{2J}^+ e^{i\beta_{2J}^+}], [\delta_{3J}^- e^{i\beta_{3J}^-}, \delta_{3J}^+ e^{i\beta_{3J}^+}])$ are IVCNSs on Q and R_J such that

$$(i)\delta_{1J}^-(a,b)e^{i\beta_{1J}^-(a,b)} \le \min\{\eta_1^-(a), \eta_1^-(b)\}e^{i\min\{\alpha_1^-(a),\alpha_1^-(b)\}},$$

$$(ii)\delta_{1J}^+(a,b)e^{i\beta_{1J}^+(a,b)} \le \min\{\eta_1^+(a), \eta_1^+(b)\}e^{i\min\{\alpha_1^+(a),\alpha_1^+(b)\}},$$

$$(iii)\delta_{2J}^-(a,b)e^{i\beta_{2J}^-(a,b)} \le \max\{\eta_2^-(a), \eta_2^-(b)\}e^{i\max\{\alpha_2^-(a),\alpha_2^-(b)\}},$$

$$(iv)\delta_{2J}^+(a,b)e^{i\beta_{2J}^+(a,b)} \le \max\{\eta_2^+(a), \eta_2^+(b)\}e^{i\max\{\alpha_2^+(a),\alpha_2^+(b)\}},$$

$$(iv)\delta_{3J}^-(a,b)e^{i\beta_{3J}^-(a,b)} \le \max\{\eta_3^-(a), \eta_3^-(b)\}e^{i\max\{\alpha_3^-(a),\alpha_3^-(b)\}},$$

$$(iv)\delta_{3J}^+(a,b)e^{i\beta_{3J}^+(a,b)} \le \max\{\eta_3^+(a), \eta_3^+(b)\}e^{i\max\{\alpha_3^+(a),\alpha_3^+(b)\}},$$

$0 \le (\delta_{1J}^+(a,b)) + (\delta_{2J}^+(a,b)) + (\delta_{3J}^+(a,b)) \le 3$ and $(\beta_{1J}^+(ab)), (\beta_{2J}^+(ab)), (\beta_{3J}^+(ab)) \in [0,2\pi]\ \forall\ ab \in R_J, J = 1,2,\dots,k$.

Note : δ_{1J}^-, δ_{1J}^+, δ_{2J}^- , δ_{2J}^+ and δ_{3J}^- , δ_{3J}^+ are function from R_J to [0,1] such that $\delta_{1J}^-(a,b) \le \delta_{1J}^+(a,b)$, $\delta_{2J}^-(a,b) \le \delta_{2J}^+(a,b)$, $\delta_{3J}^-(a,b) \le \delta_{3J}^+(a,b)$, $\beta_{1J}^-(a,b) \le \beta_{1J}^+(a,b)$, $\beta_{2J}^-(a,b) \le \beta_{2J}^+(a,b)$ and $\beta_{3J}^-(a,b) \le \beta_{3J}^+(a,b)$ for all $(a,b) \in R_J, J = 1,2,\dots,k$.

Definition 6. The adjacency matrix $A\zeta = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ of a IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$, where $A\delta_J, (J = 1,2,\dots,k)$ is a square matrix as $[u_{jk}]$ in which $u_{jk} = ([\delta_{1J}^-(u_ju_k)e^{i\beta_{1J}^-(u_ju_k)}, \delta_{1J}^+(u_ju_k)e^{i\beta_{1J}^+(u_ju_k)}], [\delta_{2J}^-(u_ju_k)e^{i\beta_{2J}^-(u_ju_k)}, \delta_{2J}^+(u_ju_k)e^{i\beta_{2J}^+(u_ju_k)}],$

$[\delta_{3J}^-(u_ju_k)e^{i\beta_{3J}^-(u_ju_k)}, \delta_{3J}^+(u_ju_k)e^{i\beta_{3J}^+(u_ju_k)}])$, where $\delta_{1J}^-(u_ju_k), \delta_{1J}^+(u_ju_k)$ is represent the strength of interval-valued truth membership amplitude term and $\delta_{2J}^-(u_ju_k), \delta_{2J}^+(u_ju_k)$ is represent the strength of interval-valued indeterminate membership amplitude term between u_j and u_k and $\delta_{3J}^-(u_ju_k), \delta_{3J}^+(u_ju_k)$ is represent the strength of interval-valued false membership amplitude term between u_j and u_k and $\beta_{1J}^-(u_ju_k), \beta_{1J}^+(u_ju_k)$ is represent the strength of interval-valued truth membership phase term and $\beta_{2J}^-(u_ju_k), \beta_{2J}^+(u_ju_k)$ is represent the strength of interval-valued indeterminate membership phase term between u_j and u_k and $\beta_{3J}^-(u_ju_k), \beta_{3J}^+(u_ju_k)$ is represent the strength of interval-valued false membership phase term between u_j and u_k.

Definition 7. The adjacency matrix $A\zeta = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ of a IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. Then the δ_J - degree of vertex u in $A(\zeta)$ is defined as $Ad_{\delta_J}(u)e^{iAd_{\beta_J}(u)} =$

$([Ad_{\delta_{1J}^-}(u)e^{iAd_{\beta_{1J}^-}(u)}, Ad_{\delta_{1J}^+}(u)e^{iAd_{\beta_{1J}^+}(u)}],$

$$[Ad_{\delta_{2J}^-}(u)e^{iAd_{\beta_{2J}^-}(u)}, Ad_{\delta_{2J}^+}(u)e^{iAd_{\beta_{2J}^+}(u)}], [Ad_{\delta_{3J}^-}(u)e^{iAd_{\beta_{3J}^-}(u)}, Ad_{\delta_{3J}^+}(u)e^{iAd_{\beta_{3J}^+}(u)}]),$$

$$Ad_{\delta_{1J}^-}(u)e^{iAd_{\beta_{1J}^-}(u)} = \left(\sum_{z=1}^{k} \delta_{1J}^-\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{1J}^-(u_{jz})}, Ad_{\delta_{1J}^+}(u)e^{iAd_{\beta_{1J}^+}(u)} = \left(\sum_{z=1}^{k} \delta_{1J}^+\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{1J}^+(u_{jz})},$$

$$Ad_{\delta_{2J}^-}(u)e^{iAd_{\beta_{2J}^-}(u)} = \left(\sum_{z=1}^{k} \delta_{2J}^-\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{2J}^-(u_{jz})}, Ad_{\delta_{2J}^+}(u)e^{iAd_{\beta_{2J}^+}(u)} = \left(\sum_{z=1}^{k} \delta_{2J}^{\mp}\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{2J}^+(u_{jz})},$$

$$Ad_{\delta_{3J}^-}(u)e^{iAd_{\beta_{3J}^-}(u)} = \left(\sum_{z=1}^{k} \delta_{3J}^-\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{3J}^-(u_{jz})}, Ad_{\delta_{3J}^+}(u)e^{iAd_{\beta_{3J}^+}(u)} = \left(\sum_{z=1}^{k} \delta_{3J}^+\left(u_{jz}\right)\right)e^{\sum_{z=1}^{k}\beta_{3J}^+(u_{jz})},$$

$\forall J = 1,2,\dots,k.$

Example 1. An IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ of a GS $\zeta^* = (Q, R_1, R_2)$ given Figure 3 is a IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ such that $\eta = \{u_1([.4e^{i.3\pi}, .7e^{i.4\pi}], [.3e^{i.1\pi}, .6e^{i.3\pi}], [.2e^{i.1\pi}, .4e^{i.3\pi}]),$

$u_2([.4e^{i.2\pi}, .6e^{i.4\pi}], [.3e^{i..5\pi}, .5e^{i.5\pi}], [.4e^{i.3\pi}, .6e^{i.4\pi}]), u_3([.5e^{i.3\pi}, .6e^{i.4\pi}], [.5e^{i.1\pi}, .7e^{i.2\pi}], [.3e^{i.4\pi}, .4e^{i.5\pi}]),$

$u_4([.3e^{i.6\pi}, .6e^{i.7\pi}], [.4e^{i.4\pi}, .5e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]).$

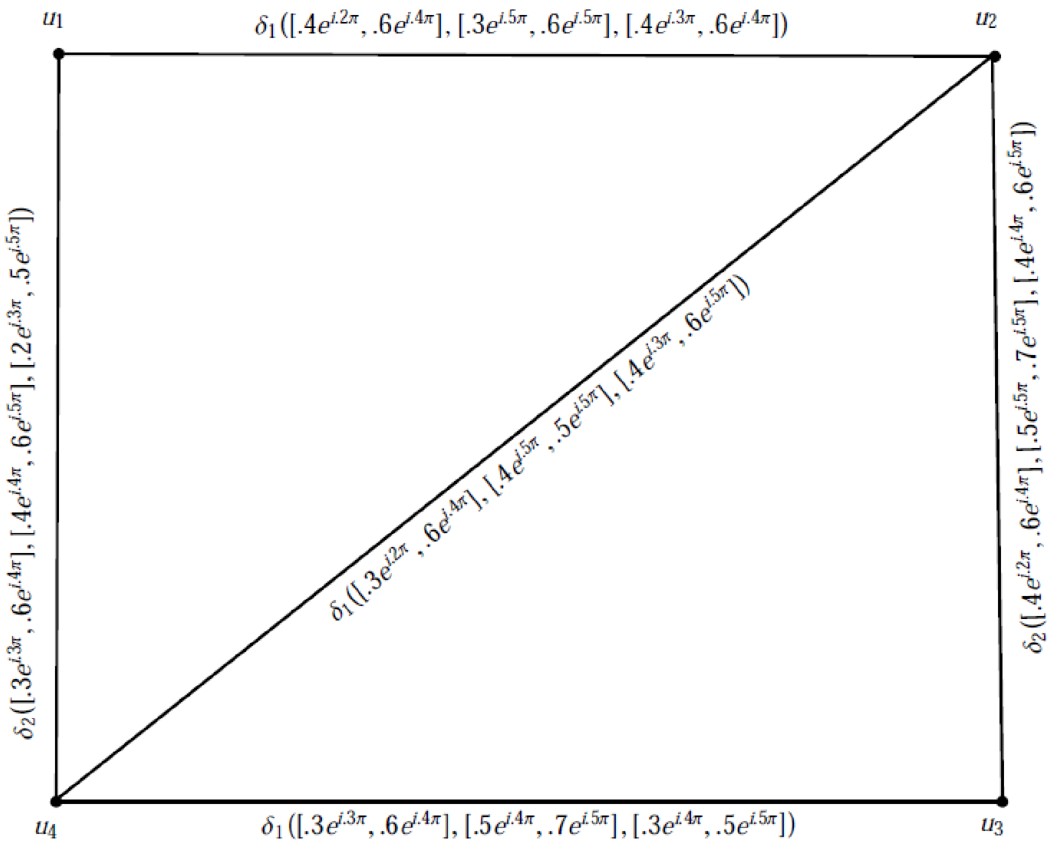

Figure 3. The adjacency matrix of the amplitude term of an IVCNGS.

The adjacency matrix of the amplitude term of an IVCNGS given in Figure 3 is:

$$A\delta_1 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .5 \\ .4 & .6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .5 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The adjacency matrix of the amplitude term of an IVCNGS given in Figure 3 is
The $\delta_1 -$ degree of vertex u_i in $A(\zeta)$ is (i=1, 2, 3, 4).

$$Ad_{\delta_1}(u_1) = \left(\left[Ad_{\delta_{11}^-}(u_1), Ad_{\delta_{11}^+}(u_1)\right], \left[Ad_{\delta_{21}^-}(u_1), Ad_{\delta_{21}^+}(u_1)\right], \left[Ad_{\delta_{31}^-}(u_1), Ad_{\delta_{31}^+}(u_1)\right]\right)$$

$$Ad_{\delta_1}(u_1)=([.4e^{i.2\pi}, .6e^{i.4\pi}], [.3e^{i.5\pi}, .6e^{i.5\pi}], [.4e^{i.3\pi}, .6e^{i.4\pi}]),$$

$$Ad_{\delta_1}(u_2) = ([.7e^{i.4\pi}, 1.2e^{i.8\pi}], [.7e^{i1.0\pi}, 1.1e^{i1.0\pi}], [.8e^{i.6\pi}, 1.2e^{i1.0\pi}]),$$

$$Ad_{\delta_1}(u_3) = ([.3e^{i.3\pi}, .6e^{i.4\pi}], [.5e^{i.4\pi}, .7e^{i.5\pi}], [.3e^{i.4\pi}, .5e^{i.5\pi}]),$$

The adjacency matrix of the phase term of an IVCNGS given in Figure 3 is:

$$A\beta_1 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .6 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

$$Ad_{\delta_1}(u_4) = ([.6e^{i.5\pi}, 1.2e^{i.8\pi}], [.9e^{i.9\pi}, 1.2e^{i1.0\pi}], [.7e^{i.7\pi}, 1.1e^{i1.1\pi}]).$$

Similarly, we calculate, the adjacency matrix of adjacency matrix of amplitude term of an IVCNGS given in Figure 3 is:

$$A\delta_2 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .6 \\ .2 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .5 & .7 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .5 & .7 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .3 & .6 \\ .4 & .6 \\ .2 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The adjacency matrix of the phase term of an IVCNGS given in Figure 3 is:

$$A\beta_2 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .3 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The $\delta_2 -$ degree of vertex u_i in $A(\zeta)$ is (i=1, 2, 3, 4).

$$Ad_{\delta_2}(u_1) = \left(\left[Ad_{\delta_{12}^-}(u_1), Ad_{\delta_{12}^+}(u_1)\right], \left[Ad_{\delta_{22}^-}(u_1), Ad_{\delta_{22}^+}(u_1)\right], \left[Ad_{\delta_{32}^-}(u_1), Ad_{\delta_{32}^+}(u_1)\right]\right)$$

$$Ad_{\delta_2}(u_1) = ([.3e^{i.3\pi}, .6e^{i.4\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_2) = ([.4e^{i.2\pi}, .6e^{i.4\pi}], [.5e^{i.5\pi}, .7e^{i.5\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_3) = ([.4e^{i.2\pi}, .6e^{i.4\pi}], [.5e^{i.5\pi}, .7e^{i.5\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_4) = ([.3e^{i.3\pi}, .6e^{i.4\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]).$$

Definition 8. The spectrum of an adjacency matrix of an IVCNGS is defined as $\langle P_1, Q_1, P_2, Q_2, P_3, Q_3 \rangle$, where $P_1, Q_1, P_2, Q_2, P_3, Q_3$ is the amplitude term of the set eigenvalues of $A(\zeta)$ and $\langle P_1', Q_1', P_2', Q_2', P_3', Q_3' \rangle$, where $P_1', Q_1', P_2', Q_2', P_3', Q_3'$ is the phase term of the set eigenvalues of $A(\zeta)$ respectively.

Example 2. The spectrum of IVCPFGS, given in Figure 3 follows.

$$Spec\left(A\delta_{11}^-(u_j, u_k)\right) = \{-0.5389, -0.2227, 0.2227, 0.5389\},$$

$$Spec\left(A\delta_{11}^+(u_j, u_k)\right) = \{-0.9708, -0.3708, 0.3708, 0.9708\},$$

$$Spec\left(A\delta_{21}^-(u_j, u_k)\right) = \{-0.6708, -0.2236, 0.2236, 0.6708\},$$

$$Spec\left(A\delta_{21}^+(u_j, u_k)\right) = \{-0.9514, -0.4415, 0.4415, 0.9514\},$$

$$Spec\left(A\delta_{31}^-(u_j, u_k)\right) = \{-0.6093, -0.1970, 0.1970, 0.6093\},$$

$$Spec\left(A\delta_{31}^+(u_j, u_k)\right) = \{-0.9306, -0.3224, 0.3224, 0.9306\},$$

$$Spec\left(A\beta_{11}^-(u_j, u_k)\right) = \{-0.3811, -0.1575, 0.1575, 0.3811\},$$

$$Spec\left(A\beta_{11}^+(u_j, u_k)\right) = \{-0.6472, -0.2472, 0.2472, 0.6472\},$$

$$Spec\left(A\beta_{21}^-(u_j, u_k)\right) = \{-0.7697, -0.2598, 0.2598, 0.7697\},$$

$$Spec\left(A\beta_{21}^+(u_j, u_k)\right) = \{-0.8090, -0.3090, 0.3090, 0.8090\},$$

$$Spec\left(A\beta_{31}^-(u_j, u_k)\right) = \{-0.5389, -0.2227, 0.2227, 0.5389\},$$

$$Spec\left(A\beta_{31}^+(u_j, u_k)\right) = \{-0.8450, -0.2367, 0.2367, 0.8450\}.$$

Therefore, the spectrum of amplitude term is

$$Spec(A(\delta_1)) = \{\langle -0.5389, -0.9708, -0.6708, -0.9514, -0.6093, -0.9306 \rangle,$$

$$\langle -0.2227, -0.3708, -0.2236, 0.4415, -0.1970, -0.3224 \rangle,$$
$$\langle 0.2227, 0.3708, 0.2236, 0.4415, 0.1970, 0.3224 \rangle,$$
$$\langle 0.5389, 0.9708, 0.6708, 0.9514, 0.6093, 0.9306 \rangle\}$$

The spectrum of phase terms is

$$Spec(A(\beta_1)) = \{\langle -0.3811, -0.6472, -0.7697, -0.8090, -0.5389, -0.8450 \rangle,$$
$$\langle -0.1575, -0.2472, -0.2598, 0.3090, -0.2227, -0.2367 \rangle,$$
$$\langle 0.1575, 0.2472, 0.2598, 0.3090, 0.2227, 0.2367 \rangle,$$
$$\langle 0.3811, 0.6472, 0.7697, 0.8090, 0.5389, 0.8450 \rangle\}$$

Similarly, we calculate

The spectrum of amplitude term is

$$Spec(A(\delta_2)) = \{\langle -0.4000, -0.6000, -0.5000, -0.7000, -0.4000, -0.6000 \rangle,$$
$$\langle -0.3000, -0.6000, -0.4000, -0.6000, -0.2000, -0.5000 \rangle,$$
$$\langle 0.3000, 0.6000, 0.4000, 0.6000, 0.2000, 0.5000 \rangle,$$
$$\langle 0.4000, 0.6000, 0.5000, 0.6000, 0.4000, 0.6000 \rangle\}$$

The spectrum of phase terms is

$$Spec(A(\beta_2)) = \{\langle -0.3000, -0.4000, -0.5000, -0.5000, -0.5000, -0.5000 \rangle,$$
$$\langle -0.2000, -0.4000, -0.4000, -0.5000, -0.3000, -0.5000 \rangle,$$
$$\langle 0.2000, 0.4000, 0.4000, 0.5000, 0.3000, 0.5000 \rangle,$$
$$\langle 0.3000, 0.4000, 0.5000, 0.5000, 0.5000, 0.5000 \rangle\}$$

Definition 9. The energy of amplitude term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as the following;

$$\epsilon(\zeta) =< \epsilon(A\delta_1), \epsilon(A\delta_2), \ldots, \epsilon(A\delta_k) >$$
$$\epsilon(A\delta_J) = \left(\sum_{i=1}^{n}(\mu_i^-)_{\delta_J}, \sum_{i=1}^{n}(\mu_i^+)_{\delta_J}, \sum_{i=1}^{n}(\lambda_i^-)_{\delta_J}, \sum_{i=1}^{n}(\lambda_i^+)_{\delta_J}, \sum_{i=1}^{n}(\chi_i^-)_{\delta_J}, \sum_{i=1}^{n}(\chi_i^+)_{\delta_J}\right), \forall J = 1,2, \ldots, k,$$

and the energy of phase term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as the following;

$$\epsilon(\zeta) =< \epsilon(A\beta_1), \epsilon(A\beta_2), \ldots, \epsilon(A\beta_k) >$$
$$\epsilon(A\beta_J) = \left(\sum_{i=1}^{n}(\vartheta_i^-)_{\beta_J}, \sum_{i=1}^{n}(\vartheta_i^+)_{\beta_J}, \sum_{i=1}^{n}(\rho_i^-)_{\beta_J}, \sum_{i=1}^{n}(\rho_i^+)_{\beta_J}, \sum_{i=1}^{n}(\gamma_i^-)_{\beta_J}, \sum_{i=1}^{n}(\gamma_i^+)_{\beta_J}\right), \forall J = 1,2, \ldots, k.$$

Example 3. The energy of amplitude term of an IVCNGS ζ given in Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(A\delta_1), \epsilon(A\delta_2) >$$
$$\epsilon(A\delta_1) =< 1.5232, 2.6833, 1.7889, 2.7857, 1.6125, 2.5060 >$$
$$\epsilon(A\delta_2) =< 1.4000, 2.4000, 1.8000, 2.6000, 1.2000, 2.2000 >$$

The energy of phase term of an IVCNGS ζ given in Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(A\beta_1), \epsilon(A\beta_2) >$$
$$\epsilon(A\beta_1) =< 1.0770, 1.7889, 2.0591, 2.2361, 1.5232, 2.1633 >$$
$$\epsilon(A\beta_2) =< 1.0000, 1.6000, 1.8000, 2.0000, 1.6000, 2.0000 >$$

Theorem 10. Let $A(\zeta) = \{A\delta_1, A\delta_2, \ldots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \geq (\mu_2^-)_{\delta_J} \geq \ldots \geq (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \geq (\mu_2^+)_{\delta_J} \geq \ldots \geq (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \geq (\lambda_2^-)_{\delta_J} \geq \ldots \geq (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \geq (\lambda_2^+)_{\delta_J} \geq \ldots \geq (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \geq (\chi_2^-)_{\delta_J} \geq \ldots \geq (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \geq (\chi_2^+)_{\delta_J} \geq \ldots \geq (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms, $(\vartheta_1^-)_{\beta_J} \geq (\vartheta_2^-)_{\beta_J} \geq \ldots \geq (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \geq (\vartheta_2^+)_{\beta_J} \geq \ldots \geq (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \geq (\rho_2^-)_{\beta_J} \geq \ldots \geq (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \geq (\rho_2^+)_{\beta_J} \geq \ldots \geq (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \geq (\gamma_2^-)_{\beta_J} \geq \ldots \geq (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \geq (\gamma_2^+)_{\beta_J} \geq \ldots \geq (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms. Then (i). $\sum_{i=1}^{n}(\mu_i^-)_{\delta_J} = \sum_{i=1}^{n}(\mu_i^+)_{\delta_J} = \sum_{i=1}^{n}(\lambda_i^-)_{\delta_J} = \sum_{i=1}^{n}(\lambda_i^+)_{\delta_J} = \sum_{i=1}^{n}(\chi_i^-)_{\delta_J} = \sum_{i=1}^{n}(\chi_i^+)_{\delta_J} = 0$ and $\sum_{i=1}^{n}(\vartheta_i^-)_{\beta_J} = \sum_{i=1}^{n}(\vartheta_i^+)_{\beta_J} = \sum_{i=1}^{n}(\rho_i^-)_{\beta_J} = \sum_{i=1}^{n}(\rho_i^+)_{\beta_J} =$

$$\sum_{i=1}^{n}(\gamma_i^-)_{\beta_J} = \sum_{i=1}^{n}(\gamma_i^+)_{\beta_J} = 0$$

(ii). $\sum_{i=1}^{n}(\mu_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{1J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\mu_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{1J}^+(u_j,u_k)\right)^2,$

$$\sum_{i=1}^{n}(\lambda_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\lambda_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\chi_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\chi_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^+(u_j,u_k)\right)^2, and$$

$$\sum_{i=1}^{n}(\vartheta_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\vartheta_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\rho_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\rho_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\gamma_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\gamma_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^+(u_j,u_k)\right)^2,$$

$$\forall J = 1,2,\ldots,k.$$

Proof (i) since $A(\zeta)$ is a symmetric matrix with zero trace, its eigenvalues are real and have a total value of zero. (ii) By the trace properties of the matrix, we have:

$$tr\left(\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)^2\right) = \sum_{i=1}^{n}(\mu_i^S)^2_{\delta_J}, \text{ where}$$

$$tr\left(\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)^2\right) = (0 + \left(\delta_{1J}^S(u_1u_2)\right)^2 + \cdots + \left(\delta_{1J}^S(u_1u_n)\right)^2,$$

$$+\left(\delta_{1J}^S(u_2u_1)\right)^2 + \cdots + \left(\delta_{1J}^S(u_1u_n)\right)^2,$$

$$\vdots$$

$$+\left(\delta_{1J}^S(u_nu_1)\right)^2 + \left(\delta_{1J}^S(u_nu_2)\right)^2 + \cdots + 0)$$

$$= 2\sum_{1\le j<k\le n}\left(\delta_{1J}^S(u_j,u_k)\right)^2$$

Similarly, we prove that

$$\sum_{i=1}^{n}(\lambda_i^S)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^S(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\chi_i^S)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{3J}^S(u_j,u_k)\right)^2$$

$$\text{and} \sum_{i=1}^{n}(\vartheta_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^S(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\rho_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^S(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\gamma_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{3J}^S(u_j,u_k)\right)^2, \forall\, S = -,+ \text{ and } J = 1,2,\ldots,k.$$

Example 4. Next, we show the example of the above Theorem 10. Let us consider $A(\zeta) = \{A\delta_1, A\delta_2\}$ be an adjacency matrix of an IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ as shown in Figure 3 in Example 1. Then:

(i). $\sum_{i=1}^{n}(\mu_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(\lambda_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(\chi_i^S)_{\delta_J} = 0$ and

$$\sum_{i=1}^{n}(\vartheta_i^S)_{\beta_J} = 0, \sum_{i=1}^{n}(\rho_i^S)_{\beta_J} = 0, \sum_{i=1}^{n}(\gamma_i^S)_{\beta_J} = 0, \forall\, S = -,+ \text{ and } J = 1,2.$$

(ii). $\sum_{u_ju_k\in R_1}(\mu_i^-)_{\delta_1}^2 = 0.6800 = 2(0.34) = 2\sum_{u_ju_k\in R_1}\left(\delta_{11}^-(u_j,u_k)\right)^2,$

$$\sum_{u_ju_k\in R_1}(\mu_i^+)_{\delta_1}^2 = 2.1600 = 2(1.08) = 2\sum_{u_ju_k\in R_1}\left(\delta_{11}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\lambda_i^-)_{\delta_1}^2 = 1.0000 = 2(0.5) = 2\sum_{u_ju_k\in R_1}\left(\delta_{21}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\lambda_i^+)_{\delta_1}^2 = 2.2000 = 2(1.1) = 2\sum_{u_ju_k\in R_1}\left(\delta_{21}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\chi_i^-)_{\delta_1}^2 = 0.8200 = 2(0.41) = 2\sum_{u_ju_k\in R_1}\left(\delta_{31}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\chi_i^+)_{\delta_1}^2 = 1.9400 = 2(0.97) = 2\sum_{u_ju_k\in R_1}\left(\delta_{31}^+(u_j,u_k)\right)^2, \text{and}$$

$$\sum_{u_ju_k\in R_1}(\vartheta_i^-)_{\beta_1}^2 = 3.4000 = 2(0.17) = 2\sum_{u_ju_k\in R_1}\left(\beta_{11}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\vartheta_i^+)_{\beta_1}^2 = 0.9600 = 2(0.48) = 2\sum_{u_ju_k\in R_1}\left(\beta_{11}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\rho_i^-)_{\beta_1}^2 = 1.3200 = 2(0.66) = 2\sum_{u_ju_k\in R_1}\left(\beta_{21}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\rho_i^+)_{\beta_1}^2 = 1.5000 = 2(0.75) = 2\sum_{u_ju_k\in R_1}\left(\beta_{21}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\gamma_i^-)_{\beta_1}^2 = 0.6800 = 2(0.34) = 2\sum_{u_ju_k\in R_1}\left(\beta_{31}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_ju_k\in R_1}(\gamma_i^+)_{\beta_1}^2 = 1.3200 = 2(0.66) = 2\sum_{u_ju_k\in R_1}\left(\beta_{31}^+(u_j,u_k)\right)^2$$

Similarly, we calculate $J = 2$.

Theorem 11. Let $A(\zeta) = \{A\delta_1, A\delta_2, \ldots, A\delta_k\}$ be an adjacency matrix of an IVCPFGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. Then:

(i). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{1J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2},$

(ii). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{2J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{2J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2},$

(iii). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{3J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{3J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2},$

(iv). $\sqrt{2\sum_{u_j u_k \in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2 + n(n-1)\ \mathrm{mod}\left(det\left(A\left(\beta_{1J}^{S}(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \leq \epsilon\left(\beta_{1J}^{S}(u_j,u_k)\right) \leq \sqrt{2n\sum_{u_j u_k \in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2}$,

(v). $\sqrt{2\sum_{u_j u_k \in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2 + n(n-1)\ \mathrm{mod}\left(det\left(A\left(\beta_{2J}^{S}(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \leq \epsilon\left(\beta_{2J}^{S}(u_j,u_k)\right) \leq \sqrt{2n\sum_{u_j u_k \in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2}$,

(vi). $\sqrt{2\sum_{u_j u_k \in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2 + n(n-1)\ \mathrm{mod}\left(det\left(A\left(\beta_{3J}^{S}(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \leq \epsilon\left(\beta_{3J}^{S}(u_j,u_k)\right) \leq \sqrt{2n\sum_{u_j u_k \in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2}$, $\forall S = -,+$ and $J = 1,2,\ldots,k$.

Proof. (i) Upper bound:

The following results are obtained by applying the Cauchy-Schwarz inequality to the vectors $(1,1,\ldots,1)$ and $\left(\mathrm{mod}\,(\mu_1^S),\ \mathrm{mod}\,(\mu_2^S),\ldots,\ \mathrm{mod}\,(\mu_n^S)\right)$ with n entries, we get:

$$\sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S) \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S)^2} \quad (1)$$

$$\left(\sum_{i=1}^{n}\mu_i^S\right)^2 = \sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S)^2 + 2\sum_{1\leq i<j\leq n}\mu_i^S\mu_j^S \quad (2)$$

By comparing the coefficients of $(\mu^S)^{n-2}$ in the characteristic polynomial:

$$\prod_{i=1}^{n}(\mu^S - \mu_i^S) = \mathrm{mod}\,(A(\zeta) - \mu^S I), \text{ we have:}$$

$$\sum_{1\leq i\leq j\leq n}\mu_i^S\mu_j^S = -\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 \quad (3)$$

Substituting 3 in 2, we obtain:

$$\sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S)^2 = 2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 \quad (4)$$

Substituting 4 in 1, we obtain:

$$\sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S) = \sqrt{n}\sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2} = \sqrt{2n\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}$$

$$\text{Therefore, } \epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \leq \sqrt{2n\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}$$

Lower bound:

$$\left(\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right)\right)^2 = \left(\sum_{i=1}^{n}\mu_i^S\right)^2 = \sum_{i=1}^{n}\mathrm{mod}\,(\mu_i^S)^2 + 2\sum_{1\leq i<j\leq n}\mathrm{mod}\,\left(\mu_i^S\mu_j^S\right)$$

$$2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 + \frac{2n(n-1)}{2}AM\{\ \mathrm{mod}\,\left(\mu_i^S\mu_j^S\right)\}$$

$$\text{Since, } AM\{\ \mathrm{mod}\,\left(\mu_i^S\mu_j^S\right)\} \geq GM\{\ \mathrm{mod}\,\left(\mu_i^S\mu_j^S\right)\},\ 1\leq i\leq j\leq n,$$

$$\text{So, } \epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \geq \sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n(n-1)GM\{\bmod\left(\mu_i^S\mu_j^S\right)\}}$$

Also since:

$$GM\{\bmod\left(\mu_i^S\mu_j^S\right)\} = \left(\prod_{1\leq i<j\leq n}\bmod\left(\mu_i^S\mu_j^S\right)\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n}\bmod\left(\mu_i^S\right)^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$\left(\prod_{i=1}^{n}\bmod\left(\mu_i^S\right)\right)^{\frac{2}{n}} = \bmod\left(det\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)\right)^{\frac{2}{n}}$$

$$\text{Therefore } \epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \geq \sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)\right)^{\frac{2}{n}}}$$

$$\text{Thus, } \sqrt{2\sum_{u_j,u_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{1J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \leq$$

$$\epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \leq \sqrt{2n\sum_{u_j,u_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2}, \forall S = -,+ \text{ and } J = 1,2,\dots,k.$$

Likewise, we can demonstrate that (ii), (iii), (iv), (v), and (vi).

Theorem 12. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. If $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2$, and $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2$, Then:

(i). $$\epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(ii). $$\epsilon\left(\delta_{2J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(iii). $$\epsilon\left(\delta_{3J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(iv). $$\epsilon\left(\beta_{1J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(v). $\epsilon\left(\beta_{2J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$

(vi). $\epsilon\left(\beta_{3J}^S(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2}{n}\right)^2\right\}}$

$\forall\ S=-,+$ and $J=1,2,\ldots,k$.

Proof. If $A=\left[a_{jk}\right]_{n\times n}$ is a symmetric matrix with zero trace, then $\mu_{max}^S \geq \frac{2\sum_{u_ju_k\in R_J} a_{jk}}{n}$, where μ_{max}^S is the maximum eigenvalue of A. If $A(\zeta)$ is the adjacency matrix of an IVCNGS ζ, then $\mu_1^S \geq \frac{2\sum_{u_ju_k\in R_J}\delta_{1J}^S(u_j,u_k)}{n}$, where $\mu_1^S \geq \mu_2^S \geq \ldots \geq \mu_n^S$.

$$\text{Moreover, since} \sum_{i=1}^{n}(\mu_i^S)^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2$$

$$\sum_{i=2}^{n}(\mu_i^S)^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 - (\mu_1^S)^2 \qquad (5)$$

With the vectors $(1,1,\ldots,1)$ and $\left(\bmod(\eta_1^S), \bmod(\eta_2^S), \ldots, \bmod(\eta_n^S)\right)$ with $n-1$ entries, the Cauchy-Schwarz inequality is applied, and the following result is obtained:

$$\epsilon\left(\delta_{1J}^S(u_j,u_k)\right) - \mu_1^S = \sum_{i=2}^{n}\bmod(\mu_i^S) \leq \sqrt{(n-1)\sum_{i=2}^{n}\bmod(\mu_1^S)^2} \qquad (6)$$

Substituting 5 in 6, we must have:

$$\epsilon\left(\delta_{1J}^S(u_j,u_k)\right) - \mu_1^S \leq \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 - (\mu_1^S)^2\right)}$$

$$\epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \leq \mu_1^S + \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 - (\mu_1^S)^2\right)} \qquad (7)$$

Now, since the function:

$$F(u) = u + \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 - u^2\right)}$$

decreases on the interval:

$$\left(\sqrt{\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2}{n}}\ \sqrt{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2}\right),$$

$$Also, n \leq 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2, 1 \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}. Therefore,$$

$$\sqrt{\frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}} \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n} \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)}{n}$$
$$\leq \mu_1^S \leq \sqrt{2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}.$$

Therefore, Eq. (7) implies:

$$\epsilon\left(\delta_{1J}^S(u_j, u_k)\right) \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n} +$$
$$\sqrt{(n-1)\left\{2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 - \left(\frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}\right)^2\right\}}, \forall\, S = -, + \text{ and } J = 1,2, \dots, k.$$

Likewise, we can demonstrate that (ii), (iii), (iv), (v), and (vi).

Theorem 13. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. Then, $\epsilon(\zeta) \leq \frac{n}{2}(1 + \sqrt{n})$.

Proof. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. If $n \leq 2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 = 2z$, it is simple to demonstrate using standard calculus that $f(z) = \frac{2y}{n} + \sqrt{(n-1)(2z - \left(\frac{2z}{n}\right)^2}$ is maximized when $z = \frac{n^2 + n\sqrt{n}}{4}$. We must have $\epsilon\left(\delta_{1J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$ if we replace this value of z with $z = \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2$ in Theorem 12. Similarly, to that, it is simple to demonstrate that $\epsilon\left(\delta_{2J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\delta_{3J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{1J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{2J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{3J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\forall\, S = -, +$ and $J = 1,2, \dots, k$. Hence, $\epsilon(\zeta) \leq \frac{n}{2}(1 + \sqrt{n})$.

4. Laplacian Energy of IVCNGS

The Laplacian energy of an IVCNGS is defined and examined, and its specific properties are given in this section.

Definition 14. Let $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ be an IVCNGS on n vertices. The degree matrix in amplitude term $D\delta_J(\zeta) = \left([D\delta_{1J}^-(u_i u_j), D\delta_{1J}^+(u_i u_j)], [D\delta_{2J}^-(u_i u_j), D\delta_{2J}^+(u_i u_j)], [D\delta_{3J}^-(u_i u_j), D\delta_{3J}^+(u_i u_j)]\right) = D\delta_J(ij)$

The degree matrix in amplitude term

$$D\beta_J(\zeta) = \left([D\beta_{1J}^-(u_i u_j), D\beta_{1J}^+(u_i u_j)], [D\beta_{2J}^-(u_i u_j), D\beta_{2J}^+(u_i u_j)], [D\beta_{3J}^-(u_i u_j), D\beta_{3J}^+(u_i u_j)]\right) = D\beta_J(ij)$$

ζ is an $n \times n$ diagonal matrix of amplitude term, which is defined as $D\delta_J(ij) = \begin{cases} d_{\delta_J}(u_i), & i = j \\ 0, & i \neq j \end{cases}$

ζ is an $n \times n$ diagonal matrix of phase term, which is defined as $D\beta_J(ij) = \begin{cases} d_{\beta_J}(u_i), & i = j \\ 0, & i \neq j \end{cases}$

Definition 15. The Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is defined as $L(\zeta) = (L\delta_1, L\delta_2, \dots, L\delta_k)$, where $L\delta_J = D\delta_J - A\delta_J$, and $D\delta_J$ is a degree matrix of an IVCNGS ζ and $A\delta_J$ is an adjacency matrix for all $J = 1,2, \dots, k$.

Example 5. The Laplacian matrix of IVCNGS is shown in Figure 3 in Example 1. The $D\lambda_1$ degree matrix of amplitude term of an IVCNGS

$$D\delta_1 = \begin{bmatrix} \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .7 & 1.2 \\ .7 & 1.1 \\ .8 & 1.2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .6 & 1.2 \\ .9 & 1.2 \\ .7 & 1.1 \end{pmatrix} \end{bmatrix}$$

The $D\lambda_1$ degree matrix of phase term of an IVCNGS

$$D\beta_1 = \begin{bmatrix} \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .8 \\ 1 & 1 \\ .6 & .9 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .5 & .8 \\ .9 & 1 \\ .7 & 1 \end{pmatrix} \end{bmatrix}$$

The Laplacian matrix of amplitude term of an IVCNGS is

$$L\delta_1 = \begin{bmatrix} \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} -.4 & -.6 \\ -.3 & -.6 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -.4 & -.6 \\ -.3 & -.6 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} .7 & 1.2 \\ .7 & 1.1 \\ .8 & 1.2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.4 & -.5 \\ -.4 & -.6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.5 & -.7 \\ -.3 & -.5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.4 & -.5 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.5 & -.7 \\ -.3 & -.5 \end{pmatrix} & \begin{pmatrix} .6 & 1.2 \\ .9 & 1.2 \\ .7 & 1.1 \end{pmatrix} \end{bmatrix}$$

Laplacian matrix of phase term of an IVCNGS is

$$L\beta_1 = \begin{bmatrix} \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.4 \end{pmatrix} & \begin{pmatrix} .4 & .8 \\ 1 & 1 \\ .6 & .9 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} -.3 & -.4 \\ -.4 & -.5 \\ -.4 & -.5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.6 \end{pmatrix} & \begin{pmatrix} -.3 & -.4 \\ -.4 & -.5 \\ -.4 & -.5 \end{pmatrix} & \begin{pmatrix} .5 & .8 \\ .9 & 1 \\ .7 & 1 \end{pmatrix} \end{bmatrix}$$

Similarly, we can calculate $L\delta_2$ and $L\beta_2$ Laplacian matrix

Definition 16. The spectrum of the Laplacian matrix of an IVCNGS is defined as $\langle P_{1L}, Q_{1L}, P_{2L}, Q_{2L}, P_{3L}, Q_{3L}, \rangle$, where $P_{1L}, Q_{1L}, P_{2L}, Q_{2L}, P_{3L}, Q_{3L}$ is the amplitude term of the set eigenvalues of $L(\zeta)$ and $\langle P'_{1L}, Q'_{1L}, P'_{2L}, Q'_{2L}, P'_{3L}, Q'_{3L} \rangle$, where $P'_{1L}, Q'_{1L}, P'_{2L}, Q'_{2L}, P'_{3L}, Q'_{3L}$ is the phase term of the set eigenvalues of $L(\zeta)$ respectively.

Example 6. The Laplacian spectrum of an IVCNGS shown in Figure 3 in Example 1 are as follows:

$$\text{Laplacian Spectrum } \left(L(\delta_{11}^-)\right) = (0.0000, 0.1866, 0.6819, 1.1314),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{11}^+)\right) = (-0.0000, 0.3515, 1.2000, 2.0485),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{21}^-)\right) = (-0.0000, 0.2236, 0.7553, 1.4211),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{21}^+)\right) = (-0.0000, 0.3289, 1.2879, 1.9832),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{31}^-)\right) = (-0.0000, 0.2149, 0.6896, 1.2955),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{31}^+)\right) = (-0.0000, 0.3339, 1.0925, 1.9735),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{11}^-)\right) = (0.0000, 0.1268, 0.4732, 0.8000),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{11}^+)\right) = (0.0000, 0.2343, 0.8000, 1.3657),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{21}^-)\right) = (0.0000, 0.2746, 0.8913, 1.6341),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{21}^+)\right) = (0.0000, 0.2929, 1.0000, 1.7071)$$
$$\text{Laplacian Spectrum } \left(L(\beta_{31}^-)\right) = (0.0000, 0.1866, 0.6819, 1.1314),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{31}^+)\right) = (-0.0528, 0.2861, 0.8466, 1.7200)$$

Therefore, the Laplacian spectrum of amplitude term is Laplacian

$$\text{spec}(L\lambda_1) = \{(0,-0,-0,-0,-0,-0), (0.1866, 0.3515, 0.2236, 0.3289, 0.2149, 0.3339), (0.6819, 1.2000, 0.7553, 1.2879, 0.6896, 1.0925), (1.1314, 2.0485, 1.4211, 1.9832, 1.2955, 1.9735)\}$$

And the Laplacian spectrum of phase term is

$$\text{spec}(L\beta_1) = \{(0,0,0,0,0,-0.0528), (0.1268, 0.2343, 0.2746, 0.2929, 0.1866, 0.2861), (0.4732, 0.8000, 0.8913, 1.0000, 0.6819, 0.8466), (0.8000, 1.3657, 1.6341, 1.7071, 1.1314, 1.7200)\}$$

Similarly, we can calculate Laplacian $\text{spec}(L\delta_2)$ and $\text{spec}(L\beta_2)$

Example 7. The Laplacian energy of amplitude term of an IVCNGS ζ given Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(L\delta_1), \epsilon(L\delta_2) >$$
$$\epsilon(L\delta_1) = < 1.6267, 2.8971, 1.9528, 2.9423, 1.7702, 2.7321 >$$

The Laplacian energy of phase term of an IVCNGS ζ given Figure 3 are as follows:

$$\epsilon(\zeta) = <\epsilon(L\delta_1), \epsilon(L\delta_2)>$$
$$\epsilon(L\delta_1) = <1.1464, 1.9314, 2.2507, 2.4142, 1.6267, 2.3334>$$

Similarly, we can calculate Laplacian $\epsilon(L\delta_2)$ *and* $\epsilon(L\beta_2)$

Theorem 17. Let $L(\zeta) = \{L\delta_1, L\delta_2, \dots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \ge (\mu_2^-)_{\delta_J} \ge \dots \ge (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \ge (\mu_2^+)_{\delta_J} \ge \dots \ge (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \ge (\lambda_2^-)_{\delta_J} \ge \dots \ge (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \ge (\lambda_2^+)_{\delta_J} \ge \dots \ge (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \ge (\chi_2^-)_{\delta_J} \ge \dots \ge (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \ge (\chi_2^+)_{\delta_J} \ge \dots \ge (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms, $(\vartheta_1^-)_{\beta_J} \ge (\vartheta_2^-)_{\beta_J} \ge \dots \ge (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \ge (\vartheta_2^+)_{\beta_J} \ge \dots \ge (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \ge (\rho_2^-)_{\beta_J} \ge \dots \ge (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \ge (\rho_2^+)_{\beta_J} \ge \dots \ge (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \ge (\gamma_2^-)_{\beta_J} \ge \dots \ge (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \ge (\gamma_2^+)_{\beta_J} \ge \dots \ge (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms. Then

(i) $$\sum_{i=1,(\mu_i^-)_{\delta_J} \in P_{1L}}^{n} (\mu_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\mu_i^+)_{\delta_J} \in Q_{1L}}^{n} (\mu_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\lambda_i^-)_{\delta_J} \in R_L}^{n} (\lambda_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\lambda_i^+)_{\delta_J} \in S_L}^{n} (\lambda_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\chi_i^-)_{\delta_J} \in R_L}^{n} (\chi_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{3J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\chi_i^+)_{\delta_J} \in S_L}^{n} (\chi_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{3J}^+(u_j, u_k)\right), and$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J} \in P'_{1L}}^{n} (\vartheta_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{1J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\vartheta_i^+)_{\beta_J} \in Q'_{1L}}^{n} (\vartheta_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{1J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J} \in P'_{2L}}^{n} (\rho_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{2J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\rho_i^+)_{\beta_J} \in Q'_{2L}}^{n} (\rho_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{2J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\gamma_i^-)_{\beta_J} \in P'_{3L}}^{n} (\gamma_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{3J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\gamma_i^+)_{\beta_J} \in Q'_{3L}}^{n} (\gamma_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{3J}^+(u_j, u_k)\right)$$

(ii) $$\sum_{i=1,(\mu_i^-)_{\delta_J} \in P_{1L}}^{n} (\mu_i^-)^2_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^-(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\mu_i^+)_{\delta_J} \in Q_{1L}}^{n} (\mu_i^+)^2_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^+(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\lambda_i^-)_{\delta_J} \in P_{2L}}^{n} (\lambda_i^-)^2_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^-(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^-}(u_j),$$

$$\sum_{i=1,(\lambda_i^+)_{\delta_J} \in Q_{2L}}^{n} (\lambda_i^+)^2_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^+(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^+}(u_j),$$

$$\sum_{i=1,(\chi_i^-)_{\delta_J}\in P_{3L}}^{n}(\chi_i^-)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\delta_{3J}^-}(u_j),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in Q_{3L}}^{n}(\chi_i^+)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\delta_{3J}^+}(u_j)\text{, and}$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P'_{1L}}^{n}(\vartheta_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q'_{1L}}^{n}(\vartheta_i^+)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J}\in P'_{2L}}^{n}(\rho_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{2J}^-}(u_j),$$

Proof. (i) Given that L(ζ) is a symmetric matrix with positive Laplacian eigenvalues, the following is true:

$$(i)\quad \sum_{i=1,(\mu_i^-)_{\delta_J}\in P_L}^{n}(\mu_i^-)_{\delta_J} = tr\left(L\delta_{1J}^-\right) = \sum_{j=1}^{n}d_{\delta_{1J}^-}(u_j) = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)$$

Likewise, we can demonstrate that

$$\sum_{i=1,(\mu_i^+)_{\delta_J}\in Q_{1L}}^{n}(\mu_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right),\quad \sum_{i=1,(\lambda_i^-)_{\delta_J}\in R_L}^{n}(\lambda_i^-)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\lambda_i^+)_{\delta_J}\in S_L}^{n}(\lambda_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\chi_i^-)_{\delta_J}\in R_L}^{n}(\chi_i^-)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in S_L}^{n}(\chi_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P'_{1L}}^{n}(\vartheta_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q'_{1L}}^{n}(\vartheta_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\rho_i^-)_{\beta_J}\in P'_{2L}}^{n}(\rho_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\rho_i^+)_{\beta_J}\in Q'_{2L}}^{n}(\rho_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\gamma_i^-)_{\beta_J}\in P'_{3L}}^{n}(\gamma_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\gamma_i^+)_{\beta_J}\in Q'_{3L}}^{n}(\gamma_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)$$

(ii) By the Definition 15 of Laplacian matrix, we have:

$$L\delta_{1J}^{-} = \begin{bmatrix} d_{\delta_{1J}^{-}}(u_1) & -\delta_{1J}^{-}(u_1u_2) & \cdots & -\delta_{1J}^{-}(z_1z_n) \\ -\delta_{1J}^{-}(u_2u_1) & d_{\delta_{1J}^{-}}(u_2) & \cdots & -\delta_{1J}^{-}(z_2z_n) \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_{1J}^{-}(u_nu_1) & -\delta_{1J}^{-}(u_nu_2) & \cdots & d_{\delta_{1J}^{-}}(u_n) \end{bmatrix}$$

By the trace properties of a matrix, we have:

$$tr\left(\left(L(\delta_{1J}^{-})\right)^2\right) = \sum_{i=1,(\mu_i^{-})_{\delta_J}\in P_{1L}}^{n} (\mu_i^{-})_{\delta_J}^2$$

$$tr\left(\left(L(\delta_{1J}^{-})\right)^2\right) = (d_{\delta_{1J}^{-}}^2(u_1) + \left(\delta_{1J}^{-}(u_1u_2)\right)^2 + \cdots + \left(\delta_{1J}^{-}(z_1z_n)\right)^2 +$$

$$\left(\delta_{1J}^{-}(u_2u_1)\right)^2 + d_{\delta_{1J}^{-}}^2(u_2) + \cdots + \left(\delta_{1J}^{-}(z_2z_n)\right)^2 + \cdots +$$

$$\left(\delta_{1J}^{-}(u_nu_1)\right)^2 + \left(\delta_{1J}^{-}(u_nu_2)\right)^2 + \cdots + d_{\delta_{1J}^{-}}^2(u_n)$$

$$= 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{-}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{-}}(u_j)$$

Therefore, $\sum_{i=1,(\mu_i^{-})_{\delta_J}\in P_{1L}}^{n} (\mu_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{-}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{-}}(u_j)$

Likewise, we can demonstrate that

$$\sum_{i=1,(\mu_i^{+})_{\delta_J}\in Q_{1L}}^{n} (\mu_i^{+})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{+}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{+}}(u_j),$$

$$\sum_{i=1,(\lambda_i^{-})_{\delta_J}\in P_{2L}}^{n} (\lambda_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{2J}^{-}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^{-}}(u_j),$$

$$\sum_{i=1,(\lambda_i^{+})_{\delta_J}\in Q_{2L}}^{n} (\lambda_i^{+})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{2J}^{+}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^{+}}(u_j),$$

$$\sum_{i=1,(\chi_i^{-})_{\delta_J}\in P_{3L}}^{n} (\chi_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{3J}^{-}(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{3J}^{-}}(u_j),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in Q_{3L}}^{n} (\chi_i^+)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{3J}^+(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{3J}^+}(u_j), \text{ and}$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P'_{1L}}^{n} (\vartheta_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{1J}^-(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q'_{1L}}^{n} (\vartheta_i^+)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{1J}^+(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J}\in P'_{2L}}^{n} (\rho_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{2J}^-(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{2J}^-}(u_j), \forall J = 1,2,\ldots,k.$$

Definition 18. The Laplacian energy of amplitude term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as: $L\epsilon(\zeta) =< L\epsilon(\delta_1), L\epsilon(\delta_2), \ldots, L\epsilon(\delta_k) >$

$$L\epsilon(\delta_J) = (\sum_{i=1}^{n} \text{ mod } \left((L\mu_i^-)_{\delta_J}\right), \sum_{i=1}^{n} \text{ mod } \left((L\mu_i^+)_{\delta_J}\right),$$

$$\sum_{i=1}^{n} \text{ mod } \left((L\lambda_i^-)_{\delta_J}\right), \sum_{i=1}^{n} \text{ mod } \left((L\lambda_i^+)_{\delta_J}\right)), \sum_{i=1}^{n} \text{ mod } \left((L\chi_i^-)_{\delta_J}\right) \sum_{i=1}^{n} \text{ mod } \left((L\chi_i^+)_{\delta_J}\right)), \text{ where}$$

$$(L\mu_i^-)_{\delta_J} = (\mu_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)}{n}, (L\mu_i^+)_{\delta_J} = (\mu_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right)}{n},$$
$$(L\lambda_i^-)_{\delta_J} = (\lambda_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right)}{n}, (L\lambda_i^+)_{\delta_J} = (\lambda_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)}{n},$$
$$(L\chi_i^-)_{\delta_J} = (\chi_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)}{n}, (L\chi_i^+)_{\delta_J} = (\chi_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)}{n},$$

For all $J = 1,2, \ldots, k$. And the Laplacian energy of phase term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as:

$$L\epsilon(\zeta) =< L\epsilon(\beta_1), L\epsilon(\beta_2), \ldots, L\epsilon(\beta_k) >$$

$$L\epsilon(\beta_J) = (\sum_{i=1}^{n} \text{ mod } \left((L\vartheta_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{ mod } (L\vartheta_i^+)_{\beta_J}),$$

$$\sum_{i=1}^{n} \text{ mod } \left((L\rho_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{ mod } \left((L\rho_i^+)_{\beta_J}\right)),$$

$$\sum_{i=1}^{n} \text{ mod } \left((L\gamma_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{ mod } \left((L\gamma_i^+)_{\beta_J}\right)), \text{ where}$$

$$(L\vartheta_i^-)_{\beta_J} = (\vartheta_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)}{n}, (L\vartheta_i^+)_{\beta_J} = (\vartheta_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)}{n},$$

$$(L\rho_i^-)_{\beta_J} = (\rho_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)}{n}, (L\rho_i^+)_{\beta_J} = (\rho_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)}{n},$$

$$(L\gamma_i^-)_{\beta_J} = (\gamma_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right)}{n}, (L\gamma_i^+)_{\beta_J} = (\gamma_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)}{n},$$

$\forall J = 1,2,\ldots,k.$

Example 8. In Example 6, the Laplacian spectrum is found. An IVCNGS is Laplacian energy is shown in Figure 3 as follows:

$$(L\mu_i^-)_{\delta_1} = \bmod\left(0 - \frac{2(1.0)}{4}\right) + \bmod\left(0.1866 - \frac{2(1.0)}{4}\right) + \bmod\left(0.6819 - \frac{2(1.0)}{4}\right) + \bmod\left(1.1314 - \frac{2(1.0)}{4}\right) = 1.6267$$

$$(L\mu_i^+)_{\delta_1} = \bmod\left(0 - \frac{2(1.8)}{4}\right) + \bmod\left(0.3515 - \frac{2(1.8)}{4}\right) + \bmod\left(1.2000 - \frac{2(1.8)}{4}\right) + \bmod\left(2.0485 - \frac{2(1.8)}{4}\right) = 2.897$$

$$(L\lambda_i^-)_{\delta_1} = \bmod\left(0 - \frac{2(1.2)}{4}\right) + \bmod\left(0.2236 - \frac{2(1.2)}{4}\right) + \bmod\left(0.7553 - \frac{2(1.2)}{4}\right) + \bmod\left(1.4211 - \frac{2(1.2)}{4}\right) = 1.9528$$

$$(L\lambda_i^+)_{\delta_1} = \bmod\left(0 - \frac{2(1.8)}{4}\right) + \bmod\left(0.3289 - \frac{2(1.8)}{4}\right) + \bmod\left(1.2879 - \frac{2(1.8)}{4}\right) + \bmod\left(1.9832 - \frac{2(1.8)}{4}\right) = 2.9422$$

$$(L\chi_i^-)_{\delta_1} = \bmod\left(0 - \frac{2(1.1)}{4}\right) + \bmod\left(0.2149 - \frac{2(1.1)}{4}\right) + \bmod\left(0.6896 - \frac{2(1.1)}{4}\right) + \bmod\left(1.2955 - \frac{2(1.1)}{4}\right) = 1.7702$$

$$(L\chi_i^+)_{\delta_1} = \bmod\left(0 - \frac{2(1.7)}{4}\right) + \bmod\left(0.3339 - \frac{2(1.7)}{4}\right) + \bmod\left(1.0925 - \frac{2(1.7)}{4}\right) + \bmod\left(1.9735 - \frac{2(1.7)}{4}\right) = 2.7321$$

$$(L\vartheta_i^-)_{\beta_1} = \bmod\left(0 - \frac{2(.7)}{4}\right) + \bmod\left(0.1268 - \frac{2(.7)}{4}\right) + \bmod\left(0.4732 - \frac{2(.7)}{4}\right) + \bmod\left(0.8000 - \frac{2(.7)}{4}\right) = 1.1464$$

$$(L\vartheta_i^+)_{\beta_1} = \bmod\left(0-\frac{2(1.2)}{4}\right)+\bmod\left(0.2343-\frac{2(1.2)}{4}\right)+\bmod\left(0.8000-\frac{2(1.2)}{4}\right)$$
$$+\bmod\left(1.3657-\frac{2(1.2)}{4}\right)=1.9314$$

$$(L\rho_i^-)_{\beta_1} = \bmod\left(0-\frac{2(1.4)}{4}\right)+\bmod\left(0.2746-\frac{2(1.4)}{4}\right)+\bmod\left(0.8913-\frac{2(1.4)}{4}\right)$$
$$+\bmod\left(1.6341-\frac{2(1.4)}{4}\right)=2.2508$$

$$(L\rho_i^+)_{\beta_1} = \bmod\left(0-\frac{2(1.5)}{4}\right)+\bmod\left(0.2929-\frac{2(1.5)}{4}\right)+\bmod\left(1.0000-\frac{2(1.5)}{4}\right)$$
$$+\bmod\left(1.7071-\frac{2(1.5)}{4}\right)=2.4142$$

$$(L\gamma_i^-)_{\beta_1} = \bmod\left(0-\frac{2(1)}{4}\right)+\bmod\left(0.1866-\frac{2(1)}{4}\right)+\bmod\left(0.6819-\frac{2(1)}{4}\right)$$
$$+\bmod\left(1.1314-\frac{2(1)}{4}\right)=1.6267$$

$$(L\gamma_i^+)_{\beta_1} = \bmod\left(-0.0528-\frac{2(1.4)}{4}\right)+\bmod\left(0.2861-\frac{2(1.4)}{4}\right)+\bmod\left(0.8466-\frac{2(1.4)}{4}\right)$$
$$+\bmod\left(1.7200-\frac{2(1.4)}{4}\right)=2.2805$$

Theorem 19. Let $L(\zeta)=\{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta=\{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \geq (\mu_2^-)_{\delta_J} \geq \ldots \geq (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \geq (\mu_2^+)_{\delta_J} \geq \ldots \geq (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \geq (\lambda_2^-)_{\delta_J} \geq \ldots \geq (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \geq (\lambda_2^+)_{\delta_J} \geq \ldots \geq (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \geq (\chi_2^-)_{\delta_J} \geq \ldots \geq (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \geq (\chi_2^+)_{\delta_J} \geq \ldots \geq (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms $L\delta_{1J}^-(u_ju_k), L\delta_{1J}^+(u_ju_k), L\delta_{2J}^-(u_ju_k)$, $L\delta_{2J}^+(u_ju_k)$ and $L\delta_{3J}^-(u_ju_k)$, $L\delta_{3J}^+(u_ju_k)$ respectively, and $(\vartheta_1^-)_{\beta_J} \geq (\vartheta_2^-)_{\beta_J} \geq \ldots \geq (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \geq (\vartheta_2^+)_{\beta_J} \geq \ldots \geq (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \geq (\rho_2^-)_{\beta_J} \geq \ldots \geq (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \geq (\rho_2^+)_{\beta_J} \geq \ldots \geq (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \geq (\gamma_2^-)_{\beta_J} \geq \ldots \geq (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \geq (\gamma_2^+)_{\beta_J} \geq \ldots \geq (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms $L\beta_{1J}^-(u_ju_k), L\beta_{1J}^+(u_ju_k), L\beta_{2J}^-(u_ju_k)$, $L\beta_{2J}^+(u_ju_k)$ and $L\beta_{3J}^-(u_ju_k)$, $L\beta_{3J}^+(u_ju_k)$ respectively,

$$(L\mu_i^-)_{\delta_J} = (\mu_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)}{n},\ (L\mu_i^+)_{\delta_J} = (\mu_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right)}{n},$$

$$(L\lambda_i^-)_{\delta_J} = (\lambda_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right)}{n},\ (L\lambda_i^+)_{\delta_J} = (\lambda_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)}{n},$$

$$(L\chi_i^-)_{\delta_J} = (\chi_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)}{n},\ (L\chi_i^+)_{\delta_J} = (\chi_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)}{n}$$

$$and\ (L\vartheta_i^-)_{\beta_J} = \mathrm{s}(\vartheta_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)}{n},\ (L\vartheta_i^+)_{\beta_J}(\vartheta_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)}{n},$$

$$(L\rho_i^-)_{\beta_J} = (\rho_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)}{n},\ (L\rho_i^+)_{\beta_J} = (\rho_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)}{n}$$

$$(L\gamma_i^-)_{\beta_J} = (\gamma_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right)}{n},\ (L\gamma_i^+)_{\beta_J} = (\gamma_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)}{n},$$

then: $\sum_{i=1}^{n}(L\mu_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(L\lambda_i^S)_{\delta_J} = 0,\ \sum_{i=1}^{n}(L\chi_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(L\vartheta_i^S)_{\beta_J} = 0,$

$$\sum_{i=1}^{n}(L\rho_i^S)_{\beta_J} = 0, \sum_{i=1}^{n}(L\gamma_i^S)_{\beta_J} = 0,$$

$$\sum_{i=1}^{n}(L\mu_i^S)_{\delta_J}^2 = 2M_{\delta_{1J}^S}, \sum_{i=1}^{n}(L\lambda_i^S)_{\delta_J}^2 = 2M_{\delta_{2J}^S}, \sum_{i=1}^{n}(L\chi_i^S)_{\delta_J}^2 = 2M_{\delta_{3J}^S}$$

$$\sum_{i=1}^{n}(L\vartheta_i^S)_{\beta_J}^2 = 2M_{\beta_{1J}^S}, \sum_{i=1}^{n}(L\rho_i^S)_{\beta_J}^2 = 2M_{\beta_{2J}^S}, \sum_{i=1}^{n}(L\gamma_i^S)_{\beta_J}^2 = 2M_{\beta_{3J}^S}$$

where:

$$M_{\delta_{1J}^S} = \sum_{u_ju_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\delta_{2J}^S} = \sum_{u_ju_k \in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\delta_{3J}^S} = \sum_{u_ju_k \in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{1J}^S} = \sum_{u_ju_k \in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{1J}^S} = \sum_{u_ju_k \in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{2J}^S} = \sum_{u_ju_k \in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{3J}^S} = \sum_{u_ju_k \in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$\forall\ S = -,+$ and $J = 1,2,\ldots,k$.

Theorem 20. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ on n vertices. Then,

(i) $$(L\mu_i^S)_{\delta_J} \le \sqrt{2n\sum_{u_ju_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$$

(ii) $(L\lambda_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)}{n}\right)}$

(iii) $(L\chi_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)}{n}\right)}$

(iv) $(L\vartheta_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)}$

(v) $(L\rho_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)}{n}\right)}$

(vi) $(L\gamma_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)}{n}\right)}$

$\forall\, S = -, +$ and $J = 1,2,\ldots,k$.

Proof. (i) By applying Cauchy-Schwarz inequality to the n numbers 1,1,…,1 and mod $\left((L\mu_1^S)_{\delta_J}\right)$, mod $\left((L\mu_2^S)_{\delta_J}\right)$,…, mod $\left((L\mu_n^S)_{\delta_J}\right)$, we have:

$$\sum_{i=1}^{n}\text{mod}\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}\text{mod}\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$(L\mu_i^S)_{\delta_J} \leq \sqrt{n}\sqrt{2M_{\delta_{1J}^S}} = \sqrt{2nM_{\delta_{1J}^S}}$$

Since, $M_{\delta_{1J}^S} = \sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2$,

Therefore,

$$(L\mu_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)} \text{ for all}$$

$S = -, +$ and $J = 1,2,\ldots,k$.

We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

Theorem 21. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ on n vertices. Then,

(i) $(L\mu_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right)}{n}\right)}$

(ii) $(L\lambda_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\delta_{2J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{2J}^S(u_j, u_k)\right)}{n}\right)}$

(iii) $(L\chi_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right)}{n}\right)}$

(iv) $(L\vartheta_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\beta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\beta_{1J}^S(u_j, u_k)\right)}{n}\right)}$

(v) $(L\rho_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\beta_{2J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\beta_{2J}^S(u_j, u_k)\right)}{n}\right)}$

(vi) $(L\gamma_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J} \left(\beta_{3J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\beta_{3J}^S(u_j, u_k)\right)}{n}\right)}$

$\forall\, S = -, +$ and $J = 1,2,\ldots,k$.

Proof. (i)

$$\left(\sum_{i=1}^{n} \text{mod}\left((L\mu_i^S)_{\delta_J}\right)\right)^2 = \sum_{i=1}^{n} \text{mod}\left((L\mu_i^S)_{\delta_J}\right)^2 + 2\sum_{u_j u_k \in R_J} \text{mod}\left((L\mu_i^S)_{\delta_J}(L\mu_j^S)_{\delta_J}\right) \geq 4M_{\delta_{1J}^S}$$

$$(L\mu_i^S)_{\delta_J} \geq 2\sqrt{M_{\delta_{1J}^S}}$$

Since, $M_{\delta_{1J}^S} = \sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2$

Therefore, $(L\mu_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$

for all $S = -, +$ and $J = 1,2,\ldots,k$. We can verify the other section (ii),(iii),(iv),(v), and (vi) in a similar manner.

Theorem 22. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ on n vertices. Then,

(i) $$(L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{1J}^S} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right)^2\right)}$$

(ii) $$(L\lambda_i^S)_{\delta_J} \leq \text{ mod }\left((L\delta_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{2J}^S} - \text{ mod }\left((L\lambda_1^S)_{\delta_J}\right)^2\right)}$$

(iii) $$(L\chi_i^S)_{\delta_J} \leq \text{ mod }\left((L\delta_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{3J}^S} - \text{ mod }\left((L\chi_1^S)_{\delta_J}\right)^2\right)}$$

(iv) $$(L\vartheta_i^S)_{\beta_J} \leq \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{1J}^S} - \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right)^2\right)}$$

(v) $$(L\rho_i^S)_{\beta_J} \leq \text{ mod }\left((L\rho_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{2J}^S} - \text{ mod }\left((L\rho_1^S)_{\beta_J}\right)^2\right)}$$

(vi) $$(L\gamma_i^S)_{\beta_J} \leq \text{ mod }\left((L\gamma_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{3J}^S} - \text{ mod }\left((L\rho_1^S)_{\beta_J}\right)^2\right)}$$

$\forall\, S = -, +$ and $J = 1,2, \ldots, k$.

Proof. (i)

$$\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{n\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{(n-1)\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$(L\mu_i^S)_{\delta_J} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) \leq \sqrt{(n-1)\left(2M_{\delta_{1J}^S} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right)^2\right)}$$

That is $M_{\delta_{1J}^S} = \sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$

Therefore, $(L\mu_i^S)_{\delta_J} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{1J}^S} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right)^2\right)}$ for all $S = -, +$ and $J = 1,2, \dots, k$. We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

Theorem 23. If the IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is regular, then:

(i) $(L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right) - (L\mu_1^S)^2_{\delta_J}\right)}$;

(ii) $(L\lambda_i^S)_{\delta_J} \leq \text{ mod }\left((L\lambda_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{2J}^S(u_j, u_k)\right) - (L\lambda_1^S)^2_{\delta_J}\right)}$;

(ii) $(L\chi_i^S)_{\delta_J} \leq \text{ mod }\left((L\chi_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{3J}^S(u_j, u_k)\right) - (L\chi_1^S)^2_{\delta_J}\right)}$;

(iv) $(L\vartheta_i^S)_{\beta_J} \leq \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{1J}^S(u_j, u_k)\right) - (L\vartheta_1^S)^2_{\beta_J}\right)}$;

(v) $(L\rho_i^S)_{\beta_J} \leq \text{mod }\left((L\rho_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{2J}^S(u_j, u_k)\right) - (L\rho_1^S)^2_{\beta_J}\right)}$;

(vi) $(L\gamma_i^S)_{\beta_J} \leq \text{mod }\left((L\gamma_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{3J}^S(u_j, u_k)\right) - (L\gamma_1^S)^2_{\beta_J}\right)}$; $\forall\, S = -, +$ and $J = 1,2, \dots, k$.

Proof.

$$d_{\delta_{1J}^S}(u_j) = \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right)}{n} \qquad (9)$$

Substituting 9 in 8, we get

$$(L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j, u_k)\right) - (L\mu_1^S)^2_{\delta_J}\right)};$$

We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

5. Application

We evaluate the effectiveness of the proposed IVCNGS policies with real-world examples of medicine resource analyses based on the clinical field. The modern human life is heavily reliant on medicine. In the present context, it is the most important essential in the world. In our daily lives, we use a variety of medications, including herbal, homeopathic, and generic medications. Satisfying human demand and supplying a sufficient number of medicines at a reasonable cost to the market is extremely significant for the pharmaceutical industry.

Let's investigate how our IVCNGS concepts are applied to the pharmaceutical industry, which encompasses generic, homeopathic, herbal, and allopathic products, to explain its exceptional performance. The vertices in this example represent generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4) . We examine the network analysis of best best-edition drugs in the pharmaceutical industry. The two intended relationships between the pharmaceutical effect of the introduced unit's impact on global demand $R_1$and the damage to medications R_2. According to the provided definition 5, impact on global demand R_1 and the damage to medications R_2. The offered definition 5 can be applied in any situation because it helps to take into account everything that has an uncertain value. In this case, a set of relations R_J and a vertex set Q are considered. Examine Q ={ generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic(u_4)}, as well as the effects on global demand R_1 and the relationships between components in the pharmaceutical industry that affect medication damage R_2. We assume in Example 1 and Figure 3 $\zeta = (\eta, \delta_1, \delta_2)$ is IVCNGS of a GS $\zeta^* = (Q, R_1, R_2)$. The greatest value of an IVCNGS amplitude term's energy ζ is $max(\epsilon(\zeta))$ =2.6833 and the greatest value of an IVCNGS phase term's energy ζ is $max(\epsilon(\zeta)) = 2.2361$. In this illustration, it is obvious that the components have a greater impact on each other when there is a greater quantity of energy present in their relationships. It is obvious that more energy exists in R_1. As a result, generic, homeopathic, herbal, and allopathic all have a greater impact on one another.

To assess each of these, we instructed two relation $e_k (k = 1,2)$ Interval-valued complex Neutrosophic Preference Relations (IVCNPRs) [32] to increase the degree of components in the pharmaceutical industry. Following is a formula to determine each expert's weight:

$$w_J = \left(\frac{\epsilon(A\delta_{1J}^S)}{\sum_{J=1}^{2} \epsilon\left(A\delta_{1J}^S\right)} e^{i\frac{\epsilon(A\beta_{1J}^S)}{\sum_{J=1}^{2}\epsilon(A\beta_{1J}^S)}}, \frac{\epsilon(A\delta_{2J}^S)}{\sum_{J=1}^{2} \epsilon\left(A\delta_{2J}^S\right)} e^{i\frac{\epsilon(A\beta_{2J}^S)}{\sum_{J=1}^{2}\epsilon(A\beta_{2J}^S)}}, \frac{\epsilon(A\delta_{3J}^S)}{\sum_{J=1}^{2} \epsilon\left(A\delta_{3J}^S\right)} e^{i\frac{\epsilon(A\beta_{3J}^S)}{\sum_{J=1}^{2}\epsilon(A\beta_{3J}^S)}} \right),$$
$$\forall S = -, + \text{ and } J = 1,2.$$

Amplitude term of IVCNGS:

$$W_1 = ((0.5210,0.5278), (0.4984,0.5172), (0.5733,0.5325)),$$
$$W_2 = ((0.4789,0.4721), (0.5015,0.4827), (0.4266,0.4674))$$

Phase term of IVCNGS:

$$W_1 = ((0.5185,0.5278), (0.5335,0.5278), (0.4877,0.5196)),$$
$$W_2 = ((0.4814,0.4721), (0.4664,0.4721), (0.5122,0.4803))$$

By using the Interval-Valued Complex Neutrosophic Averaging (IVCNA) \label{0.1} operator, compute the averaged Interval-Valued Complex Neutrosophic element (IVCNE) u_i^k of the pharmaceutical industry u_i= {generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4)} over all other testing venues for the experts e_k *(k=1,2):*

$$u_i^k = IVCNA(u_{i1}^k, u_{i2}^k, \dots, u_{in}^k) =$$

$$\left(\sqrt{1 - \left(\prod_{i=1}^{n}\left(1 - \left(\delta_{1J}^-\right)^2{}_{ij}\right)\right)^{\frac{1}{n}}}, \left(\prod_{i=1}^{n}\left(\delta_{1J}^+\right)_{ij}\right)^{\frac{1}{n}} e^{i\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\beta_{1J}^-\right)^2{}_{ij}\right)\right)^{\frac{1}{n}}}, \left(\prod_{i=1}^{n}\left(\beta_{1J}^+\right)^2_{ij}\right)^{\frac{1}{n}}}, \right.$$

$$\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\delta_{2J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\delta_{2J}^{+}\right)_{ij}\right)^{\frac{1}{n}}e^{i\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\beta_{2J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\beta_{2J}^{+}\right)_{ij}\right)^{\frac{1}{n}}},$$

$$\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\delta_{3J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\delta_{3J}^{+}\right)_{ij}\right)^{\frac{1}{n}}e^{i\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\beta_{3J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\beta_{3J}^{+}\right)_{ij}\right)^{\frac{1}{n}}})\text{for all } J=1,2,\ldots,k.$$

Displays the findings as an aggregate Table 1 and 2. Calculate a collective IVCNE u_i $(i = 1,2,3,4)$ of the generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4) using the Interval-Valued Complex Neutrosophic Weighted Averaging (IVCNWA) Operator.

$u_i^k = IVCNA(u_i^1, u_i^2, \ldots, u_i^s) =$

$$(\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{1J}^{-}\right)_{k}\right)^{W_{1J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{1J}^{+}\right)_{k}^{w_{1J}^{+}}\right)e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{1J}^{-}\right)_{k}\right)^{W_{1J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{1J}^{+}\right)_{k}^{w_{1J}^{+}}\right)},$$

$$\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{2J}^{-}\right)_{k}\right)^{W_{2J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{2J}^{+}\right)_{k}^{w_{2J}^{+}}\right)e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{2J}^{-}\right)_{k}\right)^{W_{2J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{2J}^{+}\right)_{k}^{w_{2J}^{+}}\right)},$$

$$\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{3J}^{-}\right)_{k}\right)^{W_{3J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{3J}^{+}\right)_{k}^{w_{3J}^{+}}\right)e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{3J}^{-}\right)_{k}\right)^{W_{3J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{2J}^{+}\right)_{k}^{w_{3J}^{+}}\right)}, \forall\ J=1,2,\ldots,k.$$

Table 1 The expert aggregation results in amplitude term.

Experts	The Overall Results of the Experts
e_1	$u_1^1 = \langle 0.2065, 0.8801, 0.1526, 0.8801, 0.2065, 0.8801\rangle$
	$u_2^1 = \langle 0.2548, 0.7745, 0.2548, 0.7406, 0.2889, 0.7745\rangle$
	$u_3^1 = \langle 0.1526, 0.8801, 0.2634, 0.9146, 0.1526, 0.8408\rangle$
	$u_4^1 = \langle 0.2146, 0.7745, 0.3302, 0.7691, 0.2548, 0.7406\rangle$
e_2	$u_1^2 = \langle 0.1526, 0.8801, 0.2065, 0.8801, 0.1007, 0.8408\rangle$
	$u_2^2 = \langle 0.2065, 0.8801, 0.2634, 0.9146, 0.2065, 0.8801\rangle$
	$u_3^2 = \langle 0.2065, 0.8801, 0.2634, 0.9146, 0.2065, 0.8801\rangle$
	$u_4^2 = \langle 0.1526, 0.8801, 0.2065, 0.8801, 0.1007, 0.8408\rangle$

Table 2 The expert aggregation results in phase term.

Experts	The Overall Results of the Experts
e_1	$u_1^1 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.1526, 0.7952\rangle$
	$u_2^1 = \langle 0.1426, 0.6324, 0.3660, 0.7071, 0.2146, 0.6999\rangle$
	$u_3^1 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.2065, 0.8408\rangle$
	$u_4^1 = \langle 0.1822, 0.6324, 0.3302, 0.7071, 0.2548, 0.7400\rangle$
e_2	$u_1^2 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.1526, 0.8408\rangle$
	$u_2^2 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.2065, 0.7952\rangle$
	$u_3^2 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.2065, 0.8408\rangle$
	$u_4^2 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.1526, 0.8408\rangle$

Therefore, generic (u_1), = (0.1828, 0.8801, 0.1817, 0.8801, 0.1700, 0.8615), homeopathic (u_2), = (0.2330, 0.8226, 0.2591, 0.8200, 0.2574, 0.8222), herbal (u_3) = (0.1805, 0.8801, 0.2633, 0.9146, 0.1777, 0.8589), and allopathic (u_4) = (0.1876, 0.8226, 0.2762, 0.8208, 0.2047, 0.7858). Evaluate the score function $S(u_k) = ((Ad^-_{\delta_{1J}})^2 - (Ad^+_{\delta_{1J}})^2) + ((Ad^-_{\delta_{2J}})^2 - (Ad^+_{\delta_{2J}})^2) + ((Ad^-_{\delta_{3J}})^2 - (Ad^+_{\delta_{3J}})^2)$ \cite{A1} of u_k $(k = 1,2,3,4)$ and rated all the testing venues $(u_i), i = 1,2,3,4$.

$$S(u_1) = -2.1960,\ \ S(u_2) = -1.8374,\ \ S(u_3) = -2.2152,\ \ S(u\_4) = -1.8144.$$

Then $S(u_4) > S(u_2) > S(u_1) > S(u_3)$. Therefore, $S(u_4)$ is the best test venue.

Phase terms: Similarly, We can verify the phase terms.

5.1 Algorithm

We now explain our method's step-by-step computation process, which is used in the algorithm that follows.

(i). Input the set $Q = \{a_1, a_2, \dots, a_n\}$ use a variety of medications (vertices) and put the membership values $\eta = (\eta_1, \eta_2, \eta_3) = ([\eta_1^- e^{i\alpha_1^-}, \eta_1^+ e^{i\alpha_1^+}], [\eta_2^- e^{i\alpha_2^-}, \eta_2^+ e^{i\alpha_2^+}], [\eta_3^- e^{i\alpha_3^-}, \eta_3^+ e^{i\alpha_3^+}])$ of the nodes a_i's, $\eta_1^S, \eta_2^S, \eta_3^S \in [0,1]$ and $\alpha_1^S, \alpha_2^S, \alpha_3^S \in [0,2\pi]$ for all $S = -, +$.

(ii). Input the membership values $\delta_J = (\delta_{1J}, \delta_{2J}, \delta_{3J}) = ([\delta_{1J}^- e^{i\beta_1^-}, \delta_{1J}^+ e^{i\beta_1^+}], [\delta_{2J}^- e^{i\beta_2^-}, \delta_{2J}^+ e^{i\beta_2^+}])$ of the edges $a_i a_j \in R_J$ such that

$$\delta_{1J}^S(a_i a_j) e^{i\beta_{1J}^S(a_i a_j)} \le min\{\eta_1^S(a_i), \eta_1^S(a_j)\} e^{imin\{\alpha_1^S(a_i), \alpha_1^S(a_j)\}}$$
$$\delta_{2J}^S(a_i a_j) e^{i\beta_{2J}^S(a_i a_j)} \le max\{\eta_2^S(a_i), \eta_2^S(a_j)\} e^{imax\{\alpha_2^S(a_i), \alpha_2^S(a_j)\}}$$
$$\delta_{3J}^S(a_i a_j) e^{i\beta_{3J}^S(a_i a_j)} \le max\{\eta_3^S(a_i), \eta_3^S(a_j)\} e^{imax\{\alpha_3^S(a_i), \alpha_3^S(a_j)\}}$$

$0 \le (\delta_{1J}^S(a_i a_j)) + (\delta_{2J}^S(a_i a_j)) + (\delta_{3J}^S(a_i a_j)) \le 3$ and $(\beta_{1J}^S(a_i a_j)), (\beta_{2J}^S(a_i a_j)), (\beta_{3J}^S(a_i a_j)) \in$ $[0,2\pi]$ $\forall\ S = -, +$ and $a_i a_j \in R_J$, $J = 1,2, \dots, k$.

(iii). On the set used variety of medications Q, develop mutually disjoint, irreflexive, symmetric relations $R_1, R_2, \dots, R_k$. Give each relation an identity that reflects a particular stage of development between the two types of medications it represents.

(iv). Construct a graph structure on a set of medications with relation, then calculate the energy of each $A\eta_1, A\eta_2, \dots, A\eta_k$.

(v). Input a calculation like IVCNPRs

$$w_J = \left(\frac{\epsilon(A\delta_{1J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{1J}^S)} e^{i\frac{\epsilon(A\beta_{1J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{1J}^S)}}, \frac{\epsilon(A\delta_{2J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{2J}^S)} e^{i\frac{\epsilon(A\beta_{2J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{2J}^S)}}, \frac{\epsilon(A\delta_{3J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{3J}^S)} e^{i\frac{\epsilon(A\beta_{3J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{3J}^S)}} \right),$$

$\forall\ S = -, + \ and\ J = 1,2.$

(vi). Calculate IVCNA and IVCNWA

(vii). Evaluate the score function $S(u_k) = ((Ad^{-}_{\delta_{1J}})^2 - (Ad^{+}_{\delta_{1J}})^2) + ((Ad^{-}_{\delta_{2J}})^2 - (Ad^{+}_{\delta_{2J}})^2) + ((Ad^{-}_{\delta_{3J}})^2 - (Ad^{+}_{\delta_{3J}})^2)$

(viii). Provide an optimal testing venue output.

6. Conclusions and Future Works

The idea of IVCNGS has been developed in this research article by the authors. A more realistic description of uncertainty is offered by the Set IVCNS, an extension of the CNS and IVNS, compared to conventional fuzzy sets. It can be applied in many different contexts through fuzzy control. Many of the mathematical properties of the energy graph have been studied. The integration of the adjacency matrix IVCNGS, the energy of IVCNGS, and Laplacian energy IVCNGS with their intriguing properties has been proposed in this paper. Using the adjacency matrix's eigenvalues, we computed the IVCNGS's spectrum and determined its energy. Moreover, we presented the application of the energy IVCNGS in decision-making, specifically in determining the optimal level of pharmaceutical sources. If the adjacency matrix IVCNGS is used, there are several possible directions for this field's further investigation. Extension of the graph Structures energy to Complex Bipolar Picture Fuzzy Graph Structures, Interval-Valued Spherical Fuzzy Graph Structures, and dominating Complex bipolar neutrosophic graph structures are recommended areas of future research. Some of the limitations of this work are as follows:

- IVCNGS was the main focus of the study and related network systems.
- This approach is only applicable when there are symmetric, irreflexive, and mutually disjoint relations on the IVCNGS.
- The IVCNGS idea is not relevant if the membership values of the characters are provided in distinct environments.
- Obtaining accurate data could sometimes not be possible.

Acknowledgments
The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L. A. (1965). Fuzzy sets. Inf. Control, 8: 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X.
2. Turksen. (1986). Interval-valued fuzzy sets based on normal forms. Fuzzy Sets Syst, 20: 19110; https://doi.org/1016/0165-0114(86)90077-1.
3. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets & Syst, 20(1), 87-96: https://doi.org/10.1016/S0165-0114(86)80034-3.

4. Atanassove, K.T. and Gargov, G. (2004). Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application fuzzy sets. Fuzzy Sets and Systems, Pages 55-95. https://doi.org/10.1016/S0888-613X(03)00072-0.
5. Ramot, D., Friedman, M., Langholz G.A., & Kandel. (2003). Complex fuzzy logic. IEEE Trans. Fuzzy Syst, 11: 450?61. DOI: 10.1109/TFUZZ.2003.814832.
6. Alkouri, A., Salleh, A. (2012). Complex intuitionistic fuzzy sets. AIP Conf. Proc, 14: 464–470. https://doi.org/10.1063/1.4757515.
7. F. Smarandache. (2010). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. International Journal of Pure and Applied Mathematics, 24: 289-297.
8. F. Smarandache, Neutrosophic Graphs, in his book Symbolic Neutrosophic Theory, Europa, Nova.
9. F. Smarandache. (2005). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter. J.Pure Appl. Math, 24: 287-297.
10. F. Smarandache. (1999). A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability. Set and Logic. Rehoboth: American Research Press.
11. F. Smarandache. (1998). Neutrosophy, Neutrosophic Probability, Set, and Logic. Amer. Res. Press, Rehoboth, USA, 105 pages. http://fs.gallup.unm.edu/eBookneutrosophics4.pdf(4th edition).
12. F. Smarandache., M. Ali. (2017). Complex neutrosophic set. Neural Comput. APPl., 28(7): 1817-1834. https://doi.org/10.1007/s00521-015-2154-y.
13. Atiqe U. R., Muhammad S., F. Smarandache., & Muhammad. R. A. (2020). Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set. Neutrosophic Sets and Systems, Vol. 38.
14. Ivan Gutman., Bo Zhou. (2006). Laplacian energy of a graph. Linear Algebra and its Applications, Volume 414, Issue 1, 1April, Pages 29-37. https://doi.org/10.1016/j.laa.2005.09.008.
15. Rosenfeld, A. (1975). Fuzzy graphs. Zadeh, L.A., K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and their Applications, Academic Press, New York, pp.77-95.
16. Bhattacharya, P. (1987) Some remarks on fuzzy graphs. Pattern Recognit Lett, 6: 297–302. https://doi.org/10.1016/0167-8655(87)90012-2.
17. Rashmanlou, H. & M. Pal. (2013). Some properties of highly irregular interval-valued fuzzy graphs. World Applied Sciences Journal, 27(12): 1756–1773. https://doi.org/10.5829/idosi.wasj.2013.27.12.1316.
18. Rashmanlou, H., & Pal, M. (2014). Antipodal interval-valued fuzzy graphs. https://doi.org/10.48550/arXiv.1407.6190
19. Karunambigai, M. G., Palanivel, K., & Sivasankar, S. (2015). Edge regular intuitionistic fuzzy graph. Advances in Fuzzy Sets and Systems, 20(1), 25-46. http://dx.doi.org/10.17654/AFSSSep2015_025_046.
20. Thirunavukarasu, P., Suresh, R., & Viswanathan, K. K. (2016). Energy of a complex fuzzy graph. Int. J. Math. Sci. Eng. Appl, 10(1), 243-248.
21. Yaqoob. N., Gulistan M., Kadry S., & Wahab H. (2019). Complex intuitionistic fuzzy graphs with application incellular network provider companies. Mathematics, 7: 35. https://doi.org/10.3390/MATH7010035.
22. Yaqoob. N., Akram .M. (2018). Complex neutrosophic graphs. Bull. Comput. Appl. Math, 6: 85–109.
23. Anam Luqman., Akram, M., & F. Smarandache. (2019). Complex Neutrosophic Hypergraphs: New Social Network Models. Algorithms, 12: 234. https://doi.org/10.3390/a12110234.
24. Shabaf, S. R., Fayazi, F. (2014). Laplacian Energy of a Fuzzy Graph, Iran. J. Math. Chem, 5, 1–10.
25. Soumitra Poulik., Ganesh Ghorai. (2020). Detour g-interior nodes and detour g-boundary nodes in bipolar fuzzy graph with applications. Hacettepe Journal of Mathematics \& Statistics, Volume 49 (1): 106 – 119. https://doi.org/10.15672/HJMS.2019.666.
26. Soumitra Poulik., Ganesh Ghorai., & Qin Xin. (2021). Pragmatic results in Taiwan education system based IVFG \& IVNG, 25, 711–724. DOI: https://doi.org/10.1007/s00500-020-05180-4.
27. Poulik, S., & Ghorai, G. (2020). Empirical results on operations of bipolar fuzzy graphs with their degree. Missouri Journal of Mathematical Sciences, 32(2), 211-226. https://doi.org/10.35834/2020/3202211.
28. Poulik, S., & Ghorai, G. (2020). Note on “Bipolar fuzzy graphs with applications”. Knowledge-Based Systems, 192, 105315. https://doi.org/10.1016/j.knosys.2019.105315.

29. Sampathkumar, E. (2006). Generalized graph structures. Bull. Kerala Math. Assoc, 3(2), 65-123.
30. Dinesh, T., Ramakrishnan, T.V. (2011). Generalised Fuzzy Graph Structures. Applied Mathematical Sciences, 5(4), 173 -180.
31. Akram, M., & R. Akmal. (2016). operations on Intuitionistic Fuzzy Graph Structures. Fuzzy Inf.Eng, 8: 389-410. https://doi.org/10.1016/J.FIAE.2017.01.001.
32. Mohamad, S. N. F., Hasni, R., Smarandache, F., & Yusoff, B. (2021). Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. Axioms, 10(3), 172. https://doi.org/10.3390/axioms10030172.

Received: 28 Nov 2023, **Revised:** 29 Nov 2023,

Accepted: 04 Jan 2024, **Available online:** 06 Jan 2024.