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Domination on Bipolar Fuzzy Graph Operations: Principles, Proofs, and Examples

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Abstract: Bipolar fuzzy graphs, capable of capturing situations with both positive and negative memberships, have found diverse applications in various disciplines, including decision-making, computer science, and social network analysis. This study investigates the domain of domination and global domination numbers within bipolar fuzzy graphs, owing to their relevance in these aforementioned practical fields. In this study, we introduce certain operations on bipolar fuzzy graphs, such as intersection$(\Gamma_1 \cap \Gamma_2)$, join $(\Gamma_1 + \Gamma_2)$, and union $(\Gamma_1 \cup \Gamma_2)$ of two graphs. Furthermore, we analyze the domination number $\gamma(\Gamma)$ and the global domination number $\gamma_g(\Gamma)$ for various operations on bipolar fuzzy graphs, including intersection, join, and union of fuzzy graphs and their complements.

Keywords: Bipolar Fuzzy Graph; Global Domination; Domination Number; Operations Fuzzy Graphs.

1. Introduction

L.A. Zadeh, who created the fuzzy set theory and fuzzy logic, originally suggested and studied the idea of "fuzzy sets" in 1965 [1]. By giving each element in a subset of a universal set a specific value in the closed interval [0, 1], this theory suggests a graded membership for each of such elements. Many scientific disciplines, including the fields of computer science, machine learning, analysis of decisions, the science of information, system sciences, controlling engineering, expertise systems, recognition of patterns, management science, and operation research, as well as a number of mathematical disciplines, including topology, algebra, geometry, graph theory, and analysis, have used Zadeh's ideas.

Rosenfeld 1975 [2] studied the notion of fuzzy graphs and numerous fuzzy analogs of graph-theoretic notions, such as the path, cycles, and connectedness. Zadeh 1987 [3] investigated the fuzzy relationship as well. Ore studied the mathematical definition of dominance in the graph in 1962 [4], while A. Somasundaram and S. Somasundaram examined various concepts of domination in fuzzy graphs [5].

Sampat-Kumar presented the first concept of global dominant sets in graphs in 1989 [6]. The notions of domination and global domination of some operations in fuzzy product graphs were presented by Haifa A. and Mahioub S. in [6], while the concepts of global domination number, domatic number, and global domatic number were introduced by Mahioub in [7]. Mordeson, J.N., and Peng C-S introduced and analyzed operations on the fuzzy graph in 1994 [8], and also in 2017 [9] Somasundaram presented more notions on domination in fuzzy graphs.

The study of domination and global domination numbers in bipolar fuzzy graphs has implications in fields such as operations research, game theory, and graph theory. By studying these important parameters, researchers can gain insight into the properties and behavior of complex systems modeled by bipolar fuzzy graphs.

Bipolar fuzzy graphs are a type of fuzzy graph where each vertex is assigned a pair of values that represent its positive and negative degrees. This paper studies the domination and global domination properties of these graphs, which are important concepts in network analysis. Domination refers to the minimum number of vertices needed to control or influence the entire graph, while global domination refers to the minimum number of vertices needed to control or influence any vertex in the graph. Additionally, Crisp graphs, being a fundamental mathematical construct, exhibit a plethora of operations that allow for their manipulation and analysis. These operations include but are not limited to, union, intersection, join, tensor product, Cartesian product, composition, strong product, disjunction, and symmetric difference of graphs. A comprehensive treatment of these operations is provided in [10-16]. Tobaili et al. [17] investigated hub number properties within the context of fuzzy graph structures. Further exploration into domination parameters on product fuzzy graphs was conducted by Ahmed and Alsharafi [18], with a specific focus on the semi-global domination number. In this study, we focus our attention on some of these operations, namely union, intersection, and join on bipolar fuzzy graphs, and discuss theorems and bounds of domination and global domination in such operations of the bipolar fuzzy graph.

Bipolar fuzzy graphs (BFG) are an extension of fuzzy graphs that can effectively capture uncertain or imprecise information in various applications. BFGs are used to define concepts such as covering, matching, and domination in graph theory when the vertices and edges are uncertain or imprecise. BFGs have been used in various domains, including disaster management, location selection, and medical diagnosis. The energy of a directed bipolar fuzzy graph is calculated as the sum of the absolute values of the eigenvalues of its adjacency matrix, and it can be used in solving multi-criteria decision-making problems [19-22].

Inverse domination in bipolar fuzzy graphs refers to the idea of an inverse dominating set (IDS) in which a set I is a dominating set of the complement of the dominating set D. The least IDS is called the inverse domination number. In addition, inverse domination has also been defined and studied in interval-valued fuzzy graphs, with bounds on the inverse domination number provided for different types of interval-valued fuzzy graphs. Furthermore, a new definition of inverse domination number has been introduced in the graphs, with bounds and results established for this parameter [23]. The cardinality, dominating set, independent set, total dominating number, independent dominating number, and redundancy number of bipolar fuzzy graphs have been introduced and investigated in [24]. The concept of domination in fuzzy graphs has been extended to bipolar frameworks, and various expanded concepts of bipolar fuzzy graphs have been obtained in [25].

This study suggests exploring the concepts of domination and global domination in some bipolar fuzzy graph operations. There are a few points that we want to highlight about the motivation and applications;

Bipolar fuzzy graphs offer a comprehensive approach to representing complex systems in which relationships can have both positive and negative aspects, unlike graphs that only consider positive membership.

Domination and global domination are concepts in graph theory that have applications in decision-making, computer science, and social network analysis. By studying these properties in graphs, we can gain fresh perspectives.

Investigating domination numbers helps us to understand how efficiently a set of vertices can control a graph. This has implications for modeling influence and control in world systems.

Analyzing operations like union, intersection, and join on graphs provides us with a mathematical framework to examine and manipulate these graphical models. This can be useful for algorithm development in data processing.

The insights obtained from this research, such as establishing bounds on domination numbers after operations, could reveal connections within fuzzy graph models of complex data.

The findings could have implications for fields such as machine learning, data mining, pattern recognition, and other disciplines dealing with data sets that require representation using bipolar fuzzy graphs.

2. Preliminaries

In this section, we review some definitions of graphs, fuzzy graphs, bipolar fuzzy graphs, and domination numbers in a bipolar fuzzy graph [7-10].

Definition 2.1: A crisp graph Γ is defined as an ordered pair $\Gamma = (V, E)$, where V is a set of vertices E is a set of edges, and each edge is a two-element subset of V. The edges of a crisp graph are present or absent, and there is no ambiguity or uncertainty about their existence. A fuzzy graph $\Gamma = (\lambda, \tau)$ is defined as:

Definition 2.2: A set V of vertices, where each vertex s is associated with a membership function $\lambda_v(s)$ that assigns a degree of membership to each element s in V. The membership function maps each element to a value between 0 and 1, where 0 indicates no membership, and 1 indicates full membership.

Definition 2.3: A set E of edges, where each edge e is associated with a membership function $\tau_e(s,t)$ that assigns a degree of membership to each pair of vertices (s,t) in E. The membership function maps each pair of vertices to a value between 0 and 1, where 0 indicates no membership and 1 indicates full membership.

Definition 2.4: The order and size of a fuzzy graph $\Gamma = (\lambda, \tau)$ are defined as follows:

The order of Γ is the sum of the degrees of membership of all vertices in Γ, that is, $p = \sum_{s \in V} \lambda(s)$.
The size of Γ is the sum of the degrees of membership of all edges in Γ, that is, $q = \sum_{(s,t) \in E} \tau(s,t)$.

Definition 2.5: The complement of a fuzzy graph $\Gamma = (\lambda, \tau)$ is another fuzzy graph $\overline{\Gamma} = (\overline{\lambda}, \overline{\tau})$, defined as follows:
The vertex set of $\overline{\Gamma}$ is the same as the vertex set of Γ, i.e., $V(\overline{\Gamma}) = V(\Gamma)$.
The degree of membership of each vertex in $\overline{\Gamma}$ is the same as in Γ, that is, $\overline{\lambda}(t) = \lambda(t)$ for all $t \in V(\Gamma)$.

Definition 2.6: The degree of membership of each edge in $\overline{\Gamma}$ is the complement of the degree of membership of the corresponding edge in Γ, i.e., $\overline{\tau}(s,t) = \lambda(s) \wedge \lambda(t) - \tau(s,t)$ for all $(s,t) \in E(\Gamma)$.

Definition 2.7: A dominating set D of a fuzzy graph $\Gamma = (\lambda, \tau)$ is a subset of vertices such that every vertex $t \in V(\Gamma) - D$ is dominated by at least one vertex $s \in D$. In other words, for every vertex $t \in V(\Gamma) - D$, there exists a vertex $s \in D$ such that $\tau(s,t) \geq \lambda(s)$. So, a dominating set in a fuzzy graph is a subset of vertices that "control" the graph, in the sense that every non-dominated vertex is within a certain distance from a vertex in the dominating set.

Definition 2.8: A dominating set D of a fuzzy graph $\Gamma = (\lambda, \tau)$ is called a minimal dominating set if no proper subset of D is a dominating set of Γ. In other words, for every $t \in D$, the set $D - t$ is not a dominant set of Γ. Thus, a minimal dominating set is a dominating set that cannot be reduced in size while still maintaining the property of domination. It is the "smallest" dominating set possible for the given fuzzy graph.

Definition 2.9: The domination number of a fuzzy graph $\Gamma = (\lambda, \tau)$, denoted by $\gamma(\Gamma)$, is defined as the minimum fuzzy cardinality of all minimal dominating sets in Γ. In other words, $\gamma(\Gamma)$ is the smallest possible value of $\sum_{t \in D} \lambda(t)$ on all minimal dominating sets D of Γ. Intuitively, the domination number of a fuzzy graph measures the "influence" of the graph in the sense that it represents the minimum number of vertices needed to control the graph. A smaller number of dominations indicates a more efficient control structure, where a smaller number of vertices can dominate the entire graph. A vertex subset D of V in a fuzzy graph $\Gamma = (\lambda, \tau)$ is said to be a global dominating set of Γ if it is a dominating set of both Γ and its complement $\overline{\Gamma}$. In other words, every vertex in $V(\Gamma) - D$ is dominated by at least one vertex in D, and every vertex in $V(\overline{\Gamma}) - D$ is dominated by at least one vertex in D. So, a global dominating set in a fuzzy graph is a subset of vertices that control both the presence and absence of edges in the graph. It is a more stringent condition than a dominating set or an independent set, as it requires that the set dominates both the original graph and its complement.

Definition 2.10: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ denote two fuzzy graphs. We consider their join $\Gamma = \Gamma_1 + \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs, where E' is defined as the set of all edges joining the nodes of V_1 and V_2, under the assumption that $V_1 \cap V_2 \neq \phi$.

Furthermore, let us assume that Γ_1 and Γ_2 are fuzzy graphs. In this context, we define the joining of the two product fuzzy graphs denoted by $\Gamma = \Gamma_1 + \Gamma_2 : (\lambda_1 + \lambda_2, \tau_1 + \tau_2)$, as follows:

$$(\lambda_1 + \lambda_2)(s) = \begin{cases} (\lambda_1 \cup \lambda_2) & \text{if } \ s \in V_1 \cap V_2 \\ \lambda_1(s); s \in V_1 - V_2 \\ \lambda_2(s); s \in V_2 - V_1 \end{cases} \quad (1)$$

and

$$(\tau_1 + \tau_2)(st) = \begin{cases} (\tau_1 \cup \tau_2) & \text{if } \ st \in E_1 \cap E_2 \\ \tau_1(st); st \in E_1 - E_2 \\ \tau_2(st); st \in E_2 - E_1 \end{cases} \quad (2)$$

Definition 2.11: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ denote two fuzzy graphs. We consider their intersection $\Gamma^* = \Gamma_1^* \cap \Gamma_2^* = (V_1 \cap V_2, E_1 \cap E_2)$ of graphs, under the assumption that $V_1 \cap V_2 \neq \phi$.

Moreover, let us consider Γ_1 and Γ_2 as fuzzy graphs and define their intersection, denoted by $\Gamma = \Gamma_1 \cap \Gamma_2 : (\lambda_1 \cap \lambda_2, \tau_1 \cap \tau_2)$, as a product fuzzy graph. The intersection is defined as follows:

$$\lambda_1 \cap \lambda_2 = \{\min(\lambda_1, \lambda_2) \quad \text{ifs} \in V_1 \cap V_2. \quad (3)$$

and

$$\tau_1 \cap \tau_2 = \{\min(\tau_1, \tau_2) \quad \text{ifs} \in E_1 \cap E_2 \quad (4)$$

Definition 2.12: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ be two fuzzy graphs considering the union $\Gamma = \Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2)$ of graphs, where $V_1 \cap V_2 \neq \phi$. Then the union of two fuzzy graphs Γ_1 and Γ_2 is a fuzzy graph $\Gamma = \Gamma_1 \cup \Gamma_2 : (\lambda_1 \cup \lambda_2 \ , \ \tau_1 \cup \tau_2)$ defined as follows:

$$(\lambda_1 \cup \lambda_2)(s) = \begin{cases} \max(\lambda_1, \lambda_2) & \text{ifs} \in V_1 \cap V_2 \\ \lambda_1(s); s \in V_1 - V_2 \\ \lambda_2(s); s \in V_2 - V_1 \end{cases} \quad (5)$$

and

$$(\tau_1 \cup \tau_2)(st) = \begin{cases} \max(\tau_1, \tau_2) & \text{ifst} \in E_1 \cap E_2 \\ \tau_1(st); st \in E_1 - E2 \\ \tau_2(st); st \in E_2 - E_1 \end{cases} \quad (6)$$

3. Results

This section studies some bipolar fuzzy graph operations and domination and global domination numbers on bipolar fuzzy graph operations.

3.1 Some Bipolar Fuzzy Graph Operations

The study of operations on bipolar fuzzy graphs can yield several potential benefits. Firstly, these operations can facilitate the analysis and interpretation of complex data sets that are difficult to model using traditional graphs. Second, by providing a mathematical framework for the manipulation of bipolar fuzzy graphs, these operations can aid in the development of algorithms for processing and analyzing large amounts of data. Finally, the study of operations on bipolar fuzzy graphs can lead to the discovery of new insights and relationships within data sets, which can have practical applications in fields such as machine learning, data mining, and pattern recognition. Within this section, we shall commence an exploration of certain operations on bipolar fuzzy graphs, namely, the intersection, the join, and the union.

Definition 3.1.1. Let $\Gamma_1 = (A_1, B_1)$ and $\Gamma_2 = (A_2, B_2)$ be two bipolar fuzzy graphs, where $A_1 = (\lambda_1^+, \lambda_1^-), B_1 = (\tau_1^+, \tau_1^-), A_2 = ((\lambda_2^+, \lambda_2^-)$ and $B_2 = (\tau_2^+, \tau_2^-)$ consider the intersection $\Gamma = \Gamma_1 \cap \Gamma_2 = (A_1 \cap A_2, B_1 \cap B_2)$ of graphs. Suppose that $V_1 \cap V_2 \neq \phi$, then the intersection of two bipolar fuzzy graphs Γ_1 & Γ_2 is a bipolar fuzzy graph $\Gamma = \Gamma_1 \cap \Gamma_2 = (A_1 \cap A_2, B_1 \cap B_2)$ defined as follows:

$$A_1 \cap A_2 = \begin{cases} (\lambda_1^+ \cap \lambda_2^+)(s) = \min(\lambda_1^+, \lambda_2^+)(s) & \text{if} s \in V_1 \cap V_2 \\ (\lambda_1^- \cap \lambda_2^-)(s) = \max(\lambda_1^-, \lambda_2^-)(s) & \text{if} s \in V_1 \cap V_2. \end{cases} \quad (7)$$

and

$$B_1 \cap B_2 = \begin{cases} (\tau_1^+ \cap \tau_2^+)(st) = \min(\tau_1^+, \tau_2^+)(st) & \text{if} st \in E_1 \cap E_2 \\ (\tau_1^- \cap \tau_2^-)(st) = \max(\tau_1^-, \tau_2^-)(st)) & \text{if} st \in E_1 \cap E_2 \end{cases} \quad (8)$$

Example 1. Let Γ_1 and Γ_2 be two bipolar fuzzy graphs such that $(\tau_1^+ \cap \tau_2^+)(st) = \min(\tau_1^+, \tau_2^+)(st)$ and $(\tau_1^- \cap \tau_2^-)(st) = \max(\tau_1^-, \tau_2^-)(st)$ given in Figure 1 and their intersection.

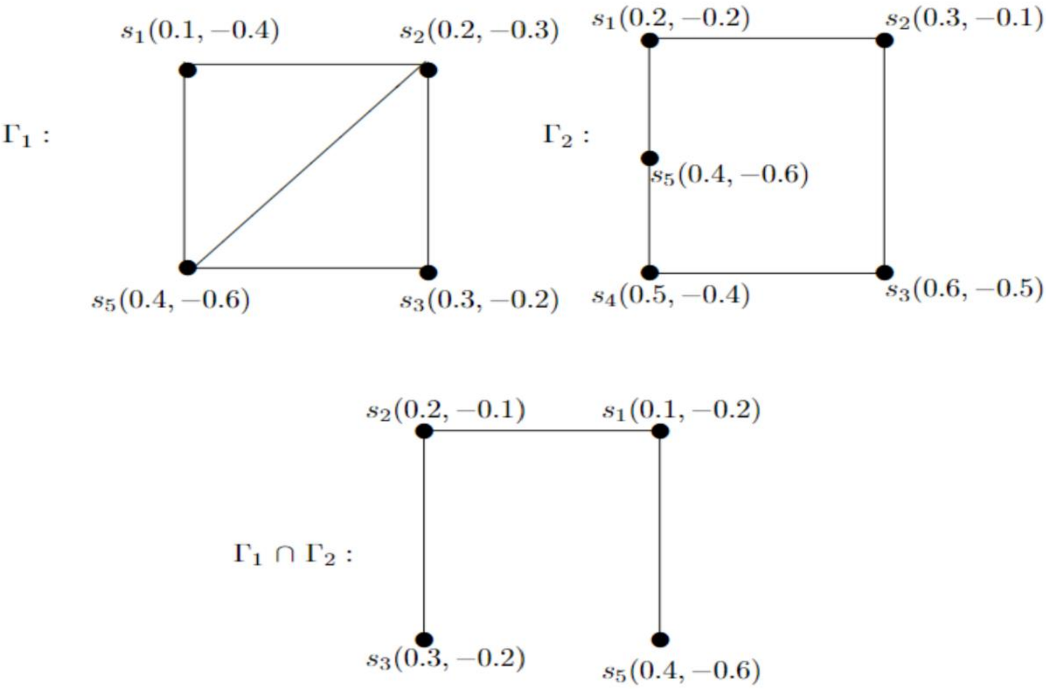

Figure 1. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 \cap \Gamma_2$.

Definition 3.1.2. Let $A_1 = (\lambda_1^+, \lambda_1^-)$ and $A_2 = (\lambda_2^+, \lambda_2^-)$ be bipolar fuzzy graphs subset of V_1 and V_2 and $B_1 = (\tau_1^+, \tau_1^-), B_2 = (\tau_2^+, \tau_2^-)$ be bipolar fuzzy graphs subset of $V_1 \times V_2$, and assume that $V_1 \cap V_2 \neq \phi$, then the join $\Gamma = (\Gamma_1 + \Gamma_2) = (A_1 + A_2, B_1 + B_2)$ is defined as follows:

$$(A_1 + A_2)(s) = \begin{cases} (\lambda_1^+ + \lambda_2^+)(s) = \max(\lambda_1^+, \lambda_2^+)(s) & \text{if} s \in V_1 \cap V_2 \\ (\lambda_1^- + \lambda_2^-)(s) = \min(\lambda_1^-, \lambda_2^-)(s) & \text{if} s \in V_1 \cap V_2 \\ (\lambda_1^+, \lambda_1^-)(s) & \text{if} s \in V_1 - V_2 \\ (\lambda_2^+, \lambda_2^-)(s) & \text{if} s \in V_2 - V_1 \end{cases} \quad (9)$$

and

$$(B_1 + B_2)(st) = \begin{cases} (\tau_1^+ + \tau_2^+)(st) = \max(\tau_1^+, \tau_2^+)(st) & \text{if} st \in E_1 \cap E_2 \\ (\tau_1^- + \tau_2^-(st)) = \min(\tau_1^-, \tau_2^-)(st) & \text{if} st \in E_1 \cap E_2 \\ (\tau_1^+, \tau_1^-)(st) & \text{if} st \in E_1 - E2 \\ (\tau_2^+, \tau_2^-)(st) & \text{if} st \in E_2 - E_1 \\ \max(\tau_1^+, \tau_2^+)(st) & \text{if} st \in E' \\ \min(\tau_1^-, \tau_2^-)(st) & \text{if} st \in E' \end{cases} \quad (10)$$

Example 2. For two bipolar fuzzy graphs Γ_1 and Γ_2, the joint $\Gamma_1 + \Gamma_2$ is given in Figure 2.

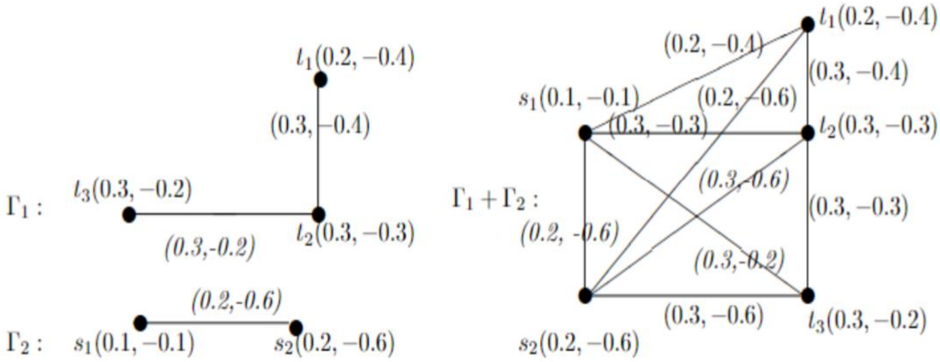

Figure 2. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 + \Gamma_2$.

Theorem 3.1.1. Let Γ_1 and Γ_2 be two bipolar fuzzy graphs, then, $\overline{\Gamma_1 + \Gamma_2} \neq \overline{\Gamma_1} + \overline{\Gamma_2}$.

The theorem states that the complement of the sum of two bipolar fuzzy graphs Γ_1 and Γ_2 is not equal to the sum of the complements of Γ_1 and Γ_2. In other words, De Morgan's laws of complementation do not hold for bipolar fuzzy graphs. Since the complement of the sum is a complete graph while the sum of the complements is a disjoint graph, we can conclude that De Morgan's laws of complementation do not hold for bipolar fuzzy graphs. This can be shown by considering a counterexample:

Example 3. Consider the bipolar fuzzy graphs $\Gamma_1 = (V_1, A_1, B_1)$, and $\Gamma_2 = (V_2,\ A_2,\ B_2)$, then $\Gamma_1 + \Gamma_2 = (V, A_1 + A_2, B_1 + B_1)$, and $\overline{\Gamma_1 + \Gamma_2} = (V, \overline{A_1 + A_2}, \overline{B_1 + B_1})$. Note that $\overline{\Gamma_1} = (V, \overline{A_1}, \overline{B_1})$, and $\overline{\Gamma_2} = (V, \overline{A_2}, \overline{B_2})$, where $A_1 = (\lambda_1^+, \lambda_1^-), B_1 = (\tau_1^+, \tau_1^-), A_2 = (\lambda_2^+, \lambda_2^-)$ and $B_2 = (\tau_2^+, \tau_2^-)$, such that $\tau_1^+(s,t) = \max(\lambda_1^+(s), \lambda_1^+(t))$, $\tau_1^-(s,t) = \min(\lambda_1^-(s), \lambda_1^-(t))$ $\forall$ $(s,t) \in E_1$, $\tau_2^+(s,t) = \max(\lambda_2^+(s), \lambda_2^+(t))$, $\tau_2^-(s,t) = \min(\lambda_2^-(s), \lambda_2^-(t))$, $\forall$ $(s,t) \in E_2$ which are respectively given in Figure 3, with $\overline{\Gamma_1} + \overline{\Gamma_2} = (V, \overline{A_1} + \overline{A_2}, \overline{B_1} + \overline{B_2})$.

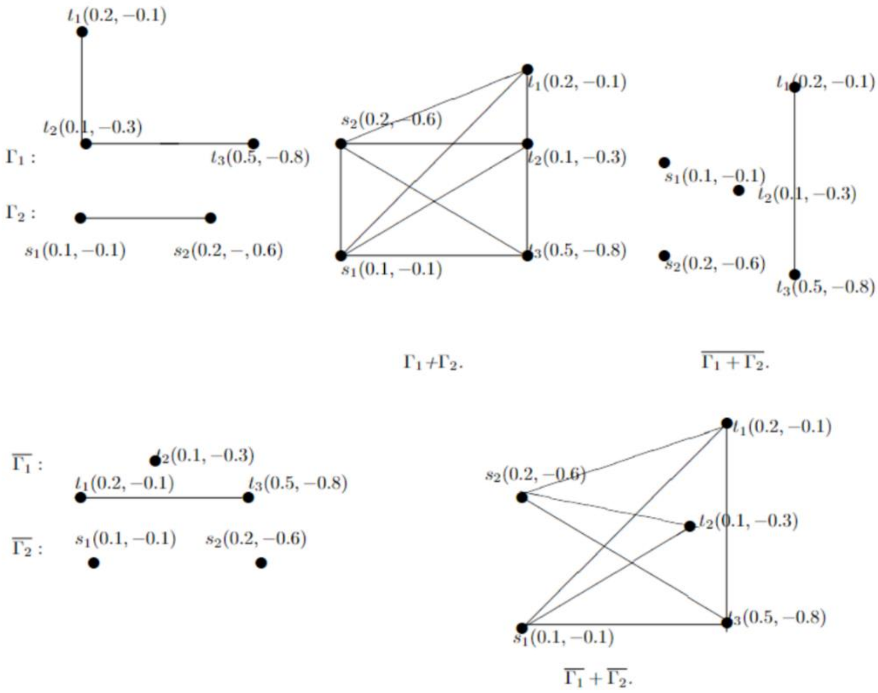

Figure 3. Graphs of Γ_1, Γ_2, $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and their $\Gamma_1+\Gamma_2$, $\overline{\Gamma_1+\Gamma_2}$, and $\overline{\Gamma}_1+\overline{\Gamma}_2$.

Definition 3.1.3. The union of two bipolar fuzzy graphs Γ_1 and Γ_2 is a new bipolar fuzzy graph $\Gamma=\Gamma_1\cup\Gamma_2=(V_1\cup V_2,E_1\cup E_2)$: $(\lambda_1^+\cup\lambda_2^+)(s),(\lambda_1^-\cup\lambda_2^-)(s),(\tau_1^+\cup\tau_2^+)(st),(\tau^-\cup\tau_2^-)(st)$ such that $V_1\cap V_2\neq\phi$. These membership and relation grades are defined in the following way: For each vertex s in the union of V_1 and V_2, the positive and negative membership grades of s in Γ are defined as $(\lambda_1^+\cup\lambda_2^+)(s)$ and $(\lambda_1^-\cup\lambda_2^-)(s)$, respectively. These grades are determined based on whether s is present in both Γ_1 and Γ_2 or in only one of the two graphs:

$$\begin{cases}(\lambda_1^+\cup\lambda_2^+)(s)=\max(\lambda_1^+,\lambda_2^+)(s) & \text{if} s\in V_1\cap V_2\\ (\lambda_1^-\cup\lambda_2^-)(s)=\min(\lambda_1^-,\lambda_2^-)(s) & \text{if} s\in V_1\cap V_2\\ (\lambda_1^+,\lambda_1^-)(s) & \text{if} s\in V_1-V_2\\ (\lambda_2^+,\lambda_2^-)(s) & \text{if} s\in V_2-V_1\end{cases} \tag{11}$$

Similarly, for each edge st in the union of E_1 and E_2, the positive and negative relation grades of st in Γ are defined as $(\tau_1^+\cup\tau_2^+)(st)$ and $(\tau_1^-\cup\tau_2^-)(st)$, respectively. These grades are also determined based on whether st is present in both Γ_1 and Γ_2 or in only one of the two graphs.

$$\begin{cases}(\tau_1^+\cup\tau_2^+)(st)=\max(\tau_1^+,\tau_2^+)(st) & \text{if} st\in E_1\cap E_2\\ (\tau_1^-\cup\tau_2^-)(st)=\min(\tau_1^-,\tau_2^-)(st) & \text{if} st\in E_1\cap E_2\\ (\tau_1^+,\tau_2^+)(st) & \text{if} st\in E_1-E_2\\ (\tau_2^+,\tau_2^-)(st) & \text{if} st\in E_2-E_1\end{cases} \tag{12}$$

Example 4. Consider the two bipolar fuzzy graphs $\Gamma_1=((\lambda_1^+,\lambda_1^-),(\tau_1^+,\tau_1^-))$ and $\Gamma_2=((\lambda_2^+,\lambda_2^-),(\tau_2^+,\tau_2^-))$ be two bipolar fuzzy graphs such that $\tau_1^+(s,t)=\max(\lambda_1^+(s),\lambda_1^+(t)$, $\tau_1^-(s,t)=\min(\lambda_1^-(s),\lambda_1^-(t))$ $\forall$ $(s,t)\in E_1$, $\tau_2^+(s,t)=\max(\lambda_2^+(s),\lambda_2^+(t))$, $\tau_2^-(s,t)=\min(\lambda_2^-(s),\lambda_2^-(t))$, $\forall$ $(s,t)\in E_2$ and $\Gamma_1\cup\Gamma_2$ are given in Figure 4.

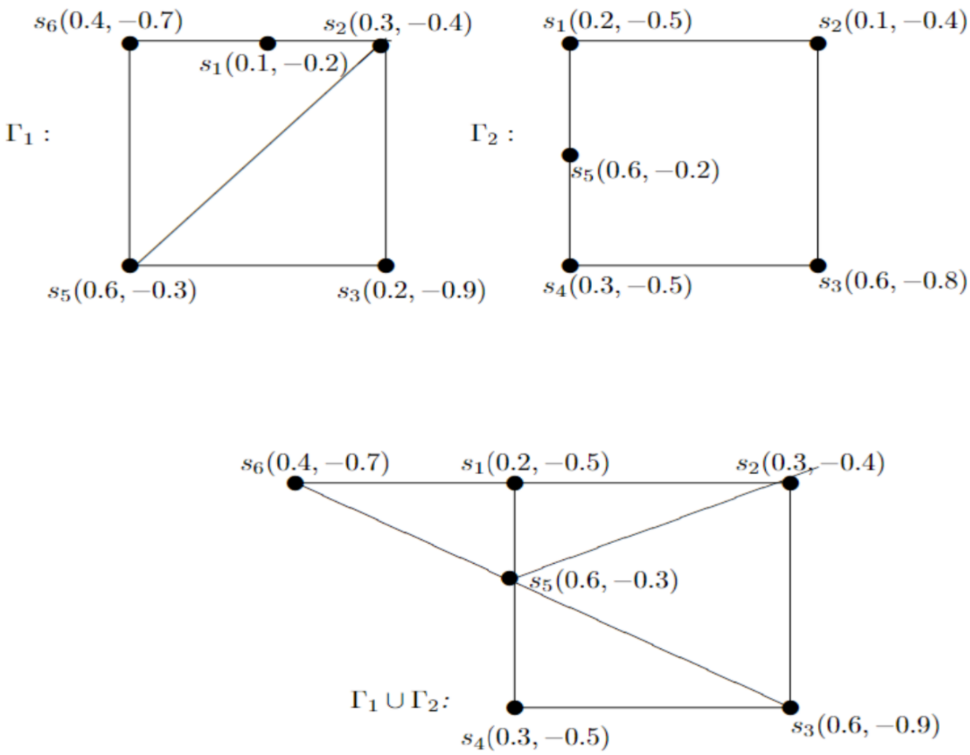

Figure 4. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 \cup \Gamma_2$.

Theorem 3.1.2. Let $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ be two bipolar fuzzy graphs, then,

(i) $(\overline{\Gamma_1 + \Gamma_2}) = \overline{\Gamma_1} \cup \overline{\Gamma_2}$

(ii) $(\overline{\Gamma_1 \cup \Gamma_2}) = \overline{\Gamma_1} + \overline{\Gamma_2}$

Proof. Consider the identity map $I: V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove (i) it is enough to prove that

$A)(i)$ $(\overline{\lambda_1^+ + \lambda_2^+})(t_i) = (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i)$ and $(\overline{\lambda_1^- + \lambda_2^-})(t_i) = (\overline{\lambda_1^-} \cup \overline{\lambda_2^-})(t_i)$

$A)(ii)$ $(\overline{\tau_1^+ + \tau_2^+})(t_i, t_j) = \overline{\tau_1^+} \cup \overline{\tau_2^+}(t_i, t_j)$and $(\overline{\tau_1^- + \tau_2^-})(t_i, t_j) = (\overline{\tau_1^-} \cup \overline{\tau_2^-})(t_i, t_j)$,

$A)(i)(\overline{\lambda_1^+ + \lambda_2^+})(t_i) = (\lambda_1^+ + \lambda_2^+)(t_i)$

$$= \begin{cases} \lambda_1^+(t_i); & t_i \in V_1 \\ \lambda_2^+(t_i); & t_i \in V_2 \end{cases} \tag{13}$$

$$= \begin{cases} \overline{\lambda_1^+}(t_i); & t_i \in V_1 \\ \overline{\lambda_2^+}(t_i); & t_i \in V_2 \end{cases} \tag{14}$$

$$= (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i).$$

Similarly $(\overline{\lambda_1^- + \lambda_2^-})(t_i) = (\overline{\lambda_1^-} \cup \overline{\lambda_2^-})(t_i)$.

$A)(ii)$ $(\overline{\tau_1^+ + \tau_1^+})(t_i, t_j) = (\lambda_1^+ + \lambda_2^+)(t_i) \wedge (\lambda_1^+ + \lambda_2^+(t_j) - (\tau_1^+ + \tau_2^+)(t_i, t_j)$

$$= \begin{cases} \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_1^+(t_i t_j) & if \quad (t_i, t_j) \in E_1 \\ \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_2^+(t_i t_j) & if \quad (t_i, t_j) \in E_2 \\ \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) - \lambda_1^+(t_i) \wedge \lambda_2(t_j) & if \quad (t_i, t_j) \in E' \end{cases} \tag{15}$$

$$= \begin{cases} \overline{\tau_1^+}(t_i, t_i) & if \quad (t_i, t_j) \in E_1 \\ \overline{\tau_2^+}(t_i, t_i)) & if \quad (t_i, t_j) \in E_2 \\ 0 \quad if \quad (t_i, t_j) \in E' \end{cases} \tag{16}$$

$$= (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i).$$

Similarly $(\overline{\tau_1^- + \tau_2^-})(t_i, t_j) = \overline{(\tau_1^-} \cup \overline{\tau_2^-})(t_i, t_j)$.

Consider the identity map $I: V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove (ii), it is enough to prove

$A)(i)$ $(\overline{\lambda_1^+ \cup \lambda_2^+})(t_i) = (\overline{\lambda_1^+} + \overline{\lambda_2^+})(t_i)$and $(\overline{\lambda_1^- \cup \lambda_2^-})(t_i)$ $=(\overline{\lambda_1^-}) + (\overline{\lambda_2^-})(t_i)A)(\mathrm{ii})$ $(\overline{\tau_1^+ \cup \tau_2^+})(t_i, t_j) = (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i, t_j)$, and $(\overline{\tau_1^- \cup \tau_2^-})(t_i, t_j) = \overline{\tau_1^-} + \overline{\tau_2^-}(t_i, t_j)$

$A)(i)$ $(\overline{\lambda_1^+ \cup \lambda_2^+})(t_i) = (\lambda_1^+ \cup \lambda_2^+)(t_i)$

$$= \begin{cases} \lambda_1^+(t_i); & t_i \in V_1 \\ \lambda_2^+(t_i); & t_i \in V_2 \end{cases} \tag{17}$$

$$= \begin{cases} \overline{\lambda_1^+}(t_i); & t_i \in V_1 \\ \overline{\lambda_2^+}(t_i); & t_i \in V_2 \end{cases} \tag{18}$$

$$= (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i) = (\overline{\lambda_1^+} + \overline{\lambda_2^+})(t_i).$$

Similarly $(\overline{\lambda_1^- \cup \lambda_2^-})(t_i)$ $=(\overline{\lambda_1^-}) + (\overline{\lambda_2^-})(t_i)$.

$A)(ii)$ $(\overline{\tau_1^+ \cup \tau_1^+})(t_i, t_j) = (\lambda_1^+ \cup \lambda_2^+)(t_i) \wedge (\lambda_1 \cup \lambda_2(t_j)) - (\tau_1 \cup \tau_2)$

$$= \begin{cases} \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_1^+(t_i t_j) & if \quad (t_i, t_j) \in E_1 \\ \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_2^+(t_i t_j) & if \quad (t_i, t_j) \in E_2 \\ \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) - \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) & if \quad (t_i, t_j) \in E' \end{cases} \tag{19}$$

$$= \begin{cases} \overline{\tau_1^+}(t_i, t_i) & if \quad (t_i, t_j) \in E_1 \\ \overline{\tau_2^+}(t_i, t_i)) & if \quad (t_i, t_j) \in E_2 \\ 0 \quad if \quad (t_i, t_j) \in E' \end{cases} \tag{20}$$

$$= (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i, t_j) = (\overline{\tau_1^+} + \overline{\tau_2^+})(t_i, t_j).$$

Similarly $(\overline{\tau_1^- \cup \tau_2^-})(t_i, t_j) = \overline{(\tau_1^-} + \overline{\tau_2^-})(t_i, t_j)$.

3.2 Domination and Global Domination Number on Bipolar Fuzzy Graph Operations

Theorem 3.2.1. Let Γ_1 and Γ_2 be two disjoint bipolar fuzzy graphs. Then

$$\gamma(\Gamma_1 \cap \Gamma_2) = 0.$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a separate bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 = \phi$. Consequently, we can deduce that $\gamma(\Gamma_1 \cap \Gamma_2) = |D_1 \cap D_2| = |\phi| = 0$, where γ denotes the cardinality of a set.

Theorem 3.2.2. Let $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ be two bipolar fuzzy graphs such that $\tau_1^+(s,t) = max(\lambda_1^+(s), \lambda_1^+(t))$, $\tau_1^-(s,t) = min(\lambda_1^-(s), \lambda_1^-(t))$ for all $(s,t) \in E_1$, $\tau_2^+(s,t) = max(\lambda_2^+(s), \lambda_2^+(t))$, $\tau_2^-(s,t) = min(\lambda_2^-(s), \lambda_2^-(t)$, for all $(s,t) \in E_2$. Then,

$$\gamma(\Gamma_1 \cup \Gamma_2) = \gamma(\Gamma_1) + \gamma(\Gamma_1).$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 \neq \phi$. Then $D_1 \cup D_2$ is a dominating set of $\Gamma_1 \cup \Gamma_2$. Consequently, we can deduce that $\gamma(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = \gamma(\Gamma_1) + \gamma(\Gamma_2)$, where γ denotes the cardinality of a set.

Theorem 3.2.3. If Γ_1 and Γ_2 be any two not disjoint bipolar fuzzy graphs, then

$$\gamma(\Gamma_1 \cup \Gamma_2) = max(\gamma(\Gamma_1), \gamma(\Gamma_2)).$$

Proof. Let D_1 be a γ_1-set of a bipolar fuzzy graph Γ_1 and let D_2 be a γ_2-set of a bipolar fuzzy graph Γ_2. Then $D_1 \cup D_2$ is a dominating set of $\Gamma_1 \cup \Gamma_2$. Since Γ_1 and Γ_2 are not disjoint, then $D_1 \cap D_2 \neq \phi$ Hence $\gamma(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = max(\gamma(\Gamma_1), \gamma(\Gamma_1))$.

Theorem 3.2.4. If $\Gamma = \Gamma_1 + \Gamma_2$ is a complete bipolar fuzzy graph, then,

$$\gamma_g(\Gamma_1 + \Gamma_2) = p.$$

Proof. Consider a complete bipolar fuzzy graph $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two disjoint bipolar fuzzy graphs. In such a graph, every vertex has $(p-1)$ neighbors, where p is the number of vertices in the graph.
Since the complement of Γ is the null graph, the set of vertices V is the only global dominating set of both Γ and its complement $\overline{\Gamma}$.
Therefore, we can conclude that the global domination number $\gamma_g(\Gamma)$ of Γ is equal to p, where p is the number of vertices in V.

Theorem 3.2.5. If $\Gamma = \Gamma_1 + \Gamma_2 = (A_1 + A_2, B_1 + B_2)$ is a complete bipolar fuzzy graph, then

$$\gamma_g(\Gamma_1 + \Gamma_2) = \gamma_g(\overline{\Gamma_1 + \Gamma_2}).$$

Proof. Consider a bipolar fuzzy graph $\Gamma = \Gamma_1 + \Gamma_2 = (A_1 + A_2, B_1 + B_2)$, where Γ_1 and Γ_2 are two disjoint bipolar fuzzy graphs. Let D be a minimal global dominating set of Γ.

It can be observed that D is a dominating set of both Γ and its complement $\overline{\Gamma}$. This is because every vertex in $V(\Gamma)\backslash D$ is adjacent to at least one vertex in D, since D is a global dominating set. Therefore, D dominates all vertices in Γ, and its complement $V(\Gamma)\backslash D$ dominates all vertices in $\overline{\Gamma}$.

Furthermore, since D is a minimal global dominating set of Γ, it is also a minimal global dominating set of $\overline{\Gamma}$. This is because any global dominating set D' of $\overline{\Gamma}$ must also be a dominating set of Γ, since $\overline{\overline{\Gamma}} = \Gamma$. Therefore, $|D'| \geq |D|$.
From the above observations, we can conclude that the global domination number of Γ is equal to the global domination number of $\overline{\Gamma}$, i.e., $\gamma_g(\Gamma) = \gamma_g(\overline{\Gamma})$.

Theorem 3.2.6. Assume that Γ_1 and Γ_2 are two dis-joint bipolar fuzzy graphs. Then

$$\gamma_g(\Gamma_1 \cap \Gamma_2) = 0.$$

Proof. Consider a bipolar fuzzy graph Γ_1 with a global domination number γ_{g1} and a γ_{g1}-set D_1, as well as a bipolar fuzzy graph Γ_2 with a global domination number γ_{g2} and a γ_{g2}-set D_2.

Since Γ_1 and Γ_2 are disjoint, the intersection of D_1 and D_2 is non-empty. Hence, the size of the intersection, denoted by $|D_1 \cap D_2|$, is equal to zero since there are no common vertices in Γ_1 and Γ_2.

Therefore, the global domination number of the intersection of Γ_1 and Γ_2, denoted by $\Gamma_1 \cap \Gamma_2$, is also equal to zero, since the size of any minimal global dominating set of $\Gamma_1 \cap \Gamma_2$ is zero. Thus, $\gamma_g(\Gamma_1 \cap \Gamma_2) = |D_1 \cap D_2| = |\phi| = 0$.

Theorem 3.2.7. Consider two bipolar fuzzy graphs $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ such that

$\tau_1^+(s,t) = max(\lambda_1^+(s), \lambda_1^+(t)$, $\tau_1^-(s,t) = min(\lambda_1^-(s), \lambda_1^-(t)$ for all $(s,t) \in E_1$, $\tau_2^+(s,t) = \max(\lambda_2^+(s), \lambda_2^+(t)$, $\tau_2^-(s,t) = \min(\lambda_2^-(s), \lambda_2^-(t)$, for all $(s,t) \in E_2$, we claim that in this case, the global domination number of the union of the two graphs, denoted by $\Gamma_1 \cup \Gamma_2$, is equal to the sum of the global domination numbers of Γ_1 and Γ_2, i.e.,

$$\gamma_g(\Gamma_1 \cup \Gamma_2) = \gamma_g(\Gamma_1) + \gamma_g(\Gamma_2).$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 = \phi$. Then $D_1 \cup D_2$ is a global dominating set of $\Gamma_1 \cup \Gamma_2$. Consequently, we can deduce that $\gamma_g(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = \gamma_g(\Gamma_1) + \gamma_g(\Gamma_2)$.

4. Conclusions

This study has explored the domain of domination and global domination numbers within the context of bipolar fuzzy graphs. We introduced and analyzed various operations on these graphs, including intersection, join, and union. Furthermore, we investigated the behavior of the domination number $\gamma(\Gamma)$ and the global domination number $\gamma_g(\Gamma)$ under these operations, encompassing not only the original graphs but also their complements. Much work still needs to be done, and here we mention some directions for future research, such as the relationship between domination and global domination numbers in bipolar fuzzy graphs under more complex operations, such as tensor product, Cartesian product, composition, strong product, disjunction, and symmetric difference of graphs. Other graph concepts like connectivity and independence numbers may also be investigated in bipolar fuzzy graphs, along with their relationships to domination measures.

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Author Contributions
All authors contributed equally to this research.

Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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The authors declare that there is no conflict of interest in the research.

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