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A Novel Approach on Energy of λ_J-dominating Single-Valued Neutrosophic Graph Structure

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Abstract: The concept of dominance is one of the most important ideas in graph theory for handling random events, and it has drawn the interest from many scholars. Research related to graph energy has garnered a lot focus recently. The application of single-valued neutrosophic graphs (SVNGs) for energy, Laplacian energy, and dominating energy has been recommended by previous studies. In this research, we apply the concepts of single-valued neutrosophic sets (SVNS) to graph structures (GSs) and investigate some intriguing features of single-valued neutrosophic graph structures (SVNGS). Moreover, the notions of λ_J-dominating energy GS in an SVNGS environment is analyzed in this study. More specifically, illustrative examples are used to develop the adjacency matrix of a λ_J-dominating SVNGS, as well as the spectrum of the adjacency matrix and their related theory. Further, the SVNGS λ_J-dominating energy is determined. We go over various characteristics and constraints for the energy of SVNGS with λ_J-dominating. Further, we introduce the idea of isomorphic and identical λ_J-dominating SVNGS energy, which has been studied using relevant examples, and some of its established properties are presented.

Keywords: SVNGS; λ_J-dominating; Energy; Isomorphic; Identical.

1. Introduction

The graph spectrum finds application in mathematical issues related to combinatorial optimization as well as statistical physics. The graph's spectrum can be more practically applied in a variety of real-world situations, including operations management, networking systems, science and technology, and medical science data held in databases. The adjacency matrix of the graph's eigenvalues is defined as the sum of their absolute values. The graph's energy is used in quantum theory and many other energy-related applications by connecting the graph's edge to a particular type of molecule's electron energy. Motivated by chemical applications, Gutman [1] first proposed this idea in 1978. The Laplacian energy of a graph was later defined by Gutman and Zhou [2] as the sum of the absolute values of the differences between the average vertex degree of G and the Laplacian eigenvalues of G. [3-6] contains information on the characteristics of Laplacian energy and graph energy.

Real-world problems with uncertainty and ambiguity are not always flexible to the common methods of classical mathematics. In 1965, Zadeh [7] developed the notion of a fuzzy set (FS) as an extension of the conventional concept of sets. Since then, other researchers have investigated the idea of fuzzy sets and fuzzy logic to solve a variety of real-world issues involving ambiguous and uncertain situations, as represented by a membership function with a value in the real unit interval [0, 1]. Because it is a single-valued function, the membership function, however, cannot always be used to collect both support and opposition evidence. Atanassov [8] developed the intuitionistic fuzzy set (IFS) as a generalization of Zadeh's fuzzy set. It is possible to create IFS, which has both a membership and a non-membership function, by deriving a new component, degree of membership and non-membership, from the characteristics of the fuzzy set. One of the most useful techniques for controlling ambiguity and unpredictability is the IFS. The magnitudes of satisfaction and discontent are included in an intuitionistic fuzzy set, which is a collection of fuzzy values. Yager [9] invented the ordered weighted average operators' concept for IFS. Weighted averaging operators were created for the first time by Xu [10] in the IFS theory. The neutrosophic set, which the author Smarandache [11-14] devised to handle the ambiguous and inconsistent data, has been extensively investigated and is used in a variety of domains. The indeterminacy value is explicitly quantified and truth membership, indeterminacy membership, and false membership are defined completely independently if the sum of all of these values in the neutrosophic set is between 0 and 3. Neutrosophy: Neutral Logic, Neutral Probability, and Neutral Set Explain in greater detail the terms neutrosophy, neutrosophic probability, set, and logic. The neutrosophic set has quickly caught the interest of many researchers due to the wide range of description situations it covers. Xindong Peng and Jingguo Dai [15] reference is also given a thorough analysis. The neutrosophic collection has undergone a bibliometric analysis from 1998 to 2017 that is presented.

Fuzzy graph theory was created by Rosenfeld [16] in 1975 and analyzed the fuzzy graphs (FGs) for which Kauffmann developed the essential idea in 1973. He researched numerous fundamental concepts in graph theory and established some of their characteristics. When there is uncertainty or ambiguity regarding the existence of an actual object or the relationship between two objects, FGs are a useful tool for representing object relationship structures. Some of the FG applications can be found in [18-20]. The idea of computing the constrained shortest path in a network with mixed fuzzy arc weights applied in wireless sensor networks was recently presented by Peng, Z., Abbaszadeh Sori, A., et al. [28]. For more information, it is recommended to read the research papers Applications of graph's total degree with bipolar fuzzy information and Estimation of most affected cycles and busiest network route based on complexity function of the graph in a fuzzy environment in 2022 by Soumitra Poulik and Ganesh Ghorai [29-31]. Further, introduced the Connectivity Concepts in Bipolar Fuzzy Incidence Graphs.

The concept of domination in graphs can be applied to a wide range of problems, such as transportation systems, combining theory, coding theory, social network analysis, communication networks, security systems, and congestion. Somasundaram and Somasundaram [21] first presented the novel ideas of domination in FGs in 1998. After that, Somasundaram [22] presented various operations on FGs and investigated domination in products of FGs. To find out more, it is suggested to study at the research papers. The following studies [23-25] provide some helpful information about these kinds of structures. In 2022, Bera, S., and Pal, M. [26] presented a new idea regarding domination in m-polar interval-valued FG. The idea of edge-vertex domination on interval graphs in 2024 was recently introduced by Shambayati, H., Shafiei Nikabadi, M., Saberi, S., et al. [27]. In the FS environment, the energy of a graph was first proposed by Anjali and Mathew [32]. Sharbaf and Fayazi [33] introduced the idea of LE of FGs and extended some results on LE bounds to FGs. The concept of energy of Pythagorean FGs with applications was recently introduced by Muhammad Akram and Sumera Naz [34]. Moreover, they presented the concept of Bipolar FG Energy and Energy of double dominating bipolar FGs [35, 36]. Many scholars have integrated the study of energy graphs, dominating sets, and NSs since the development of the NS. The dominating energy in a single-valued NG was recently proposed by Mullai and Broumi [37]. Novel Concept of Energy in Bipolar Single-Valued NGs with Applications was proposed by Mohamad, S.N.F., Hasni, R., Smarandache, et al. [38].

By generalizing an undirected graph, a Graph Structure (GS) may be produced. Signed graphs and other types of graphs can then be studied using this structure. The concept of GSs was initially introduced in Sampath Kumar's [40] work in 2006. The idea of an FGS was first put forth by T. Dinesh and T. V. Ramakrishnan [41] in their 2011 investigation. The ideas of Operations on Intuitionistic FGS and Single-Valued NGSs were recently introduced by Muhammad Akram [42, 43]. In an additional development, Bathusha, S. N. S., et al. [44, 45] presented the energy of interval-valued Complex NGS as well as the idea of interval-valued Complex Pythagorean FGS with application. A few LE-bound findings were expanded to include interval-valued Complex NGS and applications. In 2024, new works were unveiled.

Specifically, the objective of this work is to present λ_J-dominating SVNGS that were first described. After that, the energy principles for λ_J-dominant SVNGS are explained. Furthermore, we discuss several properties and restrictions for the energy of λ_J-dominating SVNGS. We also introduce the concept of identical and isomorphic λ_J-dominating SVNGS.

1.1 Motivation

Many graph theory problems consider pairwise relations of objects, and certain properties of the objects can be connected in such a way that they produce irreflexive, symmetric, and mutually disjoint relations. By graphically representing a GS, this type of information loss can be prevented. The study of SVNS as applied to the GS and their applications is motivated by the need to handle difficult decision-making situations when faced with imprecise information. Although there is some flexibility available with existing fuzzy models, SVNGSs offer a more flexible tool for controlling uncertainty. In SVNS, values for true membership, indeterminate membership, and false membership are included in consideration. The focus of our study is specifically drawn to the features and limitations of energy of λ_J-dominating SVGCS.

Once again expressed abstractly, this concept can be used in the energy of λ_J -dominating SVGCS. Figure 1 and the following describe the organizational structure of this work: Section preliminaries present the fundamental ideas of the neutrosophic set. Next, the ideas of SVNGSs and λ_J-dominating energy SVNGSs are defined. Moreover, we discussed the energy of λ_J-dominating SVNGS including its properties and bounds. In the conclusion section, we have provided an explanation of the study's future directions as well as the importance of the findings.

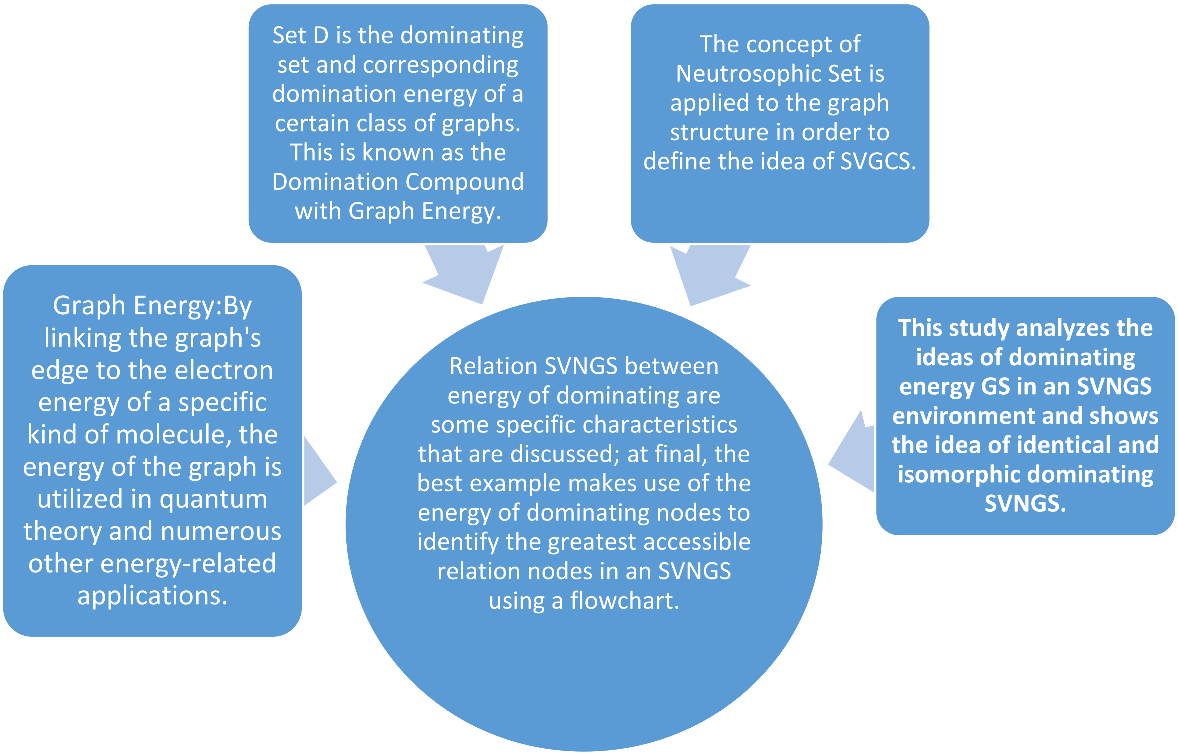

Figure 1

2. Preliminaries

The basic definitions of domination and the single-valued neutrosophic graph (SVNG) that are relevant to this study are presented in this section.

Definition 2.1. [41] Let $\zeta^* = (Q, R_1, R_2, \ldots, R_k)$ be a graph structure in which Q is a non-empty set and $R_1, R_2, \ldots, R_k$ are mutually disjoint, irreflexive, and symmetric relations on Q.

Definition 2.2. [11] Let Y be a universal set. The NS μ in Y defined membership functions $\mu_1(a), \mu_2(a),$ *and* $\mu_3(a)$ represent the true, indeterminate, and false values found in the $\mu = \{a, \mu_1(a), \mu_2(a), \mu_3(a) | a \in Y\}, where$ non-standard subset of $]0^-, 1^+[$ and the real standard, respectively, such that:
$\mu = \{a, \mu_1(a), \mu_2(a), \mu_3(a) | a \in Y\}, where \mu_1, \mu_2, \mu_3 : Y \to]0^-, 1^+[$ and $0^- \leq \mu_1(a), \mu_2(a), \mu_3(a) \leq 3^+$.

Definition 2.3. [39] Let Y be a universal set. The SVNS μ in Y is an object form $\mu_1, \mu_2, \mu_3 : Y \to [0,1]$ and $0 \leq \mu_1(a), \mu_2(a), \mu_3(a) \leq 3$.

Definition 2.4 [38] A SVNG $\zeta = (\mu, \lambda)$ is a pair, where $\mu: Q \to [0,1]$ is a SVNS on Q and $R: Q \times Q \to [0,1]$ is a SVN relation on Q such that

$$\lambda_1(a,b) \leq \min\{\mu_1(a), \mu_1(b)\},$$
$$\lambda_2(a,b) \leq \max\{\mu_2(a), \mu_2(b)\},$$
$$\lambda_3(a,b) \leq \max\{\mu_3(a), \mu_3(b)\},$$

For all $a, b \in Q$. μ *and* λ are referred to be SVN vertex set of ζ and the SVN edge set of ζ , respectively.

Definition 2.5. [37] An SVNG $\zeta = (\mu, \lambda)$ be a SVNG and $a, b \in Q$ *in* ζ, there we say that a dominates b if

$$\lambda_{1R}(a,b) \leq \mu_{1Q}(a) \wedge \mu_{1Q}(b),$$
$$\lambda_{2R}(a,b) \leq \mu_{2Q}(a), \vee \mu_{2Q}(b),$$
$$\lambda_{3R}(a,b) \leq \mu_{3Q}(a) \vee \mu_{3Q}(b).$$

3. Energy of λ_J-dominating Single-Valued Neutrosophic Graph Structure

The basic definitions of domination and the single-valued neutrosophic graph (SVNG) that are relevant to this study are presented in this section.

The energy of $\boldsymbol{\lambda_J}$-dominating GS is defined, and its properties are discussed in this section using the frameworks of SVNG theory.

Definition 3.1. Let $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ is referred to as an SVNGS of GS $\zeta^* = (Q, R_1, R_2, \ldots, R_k)$ if $\mu = \{r, \mu_1(r), \mu_2(r), \mu_3(r)\}$ is an SVN set on Q and $\lambda_J = \{rs, \lambda_{1J}(rs), \lambda_{2J}(rs), \lambda_{3J}(rs)\}$ are SVCN sets on Q and R_J such that

$$\lambda_{1J}(r,s) \leq \min\{\mu_1(r), \mu_1(s)\},$$
$$\lambda_{2J}(r,s) \leq \max\{\mu_2(r), \mu_2(s)\},$$
$$\lambda_{3J}(r,s) \leq \max\{\mu_3(r), \mu_3(s)\}$$

such that $0 \leq \lambda_{1J}(r,s) + \lambda_{2J}(r,s) + \lambda_{3J}(r,s) \leq 3$ for all $(r,s) \in R_J, J = 1,2,\ldots,k$.

Definition 3.2. The adjacency matrix $A\zeta = \{A\lambda_1, A\lambda_2, \ldots, A\lambda_k\}$ of a SVNGS $\zeta = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$, where $A\lambda_J, (J = 1,2,\ldots,k)$ is a square matrix as $[u_{jk}]$ in which

$$u_{jk} = (\lambda_{1J}(u_j u_k), \lambda_{2J}(u_j u_k), \lambda_{3J}(u_j u_k)),$$
$$\forall u_j u_k \in R_J \text{ and } J = 1,2,\ldots,k.$$

The adjacency matrix $A\zeta = \{A\lambda_1, A\lambda_2, \ldots, A\lambda_k\}$ of a SVNGS $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$. Then the λ_J degree of vertex u in $A(\zeta)$ is defined as

$$Ad_{\lambda_J}(u) = (Ad_{\lambda_{1J}}(u), Ad_{\lambda_{2J}}(u), Ad_{\lambda_{3J}}(u))$$

$$Ad_{\lambda_{1J}}(u) = (\sum_{z=1}^{k} \lambda_{1J}(u_{jz})),$$

$$Ad_{\lambda_{2J}}(u) = (\sum_{z=1}^{k} \lambda_{2J}(u_{jz})),$$

$$Ad_{\lambda_{3J}}(u) = \left(\sum_{z=1}^{k} \lambda_{2J}(u_{jz})\right), \forall J = 1,2, \ldots, k.$$

Definition 3.3. A graph structure of the form $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ is referred to as an λ_J-dominating SVNGS, where $\lambda_{1J}: Q \to [0,1]$ denoted degree of truth membership, $\lambda_{2J}: Q \to [0,1]$ denoted the degree of indeterminacy membership and $\lambda_{3J}: Q \to [0,1]$ denoted degree of false membership defined such as:

$$\lambda_{1J}(r) = \min_{(r,s)\in R_J} \left(\lambda_{1J}(r,s)\right),$$

$$\lambda_{2J}(r) = \min_{(r,s)\in R_J} \left(\lambda_{2J}(r,s)\right),$$

$$\lambda_{3J}(r) = \min_{(r,s)\in R_J} \left(\lambda_{3J}(r,s)\right), \forall J = 1,2, \ldots, k.$$

Definition 3.4. Let $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ be a λ_J-dominating SVNGS. Let $r, s \in Q$, we state that r dominates s in ζ if there exists a strong arc from r to s for all $(r,s) \in R_J$ and $J = 1,2, \ldots, k$. A subset $D_J \subseteq Q$ is referred to as a λ_J-dominant set in ζ^* if for each $s \in Q - D_J$, there exists one vertex $r \in D_J$ such that r dominates s for all $(r,s) \in R_J$ and $J = 1,2, \ldots, k$.

Definition 3.5. Let $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ be a λ_J-dominating SVNGS. The adjacency matrix of a λ_J-dominating SVNGS ζ is defined as $A_{D_J}(\zeta) = [d_{rs}]$, where

$$d_{rs} = \begin{cases} (\lambda_{1J})_{rs}, (\lambda_{2J})_{rs}, (\lambda_{3J})_{rs} & \text{if } (r,s) \in R_J \\ (1,1,1) & \text{if } r = s \text{ and } r \in D_J \\ (0,0,0) & otherwise \end{cases}$$

This adjacency matrix of a λ_J-dominating SVNGS $A_{D_J}(\zeta)$ can be written as $A_{D_J}(\zeta) = (\lambda_{1J}(\zeta), \lambda_{2J}(\zeta), \lambda_{3J}(\zeta))$ where

$$\lambda_{1J}(\zeta) = \begin{cases} (\lambda_{1J})_{rs} & \text{if } (r,s) \in R_J \\ 1 & \text{if } r = s \text{ and } r \in D_J \\ 0 & otherwise \end{cases}$$

$$\lambda_{2J}(\zeta) = \begin{cases} (\lambda_{2J})_{rs} & \text{if } (r,s) \in R_J \\ 1 & \text{if } r = s \text{ and } r \in D_J \\ 0 & otherwise \end{cases}$$

$$\lambda_{3J}(\zeta) = \begin{cases} (\lambda_{3J})_{rs} & \text{if } (r,s) \in R_J \\ 1 & \text{if } r = s \text{ and } r \in D_J \\ 0 & otherwise \end{cases}$$

Definition 3.6. The spectrum of an adjacency matrix of a λ_J-dominating SVNGS is defined as $\left\langle P_D^{\lambda_{1J}}, P_D^{\lambda_{2J}}, P_D^{\lambda_{3J}} \right\rangle$, where $P_D^{\lambda_{1J}}, P_D^{\lambda_{2J}}, P_D^{\lambda_{3J}}$ are the sets of eigenvalues of $\lambda_{1J}(\zeta), \lambda_{2J}(\zeta), \lambda_{3J}(\zeta)$, respectively.

Definition 3.7. The energy of a λ_J-dominating SVNGS $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ is defined as

$$\begin{aligned} E_D^{\lambda_J}(\zeta) &= (E(P_D^{\lambda_{1J}}), E(P_D^{\lambda_{2J}}), E(P_D^{\lambda_{3J}})) \\ &= \left\langle \sum_{i=1}^{n} |(\alpha_i)_{\lambda_J}|, \sum_{i=1}^{n} |(\beta_i)_{\lambda_J}|, \sum_{i=1}^{n} |(\gamma_i)_{\lambda_J}| \right\rangle, \end{aligned}$$

where$P_D^{\lambda_{1J}} = \{(\alpha_i)_{\lambda_J}\}_{i=1}^n$, $P_D^{\lambda_{2J}} = \{(\beta_i)_{\lambda_J}\}_{i=1}^n$ and $P_D^{\lambda_{3J}} = \{(\gamma_i)_{\lambda_J}\}_{i=1}^n$, for all $J = 1,2,\ldots,k$.

Example 3.8. An λ_J-dominating SVCNGS $\zeta = (\mu, \lambda_1, \lambda_2)$ of a GS $\zeta^* = (Q, R_1, R_2)$ given Figure 2 is a λ_J -dominating SVCNGS $\zeta = (\mu, \lambda_1, \lambda_2)$ such that $\mu = \{u_1(0.6,0.4,0.3), u_2(0.5,0.4,0.6), u_3(0.6,0.4,0.5), u_4(0.3,0.5,0.4)\}$.

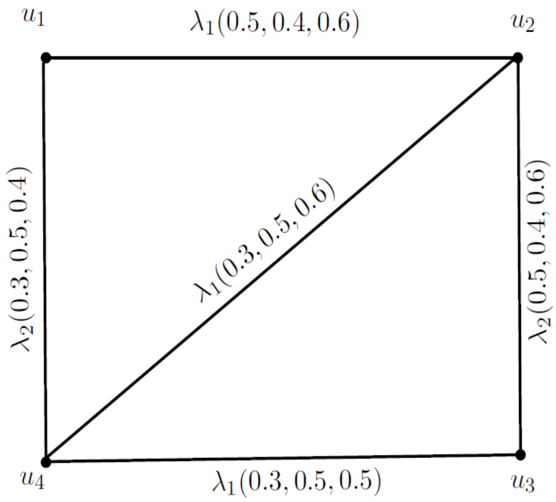

Figure 1

And λ_{1J}, $\lambda_{2J}, \lambda_{3J}$ are defined by $\lambda_{1J}: Q \to [0,1]$, $\lambda_{2J}: Q \to [0,1]$, $\lambda_{3J}: Q \to [0,1]$, as shown in Figure 2, where

$$\lambda_{11}(u_1) = \min_{(u_1,u_2)\in R_1} (\lambda_{11}(u_1, u_2)) = 0.5$$

Similarly, $\lambda_{11}(u_2) = 0.3$, $\lambda_{11}(u_3) = 0.3$, $\lambda_{11}(u_4) = 0.3$,

$$\lambda_{12}(u_1) = \min_{(u_1,u_4)\in R_2} (\lambda_{12}(u_1, u_4)) = 0.3$$

Similarly, $\lambda_{12}(u_2) = 0.5$, $\lambda_{12}(u_3) = 0.5$, $\lambda_{12}(u_4) = 0.3$.

The λ_1-dominating SVNGS

$$\lambda_1(u_1) = (0.5,0.4,0.6),\ \lambda_1(u_2) = (0.3,0.5,0.6),\ \lambda_1(u_3) = (0.3,0.5,0.5),$$

$$\lambda_1(u_4) = (0.3,0.5,0.6).$$

The λ_2-dominating SVNGS

$$\lambda_2(u_1) = (0.3,0.5,0.4),\ \lambda_2(u_2) = (0.5,0.4,0.6),\ \lambda_2(u_3) = (0.5,0.4,0.6),$$

$$\lambda_1(u_4) = (0.3,0.5,0.4).$$

Here, $u_1\lambda_1$-dominates u_2 and $u_3\lambda_1$-dominates u_4 because

$$\lambda_{11}(u_1,u_2) \le \min\{\lambda_{11}(u_1),\lambda_{11}(u_2)\},\ \lambda_{21}(u_1,u_2) \le \min\{\lambda_{21}(u_1),\lambda_{21}(u_2)\},$$
$$\lambda_{31}(u_1,u_2) \le \min\{\lambda_{31}(u_1),\lambda_{31}(u_2)\},\ \lambda_{11}(u_3,u_4) \le \min\{\lambda_{11}(u_3),\lambda_{11}(u_4)\},$$
$$\lambda_{21}(u_3,u_4) \le \min\{\lambda_{21}(u_3),\lambda_{21}(u_4)\},\ \lambda_{31}(u_3,u_4) \le \min\{\lambda_{31}(u_3),\ \lambda_{31}(u_4)\}$$

u_1 λ_2-dominates u_4 and $u_3\lambda_2$-dominates u_2 because

$$\lambda_{12}(u_1,u_4) \le \min\{\lambda_{12}(u_1),\lambda_{12}(u_4)\},\ \lambda_{22}(u_1,u_4) \le \min\{\lambda_{22}(u_1),\lambda_{22}(u_4)\},$$
$$\lambda_{32}(u_1,u_2) \le \min\{\lambda_{32}(u_1),\lambda_{32}(u_2)\}\ \lambda_{12}(u_2,u_3) \le \min\{\lambda_{12}(u_2),\lambda_{12}(u_3)\},$$
$$\lambda_{22}(u_2,u_3) \le \min\{\lambda_{22}(u_2),\lambda_{22}(u_3)\},\ \lambda_{32}(u_2,u_3) \le \min\{\lambda_{32}(u_2),\ \lambda_{32}(u_3)\}$$

Thus $D_J = \{u_1,u_3\}$ is a λ_J-dominating set because every vertex in $Q - D_J$, is λ_J-dominated by atleast one vertex in D_J for all $J = 1,2$.

The adjacency matrix of λ_1-dominating SVNGS given in Figure 2 is

$$A\lambda_1 = \begin{bmatrix} (1,1,1) & (0.5,0.4,0.6) & (0,0,0) & (0,0,0) \\ (0.5,0.4,0.6) & (0,0,0) & (0,0,0) & (0.3,0.5,0.6) \\ (0,0,0) & (0,0,0) & (1,1,1) & (0.3,0.5,0.5) \\ (0,0,0) & (0.3,0.5,0.6) & (0.3,0.5,0.5) & (0,0,0) \end{bmatrix}$$

The adjacency matrix of λ_2-dominating SVNGS given in Figure 2 is

$$A\lambda_2 = \begin{bmatrix} (1,1,1) & (0,0,0) & (0,0,0) & (0.3,0.5,0.4) \\ (0,0,0) & (0,0,0) & (0.5,0.4,0.6) & (0,0,0) \\ (0,0,0) & (0.5,0.4,0.6) & (1,1,1) & (0,0,0) \\ (0.3,0.5,0.4) & (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$$

The adjacency matrix of λ_2-dominating SVNGS given in Figure 2 is

$$A\lambda_2 = \begin{bmatrix} (1,1,1) & (0,0,0) & (0,0,0) & (0.3,0.5,0.4) \\ (0,0,0) & (0,0,0) & (0.5,0.4,0.6) & (0,0,0) \\ (0,0,0) & (0.5,0.4,0.6) & (1,1,1) & (0,0,0) \\ (0.3,0.5,0.4) & (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$$

This can be written in six different matrices as:

$$A\lambda_{11} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0.3 \\ 0 & 0.3 & 0.3 & 0 \end{bmatrix}, A\lambda_{21} = \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{bmatrix},$$

$$A\lambda_{31}=\begin{bmatrix}1&0.6&0&0\\0.6&0&0&0.6\\0&0&1&0.5\\0&0.6&0.5&0\end{bmatrix},A\lambda_{12}=\begin{bmatrix}1&0&0&0.3\\0&0&0.5&0\\0&0.5&1&0\\0.3&0&0&0\end{bmatrix},$$

$$A\lambda_{2}=\begin{bmatrix}1&0&0&0.5\\0&0&0.4&0\\0&0.4&1&0\\0.5&0&0&0\end{bmatrix},A\lambda_{2}=\begin{bmatrix}1&0&0&0.4\\0&0&0.6&0\\0&0.6&1&0\\0.4&0&0&0\end{bmatrix}.$$

Since,

$$\begin{aligned}spec(AP_D^{\lambda_{11}}(\zeta)) &= (-0.4245,0.1202,1.0801,1.2242),\\ spec(AP_D^{\lambda_{21}}(\zeta)) &= (-0.6268,0.2354,1.1173,1.2741),\\ spec(AP_D^{\lambda_{31}}(\zeta)) &= (-0.7728,0.2154,1.1647,1.3927),\\ spec(AP_D^{\lambda_{12}}(\zeta)) &= (-0.2071,-0.0831,1.0831,1.2071),\\ spec(AP_D^{\lambda_{22}}(\zeta)) &= (-0.2071,-0.1403,1.1403,1.2071),\\ spec(AP_D^{\lambda_{32}}(\zeta)) &= (-0.2810,-0.1403,1.1403,1.2810).\end{aligned}$$

Therefore,

$$spec\left(A_{P_D^{\lambda_1}}(\zeta)\right)= \{(-0.4245,-0.6268,-0.7728),(0.1202,0.2354,0.2154),$$

$$(1.0801,1.1173,1.1647),(1.2242,1.2741,1.3927)\},$$

$$spec\left(A_{P_D^{\lambda_2}}(\zeta)\right)= \{(-0.2071,-0.2071,-0.2810),(-0.0831,-0.1403,-0.1403),$$

$$(1.0831,1.1403,1.1403),(1.2071,1.2071,1.2810)\}.$$

The energy of λ_1-dominating SVNGS ζ is

$$E_D^{\lambda_1}(\zeta)=\left(E\left(P_D^{\lambda_{11}}\right),E\left(P_D^{\lambda_{21}}\right),E\left(P_D^{\lambda_{31}}\right)\right)$$

$$=\left\langle\sum_{i=1}^{n}|(\alpha_i)_{\lambda_1}|,\sum_{i=1}^{n}|(\beta_i)_{\lambda_1}|,\sum_{i=1}^{n}|(\gamma_i)_{\lambda_1}|\right\rangle=(2.6086,2.7829,3.1150)$$

The energy of λ_2-dominating SVNGS ζ is

$$E_D^{\lambda_2}(\zeta)=\left(E\left(P_D^{\lambda_{12}}\right),E\left(P_D^{\lambda_{22}}\right),E\left(P_D^{\lambda_{32}}\right)\right)$$

$$=\left\langle\sum_{i=1}^{n}|(\alpha_i)_{\lambda_2}|,\sum_{i=1}^{n}|(\beta_i)_{\lambda_2}|,\sum_{i=1}^{n}|(\gamma_i)_{\lambda_2}|\right\rangle=(2.5804,2.6948,2.8427)$$

Theorem 3.9. Let $\zeta=(\mu,\lambda_1,\lambda_2,\ldots,\lambda_k)$ be a λ_J-dominating SVNGS with n vertices and m R_J-edges. Let $D_J=\{a_1,a_2,a,\ldots,a_w\}$ be a λ_J-dominating set. If

$(\alpha_1)_{\lambda_J}, (\alpha_2)_{\lambda_J}, \ldots, (\alpha_n)_{\lambda_J}, (\beta_1)_{\lambda_J}, (\beta_2)_{\lambda_J}, \ldots, (\beta_n)_{\lambda_J}$ and $(\gamma_1)_{\lambda_J}, (\gamma_2)_{\lambda_J}, \ldots, (\gamma_n)_{\lambda_J}$ are the eigenvalues of the adjacency matrix $P_D^{\lambda_J}(\zeta)$, then

1. $\sum_{i=1}^{n}(\alpha_i)_{\lambda_{1J}} = \eta_{1J}$, $\sum_{i=1}^{n}(\beta_i)_{\lambda_{2J}} = \eta_{2J}$, $\sum_{i=1}^{n}(\gamma_i)_{\lambda_{3J}} = \eta_{3J}$.

2. $\sum_{i=1}^{n}(\alpha_i)_{\lambda_{1J}}^2 = \sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}$,

$$\sum_{i=1}^{n}(\beta_i)_{\lambda_{2J}}^2 = \sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}} P_{ji}^{\lambda_{2J}},$$

$$\sum_{i=1}^{n}(\gamma_i)_{\lambda_{3J}}^2 = \sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}} P_{ji}^{\lambda_{3J}}, \text{where } \eta_{1J} = |D_J|,\ \forall J = 1,2,\ldots,k.$$

Proof 1. The result of the matrices' trace property, we have $\sum_{i=1}^{n}(\alpha_i)_{\lambda_{1J}} = P_{ii}^{\lambda_{1J}} = \eta_{1J}$

Analogously, we can show that

$$\sum_{i=1}^{n}(\beta_i)_{\lambda_{2J}} = P_{ii}^{\lambda_{1J}} = \eta_{2J}, \sum_{i=1}^{n}(\gamma_i)_{\lambda_{3J}} = P_{ii}^{\lambda_{3J}} = \eta_{3J}$$

2. Equivalently, the sum of the square of eigenvalues of $(P_D^{\lambda_J}(\zeta))^2$

$$\sum_{i=1}^{n}(\alpha_i)_{\lambda_{1J}}^2 = \text{ trace of } (P_D^{\lambda_J}(\zeta))^2$$

$$= P_{11}^{\lambda_{1J}}P_{11}^{\lambda_{1J}} + P_{12}^{\lambda_{1J}}P_{21}^{\lambda_{1J}} + \cdots + P_{1n}^{\lambda_{1J}}P_{n1}^{\lambda_{1J}} + P_{21}^{\lambda_{1J}}P_{12}^{\lambda_{1J}} + P_{22}^{\lambda_{1J}}P_{22}^{\lambda_{1J}} + \cdots + P_{2n}^{\lambda_{1J}}P_{n2}^{\lambda_{1J}}$$
$$+ \cdots + P_{n1}^{\lambda_{1J}}P_{1n}^{\lambda_{1J}} + P_{n2}^{\lambda_{1J}}P_{2n}^{\lambda_{1J}} + \cdots + P_{nn}^{\lambda_{1J}}P_{nn}^{\lambda_{1J}}$$

$$= \sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}$$

Analogously, we can show that

$$\sum_{i=1}^{n}(\beta_i)_{\lambda_{2J}}^2 = \sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}} P_{ji}^{\lambda_{2J}},$$
$$\sum_{i=1}^{n}(\gamma_i)_{\lambda_{3J}}^2 = \sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}} P_{ji}^{\lambda_{3J}}.$$

Theorem 3.10. Let $\zeta = (\mu, \lambda_1, \lambda_2, \ldots, \lambda_k)$ be a λ_J-dominating SVNGS with n vertices and m R_J-edges. Let D_J is the λ_J-dominating set, then

1. $\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}} + n(n-1)(\delta_{1J})^{\frac{2}{n}}} \le$

$$E(P_D^{\lambda_{1J}} \le \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}\right)},$$

2. $\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}} P_{ji}^{\lambda_{2J}} + n(n-1)(\delta_{2J})^{\frac{2}{n}}} \le$

$$E(P_D^{\lambda_{2J}} \le \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}} P_{ji}^{\lambda_{2J}}\right)},$$

3. $\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}} P_{ji}^{\lambda_{3J}} + n(n-1)(\delta_{3J})^{\frac{2}{n}}} \le$

$$E(P_D^{\lambda_{3J}} \le \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}} P_{ji}^{\lambda_{3J}}\right)}$$

Where $\chi_{xJ} = det(P_D^{\lambda_J}(\zeta))$ and $\delta_{xJ} = |\chi_{xJ}|$, $x = 1,2,3$ and $J = 1,2,\dots,k$.

Proof. According to Cauchy Schwarz inequality, $(\sum_{(u_i,v_i)\in R_J} u_i, v_i)^2 \le (\sum_{(u_i,v_i)\in R_J} u_i^2)(\sum_{(u_i,v_i)\in R_J} v_i^2)$

Upper bound

If $u_i = 1$ and $v_i = |(\alpha_i)_{\lambda_{1J}}|$, then $(\sum_{(u_i,v_i)\in R_J} |(\alpha_i)_{\lambda_{1J}}|)^2 \le (\sum_{(u_i,v_i)\in R_J} 1)(\sum_{(u_i,v_i)\in R_J} \alpha_i)_{\lambda_{1J}}^2)$

$$(E_D^{\lambda_{1J}}(\zeta))^2 \le n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}\right)$$

$$(E_D^{\lambda_{1J}}(\zeta)) \le \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}\right)}$$

Lower bound

$$(E_D^{\lambda_{1J}}(\zeta))^2 = (\sum_{(u_i,v_i)\in R_J} |(\alpha_i)_{\lambda_{1J}}|)^2 = \left(\sum_{i=1}^{n}|P_{ii}^{\lambda_{1J}}|^2 + 2\sum_{1\le i<j\le n} |P_{ij}^{\lambda_{1J}}||P_{ji}^{\lambda_{1J}}|\right)$$

$$= \left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}}\right) + 2\frac{n(n-1)}{2} AM_{1\le i<j\le n}\left\{|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right\}.$$

But,

$$AM_{1\le i<j\le n}\left\{|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right\} \ge GM_{1\le i<j\le n}\left\{|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right\}.$$

Therefore,

$$(E_D^{\lambda_{1J}}(\zeta)) \geq \sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}} + n(n-1)GM_{1\le i<j\le n}\left\{|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right\}}$$

$$GM_{1\le i<j\le n}\left\{|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right\} = \left(\prod_{1\le i<j\le n}|(\alpha_i)_{\lambda_{1J}}||(\alpha_j)_{\lambda_{1J}}|\right)^{\frac{2}{n(n-1)}}$$

$$= \left(\prod_{1\le i<j\le n}|(\alpha_i)_{\lambda_{1J}}|^{n-1}\right)^{\frac{2}{n(n-1)}} = \left(\prod_{1\le i<j\le n}|(\alpha_i)_{\lambda_{1J}}|\right)^{\frac{2}{n}} = \delta_{1J}^{\frac{2}{n}}$$

$$\left(E_D^{\lambda_{1J}}(\zeta)\right) = \sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}} + n(n-1)(\delta_{1J})^{\frac{2}{n}}}$$

Combining these bounds, we have

1. $$\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}} P_{ji}^{\lambda_{1J}} + n(n-1)(\delta_{1J})^{\frac{2}{n}}}$$

$$\leq \quad E(P_D^{\lambda_{1J}}(\zeta) \leq \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{1J}}P_{ji}^{\lambda_{1J}}\right)}$$

Analogously, we can show that

2. $$\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}} P_{ji}^{\lambda_{2J}} + n(n-1)(\delta_{2J})^{\frac{2}{n}}}$$

$$\leq \quad E(P_D^{\lambda_{2J}}(\zeta) \leq \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{2J}}P_{ji}^{\lambda_{2J}}\right)}$$

3. $$\sqrt{\sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}} P_{ji}^{\lambda_{3J}} + n(n-1)(\delta_{3J})^{\frac{2}{n}}}$$

$$\leq \quad E(P_D^{\lambda_{3J}}(\zeta) \leq \sqrt{n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + 2\sum_{1\le i<j\le n} P_{ij}^{\lambda_{3J}}P_{ji}^{\lambda_{3J}}\right)}$$

Theorem 3.11. Let $A\zeta = (A\lambda_1, A\lambda_2, \dots, A\lambda_k)$ be the adjacency matrix of ζ. Let $\zeta = (\mu, \lambda_1, \lambda_2, \dots, \lambda_k)$ be a λ_J-dominating SVNGS and $A_{D_J}(\zeta)$ be a λ_J-dominating SVNGS adjacency matrix of ζ. Then

1. $(E_D^{\lambda_{1J}}(\zeta))^2 \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + (E(P_D^{\lambda_{1J}})(\zeta))^2\right)$,

2. $\left(E_D^{\lambda_{2J}}(\zeta)\right) \leq \left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + (E(P_D^{\lambda_{2J}})(\zeta))^2\right)$,

3. $(E_D^{\lambda_{2J}}(\zeta)) \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + (E(P_D^{\lambda_{2J}})(\zeta))^2\right)$, $\forall\, J = 1,2,\ldots,k$.

Proof. 1. $(E\left(P_D^{\lambda_{1J}}\right)(\zeta))^2 \geq 2\sum_{1\leq i<j\leq n} P_{ij}^{\lambda_{1J}}\, P_{ji}^{\lambda_{1J}} + n(n-1)(\delta_{1J})^{\frac{2}{n}}$

$$\geq 2\sum_{1\leq i<j\leq n} P_{ij}^{\lambda_{1J}}\, P_{ji}^{\lambda_{1J}}$$

$$i.e\ 2\sum_{1\leq i<j\leq n} P_{ij}^{\lambda_{1J}}\, P_{ji}^{\lambda_{1J}} \leq (E(P_D^{\lambda_{1J}})(\zeta))^2 \quad (1)$$

Now,

$$(E_D^{\lambda_{1J}}(\zeta))^2 \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + 2\sum_{1\leq i<j\leq n} P_{ij}^{\lambda_{1J}}\, P_{ji}^{\lambda_{1J}}\right)$$

$$(E_D^{\lambda_{1J}}(\zeta))^2 \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{1J}})^2 + (E(P_D^{\lambda_{1J}})(\zeta))^2\right) \text{ by eq-(1)}$$

Analogously, we can show that

$$(E_D^{\lambda_{2J}}(\zeta)) \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{2J}})^2 + (E(P_D^{\lambda_{2J}})(\zeta))^2\right)$$

$$(E_D^{\lambda_{3J}}(\zeta)) \leq n\left(\sum_{i=1}^{n}(P_{ii}^{\lambda_{3J}})^2 + (E(P_D^{\lambda_{3J}})(\zeta))^2\right),\ \forall\, J = 1,2,\ldots,k.$$

Definition 3.12. Let $\zeta = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a λ_J-dominating SVNGS of GS $\zeta^* = (Q, R_1, R_2, \ldots, R_k)$ is isomorphic to λ_J-dominating SVNGS $\zeta_S = \{\mu', \lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$ of GS $\zeta_S^* = \{Q', R'_1, R'_2, \ldots, R'_k\}$ if we have (f, Ψ) where $f: Q \to Q'$ is bijection and Ψ is a permutation on set $\{1,2,\ldots,k\}$ and following relation are satisfied
$\mu_1(u) = \mu'_1(f(u))$, $\mu_2(u) = \mu'_2(f(u))$ and $\mu_3(u) = \mu'_3(f(u))$ for all $u \in N$ and $\lambda_{1J}(uv) = \lambda'_{1\Psi(J)}(f(u)f(v)), \lambda_{2J}(uv) = \lambda'_{2\Psi(J)}(f(u)f(v))$ and $\lambda_{3J}(uv) = \lambda'_{3\Psi(J)}(f(u)f(v))$ for all $(u,v) \in R_J$, $J = 1,2,\ldots,k$.

Remark 3.13.] If two adjacency matrices of λ_J-dominating SVNGS are $A(\zeta_1) = \{A\lambda_1, A\lambda_2, \ldots, A\lambda_k\}$ and $A(\zeta_2) = \{A\lambda'_1, A\lambda'_2, \ldots, A\lambda'_k\}$, they are isomorphic if

1. $\sum_{i=1}^{n} Ad_{\lambda_{1J}}(u_i) = \sum_{i=1}^{n} Ad_{\lambda'_{1J}}(f(u_i)), \sum_{i=1}^{n} Ad_{\lambda_{2J}}(u_i) = \sum_{i=1}^{n} Ad_{\lambda'_{2J}}(f(u_i))$

$$\sum_{i=1}^{n} Ad_{\lambda_{3J}}(u_i) = \sum_{i=1}^{n} Ad_{\lambda'_{3J}}(f(u_i)), \forall J = 1,2,\ldots,k.$$

2. The energy of SVNGS $\zeta_1 = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ is equal to the energy of SVNGS $\zeta_1 = \{\mu', \lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$.

$$i.e \quad \epsilon(\zeta_1) = \epsilon(\zeta_2)$$

$$< \epsilon(\lambda_1), \epsilon(\lambda_2), \ldots, \epsilon(\lambda_k) >=< \epsilon(\lambda'_1), \epsilon(\lambda'_2), \ldots, \epsilon(\lambda'_k) >$$

Example 3.14. The adjacency matrix $A(\zeta) = \{A\lambda_1, A\lambda_2\}$ of λ_J-dominating SVNGS $\zeta = \{\mu, \lambda_1, \lambda_2\}$ is isomorphic to the adjacency matrix $A(\zeta_2) = \{A\lambda'_1, A\lambda'_2\}$ of λ_J-dominating SVNGS $\zeta_2 = \{\mu', \lambda'_1, \lambda'_2\}$ as shown in Figure 2 and Figure 3 under (f, Ψ) where $f: Q \to Q'$ is a bijection and Ψ is a permutation on set $\{1,2\}$ defined as $\Psi(1) = 2, \Psi(2) = 1$ and following relations are satisfied, by above Definition-[d8], Remark-[r1]. Such that $\mu = \{u_4(0.3,0.5,0.4), u_3(0.6,0.4,0.5), u_2(0.5,0.4,0.6), u_1(0.6,0.4,0.3)\}$ As shown in Figure 3, we can easy to verify

$$\sum_{i=1}^{4} Ad_{\lambda_{1J}}(u_i) = \sum_{i=1}^{4} Ad_{\lambda'_{1J}}(f(u_i)), \sum_{i=1}^{4} Ad_{\lambda_{2J}}(u_i) = \sum_{i=1}^{4} Ad_{\lambda'_{2J}}(f(u_i)),$$

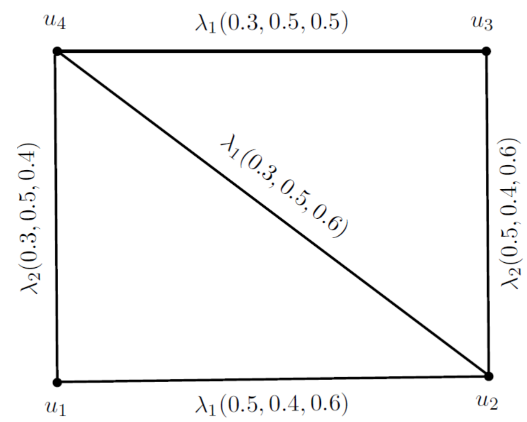

Figure 2

$$\sum_{i=1}^{4} Ad_{\lambda_{3J}}(u_i) = \sum_{i=1}^{4} Ad_{\lambda'_{3J}}(f(u_i)), \forall J = 1,2.$$

The $D_J = \{u_1, u_3\}$ is a λ_J-dominating SVNGS ζ is equal to the $D'_J = \{u_1, u_3\}$ is a λ_J-dominating SVNGS ζ_2 for all $J = 1,2$. The energy of λ_J-dominating SVNGS ζ is equal to the energy of λ_J-dominating SVNGS ζ_2 for all $J = 1,2$.

$$\epsilon(\zeta) = \epsilon(\zeta_2)$$

$$< \epsilon(\lambda_1), \epsilon(\lambda_2) >=< \epsilon(\lambda'_1), \epsilon(\lambda'_2) >$$

Definition 3.15. Let $\zeta = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a λ_J-dominating SVNGS of GS $\zeta^* = \{Q, R_1, R_2, \ldots, R_k\}$ is identical to SVNGS $\zeta_S = \{\mu', \lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$ of GS $\zeta^*_S = \{Q', R'_1, R'_2, \ldots, R'_k\}$ is a bijection and the following relations are satisfied.
$\mu_1(u) = \mu'_1(f(u))$, $\mu_2(u) = \mu'_2(f(u))$ and $\mu_3(u) = \mu'_3(f(u))\ \forall u \in Q$ and $\lambda_{1J}(uv) = \lambda'_{1J}(f(u)f(v))$, $\lambda_{2J}(uv) = \lambda'_{2J}(f(u)f(v))$, $\lambda_{3J}(uv) = \lambda'_{3J}(f(u)f(v))\ \forall\ (uv) \in R_J$, $J = 1,2, \ldots, k$.

Example 3.16. Let $\zeta = \{\mu, \lambda_1, \lambda_2\}$ and $\zeta_S = \{\mu', \lambda'_1, \lambda'_2\}$ be a two λ_J-dominating SVNGS of GS $\zeta^* = \{Q, R_1, R_2\}$ and $\zeta^*_S = \{Q', R'_1, R'_2\}$ respectively, as shown in Figure 4. SVNGS of GS ζ^* is identical to ζ^*_S under $f: Q \to Q'$ define as
$f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_1$, $f(u_4) = v_2$ and $\mu_1(u_i) = \mu'_1(f(u_i)), \forall\ u_i \in Q$ and $\lambda_{1J}(u_i v_j) = \lambda'_{1J}(f(u_i)f(v_j)), \lambda_{2J}(u_i v_j) = \lambda'_{3J}(f(u_i)f(v_j))$ and $\lambda_{3J}(u_i v_j) = \lambda'_{3J}(f(u_i)f(v_j))$, for all $(u_i v_j) \in R_J$ and $J = 1,2$.

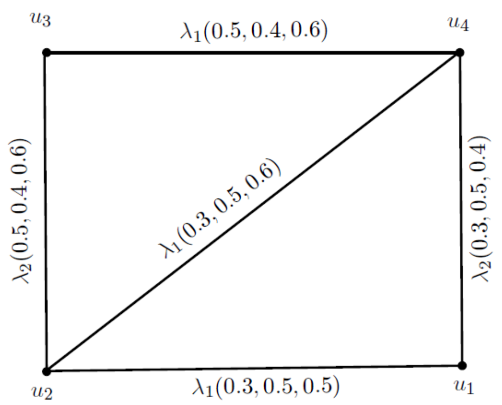

Figure 3

Hence, ζ^* is identical to ζ^*_S

Remark 3.17. If two adjacency matrices of λ_J- dominating SVNGS are $A(\zeta_1) = \{A\lambda_1, A\lambda_2, \ldots, A\lambda_k\}$ and $A(\zeta_S) = \{A\lambda'_1, A\lambda'_2, \ldots, A\lambda'_k\}$. then ζ^* is identical to ζ^*_S which is satisfied 1 to 3 condition in remark-3.13.

Theorem 3.18 Let $\zeta = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $\zeta_S = \{\mu', \lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$ be two σ_J-dominating SVNGS of GS $\zeta^* = \{Q, R_1, R_2, \ldots, R_k\}$ and $\zeta^*_S = \{Q', R'_1, R'_2, \ldots, R'_k\}$. Then ζ is identical to ζ_S under $f: Q \to Q'$.

Proof. Let $\zeta = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $\zeta_S = \{\mu', \lambda'_1, \lambda'_2, \ldots, \lambda'_k\}$ be two σ_J-dominating SVNGS of GS $\zeta^* = \{Q, R_1, R_2, \ldots, R_k\}$ and $\zeta^*_S = \{Q', R'_1, R'_2, \ldots, R'_k\}$.

$$\lambda_{1J}(u_i v_j) = \min\{\mu_1(u_i), \mu_1(u_j)\} = \min\left\{\mu'_1\left(f(u_j)\right), \mu'_1(f(u_i))\right\} = \mu'_1(f(u_j)f(u_i))$$

Therefore, $\lambda_{1J}(u_i v_j) = \mu'_1(f(u_j)f(u_i))$

Similarly, we derive the equation $\lambda_{2J}(u_i v_j) = \mu'_2\left(f(u_j)f(u_i)\right), \lambda_{3J}(u_i v_j) = \mu'_3(f(u_j)f(u_i))$, $\forall J = 1,2, \ldots, k$.

Hence, ζ is identical to ζ_S under $f: Q \to Q'$.

3.1 Flow chart for Computing the Energy of λ_J-dominating SVNGS

The proposed flowchart in Figure 5, for calculating the energy of λ_J-dominating SVNGS are presented in this section. Using the energy of λ_J-dominating nodes, a flowchart is used to identify the greatest accessible λ_J-relation nodes in an SVNGS.

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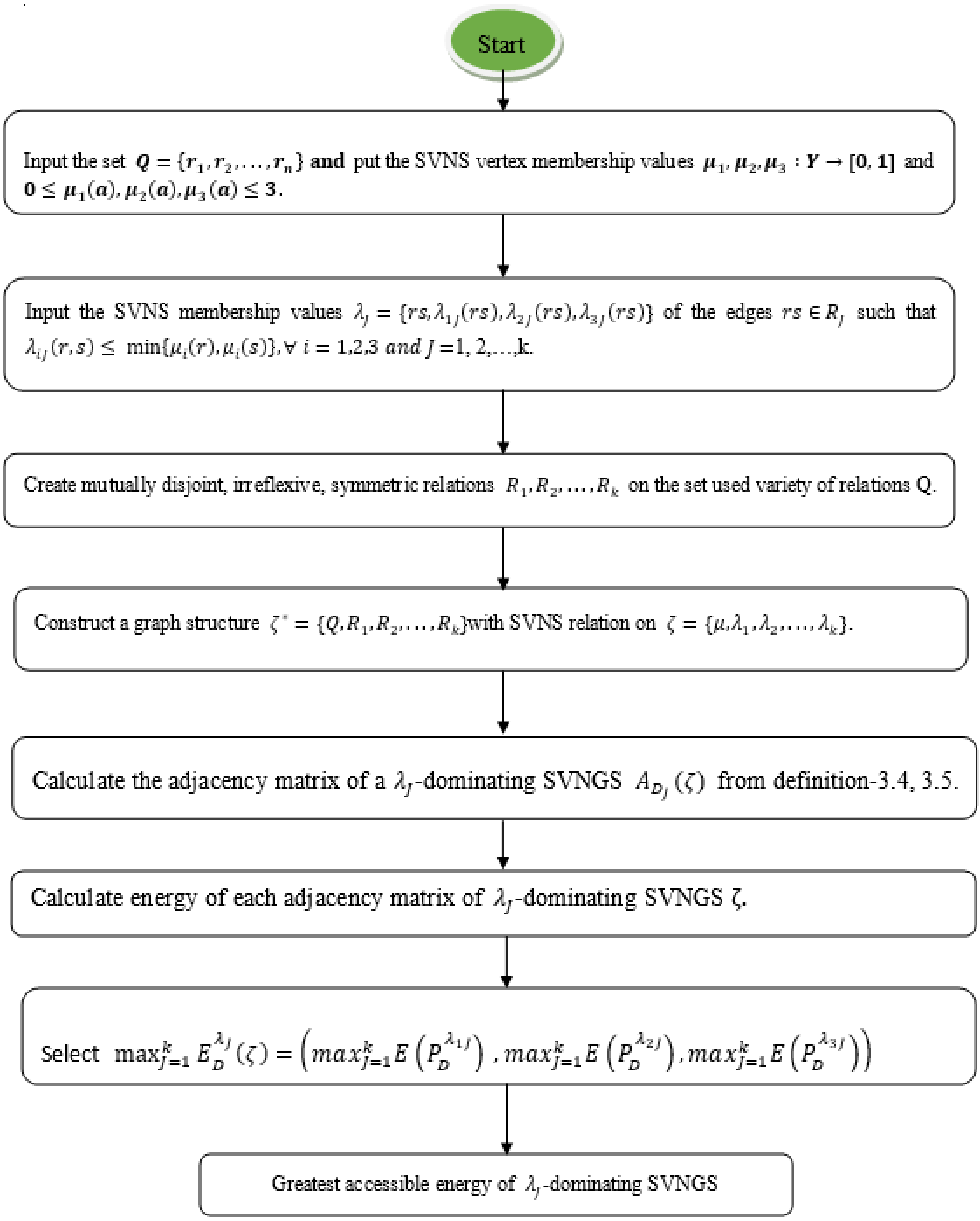

Figure 5. Flow chart for computing the energy of λ_J-dominating SVNGS.

4. Discussion

The concept of Generalized GSs was first proposed by E. Sampathkumar [40], and Generalized FGSs were then proposed by T. Dinesh et al. [41]. Recently, Akram [43] developed the SVNGS concept. S. Mathew [32] introduced the idea of an FG's energy, and Mullai, M., and Broumi, S. (2020) introduced the idea of Dominating Energy in NGs. But, when a SVNGS comes in a real-life situation, it is very necessary to know about the SVNGS with its energy of λ_J-dominating. The results of this research may reveal its applications in many ways, especially in finding optimal functions. Its manifestations can be found in our flowchart section. The studies presented so far have only addressed fuzzy with graph energy and graph dominating energy; we now find results for SVN with graph structures energy of λ_J-dominating, isomorphic, and identical as an improvement, which makes their expression even more flexible in many applications.

5. Advantages and Limitations

The main advantages of the proposed method are as follows:

- Its advantage is that it allows us to detect a specific relationship between the SVNGS and its energy of λ_J-dominating. It means $E_D^{\lambda_J}(\zeta) = (E(P_D^{\lambda_{1J}}), E(P_D^{\lambda_{2J}}), E(P_D^{\lambda_{3J}}))$ for all J=1,2,…,k. This can be calculated by the energy of λ_J-dominating SVNGS ζ.
- Also in this study, we have found some properties of isomorphic and identical energy of λ_J-dominating for specific relationship and their advantages.

Some of the work limitations are as follows:

- The energy of λ_J-dominating SVNGS was the main goal of the investigation and related network systems.
- This approach is only applicable to the SVNGS in an environment of symmetric, irreflexive, and mutually disjoint relations.
- There is no significance to the SVNGS concept if the characters' membership values are given in disparate environments.
- It may not always be possible to get trustworthy results.

6. Conclusion and Future Works

The concept of λ_J-dominating SVNGS energy is elaborated by the authors in this study. Beyond conventional fuzzy graph energy and dominating properties, the concept of λ_J-dominating SVNGS energy offers even more flexibility in describing uncertainty. It is an extension of fuzzy graph energy and dominating fuzzy graph energy. It also provides definitions that are important for comprehending the main results. Further, the energy of λ_J-dominating SVNGS was also investigated, along with some of its properties and bounds. We also present the notion of isomorphic and identical λ_J-dominating SVNGS. There are many different directions that future research in this field could go if the adjacency matrix SVNGS is used. Utilizing SVNGS, determine the properties of the edge regular, connectivity index, and Wiener index. Further research is suggested in the following areas, which we intend to expand on: complex bipolar neutrosophic graph structures; complex q-rung orthopair fuzzy graph structures; and complex interval-valued spherical fuzzy graph structures.

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Author Contributions
All authors contributed equally to this research.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

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