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Fixed Point Results in Complex Valued Neutrosophic b-Metric Spaces with Application

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Abstract: In this manuscript, we introduce the idea of complex-valued Neutrosophic b-metric spaces along with numerous significant illustrations. We provide fixed-point results for contraction maps. To support the main result, we establish the existence and uniqueness of solutions for nonlinear integral equations after the work.

Keywords: Fuzzy Metric; Complex Valued Neutrosophic Metric Space; Fixed Point; Contractive Map; Unique Solution.

1. Introduction

Azam et al. [1] pioneered the idea of complex-valued metric spaces in 2011. Rouzkard et al. [2] studied and extended the conclusions of [1] by investigating numerous common fixed point theorems in this space. Many standard fixed point solutions in such space for mappings satisfying rational expressions on a closed ball were examined by Ahmad et al. [3]. Common fixed point theorem in complex-valued b-metric established by Rao et al. [4]. Following the development of this concept, Mukheimer [5] discovered common fixed point outcomes of a pair of self-mappings meeting a rational inequality in complex-valued b-metric space. Zadeh [6] established the basis for fuzzy mathematics in 1965. Kramosil and Michalek [7] initially brought up the concept of fuzzy metric-like space and then modified it by George and Veeramani [8]. Atanassov [9] stirred things up by adding the idea of a non-membership grade of fuzzy set theory. Fuzzy metric space has been widened to Intuitionistic fuzzy metric space by Park [10]. Park used continuous triangular norm as well as continuous triangular conorm to describe this idea. Smarandache [11] described the concept of neutrosophic logic and neutrosophic sets in 1998.

This study aims to present the concept of Complex Valued Neutrosophic b-metric Space. In addition, this research expands on previous fixed-point findings over contractions. To strengthen, we finish our work with an application to integral equations and an example illustrating the applicability of our main results.

2. Preliminaries

This study will require the following definitions and results.

$\mathbb{C}$ denotes the set of complex numbers.

We set $\mathfrak{H} = \{(p, q): 0 \leq p < \infty, 0 \leq q < \infty\} \subset \mathbb{C}$.

A partial ordering $\precsim$ on $\mathbb{C}$ is defined by $\tau_1 \precsim \tau_2$ (equivalently, $\tau_2 \precsim \tau_1$) $\Leftrightarrow$ $Re(\tau_1) \leq Re(\tau_2)$ and $Im(\tau_1) \leq Im(\tau_2)$. The closed unit complex interval is defined as $\mathfrak{F} = \{(p, q): 0 \leq p < 1, 0 \leq q < 1\}$ and the open unit complex interval by $\mathfrak{F}_{\ddot{o}} = \{(p, q): 0 < p < 1, 0 < q < 1\}$.

The set $\{(\wp, q): 0 < \wp < \infty, 0 < q < \infty\}$ denoted by $\mathfrak{H}_{\ddot{o}}$. The elements $(1, 1), (0,0) \in \mathfrak{H}$ are indicated by ℓ and ö, respectively.

Remark 2.1[12]. Let $\{\tau_\iota\}$ be a sequence in $\mathfrak{H}$. Then,

(i) If $\{\tau_\iota\}$ is monotonic in $\mathfrak{H}$ and there exists $\rho, \sigma \in \mathfrak{H}$ such that $\rho \precsim \tau_\iota \precsim \sigma$, for every $\iota \in \mathrm{N}$, then there exists a $\tau \in \mathfrak{H}$ such that $\lim_{\iota\to\infty} \tau_\iota = \tau$.

(ii) $\Theta \subset \mathbb{C}$ is that there exists $\rho, \sigma \in \mathbb{C}$ with $\rho \precsim \mathbb{C} \precsim \sigma$ for all $\theta \in \Theta$, then $\inf \Theta$ and $\sup \Theta$ both exist.

Remark 2.2 [12]. Let $\tau_\iota, \tau'_\iota, \eta \in \mathfrak{H}$ for every $\iota \in \mathrm{N}$. Then,

(i) If $\tau_\iota \precsim \tau'_\iota \precsim \ell$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota\to\infty} \tau_\iota = \ell$, then $\lim_{\iota\to\infty} \tau'_\iota = \ell$.

(ii) If $\tau_\iota \precsim \eta$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota\to\infty} \tau_\iota = \tau \in \mathfrak{H}$, then $\iota \precsim \eta$.

(iii) If $\eta \precsim \tau_\iota$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota\to\infty} \tau_\iota = \tau \in \mathfrak{H}$, then $\eta \precsim \iota$.

Definition 2.3 [12]. Let $\{\tau_\iota\}$ be a sequence in $\mathfrak{H}$**.** If for all $\tau \in \mathfrak{H}$ there exists an $\iota_0 \in \mathrm{N}$ such that $\tau \precsim \tau_\iota$ for all $\iota > \iota_0$. Then $\{\tau_\iota\}$ is named to be diverged to ∞ as $\iota \to \infty$, and we write $\lim_{\iota\to\infty} \tau_\iota = \infty$.

Definition 2.4 [12]**.** A binary operation $* : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ is named a complex-valued t-norm, if for all $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathfrak{F}$

(i) $\tau_1 * \tau_2 = \tau_2 * \tau_1$;

(ii) $\tau * \ddot{o} = \ddot{o}$, $\tau * \ell = \tau$;

(iii) $\tau_1 * (\tau_2 * \tau_3) = (\tau_1 * \tau_2) * \tau_3$;

(iv) $\tau_1 * \tau_2 \precsim \tau_3 * \tau_4$ whenever $\tau_1 \precsim \tau_3$, $\tau_2 \precsim \tau_4$.

Example 2.5 [12].

(i) $\tau_1 * \tau_2 = (\wp_1\wp_2, q_1q_2)$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(ii) $\tau_1 * \tau_2 = (\min\{\wp_1, \wp_2\}, \min\{q_1, q_2\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(iii) $\tau_1 * \tau_2 = (\max\{\wp_1 + \wp_2 - 1, 0\}, \max\{q_1 + q_2 - 1, 0\})$,
for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$.
These are examples of complex-valued t-norm.

Example 2.6 [12]. The following are examples of complex-valued t-conorm:

(i) $\tau_1 \star \tau_2 = (\max\{\wp_1, \wp_2\}, \max\{q_1, q_2\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(ii) $\tau_1 \star \tau_2 = (\min\{\wp_1 + \wp_2, 1\}, \min\{q_1 + q_2, 1\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$.

Definition 2.7. Let Ξ be a nonvoid set, $*$, $\star$ are complex-valued continuous t-norm and t-conorm, $\widetilde{\mathfrak{P}}$, $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{Q}}$ are complex fuzzy sets on $\Xi^2 \times \mathfrak{H}_{\ddot{o}}$ fulfilling the following assertions:

(1) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) + \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) + \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau) \precsim 3$;

(2) $\ddot{o} \prec \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau)$;

(3) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \ell$ for every $\tau \in \mathfrak{H}_{\ddot{o}} \Leftrightarrow$ if $\mathfrak{u} = \mathfrak{v}$;

(4) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \widetilde{\mathfrak{P}}(\mathfrak{v}, \mathfrak{u}, \tau)$;

(5) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) * \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{w},\tau') \lesssim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{w}, \tau+\tau')$;

(6) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(7) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(8) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u}=\mathfrak{v}$;

(9) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{u},\tau)$;

(10) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{w},\tau') \gtrsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{w}, \tau+\tau')$;

(11) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(12) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(13) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u}=\mathfrak{v}$;

(14) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{u},\tau)$;

(15) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{w},\tau') \gtrsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{w}, \tau+\tau')$;

(16) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous.

The Triplet $(\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}})$ is called a Complex Valued Neutrosophic Metric Space (CVNMS).

Definition 2.8. Let Ξ be a nonvoid set, $\theta \geq 1$ be a given real number, $*$, $\star$ are complex-valued continuous t-norm and t- conorm , $\widetilde{\mathfrak{P}}$, $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{Q}}$ are complex fuzzy sets on $\Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}}$ fulfilling the following assertions. Then $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*, \star,\theta)$ is called a Complex Valued Neutrosophic b-Metric Space (CVNbMS). For all $\mathfrak{u},\mathfrak{v},\mathfrak{w} \in \Xi$ and $\tau,\tau' \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$.

(1) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) + \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) + \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \lesssim 3$;

(2) $\ddot{\mathfrak{o}} \prec \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)$;

(3) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \ell$ for every $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}} \Leftrightarrow \mathfrak{u}=\mathfrak{v}$;

(4) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{u},\tau)$;

(5) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) * \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{w},\tau') \lesssim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{w}, \theta(\tau+\tau'))$;

(6) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(7) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(8) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u}=\mathfrak{v}$;

(9) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{u},\tau)$;

(10) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{w},\tau') \gtrsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{w}, \theta(\tau+\tau'))$;

(11) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(12) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(13) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u}=\mathfrak{v}$;

(14) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{u},\tau)$;

(15) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{w},\tau') \gtrsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{w}, \theta(\tau+\tau'))$;

(16) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous.

Example 2.9 Let (Ξ,ρ,θ) be a b-Metric Space (bMS). Let $\tau_1 * \tau_2 = (\min\{\wp_1,\wp_2\},\min\{q_1,q_2\})$, $\tau_1 \star \tau_2 = (\max\{\wp_1,\wp_2\},\max\{q_1,q_2\})$ for all $\tau_1 = (\wp_1,q_1), \tau_2 = (\wp_2,q_2) \in \mathfrak{F}$. Let us consider the Complex

Fuzzy Sets[*CFS*] $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}} : \Xi^2 \times \mathfrak{H}_{\ddot{o}} \to \mathfrak{F}$ such that $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{pq}{pq+\rho(\mathfrak{u},\mathfrak{v})}\ell, \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{\rho(\mathfrak{u},\mathfrak{v})}{pq+\rho(\mathfrak{u},\mathfrak{v})}\ell,\ \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau) =$

$\frac{\rho(\mathfrak{u},\mathfrak{v})}{pq}\ell$, where $\tau = (p, q) \in \mathfrak{H}_{\ddot{o}}$. Then, $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is a *CVNbMS.*

Lemma 2.10 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* and $\tau_1, \tau_2 \in \mathbb{C}$. If $\tau_1 \prec \tau_2$, then $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \precsim \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2)$, $\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \succsim \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2)$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \succsim \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2)$ for all $\mathfrak{u}, \mathfrak{v} \in \Xi$.

Proof. Let $\tau_1, \tau_2 \in \mathfrak{H}_{\ddot{o}}$ be such that $\tau_1 \prec \tau_2$.

Therefore, $\tau_2 - \tau_1 \in \mathfrak{H}_{\ddot{o}}$ and so that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$, we get $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau_1) = \ell * \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau_1) = \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{u}, \tau_2 - \tau_1) * \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \precsim \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2)$

$\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2) \precsim \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{u}, \tau_2 - \tau_1) \star \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \precsim 0 \star \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau_1)$ and

$\widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \theta\tau_2) \precsim \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{u}, \tau_2 - \tau_1) \star \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau_1) \precsim 0 \star \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau_1)$.

Definition 2.11 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* and $\{\mathfrak{u}_\iota\}$ be a sequence in Ξ.

(i) $\{\mathfrak{u}_\iota\}$ converges to $\mathfrak{u} \in \Xi$ if for every $\gamma \in \mathfrak{F}_{\ddot{o}}$ and every $\tau \in \mathfrak{H}_{\ddot{o}}$, there exists $\iota_0 \in \mathbb{N}$ such that, for every $\iota > \iota_0$, $\ell - \gamma \prec \widetilde{\mathfrak{P}}(\mathfrak{u}_\iota, \mathfrak{u}, \tau)$, $\widetilde{\mathfrak{L}}(\mathfrak{u}_\iota, \mathfrak{u}, \tau) \prec \gamma$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u}_\iota, \mathfrak{u}, \tau) \prec \gamma$. We denote this by $\lim_{\iota\to\infty} \mathfrak{u}_\iota = \mathfrak{u}$.

(ii) $\{\mathfrak{u}_\iota\}$ in Ξ is named to be a Cauchy sequence in $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ if for every $\tau \in \mathfrak{H}_{\ddot{o}}$, $\lim_{\iota\to\infty} inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ell$, $\lim_{\iota\to\infty} sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ddot{o}$ and $\lim_{\iota\to\infty} sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ddot{o}$.

(iii) $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is known to be a complete *CVNbMS* if for every Cauchy sequence $\{\mathfrak{u}_\iota\}$ in $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$, there exists an $\mathfrak{u} \in \Xi$ such that $\lim_{\iota\to\infty} \mathfrak{u}_\iota = \mathfrak{u}$.

Lemma 2.12 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS*. A sequence $\{\mathfrak{u}_\iota\}$ in Ξ converge to $\mathfrak{u} \in \Xi \Leftrightarrow \lim_{\iota\to\infty} \widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ell$, $\lim_{\iota\to\infty} \widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ddot{o}$ and $\lim_{\iota\to\infty} \widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) = \ddot{o}$ holds for all $\tau \in \mathfrak{H}_{\ddot{o}}$.

3. Main Results

Theorem 3.1 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* such that, for every sequence $\{\tau_\iota\}$ in $\mathfrak{H}_{\ddot{o}}$ with $\lim_{\iota\to\infty} \tau_\iota = \infty$, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau_\iota) = \ell$, $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau_\iota) = \ddot{o}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau_\iota) = \ddot{o}$ for all $\mathfrak{u} \in \Xi$. Let $\mathfrak{k} : \Xi \to \Xi$ be a mapping satisfying

$$\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau),\ \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) \text{ and } \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau) \quad (3.1.1)$$

For all $\mathfrak{u}, \mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{o}}$ where $\delta \in (0, 1)$. Then $\mathfrak{k}$ has a unique fixed point in Ξ.

Proof:

Let $\mathfrak{u}_0$ be a random element of Ξ and define the sequence $\{\mathfrak{u}_\iota\}$ in Ξ by the iterative method $\mathfrak{u}_\iota = \mathfrak{k}\mathfrak{u}_{\iota-1}$ for every $\iota \in \mathrm{N}$. If $\mathfrak{u}_\iota = \mathfrak{u}_{\iota-1}$ for some $\iota \in \mathrm{N}$, then $\mathfrak{u}_\iota$ is a fixed point of $\mathfrak{k}$.

So $\mathfrak{u}_\iota \neq \mathfrak{u}_{\iota-1}$ for every $\iota \in \mathrm{N}$. We claim that $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Define $\mathfrak{W}_\iota = \{\widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) : m > \iota\}$, $\mathfrak{N}_\iota = \{\widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) : m > \iota\}$ and $\mathfrak{O}_\iota = \{\widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) : m > \iota\}$ for all $\iota \in \mathrm{N}$ and $\tau \in \mathfrak{H}_{\ddot{o}}$.

Since $\theta \prec \widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \precsim \ell$, $\theta \prec \widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \precsim \ell$ and $\theta \prec \widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \precsim \ell$ for every $m \in \mathrm{N}$ with $m > \iota$ and from Remark (2.1)(ii), inf $\mathfrak{W}_\iota = \alpha_\iota$, sup$\mathfrak{N}_\iota = \beta_\iota$ and sup$\mathfrak{O}_\iota = \varrho_\iota$ exists for all $\iota \in \mathrm{N}$.

Using Lemma (2.10) and (3.1.1), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \precsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) = \widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \tag{3.1.2}$$

$$\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \succsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) = \widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \tag{3.1.3}$$

and $$\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \succsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) = \widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \tag{3.1.4}$$

for $\tau \in \mathfrak{H}_{\ddot{o}}$ and $m,\iota \in \mathrm{N}$ with $m > \iota$.

Since $\ddot{o} \precsim \alpha_\iota \precsim \alpha_{\iota+1} \precsim \ell$, $\ell \succsim \beta_\iota \succsim \beta_{\iota+1} \succsim \ddot{o}$ and $\ell \succsim \varrho_\iota \succsim \varrho_{\iota+1} \succsim \ddot{o}$ for all $\iota \in \mathrm{N}$ it follows that $\{\alpha_\iota\}$,$\{\beta_\iota\}$ and $\{\varrho_\iota\}$ are monotonic sequences in $\mathfrak{H}$.

Utilizing Remark (2.1)(i), there exists ℓ_0, ℓ' and $\bar{\ell} \in \mathfrak{H}$ such that

$$\lim_{\iota\to\infty} \alpha_\iota = \ell_0,\ \lim_{\iota\to\infty} \beta_\iota = \ell' \quad \text{and} \quad \lim_{\iota\to\infty} \varrho_\iota = \bar{\ell}. \tag{3.1.5}$$

Now, by repeatedly using the contractive condition (3.1.1), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m-1},\mathfrak{u}_{\iota-1},\frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\delta^2\tau}{\theta^2}\right) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m-2},\mathfrak{u}_{\iota-2},\frac{\delta^3\tau}{\theta^3}\right) \succsim \cdots \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m-1},\mathfrak{u}_{\iota-1},\frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\delta^2\tau}{\theta^2}\right) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m-2},\mathfrak{u}_{\iota-2},\frac{\delta^3\tau}{\theta^3}\right) \precsim \cdots \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m,\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m-1},\mathfrak{u}_{\iota-1},\frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\delta^2\tau}{\theta^2}\right) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m-2},\mathfrak{u}_{\iota-2},\frac{\delta^3\tau}{\theta^3}\right) \precsim \cdots \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right).$$

for $\tau \in \mathfrak{H}_{\ddot{o}}$ and $m,\iota \in \mathrm{N}$ with $m > \iota$.

Thus, $\alpha_{\iota+1} = inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \succsim inf_{m>\iota}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \succsim inf_{\mathfrak{v}\in\Xi.}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$,

$\beta_{\iota+1} = sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \precsim sup_{m>\iota}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$ and

$\varrho_{\iota+1} = sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1},\mathfrak{u}_{\iota+1},\tau) \precsim sup_{m>\iota}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$.

Since $\lim_{\iota\to\infty} \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}} = \infty$, by using the hypothesis along with (3.1.5), we obtain

$\ell_0 \succsim \lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ell,\ \ell' \precsim \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ddot{o}$ and

$\bar{\ell} \precsim \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{v},\frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ddot{o}$.

This indicates that $\ell_0 = \ell$, $\ell' = \ddot{o}$ and $\bar{\ell} = \ddot{o}$. Thus, $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Since $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\ \star,\theta)$ is a *CVNbMS,* by Lemma (2.12), there exists a $\mathfrak{d} \in \Xi$ such that for all $\tau \in \mathfrak{H}_{\ddot{o}}$,

$$\lim_{\iota\to\infty} \widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{d}, \tau) = \ell,\ \lim_{\iota\to\infty} \widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{d}, \tau) = \ddot{\mathfrak{o}} \quad \text{and} \quad \lim_{\iota\to\infty} \widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{d}, \tau) = \ddot{\mathfrak{o}}. \qquad (3.1.6)$$

We will demonstrate that $\mathfrak{d}$ is the fixed point of $\mathfrak{k}$. As a result of (5), (10) and (15) of definition (2.8), the contractive condition (3.1.1) we get,

$$\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m+1}, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right)$$

$$\succsim \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{u}_m, \mathfrak{d}, \frac{\tau}{2\delta}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m+1}, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right)$$

$$\precsim \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{u}_m, \mathfrak{d}, \frac{\tau}{2\delta}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m+1}, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_m, \mathfrak{k}\mathfrak{d}, \frac{\tau}{2\theta}\right)$$

$$\precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{u}_{\iota+1}, \frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m, \mathfrak{d}, \frac{\tau}{2\delta}\right)$$

for any $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$. Taking the limit as $\iota \to \infty$, by (3.1.6) and Remark (2.2)(ii), we obtain $\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) = \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{k}\mathfrak{d}, \tau) = \ddot{\mathfrak{o}}$ and for all $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$, which gives $\mathfrak{d} = \mathfrak{k}\mathfrak{d}$.

To show that the fixed point $\mathfrak{d}$ is unique. Let $\mathfrak{z}$ be another fixed point of $\mathfrak{k}$, i.e., there is a $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$ with $\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau) \neq \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{z}, \tau) \neq \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{z}, \tau) \neq \ddot{\mathfrak{o}}$ from (3.1.1), we obtain that

$$\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau) = \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{y}, \frac{\theta\tau}{\delta}\right) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^2\tau}{\delta^2}\right) \dots \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\succsim inf_{\mathfrak{v}\in\Xi} \widetilde{\mathfrak{P}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{z}, \tau) = \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \frac{\theta\tau}{\delta}\right) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^2\tau}{\delta^2}\right) \dots \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\precsim sup_{\mathfrak{v}\in\Xi} \widetilde{\mathfrak{L}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{z}, \tau) = \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \frac{\theta\tau}{\delta}\right) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^2\tau}{\delta^2}\right) \dots \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\precsim sup_{\mathfrak{v}\in\Xi} \widetilde{\mathfrak{Q}}\left(\mathfrak{d}, \mathfrak{z}, \frac{\theta^{\iota}\tau}{\delta^{\iota}}\right), \quad \text{for all } \iota \in \mathbb{N}.$$

Hence, since $\lim_{\iota\to\infty} \frac{\delta^{\iota}\tau}{\theta^{\iota}} = \infty$, the above inequality becomes $\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau) \succsim \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{z}, \tau) \precsim \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{z}, \tau) \precsim \ddot{\mathfrak{o}}$ which leads to a contradiction. Thus, we determine that the fixed point of $\mathfrak{k}$ is unique.

Example 3.2. Let $\Xi = [0,1]$ and let $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ such that

$$\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{pq}{pq + (\mathfrak{u} - \mathfrak{v})^2}\ell, \quad \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{(\mathfrak{u} - \mathfrak{v})^2}{pq + (\mathfrak{u} - \mathfrak{v})^2}\ell \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{(\mathfrak{u} - \mathfrak{v})^2}{pq}\ell,$$

where $\tau = (p, q) \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$. Then, we can readily verify that $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is a *CVNbMS* with $\theta = 2$. We conclude that for any sequence $\{\mathfrak{u}_{\iota}\}$ in $\mathfrak{H}_{\ddot{\mathfrak{o}}}$ with $\lim_{\iota\to\infty} \tau_{\iota} = \infty$, we have

$\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)=\ell, \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)=\ddot{\mathrm{o}}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)=\ddot{\mathrm{o}}$ for all $\mathfrak{u}\in\Xi$. Let $\mathfrak{k}:\Xi\to\Xi$ be a mapping defined by $\mathfrak{k}\mathfrak{u}=\varsigma\,\mathfrak{u}^2$ where $0<\varsigma<\frac{1}{4}$. By a routine calculation, we see that

$\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)$, $\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)$ and $\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)$for every $\mathfrak{u},\mathfrak{v}\in\Xi$ and $\tau\in\mathfrak{H}_{\ddot{\mathrm{o}}}$, where $\delta=4\varsigma$ and $0<\delta<1$. All the requirements of Theorem (3.1) are fulfilled and 0 is the unique fixed point of $\mathfrak{k}$.

Theorem 3.3. Let $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\ \star,\theta)$ be a *CVNbMS* such that, for every sequence $\{\tau_\iota\}$ in $\mathfrak{H}_{\ddot{\mathrm{o}}}$ with $\lim_{\iota\to\infty}\tau_\iota=\infty$, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ell, \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ddot{\mathrm{o}}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ddot{\mathrm{o}}$, for all $\mathfrak{u}\in\Xi$. Let $\mathfrak{k},\mathfrak{h}:\Xi\to\Xi$ be a mapping satisfying the following requirements:

(i) $\mathfrak{h}(\Xi)\subseteq\mathfrak{k}(\Xi)$,

(ii) $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ,

(iii) $\mathfrak{k}$ is continuous on Ξ,

(iv) $\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$, $\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$ and $\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$for all $\mathfrak{u},\mathfrak{v}\in\Xi$ and $\tau\in\mathfrak{H}_{\ddot{\mathrm{o}}}$ where $0<\delta<1$. Then $\mathfrak{k}$ and $\mathfrak{h}$ have a unique common fixed point in Ξ.

Proof. Let $\mathfrak{u}_0\in\Xi$. Since $\mathfrak{h}(\Xi)\subseteq\mathfrak{k}(\Xi)$, we can choose an $\mathfrak{u}_1\in\Xi$ such that $\mathfrak{h}\mathfrak{u}_0=\mathfrak{k}\mathfrak{u}_1$. Repeating this procedure, we can choose $\mathfrak{u}_\iota\in\Xi$ such that $\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{u}_{\iota-1}$.

We claim that the sequence $\{\mathfrak{k}\mathfrak{u}_\iota\}$ is a Cauchy sequence. For every $\iota\in\mathrm{N}$ and $\tau\in\mathfrak{H}_{\ddot{\mathrm{o}}}$, define
$\mathfrak{W}_\iota=\{\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau):m>\iota\}$, $\mathfrak{N}_\iota=\{\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau):m>\iota\}$ and $\mathfrak{O}_\iota=\{\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau):m>\iota\}$
for every $\iota\in\mathrm{N}$ and $\tau\in\mathfrak{H}_{\ddot{\mathrm{o}}}$.

Since $\ddot{\mathrm{o}}\prec\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\lesssim\ell$, $\ddot{\mathrm{o}}\prec\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau)\lesssim\ell$ and $\ddot{\mathrm{o}}\prec\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau)\lesssim\ell$, for every $m\in\mathrm{N}$ with $m>\iota$ and from Remark (2.1)(ii), inf $\mathfrak{k}\mathfrak{W}_\iota=\alpha_\iota$, sup $\mathfrak{k}\mathfrak{N}_\iota=\beta_\iota$ and sup$\mathfrak{k}\mathfrak{O}_\iota=\varrho_\iota$ exists for every $\iota\in\mathrm{N}$. Using Lemma(2.10) and (iv), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)=\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau),$$

$$\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{v},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{v},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau),$$

for $\tau\in\mathfrak{H}_{\ddot{\mathrm{o}}}$ and $m,\iota\in\mathrm{N}$ with $m>\iota$.

Since $\ddot{\mathrm{o}}\lesssim\alpha_\iota\lesssim\alpha_{\iota+1}\lesssim\ell,\ell\gtrsim\beta_\iota\gtrsim\beta_{\iota+1}\gtrsim\ddot{\mathrm{o}}$ and $\ell\gtrsim\varrho_\iota\gtrsim\varrho_{\iota+1}\gtrsim\ddot{\mathrm{o}}$, for all $\iota\in\mathrm{N}$ it follows that $\{\alpha_\iota\}$, $\{\beta_\iota\}$ and $\{\varrho_\iota\}$ are monotonic sequences in$\mathfrak{H}$.

So, utilizing Remark (2.1) (i), there exists an $\ell_0,\ell'$and $\tilde{\ell}\in\mathfrak{H}$ satisfying

$$\lim_{\iota\to\infty}\alpha_\iota=\ell_0,\ \lim_{\iota\to\infty}\beta_\iota=\ell'\ \text{ and }\ \lim_{\iota\to\infty}\varrho_\iota=\tilde{\ell} \tag{3.3.1}$$

By applying the condition (iv), we have

$$\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\gtrsim\cdots\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\lesssim\cdots\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\widetilde{\mathfrak{Q}}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\lesssim\cdots\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right),$$

for $\tau\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$ and $m,\iota\in\mathrm{N}$ with $m>\iota$. Thus,

$$\alpha_{\iota+1}=inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\gtrsim inf_{m>\iota}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\gtrsim inf_{\mathfrak{v}\in\Xi.}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right).$$

$$\beta_{\iota+1}=sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\lesssim sup_{m>\iota}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\text{ and}$$

$$\varrho_{\iota+1}=sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\lesssim sup_{m>\iota}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right).$$

Since $\lim\limits_{\iota\to\infty}\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}=\infty$, by using the hypothesis along with (3.3.1), we obtain

$$\ell_0\gtrsim\lim_{\iota\to\infty}inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ell,\ \ \ell'\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}},$$

$$\ell'\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}}\text{ and }\bar{\ell}\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}}$$

Which implies that $\ell_0=\ell$ and $\ell'=\ddot{\mathfrak{o}}$. Thus, $\{\mathfrak{k}\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Using Lemma (2.12) and completeness of Ξ, there exists a $\mathfrak{d}\in\Xi$ such that $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{d}$.

Using (iv), we can check that the continuity of $\mathfrak{k}$ implies continuity of $\mathfrak{h}$. So, $\lim\limits_{\iota\to\infty}\mathfrak{h}\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{d}$.

Since $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ, we have $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{h}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{d}$.

Moreover, we know that $\lim\limits_{\iota\to\infty}\mathfrak{h}\mathfrak{u}_{\iota-1}=\mathfrak{d}$ so we get $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{h}\mathfrak{u}_{\iota-1}=\mathfrak{k}\mathfrak{d}$.

Based on the uniqueness of limit, we get $\mathfrak{k}\mathfrak{d}=\mathfrak{h}\mathfrak{d}$ and therefore $\mathfrak{h}\mathfrak{h}\mathfrak{d}=\mathfrak{k}\mathfrak{h}\mathfrak{d}$.

Repeated use of the condition (iv) yields

$$\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) \gtrsim \cdots \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \gtrsim inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right)$$

$$\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) \lesssim \cdots \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta\tau}{\delta}\right) \lesssim \cdots \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{h}\mathfrak{d},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right).$$

Letting the limit as $\iota \to \infty$, and applying the hypothesis we get,
$\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) = \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{h}\mathfrak{d},\tau) = \ddot{\mathfrak{o}}$ which implies that $\mathfrak{h}\mathfrak{h}\mathfrak{d} = \mathfrak{k}\mathfrak{h}\mathfrak{d} = \mathfrak{h}\mathfrak{d}$.
i.e., $\mathfrak{h}\mathfrak{d}$ is a common fixed point of $\mathfrak{k}$ and $\mathfrak{h}$.
We shall establish the uniqueness of the common fixed point $\mathfrak{h}\mathfrak{d}$.
Assume that $\mathfrak{h}\mathfrak{d}$ and $\mathfrak{z}$ are two distinct common fixed points of $\mathfrak{k}$ and $\mathfrak{h}$.
Utilizing (iv) with $\mathfrak{u} = \mathfrak{h}\mathfrak{d}$ and $\mathfrak{v} = \mathfrak{z}$, we find that,

$$\ell \gtrsim \widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{d},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{z},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \gtrsim inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

$$\ddot{\mathfrak{o}} \lesssim \widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{z},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \text{ and}$$

$$\ddot{\mathfrak{o}} \lesssim \widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{h}\mathfrak{d},\mathfrak{h}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{h}\mathfrak{d},\mathfrak{k}\mathfrak{z},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{d},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

Since $\lim_{\iota\to\infty}\frac{\theta^{\iota}\tau}{\delta^{\iota}} = \infty$, we conclude that $\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau) = \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{d},\mathfrak{z},\tau) = \ddot{\mathfrak{o}}$

Thus, $\mathfrak{h}\mathfrak{d} = \mathfrak{z}$, this concludes the proof.

Example 3.4 Let $\Xi = [0,1]$ and let $\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{o}} \to \mathfrak{F}$ such that $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = e^{-\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}}\ell$, $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = (1 - e^{-\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}})\ell$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = (e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}} - 1)\ell$ where $\tau = (\wp, q) \in \mathfrak{H}_{\ddot{o}}$. Then, we can readily verify that $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\ \star,\theta)$ is a *CVNbMS* with $\theta = 4$. On the other hand, let $\lim_{\iota\to\infty}\tau_{\iota} = \infty$ for any sequence $\{\tau_{\iota}\}$ in $\mathfrak{H}_{\ddot{o}}$, where $\tau_{\iota} = (\wp_{\iota}, q_{\iota})$. Since $(\mathfrak{u}-\mathfrak{v})^2 \leq 1$ for every $\mathfrak{u},\mathfrak{v} \in \Xi$ it follows that

$$inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_{\iota})=inf_{\mathfrak{v}\in\Xi}\ e^{-(\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}})}\ell \ = e^{-(\frac{sup_{\mathfrak{v}\in\Xi}(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}})}\ell \gtrsim e^{-(\frac{1}{\wp_{\iota}+q_{\iota}})}\ell.$$

$$sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_{\iota}) = sup_{\mathfrak{v}\in\Xi}\left\{\ell - \frac{\ell}{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}}\right\} = \ell - \frac{\ell}{e^{\frac{sup(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}} \lesssim \ell - \frac{\ell}{e^{\frac{1}{\wp_{\iota}+q_{\iota}}}} \quad \text{and}$$

$$sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_{\iota}) = sup_{\mathfrak{v}\in\Xi}\left\{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}\ell - \ell\right\} = sup_{\mathfrak{v}\in\Xi}\left\{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}\ell - \ell\right\} \lesssim e^{\frac{1}{\wp_{\iota}+q_{\iota}}}\ell.$$

Therefore, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \gtrsim \lim_{\iota\to\infty} e^{-(\frac{1}{p_\iota+q_\iota})}\ell = \ell$,

$\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \lesssim \lim_{\iota\to\infty}(\ell - \frac{\ell}{e^{\frac{1}{p_\iota+q_\iota}}}) = \ddot{\mathrm{o}}$ and

$\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\,\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \lesssim \lim_{\iota\to\infty}(e^{\frac{1}{p_\iota+q_\iota}}\ell) = \ddot{\mathrm{o}}$. Let $\mathfrak{k},\mathfrak{h} : \Xi \to \Xi$ be defined by $\mathfrak{k}\mathfrak{u} = \mathfrak{u}$ and, $\mathfrak{h}\mathfrak{u} = \frac{\mathfrak{u}}{4}$.

One can readily verify that $\mathfrak{h}(\Xi) \subseteq \mathfrak{k}(\Xi)$ and $\mathfrak{k}$ is continuous on Ξ. Furthermore, $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ. Moreover, It is simple to demonstrate that condition (iv) true for every $\mathfrak{u},\mathfrak{v} \in [0,1]$ with $\delta = \frac{1}{4}$.

Definition.3.5 Let $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\star,\theta)$ be a complete *CVNbMS.* The modified contraction condition for the mapping $\mathfrak{k} : \Xi \to \Xi$ as follows:

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \quad (1)$$

For all $\mathfrak{u},\mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}$ where $\delta \in [0,1)$.

Theorem 3.6 Let $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\star,\theta)$ be a *CVNbMS,* and $\mathfrak{k} : \Xi \to \Xi$ be a mapping fulfilling the contraction condition (I). Then, $\mathfrak{k}$ has a unique common fixed point in Ξ.

Proof: Let $\mathfrak{u}_0$ be a random element of Ξ. Using induction, we can generate a sequence $\{\mathfrak{u}_\iota\}$ in Ξ such that $\mathfrak{u}_\iota = \mathfrak{k}\mathfrak{u}_{\iota-1}$ for every $\iota \in \mathbb{N}$. Continuing from the proof of Theorem (3.1) in [12], we examine that the sequence $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ and converges to some $\mathfrak{d} \in \Xi$.

We will demonstrate that $\mathfrak{d}$ is a fixed point of $\mathfrak{k}$. By the contractive condition (I), we have

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)$$

for all $\iota \in \mathbb{N}$ and $\tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}$. The above inequality demonstrates that

$$\ell(1-\delta) + \delta\widetilde{\mathfrak{P}}(\mathfrak{u}_\iota,\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\tau), \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau). \quad (3.6.1)$$

for all $\iota \in \mathbb{N}$ and $\tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}$.

Therefore,

$$\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{\iota+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{\iota+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{\iota+1},\ \mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) \text{ for any } \tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}.$$

Taking the limit as $\iota \to \infty$, from (3.6.1) and Remark (2.2) (ii), we determine that $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ddot{\mathrm{o}}$ for all $\tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}$, which yields $\mathfrak{k}\mathfrak{d} = \mathfrak{d}$.

To prove that the fixed point of $\mathfrak{k}$ is unique, assume that there exists another $\mathfrak{z} \in \Xi$ such that $\mathfrak{k}(\mathfrak{z}) = \mathfrak{z}$. Then, there is a $\tau \in \mathfrak{H}_{\ddot{\mathrm{o}}}$ fulfilling $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$.

As a result of (I), we have

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) = \ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau).$$

Since $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$, we obtain

$Re(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$ or $Im(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$, $Re(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ or $Im(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ and $Re(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ or $Im(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$. Let $Re(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$, $Re(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ and $Re\left(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)\right) \neq 0$.

Therefore, we get

$1 - Re\left(\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau)\right) \precsim \delta\left[1 - Re(\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau))\right] \precsim 1 - Re\left(\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau)\right),$

$Re(\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau) \precsim \delta Re(\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau)) \precsim Re(\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau)) = Re(\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau))$ and

$Re(\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau) \precsim \delta Re(\widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau)) \precsim Re(\widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau)) = Re(\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d}, \mathfrak{k}\mathfrak{z}, \tau))$ which is a contradiction.

We can omit the details of the other since the other case is identical to this one.

Thus, $\widetilde{\mathfrak{P}}(\mathfrak{d}, \mathfrak{z}, \tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d}, \mathfrak{z}, \tau)=\ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d}, \mathfrak{z}, \tau)=\ddot{\mathfrak{o}}$ for all $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$ and the proof is completed.

Example: 3.7 Let $\Xi = [0,1]$ and let $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ such that

$\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \ell - \frac{(\mathfrak{u} - \mathfrak{v})^2}{1+\wp q}\ell, \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{(\mathfrak{u} - \mathfrak{v})^2}{1+\wp q}\ell$ and $\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \frac{(\mathfrak{u} - \mathfrak{v})^2}{1+\wp q-(\mathfrak{u} - \mathfrak{v})^2}\ell$ where $\tau = (\wp, q) \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$.

Define the mapping $\mathfrak{k}: \Xi \to \Xi$ by $\mathfrak{k}\mathfrak{u} = \frac{\mathfrak{u}^2}{4}$. Therefore, we have

$\frac{(\mathfrak{k}\mathfrak{u} - \mathfrak{k}\mathfrak{v})^2}{1+\wp q}\ell \precsim \delta \frac{(\mathfrak{u} - \mathfrak{v})^2}{1+\wp q}\ell$ and $\frac{(\mathfrak{k}\mathfrak{u} - \mathfrak{k}\mathfrak{v})^2}{1+\wp q-(\mathfrak{k}\mathfrak{u} - \mathfrak{k}\mathfrak{v})^2}\ell \precsim \delta \frac{(\mathfrak{u} - \mathfrak{v})^2}{1+\wp q-(\mathfrak{u} - \mathfrak{v})^2}\ell$ where $\delta \in [\frac{1}{4}, 1)$. Thus, we determine that (I) holds, all the necessary hypotheses of Theorem (3.6) are fulfilled and thus we establish the existence and uniqueness of the fixed point of $\mathfrak{k}$ and 0 is the unique fixed point of $\mathfrak{k}$.

Corollary 3.8 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* and $\mathfrak{k} : \Xi \to \Xi$ be a mapping satisfying $\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}^{\iota}\mathfrak{u}, \mathfrak{k}^{\iota}\mathfrak{v}, \tau) \precsim \delta\left[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau)\right]$, $\widetilde{\mathfrak{L}}(\mathfrak{k}^{\iota}\mathfrak{u}, \mathfrak{k}^{\iota}\mathfrak{v}, \tau) \precsim \delta\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau)$ and $\widetilde{\mathfrak{Q}}(\mathfrak{k}^{\iota}\mathfrak{u}, \mathfrak{k}^{\iota}\mathfrak{v}, \tau) \precsim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau)$ for every $\mathfrak{u}, \mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$, where $0 \le \delta < 1$. Then, $\mathfrak{k}$ has a unique common fixed point in Ξ.

Proof: By Theorem (3.6), we get a unique $\mathfrak{u} \in \Xi$ such that $\mathfrak{k}^{\iota}\mathfrak{u} = \mathfrak{u}$. Since $\mathfrak{k}^{\iota}\mathfrak{k}\mathfrak{u} = \mathfrak{k}\mathfrak{k}^{\iota}\mathfrak{u} = \mathfrak{k}\mathfrak{u}$ and from uniqueness, we get $\mathfrak{k}\mathfrak{u} = \mathfrak{u}$. This demonstrates that $\mathfrak{k}$ has a unique fixed point in Ξ.

4. Application

Applying our main results from the previous part, we analyze the existence theorem for a solution to the following integral equation in this section:

$$\mathfrak{u}(\tilde{\mathfrak{s}})= \kappa(\tilde{\mathfrak{s}}) + \sigma \int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}}, \mathfrak{u}(\bar{\bar{\theta}})\right) d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1], \quad (2)$$

where

(i) κ is a continuous real-valued function on[0,1]; $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\psi(\tilde{\mathfrak{s}}, \mathfrak{u}) \ge 0$ and there exists a $\delta \in [0,1)$ such that $|\psi(\tilde{\mathfrak{s}}, \mathfrak{u}) - \psi(\tilde{\mathfrak{s}}, \mathfrak{v})| \le \delta|\mathfrak{u} - \mathfrak{v}|$, for every $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$;

(ii) $\beth : [0,1] \times [0,1]] \to \mathbb{R}$ is a continuous at $\tilde{\mathfrak{s}} \in [0,1]$ for every $\bar{\bar{\theta}} \in [0,1]$ and measurable at $\bar{\bar{\theta}} \in [0,1]$ for every $\tilde{\mathfrak{s}} \in [0,1]$. Moreover, $\beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) \ge 0$ and $\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})d\bar{\bar{\theta}} \le \mathcal{L}$;

(iii) $\delta^2\mathcal{L}^2\sigma^2 \le \frac{1}{2}$.

Theorem 4.1. If the condition (i)-(iv) fulfilled. Then, the integral Eq. (2) has unique solution in (C[0,1], ℝ), where (C[0,1], ℝ) is the set of all continuous real valued functions on [0,1].

Proof: Let Ξ =(C[0,1], ℝ) and define a mapping $\mathfrak{k} : \Xi \to \Xi$ by

$\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}})= \kappa(\tilde{\mathfrak{s}}) + \sigma \int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}}, \mathfrak{u}(\bar{\bar{\theta}})\right) d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1]$, for all $\mathfrak{u} \in \Xi$ and for every $\tilde{\mathfrak{s}} \in [0,1]$.

We need to prove that the mapping $\mathfrak{k}$ fulfils all requirements of Theorem (3.6).

Define $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{o}} \to \mathfrak{F}$ by $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \ell - \sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\wp q}} \ell$,

$\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\wp q}} \ell$ and $\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \left(\frac{\sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\wp q}}}{1 - \sup_{\tilde{\mathfrak{t}} \in [0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\wp q}}} \right) \ell$

where $\tau = (\wp, q) \in \mathfrak{H}_{\ddot{o}}$. Clearly, $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a complete CVNbMS.
Moreover, for every $\mathfrak{u}, \mathfrak{v} \in \Xi$ and $\tilde{\mathfrak{s}} \in [0,1]$, we get

$$|\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{k}\mathfrak{v}(\tilde{\mathfrak{s}})| = \sigma \left| \int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) \psi\left(\bar{\bar{\theta}}, \mathfrak{u}(\bar{\bar{\theta}})\right) - \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) \psi\left(\bar{\bar{\theta}}, \mathfrak{v}(\bar{\bar{\theta}})\right) d\bar{\bar{\theta}} \right|$$

$$\leq \sigma \int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) \left| \psi\left(\bar{\bar{\theta}}, \mathfrak{u}(\bar{\bar{\theta}})\right) - \psi\left(\bar{\bar{\theta}}, \mathfrak{v}(\bar{\bar{\theta}})\right) \right| d\theta \leq \sigma \int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) \delta \left| \mathfrak{u}(\bar{\bar{\theta}}) - \mathfrak{v}(\bar{\bar{\theta}}) \right| d\bar{\bar{\theta}}$$

$$\leq \sigma \mathcal{L}\, \delta \sup_{\tilde{\mathfrak{s}} \in [0,1]} |\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|$$

Since, $\sup_{\tilde{\mathfrak{s}} \in [0,1]} |\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{k}\mathfrak{v}(\tilde{\mathfrak{s}})| \leq \sigma \mathcal{L}\, \delta \sup_{\tilde{\mathfrak{s}} \in [0,1]} |\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|$

We get, $\sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{k}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}} \leq \sigma^2 \mathcal{L}^2 \delta^2 \sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}} \leq \frac{1}{2} \sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}$ and

$$\left(\frac{\sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{k}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}}{1 - \sup_{\tilde{\mathfrak{t}} \in [0,1]} \frac{|\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{k}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}} \right) \leq \sigma^2 \mathcal{L}^2 \delta^2 \left(\frac{\sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}}{1 - \sup_{\tilde{\mathfrak{t}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}} \right) \leq \frac{1}{2} \frac{\sup_{\tilde{\mathfrak{s}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}}{1 - \sup_{\tilde{\mathfrak{t}} \in [0,1]} \frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\wp q}}}.$$

This establishes that the mapping $\mathfrak{k}$ fulfilling the contractive condition (1) in Theorem (3.6), and $\mathfrak{k}$ has a unique solution in (C [0,1], ℝ), i.e., the integral Eq. (2) has a unique solution in (C [0,1], ℝ).

Example 4.2 Take the integral equation

$$\mathfrak{u}(\tilde{\mathfrak{s}}) = \frac{1}{1 + \tilde{\mathfrak{s}}} + 2 \int_0^1 \frac{\bar{\bar{\theta}}^2}{\tilde{\mathfrak{s}}^2 + 2} \cdot \frac{|\cos \mathfrak{u}(\tilde{\mathfrak{s}})|}{5e^{\bar{\bar{\theta}}}} d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1], \tag{4.2.1}$$

It can observed that the above equation is of the form (II), for $\sigma = 2$, $\kappa(\tilde{\mathfrak{s}}) = \frac{1}{1+\tilde{\mathfrak{s}}}$, $\xi(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) = \frac{\theta^2}{\tilde{\mathfrak{s}}+2}$, $\psi(\tilde{\mathfrak{s}}, \mathfrak{u}) = \frac{|\cos \tilde{\mathfrak{s}}|}{5e^{\tilde{\mathfrak{s}}}}$.

Clearly, ψ is continuous on $[0,1] \times \mathbb{R}$ and we get

$\left|\psi(\bar{\bar{\theta}}, \mathfrak{u}) - \psi(\bar{\bar{\theta}}, \mathfrak{v})\right| = \frac{1}{5e^{\tilde{\mathfrak{s}}}} \left| |\cos \mathfrak{u}| - |\cos \mathfrak{v}| \right| \leq \frac{1}{5e^{\tilde{\mathfrak{s}}}} |\cos \mathfrak{u} - \cos \mathfrak{v}| \leq \frac{1}{5} |\cos \mathfrak{u} - \cos \mathfrak{v}| \leq \frac{1}{5} |\mathfrak{u} - \mathfrak{v}|$

for every $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$. Thus, ψ fulfills the condition (ii) of the integral equation (II) with $= \frac{1}{5}$. It is easy to verify that the mapping κ is continuous and $\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) d\bar{\bar{\theta}} = \int_0^1 \frac{\bar{\bar{\theta}}^2}{\tilde{\mathfrak{s}}^2 + 2} d\bar{\bar{\theta}} = \frac{1}{\tilde{\mathfrak{s}}^2 + 2} \frac{1}{3} \leq \frac{1}{6} = \mathcal{L}$, the mapping ξ meets the condition (iii). We get $\sigma^2 \mathcal{L}^2 \delta^2 \leq \frac{1}{2}$.. Thus, the hypotheses (i), (ii), (iii), and (iv) are true. Using the Theorem (3. 6) leads us to the conclusion that the integral equation (II) has a unique solution in (C [0, 1], ℝ).

5. Conclusion

In this paper, we have defined complex valued neutrosophic metric like space and we have proved fixed point theorems for mappings on complex valued neutrosophic metric like space. We hope that the results proved in this paper will form new connections for those who are working in complex valued neutrosophic metric-like spaces.

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Author Contributions
All authors contributed equally to this research.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

References

1. Azam, A., Fisher, B., & Khan, M. (2011). Common fixed point theorems in complex valued metric spaces. Numer. Funct. Anal. Optim.,.32(3), 243–253. http://doi.org/10.1080/01630563.2011.533046.
2. Rouzkard, F., & Imdad, M. (2012). Some common fixed point theorems on complex valued metric spaces. Comput. Math. Appl., 64(6), 1866–1874. http://doi.org/10.1016/j.camwa.2012.02.063.
3. Ahmad, J. A., Azam, A., & Saejung, S. (2014). Common fixed point results for contractive mappings in complex valued metric spaces. Fixed Point Theory Appl., 67(1). http://doi.org/10.1186/1687-1812-2014-67.
4. Rao, K. P., Swamy, R., & Prasad, J. R. (2013). A common fixed point theorem in complex valued *b*-metric spaces. Bull. Math. Statist. Res., 1(1), 1–8, 2013. URL: http://www.bomsr.com/4.1.16.html.
5. Mukheimer, A. A.(2014).Some common fixed point theorems in complex valued-metric spaces. Sci. World J., Vol. 2014. http://doi.org/10.1155/2014/587825.
6. Zadeh, L. A. (1965). Fuzzy sets. Inf. Control, 8(3), 338–353. http://doi.org/10.1016/S0019-9958 (65)90241-X.
7. Kramosil, I., & Michalek, J. (1975). Fuzzy metrics and statistical metric spaces. Kybernetika, 11(5), 336–344. URL: https://www.kybernetika.cz.
8. George, A. & Veeramani, P. (1997). On some results of analysis for fuzzy metric spaces. Fuzzy Sets Syst., 90(3), 365–368. http://doi.org/10.1016/S0165-0114(96)00207-2.
9. Atanassov, K. (1986). on Intuitionistic Fuzzy Sets. Fuzzy sets and systems. 20, 87-96, http://doi.org/10.1016/S0165-0114(86)80034-3.
10. Park, J.H. (2004). Intuitionistic fuzzy metric spaces. ChaosSolitons Fractals, Vol.22. https://doi.org/10.1016/j.chaos.2004.02.051.
11. Smarandache, F. (1998). Neutrosophy: Neutrosophic probability, set and logic, Rehoboth: American Research Press. http://doi: org/10.5281/zenodo.57726.
12. Shukla, S., Rodriguez-Lopez, R., & Abbas, M. (2018). Fixed point results for contractive mappings in complex valued fuzzy metric spaces. Fixed Point Theory, 19(2), 751–774. http://doi.org/ 10.24193/fpt-ro.2018.2.56.

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