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Divergence Measures and Aggregation Operators for Single-Valued Neutrosophic Sets with Applications in Decision-Making Problems

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Abstract: Single-valued neutrosophic sets (SVNSs) facilitate the representation of uncertain information more extensively than conventional methods. The study of divergence measures of SVNSs is important due to their applications in different areas like multi-criteria decision-making (MCDM), pattern recognition, cluster analysis, machine learning, etc., In this paper, we introduce a divergence measure for SVNSs. The suggested divergence measure is applied to cluster analysis for the classification of imprecise data. For establishing the reasonability and advantage of the suggested divergence measure in a clustering problem over the existing measures, a comparative assessment is also presented. Furthermore, we introduce, an inferior ratio method for handling the MCDM problem in the SVN environment. The consistency of the results of the suggested method with existing approaches also supports the credibility of its practical usage.

Keywords: Single-valued Neutrosophic Set, Aggregation Operator, Clustering Analysis, Divergence Measure, Inferior Ratio, Multi-Criteria Decision Making (MCDM).

1. Introduction

The aggregation operator is a crucial tool for unifying the information in a particular sense and provides meaningful mathematical output in an uncertain system. However, in recent years, aggregation operators played a vital role in various decision-making problems in uncertain environments. Bhatia and Singh [19] derived some fuzzy divergence measures using aggregation functions. The notion of a fuzzy set (FS) developed by Zadeh [1] is a mathematical and powerful tool to deal with imprecise and vague data in real-life problems wherein, the degree of membership or truth membership value of each element of the universe assumes a value between [0, 1]. In an FS degree of non-membership or false membership is dependent on membership value. So, to assign a degree of non-membership as an independent value, Atanassov [2] presented an Intuitionistic Fuzzy Set (IFS) as an extension of the FS, where each element of the universal set is associated with the truth membership and falsity membership value independently. However, in certain situations, FS and IFS are unable to represent some aspects of uncertainties like indeterminacy and restrictiveness, and thus, a notion of a neutrosophic set has been introduced as an extension of FS and IFS.

Smarandache [3] came up with the novel mathematical entity "Neutrosophic (NS) set". A NS assigns to each element of the universe, a membership degree (MD), non-membership degree (NMD), and, indeterminacy degree (ID). In a neutrosophic set, the values of MD, NMD, and ID are independent of each other and lie in [0, 1]. Wang et al. [4] presented a single-valued neutrosophic set (SVNS) as a subclass of the neutrosophic set (NS), to easily apply in science and engineering disciplines. Since its inception, a lot of researchers have actively contributed to the development of many versions of SVNS such as neutrosophic soft sets, picture FS, interval-valued neutrosophic sets, etc. (Maji [5]; Cuong and Kreinovich [6]; Peng and Liu [7]; Ulucay [8]). In the last decade, a lot of researchers have proposed several new measures for SVNSs such as entropy measures, distance measures, and similarity measures. Prominent research on information measures concerning FS, IFS, and SVNS and are due to (Szmidt and Kacprzyk [9]; Hwang and Yang [10]; Bhandari and Pal [11]; Xiao [12]; Wu et al. [13]; Elshabshery and Fattouh [14]; Aydogdu [15]; Qin and Wang [16]; Jin et al. [17]).

In this article, we intend to extend the formulation of SVN divergence measures using aggregation operators, and their applications. In the following, we present a literature review concerning divergence measures and applications of SVNSs to MCDM and clustering analysis.

The best option from the set of options based on a specific criterion is determined by the decision-making process and MCDM is one of its types of decision-making that consists of more than one criterion. To solve, an MCDM problem and provide the best optimal solution, the technique for order of preference by similarity to the ideal solution (TOPSIS) in a fuzzy environment was suggested in [18]. The development of divergence measures for fuzzy sets using various operators' has been considered in [13]. Divergence measure is a mathematical tool in MCDM problems and attracted the attention of many researchers in recent years (Bhatia and Singh [19]; Ohlan and Ohlan [20]; Singh and Sharma [21]). Monte et al., [22] first initiated the concept of divergence measure in intuitionistic fuzzy sets and studied the characterization of divergence measure between IFSs. Several intuitionistic fuzzy divergence measures have been proposed and useful in medical diagnosis, pattern recognition, MCDM problems, and clustering analysis over the years. Some prominent references are (Maheshwari and Srivastava [23]; Thao [24]; Boran et al. [25]; Ye [26]; and Chai et al. [43]). In the Picture fuzzy settings, Wei [27] proposed averaging and geometric averaging operators and applied them to handle MCDM problems. In some recent studies, divergence measures have been exploited for SVNSs and are effectively being used in MCDM and clustering problems. Guleria et al. [28] proposed a divergence measure using Hellinger's discrimination and discussed some of its algebraic properties. To date, several papers are available that attempted to define divergence measures for SVNSs along with their applications. Some of the important research is due to (Thao [29]; Selvachandran et al. [30]; Broumi et al. [31]; Nancy and Garg [32]). By recognizing the important applications of different aggregation operators in the FS, IFS, and, the Picture fuzzy set, we continue to extend them to construct aggregation-based divergence measures for SVNSs. Biswas et al. [33] proposed a new approach for MCDM issues by extending the conventional TOPSIS to a single-valued neutrosophic environment. The TOPSIS method has been widely used for solving MCDM problems, but in some situations, it has certain drawbacks. To overcome these limitations, Vencheh and Mirjaberi [34] introduced an inferior ratio method for the MCDM problem in the classical fuzzy environment. Ganie and Singh [35] proposed an inferior ratio method in the picture fuzzy settings and applied it to an MCDM problem.

Furthermore, for handling classification problems, clustering analysis is a fundamental tool that is widely applicable in many real-world scenarios. A lot of researchers applied different techniques to study the clustering method in FS, IFS, and SVNSs. In fuzzy sets, the fuzzy clustering approach was proposed by Ruspini [36] defining the fuzzy division concept. An intuitionistic fuzzy hierarchical algorithm for IFS was proposed by Xu [37] based on the intuitionistic fuzzy aggregation operator and the distance measures between IFSs. Based on the fuzzy C-means clustering method and distance measure between IFSs, Xu and Wu [38] proposed an intuitionistic fuzzy C – C-means algorithm. The clustering algorithms for IFSs by defining the association coefficients and the similarity measures were proposed by Zhang et al. [39] and Xu et al. [40]. In the SVN environment, Ye [41] and Huang [42] proposed an SVN clustering algorithm to cluster SVN data.

In the existing literature, various comparison measures have been suggested based on axiomatic validation. These measures had been obtained as extensions of fuzzy/ intuitionistic fuzzy comparison measures. Sometimes such formulation requires a lot of effort. This motivated us to explore an alternate method that can guarantee a larger class of comparison measures with the least effort. In this regard, we propose an aggregation operator-based approach for obtaining SVN divergence measures. This enables an experimenter to obtain a class of SVN divergence measures. Moreover, the existing SVN distance/ similarity measure produces unreasonable and counterintuitive results in some applications. Considering this situation, we are motivated to investigate the applications of the proposed methods for removing the drawbacks of the existing measures.

The innovative contribution of this paper is as.

- We propose a new aggregation-based divergence measure for SVNSs with its validity proof and discuss some of its algebraic properties.
- We show the presentation of the suggested measure to the MCDM problem and present a comparative study to justify the consistency/advantages of the suggested measures.
- We also examine the applications of the proposed divergence measure in cluster analysis and contrast it with existing measures, to assess the advantage of the suggested measure.

The remaining content of this article is prepared as follows: Section 2 provides some basic concepts associated with single-valued neutrosophic divergence measure. In section 3 we present an approach to derive divergence measures for SVNSs. Section 4 covers the application of the proposed measure to the MCDM problem and cluster analysis. Section 5 concludes the paper with some comments on the future directions.

2. Preliminaries

We present some basic concepts associated with single-valued neutrosophic sets, which have been used in this paper:

Definition 1 [3]. Let x be a generic component of the universal set X. A SVNS A in a universal set X is a triplet consists that with MD, ID, and the NMD s.t $\alpha_A(x_i): X \to [0,1]$, $\beta_A(x_i): X \to [0,1]$ and $\gamma_A(x_i): X \to [0,1]$ with condition

$$0 \le \alpha_A(x_i) + \beta_A(x_i) + \gamma_A(x_i) \le 3.$$

Note that, $\alpha_A(x_i)$, $\beta_A(x_i)$, and $\gamma_A(x_i)$ represent the MD, ID, and NMD. Here, indeterminacy gets quantified explicitly, while MD and NMD are independent. Some of the basic and important operations on SVNSs may be definite as follows:

Operations on SVNSs

Let $A = \{\langle \alpha_A(x_i), \beta_A(x_i), \gamma_A(x_i)\rangle | \; x_i \in X\}$ and $B = \{\langle \alpha_B(x_i), \beta_B(x_i), \gamma_B(x_i)\rangle | \; x_i \in X\}$ be two SVNSs, then the union, intersection, complement, equality, and inclusion of A and B are presented as follows [4]:

The Union of A and B is presented as

$A \cup B = \{< max.(\alpha_A(x_i), \alpha_B(x_i)), min.(\beta_A(x_i), \beta_B(x_i)),\ min.(\gamma_A(x_i), \gamma_B(x_i))> \mid x_i \in X\}$.

The intersection of A and B is

$A \cap B = \{< min.(\alpha_A(x_i), \alpha_B(x_i)), max.(\beta_A(x_i), \beta_B(x_i)),\ max.(\gamma_A(x_i), \gamma_B(x_i)) > \mid x_i \in X\}$.

The complement of A is defined as $A^c = \{< 1 - \alpha_A(x_i), 1 - \beta_A(x_i), 1 - \gamma_A(x_i) > \mid x_i \in X\}$.

Let $A \subseteq B$ then $\{\alpha_A(x_i) \le \alpha_B(x_i), \beta_A(x_i) \ge \beta_B(x_i), \gamma_A(x_i \ge \gamma_B(x_i) \;\forall\; x_i \in X\}$.

Definition 2 [19]. The aggregation operation on SVNSs is the operation by which several SVNSs are combined to generate a single set. Mathematically, an aggregation operation is defined by the function $A: [0,1]^n \to [0,1]$ with conditions:

Boundary conditions: $A(0,0,\dots,0) = 0, A(1,1,\dots,1) = 1$.

Monotonicity: A is monotonic in each argument.

Definition 3 [29]. The divergence measure of SVNS A and B is a function $D: SVNS(X) \times SVNS(X) \to [0,1]$ if it satisfies the next axioms:

SVNDM1: $\mathrm{D(A,B)} = D(B,A), \forall\, A, B \in NS(X)$;

SVNDM2: $D(A,B) \ge 0; D(A,B) = 0;\ if\ A = B, \forall\, A, B \in NS(X)$;

SVNDM3: $D(A \cap C, B \cap C) \le D(A,B), \forall\, A, B, C \in NS(X)$;

SVNDM4: $D(A \cup C, B \cup C) \le D(A,B), \forall\, A, B, C \in NS(X)$.

Definition 4 [44]. Degree of confidence is used to measure the confidence level of a comparison measure in classifying a pattern P_j, which belongs to a class of patterns P_i and it can be written as

$$DoC = \sum_{i=1}^{n}\sum_{i\neq j} |Measure(P_i, P_j) - Measure(P_i, P_j)|.$$

It assures that the better the degree of confidence, the more confident the outcome of the measure is.

3. Divergence Measure based on Aggregation Operator

In this section, we present a new divergence measure for single-valued neutrosophic sets based on an aggregation operator.

3.1 Background

With the help of aggregation operators, a probabilistic divergence measure for FS was proposed by Bhatia and Singh [19].

Consider $M(c, d)$ and $N(e, f)$ to be two aggregation operators then the probabilistic divergence measure is given as follows.

$$DM(P, Q) = \sum_{i=1}^{n} |M(p_i, q_i) - N(p_i, q_i)|. \quad (1)$$

Where P, Q is finite discrete probability distributions. Let us suppose

$$M(a, b) = \frac{a + b}{2}, \qquad N(a, b) = \frac{a^2 + b^2}{a + b}$$

Take $a = p_i$, $b = q_i$, Then by using (1), Bhatia and Singh [19] obtained the divergence measure

$$DM(P, \mathrm{Q}) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{2(p_i + q_i)}. \quad (2)$$

Using Eq. (2), Bhatia and Singh [19] proposed a divergence measure for the fuzzy set.

3.2 Novel SVN Divergence Measures

We suggest the following divergence measure for SVNSs:

Using arithmetic mean and geometric mean operators we propose the following divergence measure:

$$DM_1(A, B) = \sum_{i=1}^{n} \begin{bmatrix} \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\, \alpha_B(x_i)} + \\ \frac{\beta_A(x_i) + \beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\, \beta_B(x_i)} + \\ \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\, \gamma_B(x_i)} \end{bmatrix}. \quad (3)$$

Similarly, we propose another divergence measure using aggregation operators:

Using square root and arithmetic mean operator we propose divergence measure:

$$DM_2(A, B) = \sum_{i}^{n} \left[\sqrt{\frac{{\alpha_A}^2(x_i) + {\alpha_B}^2(x_i)}{2}} - \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} + \sqrt{\frac{{\beta_A}^2(x_i) + {\beta_B}^2(x_i)}{2}} - \frac{\beta_A(x_i) + \beta_B(x_i)}{2} + \sqrt{\frac{{\gamma_A}^2(x_i) + {\gamma_B}^2(x_i)}{2}} - \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2}\right]. \quad (4)$$

We can propose similarity measures using the above divergence measures.

For the divergence measure given in Eq. (3) and Eq. (4), we propose the new similarity measures for setting A and B.

$$SM_1(A, B) = 1 - \sum_{i=1}^{n} \begin{bmatrix} \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} + \\ \frac{\beta_A(x_i) + \beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\, \beta_B(x_i)} + \\ \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\, \gamma_B(x_i)} \end{bmatrix}. \quad (5)$$

$$SM_2(A,B) = 1 - \sum_{i=1}^{n} \begin{bmatrix} \sqrt{\frac{{\alpha_A}^2(x_i) + {\alpha_B}^2(x_i)}{2}} - \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} + \\ \sqrt{\frac{{\beta_A}^2(x_i) + {\beta_B}^2(x_i)}{2}} - \frac{\beta_A(x_i) + \beta_B(x_i)}{2} + \\ \sqrt{\frac{{\gamma_A}^2(x_i) + {\gamma_B}^2(x_i)}{2}} - \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} \end{bmatrix}. \tag{6}$$

Next, we prove the validity of the proposed divergence measure $DM_1(A,B)$ and $DM_2(A,B)$ for single-valued neutrosophic sets.

Theorem 3.2. The divergence measure DM_1 (A, B) given in Eq. (3) is a valid divergence measure for SVNSs.

Proof. To prove this theorem, we need to demonstrate that the divergence measure given in the Eq. (3) satisfies the 4 axioms stated in Definition 2.3.

SVNDM1: It is seen that Eq. (3) is symmetric w.r.t A and B, therefore it holds that DM_1 *(A, B)* = DM_1 *(B, A)*.

SVNDM2: Since, $0 \le \alpha_A(x_i), \beta_A(x_i), \gamma_A(x_i) \le 1$. Also, $0 \le \alpha_B(x_i), \beta_B(x_i), \gamma_B(x_i) \le 1$ which implies that $\frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} \ge \sqrt{\alpha_A(x_i)\, \alpha_B(x_i)}$; $\frac{\beta_A(x_i) + \beta_B(x_i)}{2} \ge \sqrt{\beta_A(x_i)\, \beta_B(x_i)}$; $\frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} \ge \sqrt{\gamma_A(x_i)\, \gamma_B(x_i)}$.

which implies that $DM_1(A,B) = \sum_{i=1}^{n} \begin{bmatrix} \frac{\alpha_A(x_i)+\alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_A(x_i)+\beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i)+\gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{bmatrix} \ge 0$

i.e., $DM_1(A,B) \ge 0$.

SVNDM3: We split the universal set X into 2 disjoint subsets:

$$X_1 = \begin{Bmatrix} x_i \in X \mid \alpha_A(x_i) \ge \alpha_B(x_i) \ge \alpha_C(x_i); \\ \beta_A(x_i) \le \beta_B(x_i) \le \beta_C(x_i); \\ \gamma_A(x_i) \le \gamma_B(x_i) \le \gamma_C(x_i) \end{Bmatrix} \tag{7}$$

$$X_2 = \begin{Bmatrix} x_i \in X \mid \alpha_A(x_i) \le \alpha_B(x_i) \le \alpha_C(x_i); \\ \beta_A(x_i) \ge \beta_B(x_i) \ge \beta_C(x_i); \\ \gamma_A(x_i) \ge \gamma_B(x_i) \ge \gamma_C(x_i) \end{Bmatrix} \tag{8}$$

By using Eq. (3), We have

$$DM_1(A \cap C, B \cap C) = \sum_{X_1} \begin{bmatrix} \frac{\alpha_{A\cap C}(x_i)+\alpha_{B\cap C}(x_i)}{2} - \sqrt{\alpha_{A\cap C}(x_i)\alpha_{B\cap C}(x_i)} + \\ \frac{\beta_{A\cap C}(x_i)+\beta_{B\cap C}(x_i)}{2} - \sqrt{\beta_{A\cap C}(x_i)\beta_{B\cap C}(x_i)} + \\ \frac{\gamma_{A\cap C}(x_i)+\gamma_{B\cap C}(x_i)}{2} - \sqrt{\gamma_{A\cap C}(x_i)\gamma_{B\cap C}(x_i)} \end{bmatrix} + \sum_{X_2} \begin{bmatrix} \frac{\alpha_{A\cap C}(x_i)+\alpha_{B\cap C}(x_i)}{2} - \sqrt{\alpha_{A\cap C}(x_i)\alpha_{B\cap C}(x_i)} \\ + \frac{\beta_{A\cap C}(x_i)+\beta_{B\cap C}(x_i)}{2} - \sqrt{\beta_{A\cap C}(x_i)\beta_{B\cap C}(x_i)} + \\ \frac{\gamma_{A\cap C}(x_i)+\gamma_{B\cap C}(x_i)}{2} - \sqrt{\gamma_{A\cap C}(x_i)\gamma_{B\cap C}(x_i)} \end{bmatrix}$$

Now, by using the Eqs. (7) and (8), $DM_1(A \cap C, B \cap C)$ will become

$$DM_1(A \cap C, B \cap C) = \sum_{X_1} \begin{bmatrix} \frac{\beta_A(x_i)+\beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i)+\gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{bmatrix} + \sum_{X_2} \left[\frac{\alpha_A(x_i)+\alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \right]$$

$$= \sum_{X_1 \cup X_2} \begin{bmatrix} \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_A(x_i) + \beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{bmatrix} = \sum_{i=1}^{n} \left(\sum_{x_1 \cup x_2} \begin{bmatrix} \frac{\alpha_A(x_i) + \alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_A(x_i) + \beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i) + \gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{bmatrix} \right)$$

which implies that $DM_1(A \cap C, B \cap C) \le DM_1(A,B)$.

SVNDM4: We can prove on the same lines as that of SVNDM3. Hence, $DM_1\ (A \cup C, B \cup C) \leq DM_1(A, B)$.This implies that $DM_1\ (A, B)$ is a valid divergence measure.

Similarly, we can show that the divergence measures $DM_2\ (A, B)$ given in Eq. (4) is a valid divergence measure. Now, we investigate some properties of the proposed SVNS divergence measures.

Theorem 3.3. Let $A, B, C \in SVNS(x)$, the proposed divergence measure satisfies:

$$DM_1\ (A \cup B, A \cap B\ ) = DM_1\ (A, B);$$
$$DM_1(A \cup B, A\ ) + DM_1(A \cap B, A) = 2\big(DM_1\ (A, B)\big);$$
$$DM_1(A \cup B, C\ ) + DM_1(A \cap B, C) = 2\big(DM_1(A, C) + DM_1(B, C)\big);$$
$$DM_1\ (A, A \cup B\ ) = DM_1\ (B, A \cap B);$$
$$DM_1\ (B, A \cup B\ ) = DM_1\ (A, A \cap B).$$

Proof (a). To prove this property, we divide the universal set into two disjoint subsets

$$X_1 = \left\{ \begin{matrix} x_i \in X |\ \alpha_A(x_i) \geq \alpha_B(x_i); \\ \beta_A(x_i) \leq \beta_B(x_i); \\ \gamma_A(x_i) \leq \gamma_B(x_i) \end{matrix} \right\} \tag{9}$$

$$X_2 = \left\{ \begin{matrix} x_i \in X |\ \alpha_A(x_i) \leq \alpha_B(x_i); \\ \beta_A(x_i) \geq \beta_B(x_i); \\ \gamma_A(x_i) \geq \gamma_B(x_i) \end{matrix} \right\} \tag{10}$$

Now, $DM_1(A \cup B, A \cap B) =$

$$\sum_{X_1} \begin{bmatrix} \frac{\alpha_{A\cup B}(x_i)+\alpha_{A\cap B}(x_i)}{2} - \sqrt{\alpha_{A\cup B}(x_i)\alpha_{A\cap B}(x_i)} \\ + \frac{\beta_{A\cup B}(x_i)+\beta_{A\cap B}(x_i)}{2} - \sqrt{\beta_{A\cup B}(x_i)\beta_{A\cap B}(x_i)} + \\ \frac{\gamma_{A\cup B}(x_i)+\gamma_{A\cap B}(x_i)}{2} - \sqrt{\gamma_{A\cup B}(x_i)\gamma_{A\cap B}(x_i)} \end{bmatrix} + \sum_{X_2} \begin{bmatrix} \frac{\alpha_{A\cup B}(x_i)+\alpha_{A\cap B}(x_i)}{2} - \sqrt{\alpha_{A\cup B}(x_i)\alpha_{A\cap B}(x_i)} \\ + \frac{\beta_{A\cup B}(x_i)+\beta_{A\cap B}(x_i)}{2} - \sqrt{\beta_{A\cup B}(x_i)\beta_{A\cap B}(x_i)} + \\ \frac{\gamma_{A\cup B}(x_i)+\gamma_{A\cap B}(x_i)}{2} - \sqrt{\gamma_{A\cup B}(x_i)\gamma_{A\cap B}(x_i)} \end{bmatrix}$$

Using Eqs. (9) and (10), we have

$$DM_1(A \cup B, A \cap B) = \sum_{X_1} \begin{bmatrix} \frac{\alpha_A(x_i)+\alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_B(x_i)+\beta_A(x_i)}{2} - \sqrt{\beta_B(x_i)\beta_A(x_i)} \\ + \frac{\gamma_B(x_i)+\gamma_A(x_i)}{2} - \sqrt{\gamma_B(x_i)\gamma_A(x_i)} \end{bmatrix} +$$

$$\sum_{X_2} \begin{bmatrix} \frac{\alpha_B(x_i)+\alpha_A(x_i)}{2} - \sqrt{\alpha_B(x_i)\alpha_A(x_i)} \\ + \frac{\beta_A(x_i)+\beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i)+\gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{bmatrix}$$

$$= \sum_{X_1 \cup X_2} \left[\begin{pmatrix} \frac{\alpha_A(x_i)+\alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_B(x_i)+\beta_A(x_i)}{2} - \sqrt{\beta_B(x_i)\beta_A(x_i)} \\ + \frac{\gamma_B(x_i)+\gamma_A(x_i)}{2} - \sqrt{\gamma_B(x_i)\gamma_A(x_i)} \end{pmatrix} \begin{pmatrix} \frac{\alpha_B(x_i)+\alpha_A(x_i)}{2} - \sqrt{\alpha_B(x_i)\alpha_A(x_i)} \\ + \frac{\beta_A(x_i)+\beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} \\ + \frac{\gamma_A(x_i)+\gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{pmatrix} \right] \leq$$

$$\sum_{i=1}^{n} \left(\sum_{X_1 \cup X_2} \left[\begin{pmatrix} \frac{\alpha_A(x_i)+\alpha_B(x_i)}{2} - \sqrt{\alpha_A(x_i)\alpha_B(x_i)} \\ + \frac{\beta_B(x_i)+\beta_A(x_i)}{2} - \sqrt{\beta_B(x_i)\beta_A(x_i)} \\ \frac{\gamma_B(x_i)+\gamma_A(x_i)}{2} - \sqrt{\gamma_B(x_i)\gamma_A(x_i)} \end{pmatrix} + \begin{pmatrix} \frac{\alpha_B(x_i)+\alpha_A(x_i)}{2} - \sqrt{\alpha_B(x_i)\alpha_A(x_i)} \\ \frac{\beta_A(x_i)+\beta_B(x_i)}{2} - \sqrt{\beta_A(x_i)\beta_B(x_i)} + \\ \frac{\gamma_A(x_i)+\gamma_B(x_i)}{2} - \sqrt{\gamma_A(x_i)\gamma_B(x_i)} \end{pmatrix} \right] \right).$$

which implies that $DM_1(A \cup B, A \cap B\ ) \leq D_1(A, B)$.

Proofs of $(\boldsymbol{b})$, $(\boldsymbol{c})$ **and** $(\boldsymbol{d})$ also follows on the same lines as that of (a). Similarly, we prove that the divergence measure $D_2(A, B)$ also satisfies all the properties which are given above.

4. Applications

In this part, we study the application of the suggested divergence measures in the MCDM problem and clustering analysis.

4.1 Application in MCDM Problems

In this section, we identify the weakness of the classical SVN TOPSIS method. To overcome the limitations, we introduce a new Single-Valued Neutrosophic Inferior Ratio method (SVNIR). This method utilizes the proposed divergence measures.

The weakness of classical SVN TOPSIS:

The key motive of the MCDM issue is to recognize or select the best option under different attributes /criteria. Hwang and Yoon [45] proposed the TOPSIS method which is one of the most effective and popular techniques for solving MCDM problems. The idea that came from the TOPSIS method is to choose the best alternative nearest to the positive ideal solution and farthest from the negative ideal solution. However, the chosen alternative due to TOPSIS is not farthest from the NIS for the proposed divergence measures as well as for existing measures as can be shown in the example given below:

Example1. Consider the SVN decision matrix with three alternatives $A_i(i = 1, 2, 3)$, and two attributes $C_i(i = 1, 2)$.

$$D = \begin{pmatrix} (0.2, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.4, 0.2, 0.3) & (0.1, 0.3, 0.2) \\ (0.6, 0.2, 0.2) & (0.2, 0.2, 0.3) \end{pmatrix}$$

Then the SVN positive ideal solution (SVNPIS) $Z^+$and SVN negative ideal solution (SVNNIS) Z^- are given as

$$Z^+ = \begin{pmatrix} max.\,(0.2, 0.4, 0.6) & max.\,(0.5, 0.1, 0.2) \\ min.\,(0.1, 0.2, 0.2) & min.\,(0.1, 0.3, 0.2) \\ min.\,(0.3, 0.3, 0.2) & min.\,(0.2, 0.2, 0.3) \end{pmatrix} = (0.6, 0.1, 0.2)\ (0.5, 0.1, 0.2).$$

$$Z^- = \begin{pmatrix} min.\,(0.2, 0.4, 0.6) & min.\,(0.5, 0.1, 0.2) \\ max.\,(0.1, 0.2, 0.2) & max.\,(0.1, 0.3, 0.2) \\ max.\,(0.3, 0.3, 0.2) & max.\,(0.2, 0.2, 0.3) \end{pmatrix} = (0.2, 0.2, 0.3)\ (0.1, 0.3, 0.3).$$

Now, we compute the divergence value of every option A_i from the SVNPIS and SVNNIS by using the proposed divergence measure formula given in the Eqs. (3) and (4). Also, compute the closeness coefficient of every option and obtain the ranking results in ascending order of closeness coefficient as shown in Table 1 and Table 2, respectively.

Table 1. Closeness coefficient and ranking of alternatives by using divergence measure DM_1.

	$DM_1(A_i, Z^+)$	$DM_1\ (A_i, Z^-)$	Closeness Coefficient	Ranking
A_1	0.0586	0.1168	0.6659	1
A_2	0.1269	0.0222	0.1488	3
A_3	0.0559	0.0722	0.5636	2

Table 2. Closeness coefficient and ranking of alternatives by using divergence measure DM_2.

	$DM_2(A_i, Z^+)$	$DM_2\ (A_i, Z^-)$	Closeness Coefficient	Ranking
A_1	0.1108	0.2140	0.6588	1
A_2	0.2340	0.0433	0.1561	3
A_3	0.1079	0.1375	0.5603	2

From Table 1 and Table 2, we observe that the chosen alternative A_1 is farthest from the SVNNIS for the proposed DM_1 and DM_2 i.e., $DM_1(A_1,\ Z^-) = 0.1168$, and $DM_2(A_1,\ Z^-) = 0.2140$ but it is not closest to the SVNPIS for DM_1 and DM_2 i.e., $DM_1(A_1, Z^+) = 0.0586 > DM_1(A_3, Z^+) = 0.0559$ and $DM_2(A_1, Z^+) = 0.1108 > DM_2(A_3, Z^+) = 0.1079$.

Now refer to Example 1, if we utilize the classical TOPSIS method for some existing measures DM_3 and DM_4 (Chai et al. [43]), where $DM_3 = \frac{1}{|X|}\sum_{x\in X}\left(|\alpha_A{}^2(x) - \alpha_B{}^2(x)|\vee|\beta_A{}^2(x) - \beta_B{}^2(x)|\vee|\gamma_A{}^2(x) - \gamma_B{}^2(x)|\right)$ and $DM_4 = 1 - \frac{1}{|X|}\sum_{x\in X}\left\{\frac{(\alpha_A{}^2(x)\wedge\alpha_B{}^2(x))+(\beta_A{}^2(x)\wedge\beta_B{}^2(x))+(\gamma_A{}^2(x)\wedge\gamma_B{}^2(x))}{(\alpha_A{}^2(x)\vee\alpha_B{}^2(x))+(\beta_A{}^2(x)\vee\beta_B{}^2(x))+(\gamma_A{}^2(x)\vee\gamma_B{}^2(x))}\right\}$, similar weakness has been identified as shown in Table 3 and Table 4.

Table 3. Closeness coefficient and ranking of alternatives.

	$D_3(A_i, Z^+)$	$D_3(A_i, Z^-)$	Closeness Coefficient	Ranking
A_1	0.0250	0.1200	0.8275	1
A_2	0.0650	0.0600	0.4800	3
A_3	0.0400	0.1750	0.8139	2

From Table 3, it shows that the chosen alternative A_1 is not farthest from the SVNNIS i.e., $D_3(A_1, Z^-) = 0.12 < D_3(A_3, Z^-) = 0.175$ but it is closest to the SVNPIS i.e., $D_3(A_1, Z^+) = 0.025$.

Table 4. Closeness coefficient and ranking of alternative.

	$D_4(A_i, Z^+)$	$D_4(A_i, Z^-)$	Closeness coefficient	Ranking
A_1	0.4021	0.5184	0.5270	2
A_2	0.7067	0.3384	0.3237	3
A_3	0.4156	0.5593	0.5736	1

From Table 4, it shows that the chosen alternative is farthest from the SVNNIS i.e., $D_4(A_3, Z^-) = 0.5593$, but it is not closest to the SVNPIS $D_4(A_3, Z^+) = 0.4156 > D_4(A_1, Z^+) = 0.4021$.

Example 2. Consider the SVN decision matrix with three alternatives $A_i (i = 1, 2, 3)$, and two attributes $C_i (i = 1, 2)$.

$$D = \begin{pmatrix} (0.1, 0.1, 0.2) & (0.6, 0.2, 0.3) \\ (0.5, 0.2, 0.3) & (0.2, 0.2, 0.2) \\ (0.1, 0.3, 0.3) & (0.4, 0.1, 0.3) \end{pmatrix}$$

Then the SVN positive ideal solution (SVNPIS) Z^+ and SVN negative ideal solution (SVNNIS) Z^- are given as

$$Z^+ = \begin{pmatrix} max.(0.1, 0.5, 0.1) & max.(0.6, 0.2, 0.4) \\ min.(0.1, 0.2, 0.3) & min.(0.2, 0.2, 0.1) \\ min.(0.2, 0.3, 0.3) & min.(0.3, 0.2, 0.3) \end{pmatrix} = (0.5, 0.1, 0.2)\ (0.6, 0.1, 0.2).$$

$$Z^- = \begin{pmatrix} min.(0.1, 0.5, 0.1) & min.(0.6, 0.2, 0.4) \\ max.(0.1, 0.2, 0.3) & max.(0.2, 0.2, 0.1) \\ max.(0.2, 0.3, 0.3) & max.(0.3, 0.2, 0.3) \end{pmatrix} = (0.1, 0.3, 0.3)\ (0.2, 0.3, 0.3).$$

Similarly, for another example, we calculate the divergence measure of every option A_i from the SVNPIS and SVNNIS by using the proposed divergence measure formula given in Eq. (4) and we compute the closeness coefficient of each alternative and then obtain the ranking results in ascending order of closeness coefficient.

From the closeness coefficient and ranking results of the alternatives, we observe that the chosen alternative A_1 is closest to the SVNPIS for the proposed DM_2 i.e., $DM_2(A_2, Z^+) = 0.1444$but it is not farthest from the SVNNIS for DM_2 i.e., $DM_2(A_2, Z^-) = 0.1669 < DM_2(A_1, Z^-) = 0.1712$.

Now refer to Example 2, if utilizing the classical TOPSIS method for some existing measures (Ye [41], Huang [42], Chai, et al. [43], and Nancy and Garg [32]), a similar weakness has been identified. Thus, from Examples 1 and 2, we see that the best alternative is neither closest to the SVNPIS nor farthest from the SVNNIS. To overcome this weakness of the classical TOPSIS method, we introduced the single-valued neutrosophic inferior ratio method (SVNIR) for the MCDM problem. Our proposed method constructs a compromise solution that constitutes an idea that the best alternative is closest to SVNPIS and farthest from the SVNNIS. To demonstrate this, we formulate an MCDM algorithm in the SVN environment. The flowchart of the algorithm is presented in Figure 1.

Let us consider a set of m- options i.e., $A_i = (A_1, A_2, \dots, A_m)$ and n-criterion $Z = (Z_1, Z_2, \dots, Z_n)$.

To select the best alternative, the proposed algorithm is as follows.

Algorithm

Step 1. Construct the SVN- decision matrix $E = \left(e_{ij}\right)_{m\times n}$ in which $e_{ij} = (a_{ij}, b_{ij}, c_{ij})$ is an SVN value where b_{ij} is the indeterminacy-membership value of alternative A_i , c_{ij} is the non-membership value of alternative A_i.

Step 2. Compute the normalized SVN decision matrix $N = \left(n_{ij}\right)_{m\times n}$ where

$$n_{ij} = \begin{cases} e_{ij} = \left(a_{ij}, b_{ij}, c_{ij}\right) & for\ benefit\ criteria; \\ (e_{ij})^c = \left(c_{ij}, b_{ij}, a_{ij}\right) & for\ cost\ criteria. \end{cases}$$

Step 3. Obtain the SVNPIS Z^+ and SVNNIS Z^- where $Z^+ = \{n_1{}^+, n_2{}^+, \dots, n_m{}^+\}$ and $Z^- = \{n_1{}^-, n_2{}^-, \dots, n_m{}^-\}$ with $n_j{}^+ = (max_i\ (a_{ij}), min_i(b_{ij}), min_i\ (c_{ij}))$, $n_j{}^- = (min_i\ (a_{ij}), max_i(b_{ij}), max.(c_{ij}))$, $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$.

Step 4. Calculate the divergence measure of every option A_i $(i = 1,2, \dots, m)$ from SVNPIS Z^+ and SVNNIS Z^- using our proposed divergence measure given in Eqs. (3) and (4) i.e., calculate $DM_1(A_i, Z^+)$ $DM_1(A_i, Z^-)$ and $DM_2(A_i, Z^+)$ $DM_2(A_i, Z^-)$ $(i = 1,2, \dots, m)$. The smaller of $DM_1(A_i, Z^+), DM_2(A_i, Z^+)$ and greater of $DM_1(A_i, Z^-), DM_2(A_i, Z^-)$, the better of A_i is.

Step 5. Compute $DM_1(Z^+)$ and $DM_2(Z^+)$ where $DM_1(Z^+) = min.(DM_1(A_i, Z^+))$ and $DM_2(Z^+) = min.(DM_2(A_i, Z^+))$ and hence the alternative that satisfies $DM_1(Z^+) = (DM_1(A_i, Z^+))$ and $DM_2(Z^+) = (DM_2(A_i, Z^+))$ is closest to SVNPIS.

Step 6. Similarly, compute $DM_1(Z^-)$ and $DM_2(Z^-)$ where $DM_1(Z^-) = max.(DM_1(A_i, Z^-))$ and $DM_2(Z^-) = max.(DM_2(A_i, Z^-))$ and hence the alternative that satisfies $DM_1(Z^-) = (DM_1(A_i, Z^-))$ and $DM_2(Z^-) = (DM_2(A_i, Z^-))$ is farthest from SVNNIS.

Step 7. Calculate $\rho\ (A_i)$for each alternative where, $\rho\ (A_i) = \frac{DM_1(A_i, Z^-)}{DM_1(Z^-)} - \frac{DM_1(A_i, Z^+)}{DM_1(Z^+)}$. (11)

Also, $\rho\ (A_i) = \frac{DM_2(A_i, Z^-)}{DM_2(Z^-)} - \frac{DM_2(A_i, Z^+)}{DM_2(Z^+)}$ (12)

Where $\rho\ (A_i)$ measures the degree to which an alternative is closest to SVNPIS and farthest from the SVNNIS simultaneously.

Step 8. Compute the single-valued neutrosophic inferior ratio for each alternative (SVNIR) by using Eq. (13).

$\zeta_i = \frac{\rho\ (A_i)}{min.(\rho\ (A_i))}$. (13)

Step 9. The options are ranked in ascending order of values of SVNIR (ζ_i).

Now, we use the SVNIR method for solving the MCDM problem in the SVN environment and the flowchart of the following algorithm is shown in Figure 1. For this, we consider an example as follows:

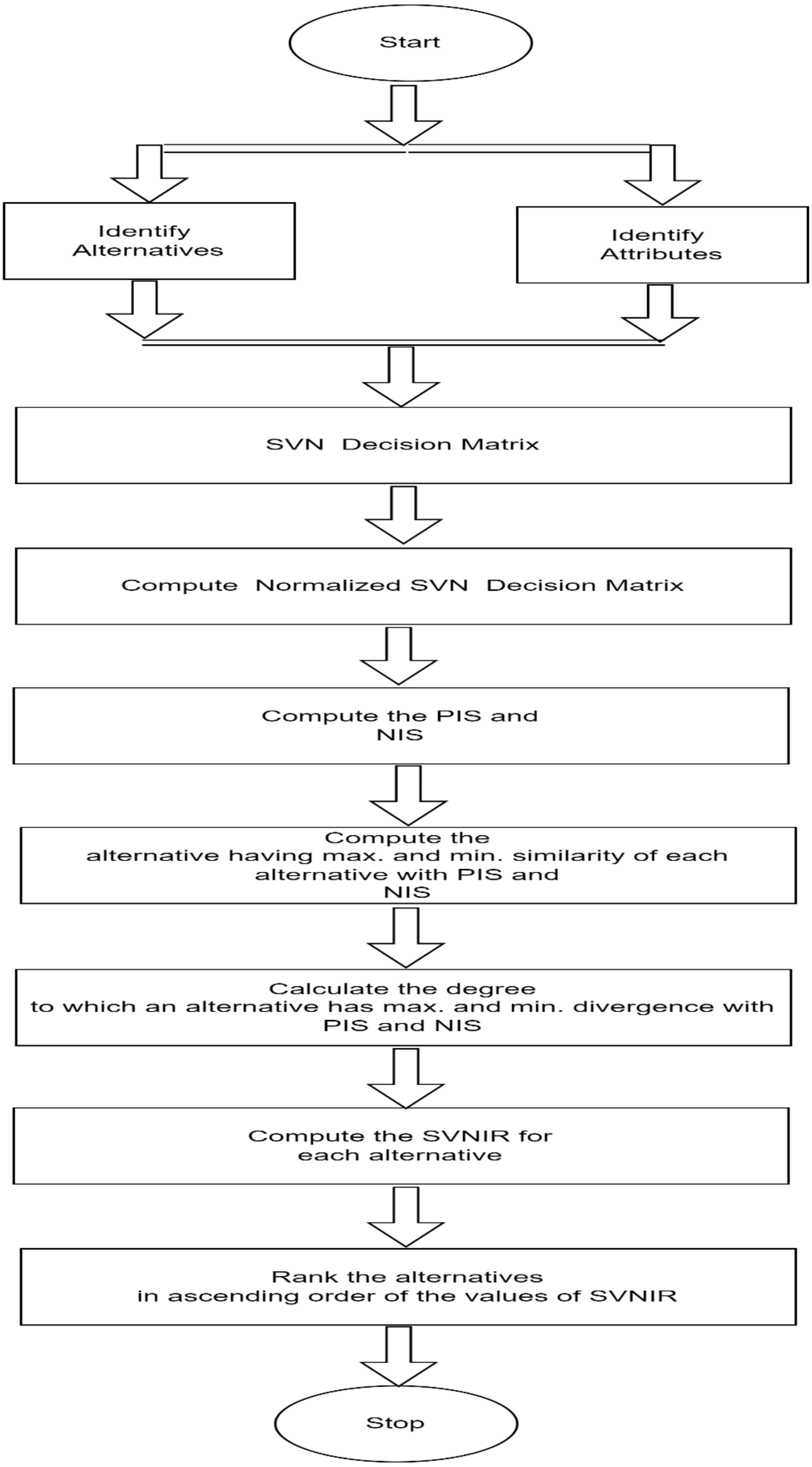

Figure 1. Flowchart of SVNIR algorithm.

Example 3 [46]. Let us assume a manufacturing corporation that needs to choose the best supplier. Let us assume that there are five available suppliers $A = (A_1, A_2, A_3, A_4, \text{and}, A_5)$. The capabilities and competencies have been computed under four criteria (the level of technology innovation, the control ability of flow, the ability of management, and the level of service) i.e., $Z = (Z_1, Z_2, Z_3, Z_4)$. By using the SVNIR method, we follow the algorithm given above:

Step 1. The information for five alternatives corresponding to four attributes is given by decision experts in the form of SVN values given in Table 5.

Table 5. SVN decision matrix.

Alternatives	Z_1	Z_2	Z_3	Z_4
A_1	(0.5, 0.2, 0.3)	(0.1, 0.4, 0.3)	(0.3, 0.2, 0.5)	(0.7, 0.8, 0.9)
A_2	(0.4, 0.3, 0.2)	(0.5, 0.6, 0.1)	(0.4, 0.3, 0.2)	(0.9, 0.1, 0.2)
A_3	(0.6, 0.4, 0.1)	(0.1, 0.3, 0.5)	(0.7, 0.8, 0.2)	(0.3, 0.5, 0.4)
A_4	(0.7, 0.8, 0.4)	(0.3, 0.5, 0.4)	(0.9, 0.3, 0.2)	(0.5, 0.3, 0.4)
A_5	(0.9, 0.5, 0.1)	(0.4, 0.3, 0.2)	(0.8, 0.8, 0.7)	(0.6, 0.3, 0.2)

Step 2. All the criteria are of the same kind in the given example, so there is no need to convert the cost criteria into benefit factors or vice-versa. Therefore, the normalized decision matrix is the same as given in Table 5.

Step 3. Obtain SVNPIS Z^+ and SVNNIS Z^- where $Z^+ = (max.(0.5, 0.4, 0.6, 0.7, 0.9)$, min. (0.2,0.3, 0.8, 0.5), $min.(0.3, 0.2, 0.1, 0.4, 0.1))$ for criteria Z_1.

$Z^- = (min.(0.5, 0.4, 0.6, 0.7, 0.9), max.(0.2, 0.3, 0.4, 0.8, 0.5), max.(0.3, 0.2, 0.1, 0.4, 0.1))$ for criteria Z_1 as shown below:

$$Z^+ = (0.9, 0.2, 0.1)(0.5, 0.3, 0.1)(0.9, 0.2, 0.2)(0.9, 0.1, 0.2)$$
$$Z^- = (0.4, 0.8, 0.4)(0.1, 0.6, 0.5)(0.3, 0.8, 0.7)(0.3, 0.8, 0.9).$$

Similarly for other criteria, we find Z^+ and Z^-.

Step 4. Calculate the divergence measure given in Eqs. (3) and (4) i.e., $DM_1(A_i, Z^+)$, $DM_1(A_i, Z^-)$ and $DM_2(A_i, Z^+)$ $DM_2(A_i, Z^-)$ $(i = 1,2,3,4,5)$. The obtained result is given in Table 6.

Table 6. SVNPIS and SVNNIS for each alternative.

Alternatives	$DM_1(A_i, Z^+)$	$DM_2(A_i, Z^-)$	$DM_2(A_i, Z^+)$	$DM_2(A_i, Z^-)$
A_1	0.5760	0.2793	1.0452	0.5208
A_2	0.1444	0.7428	0.2796	1.3496
A_3	0.4653	0.3052	0.8544	0.5803
A_4	0.3098	0.3918	0.5796	0.7432
A_5	0.2643	0.5012	0.4938	0.9350

Step 5. From Table 6 we have, $DM_1(Z^+)$ = 0.1444 and $DM_2(Z^+) = 0.2796$ which is closest to the SVNPIS.

Step 6. Also, from Table 6 we get $DM_1(Z^-)$ = 0.7428 and $DM_2(Z^-) = 1.3496$ which is farthest from SVNNIS.

Step 7. Calculate $\rho\ (A_i)$for each alternative using Eqs. (11) and (12) as given in Table 7.

Table 7. Computed values of $\rho\ (A_i)$.

$\rho\ (A_i)$	A_1	A_2	A_3	A_4	A_5
DM_1	-3.6053	0	-2.803	-1.6073	-1.1419
DM_2	-3.3523	0	-2.6258	-1.5223	-1.0733

Step 8. Calculate the SVNIR (ζ_i) for alternative $A_i (i = 1, 2, 3, 4)$ using Eq. (13) and rank the alternatives in ascending order as shown in Table 8.

Table 8. Computed values of ζ_i.

ζ_i	A_1	A_2	A_3	A_4	A_5
DM_1	1	0	0.7774	0.4458	0.3167
DM_2	1	0	0.7832	0.4541	0.3201

Ranking due to the proposed measures is $A_2 > A_5 > A_4 > A_3 > A_1$. Table 8, indicates that the best alternative is A_2 which is closest to the SVNPIS and farthest from the SVNNIS. Now, to check the reasonability and validity of the proposed measures, we apply the existing method (Nancy and Garg [32], Aydogdu [15], Shahzadi et al. [47], Ye [46], Ye [48], Broumi and Smarandache [49]) for solving the same investment problem given in Example 3, and the results are listed in Table 9.

Table 9. Ranking of existing measures with the help of the SVNIR method.

Measures	Ranking
Nancy and Garg [32]	$A_2 > A_5 > A_4 > A_3 > A_1$
Aydogdu [15]	$A_2 > A_5 > A_4 > A_3 > A_1$
Shahzadi et al. [47]	$A_2 > A_5 > A_4 > A_3 > A_1$
Ye [46]	$A_2 > A_5 > A_4 > A_1 > A_3$
Ye [48]	$A_2 > A_5 > A_4 > A_3 > A_1$
Broumi and Smarandache [49]	$A_2 > A_5 > A_4 > A_3 > A_1$

From Table 9, we observe that the prominent existing methods (Nancy and Garg [32], Aydogdu [15], Shahzadi et al. [47], Ye [46], Ye [48] Broumi, and Smarandache [49]) indicate that the best alternative is A_2, and, the same alternative is indicated by our suggested method. This implies that our suggested measure is in agreement with the existing measures in the SVN environment.

4.2 *Application of the Proposed Divergence Measure in Clustering Analysis*

Clustering analysis has a tremendous application in various fields like image processing, pattern recognition, and data analysis, etc., To date, various scholars investigated the Clustering analysis algorithm of FSs, IFSs, HFSs, SVNSs. In the following, we proposed a new distance/similarity measure-based clustering approach to cluster SVN data. A numerical example is considered to show that the proposed measure is more effective than the existing measures.

The algorithm of clustering analysis is as follows (Ye [41]):

Step 1. For SVNSs $(A_1, A_2, A_{3,} \dots, A_n)$ on X, construct an SVN similarity matrix $C = (S_{ij})_{n\times n}$, with the help of the similarity measure of SVNSs given in Eq. (5) and Eq. (6) where $S_{ij} = S_k\,(A_i, A_j)$ where *k* =*1, 2* and *i, j* =*1, 2…, n.*

Step 2. Compute the SVN matrix i.e., check whether $C^2 \subseteq C$, where $C^2 = C \circ C = (\tilde{c}_{ij})_{m\times m}$ is a composition matrix of C and, where $\tilde{c}_{ij} = max.(min.(S_{ik},\ S_{kj}));\ i,j = 1,2,3,\dots m$. The process is repeated until $C^{2^k} = C^{2^{(k+1)}}$holds, where C^{2^k} is an equivalent similarity matrix and it is denoted as $\bar{C} = \bar{S}_{ij_{n\times n}}$.

Step 3. Construct λ- cutting matrix $\bar{C}^{\lambda} = \left(\bar{S}_{ij}{}^{\lambda}\right)_{n\times n}$ for a given confidence level $\lambda \in [0,1]$.

Where $S_{ij}{}^{\lambda} = \begin{cases} 0\ if\ S_{ij}{}^{\lambda} < \lambda \\ 1\ if\ S_{ij}{}^{\lambda} \geq \lambda \end{cases}$

Step 4. Finally, we classify/identify SVNSs using the principle; if all features of the i^{th} column in C^{λ} are the same as the corresponding elements of the j^{th} column in C^{λ}, then we say SVNS A_i and A_j are in the same class.

Example 4. A classification problem adapted from (Ye [41]) is described below:

A car market is going to classify five different cars. Every car has five different evaluation factors: A_1= consumption of fuel, A_2 = degree of friction, A_3 = car price, A_4 = degree of comfort, and A_5 = design. The information of every car under each evaluation factor is represented by SVNSs, which are given below:

$$A_1 = (0.9, 0.8, 0.9), (0.4, 0.5, 0.6), (0.7, 0.3, 0.5), (0.6, 0.7, 0.6), (0.3, 0.7, 0.1)$$
$$A_2 = (0.8, 0.8, 0.2), (0.7, 0.6, 0.7), (0.9, 0.7, 0.6), (0.4, 0.6, 0.1), (0.7, 0.8, 0.5)$$
$$A_3 = (0.6, 0.6, 0.7), (0.9, 0.9, 0.2), (0.8, 0.9, 0.7), (0.4, 0.3, 0.1), (0.5, 0.4, 0.1)$$
$$A_4 = (0.7, 0.5, 0.3), (0.1, 0.9, 0.8), (0.7, 0.3, 0.2), (0.8, 0.9, 0.2), (0.8, 0.7, 0.6)$$
$$A_5 = (0.5, 0.6, 0.4), (0.6, 0.7, 0.8), (0.6, 0.6, 0.5), (0.7, 0.8, 0.1), (0.3, 0.2, 0.2)$$

Now, we calculate the similarity measure SM_1 corresponding to the divergence measure DM_1 given in Eqs. (5) and (6) between each pair of SVNSs $A_1, A_2, A_3, A_4,$ and A_5. The result is obtained in the form of a matrix C which is given below:

$$C = \begin{bmatrix} 1 & 0.560 & 0.557 & 0.543 & 0.667 \\ 0.560 & 1 & 0.658 & 0.658 & 0.729 \\ 0.557 & 0.658 & 1 & 0.229 & 0.705 \\ 0.543 & 0.658 & 0.229 & 1 & 0.611 \\ 0.667 & 0.729 & 0.705 & 0.611 & 1 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1 & 0.667 & 0.667 & 0.611 & 0.667 \\ 0.667 & 1 & 0.705 & 0.658 & 0.729 \\ 0.667 & 0.705 & 1 & 0.658 & 0.705 \\ 0.611 & 0.658 & 0.658 & 1 & 0.658 \\ 0.667 & 0.729 & 0.705 & 0.658 & 1 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 1 & 0.667 & 0.667 & 0.658 & 0.667 \\ 0.667 & 1 & 0.705 & 0.658 & 0.729 \\ 0.667 & 0.705 & 1 & 0.658 & 0.705 \\ 0.658 & 0.658 & 0.658 & 1 & 0.658 \\ 0.667 & 0.729 & 0.705 & 0.658 & 1 \end{bmatrix}$$

$$C^8 = \begin{bmatrix} 1 & 0.667 & 0.667 & 0.658 & 0.667 \\ 0.667 & 1 & 0.705 & 0.658 & 0.729 \\ 0.667 & 0.705 & 1 & 0.658 & 0.705 \\ 0.658 & 0.658 & 0.658 & 1 & 0.658 \\ 0.667 & 0.729 & 0.705 & 0.658 & 1 \end{bmatrix}$$

Here, we have $C^8 \subseteq C^4$, so C^4 is an SVN equivalent matrix. Now, to perform clustering for a confidence level λ, we construct λ- cutting matrix $C^\lambda = \left(S_{ij}{}^\lambda\right)_{m\times m}$, and based on the result we get all the possible classifications of A_j(j = 1, 2, 3, 4, 5) as shown in Table 10.

Table 10. Clustering result for SM_1.

Confidence level	Possible classification
$\mathbf{0 \le \lambda \le 0.658}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.658 \le \lambda \le 0.667}$	$\{A_1, A_2, A_3\}, \{A_4\}, \{A_5\}$
$\mathbf{0.667 \le \lambda \le 0.705}$	$\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}$
$\mathbf{0.705 \le \lambda \le 0.729}$	$\{A_1\}, \{A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.729 \le \lambda \le 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

Similarly, we find the similarity measure SM_2 corresponding to the divergence measure DM_2 between each pair of SVNSs $A_1, A_2, A_3, A_4,$ and A_5. From the calculation, we observe that $C^8 \subseteq C^4$, so C^4 is an SVN equivalent matrix. Now, for a confidence level λ, we construct λ- cutting matrix $C^\lambda =$

$\left(S_{ij}{}^{\lambda}\right)_{m\times m}$ by performing cluster analysis and based on the result we get all the classifications of A_j(j = 1, 2, 3, 4, 5) as given in Table 11.

Table 11. Clustering result for SM_2.

Confidence level	Possible classification
$\mathbf{0 \le \lambda \le 0.706}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.706 \le \lambda \le 0.710}$	$\{A_1, A_2, A_3\}, \{A_4\}, \{A_5\}$
$\mathbf{0.710 \le \lambda \le 0.732}$	$\{A_1\}, \{A_2, A_3, A_5\}, \{A_4\}$
$\mathbf{0.732 \le \lambda \le 0.755}$	$\{A_1\}, \{A_2, A_5\}\{A_3\}, \{A_4\}$
$\mathbf{0.755 \le \lambda \le 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

If we apply Shahzadi et al. [47] for the clustering of five different cars as given in Example 4. We construct the SVN similarity matrix and follow the same algorithm. Here, we have $C^4 \subseteq C^2$, so C^2 is an SVN equivalent matrix. Now, to perform clustering for a confidence level λ, we construct λ-cutting matrix $C^{\lambda} = \left(S_{ij}{}^{\lambda}\right)_{m\times m}$, and based on the result we get all the classifications of A_j(j = 1, 2, 3, 4, 5) as shown in Table 12.

Table 12. Clustering result for Shahzadi et al. [47].

Confidence level	Possible classification
$\mathbf{0 \le \lambda \le 0.753}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.753 \le \lambda \le 0.756}$	$\{A_1\}, \{A_2, A_3, A_4, A_5\}$
$\mathbf{0.756 \le \lambda \le 0.776}$	$\{A_1\}, \{A_2, A_4, A_5\}\{A_3\}$
$\mathbf{0.776 \le \lambda \le 0.783}$	$\{A_1\}, \{A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.783 \le \lambda \le 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

Now, if we utilize Aydogdu [15], Ye [48], and Sahin et al. [50] existed measures to cluster the five different cars as mentioned in Example 4. We compute the SVN similarity matrix follow the same algorithm and obtain the $C^2, C^2,$ and C^4 as SVN equivalent matrix respectively. Now, to perform clustering for a confidence level λ, we construct λ- cutting matrix $C^{\lambda} = \left(S_{ij}{}^{\lambda}\right)_{m\times m}$, and based on the result we get all the classifications of A_j(j = 1, 2, 3, 4, 5) as given in Tables 13-15.

Table 13. Clustering result for Aydogdu [15].

Confidence level	Possible classification
$\mathbf{0 \le \lambda \le 0.766}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.766 \le \lambda \le 0.773}$	$\{A_1, A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.773 \le \lambda \le 0.780}$	$\{A_1\}, \{A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.780 \le \lambda \le 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

Table 14. Clustering result for Ye [48].

Confidence level	Possible classification
$\mathbf{0 \le \lambda \le 0.922}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.922 \le \lambda \le 0.924}$	$\{A_1, A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.924 \le \lambda \le 0.926}$	$\{A_1\}, \{A_2, A_5\}, \{A_3\}, \{A_4\}$
$\mathbf{0.926 \le \lambda \le 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

Table 15. Clustering result for Sahin et al. [50].

Confidence level	Possible classification
$\mathbf{0 \leq \lambda \leq 0.824}$	$\{A_1, A_2, A_3, A_4, A_5\}$
$\mathbf{0.824 \leq \lambda \leq 0.826}$	$\{A_1, A_3\}, \{A_2, A_4, A_5\}$
$\mathbf{0.826 \leq \lambda \leq 0.893}$	$\{A_1\}, \{A_3\}, \{A_2, A_4, A_5\}$
$\mathbf{0.893 \leq \lambda \leq 0.911}$	$\{A_1\}, \{A_3\}, \{A_2, A_5\}, \{A_4\}$
$\mathbf{0.911 \leq \lambda \leq 1}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}$

Analysis: The classification result obtained from clustering analysis by using our proposed measure and some existing measures is shown above. From the classification table, it has been observed that the confidence level range in our proposed measures SM_1 and SM_2 is wider as compared to the existing measure. Due to this, classification is possible at a lower confidence level by our proposed measures. Therefore, our proposed measure is more effective than the existing measures.

5. Conclusion

This article presented an aggregation-based divergence measure for SVNSs and verified its properties. In an MCDM problem, the proposed divergence measures have been utilized to improve and/or overcome the drawbacks that are inherent in the existing classical TOPSIS method. A novel single-valued neutrosophic inferior ratio (SVNIR) method was introduced to address the drawbacks of existing measures and validated using numerical examples. Comparative studies in the context of decision-making and clustering analysis established the reasonability and superiority of the proposed method.

The main limitation of the proposed method is that it is expert-based and a non-academic decision-maker may face difficulty during its implementation. Moreover, the unavailability of real data in the neutrosophic framework is also a bottleneck to applying the method to real data-related problems.

In the future, we will focus on some other versions of neutrosophic sets by using aggregation operators like an interval-valued neutrosophic set, neutrosophic soft sets, refined neutrosophic sets, etc., and study their application to different fields of artificial intelligence.

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All authors contributed equally to this research.

Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest
The authors declare that there is no conflict of interest in the research.

References

1. Zadeh, L.A. (1965). Fuzzy sets. Information Control, 8(3), 394-432. http://dx.doi.org/10.1016/S0019-9958(65)90241-X

2. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Syst., 20(1), 87-96. https://doi:10.1016/S0165-0114(86)80034-3
3. Smarandache, F. (2005). Neutrosophic Set- A Generalization of the Intuitionistic Fuzzy Set. International Journal of Pure and Applied Mathematics, 24(3), 287-297. https://doi:10.1109/GRC.2006.1635754
4. Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single-valued neutrosophic sets. Multispace Multistruct, 4, 410-413. https://fs.unm.edu/SingleValuedNeutrosophicSets.pdf
5. Maji, P.K. (2013). Neutrosophic soft set. Annals of Fuzzy Mathematics and Informatics, 5(1), 157-168. http://www.afmi.or.kr
6. Cuong, C., & Kreinovich, V. (2014). Picture fuzzy sets. Journal of Computer Science and Cybernetics, 30(4). 409-420. https://doi:10.15625/1813-9663/30/4/5032
7. Peng, X., & Liu, C. (2017). Algorithms for neutrosophic soft decision making based on EDAS, new similarity measure and level soft set. Journal of Intelligent & Fuzzy Systems, 32(1), 955-968. https://doi:10.3233/jifs-161548
8. Ulucay, V., Sahin, M., Olgun, N., & Kilicman, A. (2017). On neutrosophic soft lattices. Afr. Mat., 28, 379-388. https://doi:10.1007/s13370-016-0447-7
9. Szmidt, E., & Kacprzyk, J. (2001). Entropy for intuitionistic fuzzy sets. Fuzzy sets and systems, 118(3), 467-477. https://doi.org/10.1016/S0165-0114(98)00402-3
10. Hwang, C.M., & Yang, M.S. (2008). On the entropy of fuzzy sets. International Journal of Uncertainty, Fuzziness and Knowledge- Based Systems, 16(4), 519-527. https://doi:org/10.1142/S021848850800539X
11. Bhandari, D., & Pal, N.R. (1993). Some new information measures for fuzzy sets. Information Sciences, 67(3), 209-228. https://doi.org/10.1016/0020-0255(93)90073-U
12. Xiao, F. (2019). A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 51(6), 3980-399. https://doi:10.1109/TSMC.2019.2958635
13. Wu, H., Yuan, Y., Wei, L., & Pei, L. (2018). On entropy, similarity measure and cross-entropy of single-valued neutrosophic sets and their application in multi-attribute decision making. Soft Computing, 22, 7367 - 7376. https://doi.org/10.1007/s00500-018-3073-5
14. Elshabshery, A., & Fattouh, M. (2021). On some Information Measures of Single–Valued Neutrosophic Sets and their Applications in MCDM Problems. Int. J. Eng. Res. Technol., 10(5) 406-415. http://www.ijert.org
15. Aydogdu, A. (2015). On Similarity and Entropy of Single Valued Neutrosophic Sets. ICSRS, 29 (1), 67 – 74. http://doi:www.geman.in
16. Qin, K., & Wang, L. (2020). New similarity and entropy measures of single–valued neutrosophic sets with applications in multi-attribute decision making. Soft Computing, 24, 16165–16176. https://doi.org/10.1007/s00500-020-04930-8
17. Jin, F., Ni, Z., Zhu, X., Chen, H., Langari, R., Mao, X., & Yuan, H. (2018). Single-valued neutrosophic entropy and similarity measure to solve supplier selection problems. Journal of Intelligent and Fuzzy Syst., 35 (6), 6513–6523. https://doi:10.3233/jifs-18854
18. Chen, C.T. (2000). Extensions of the TOPSIS for group decision making under fuzzy environment. Fuzzy Sets Syst.,114(1), 1–9. https://doi:10.1016/S0165-0114(97)00377-1
19. Bhatia, P.K., & Singh, S. (2013). A new measure of fuzzy directed divergence and its application in image segmentation. International Journal of Intelligent Systems and Applications, 5(4), 81- 89. https://doi:org/105815/ijisa.2013.04.08
20. Ohlan, A., & Ohlan, R. (2016). Generalized Hellinger's fuzzy divergence measure and its applications, Generalizations of Fuzzy Information Measures. Springer, 107–121. https://doi:10.1007/978-3-319-45928-8
21. Singh, S., & Sharma, S. (2019). On generalized fuzzy entropy and fuzzy divergence measure with applications. International Journal of Fuzzy System Applications, 8, 47-69. https://doi:10.4018/ijfsa.2019070102
22. Montes, I., Pal, N.R., Janis, V., & Montes, S. (2014). Divergence measures for intuitionistic fuzzy sets. IEEE Trans Fuzzy Syst, 23, 444–456. https://doi:10.1109/TFUZZ.2014.2315654

23. Maheshwari, S., & Srivastava, A. (2016). Study on divergence measures for intuitionistic fuzzy sets and its application in medical diagnosis. Journal of Applied Analysis and Computational, 6, 772-789. https://doi:10.11948/2016050
24. Thao, N.X. (2021). Some New Entropies and Divergence Measures of Intuitionistic Fuzzy Sets Based on Archimedean t-Conorm and Application in supplier selection. Soft Computing, 25, 5791-5805. https://doi.org/10.1007/s00500-021-05575-x
25. Boran, F.E., Genc, S., Kurt, M., & Akay, D. (2009). A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method. Expert Syst. Appl., 36, 11363–11368. https://doi:10.1016/j.eswa.2009.03.039
26. Ye, J. (2017). Intuitionistic fuzzy hybrid arithmetic and geometric aggregation operators for the decision-making of mechanical design schemes. Applied Intelligence, 47, 743-751. https://10.1007/s10489-017-0930-3
27. Wei, G. (2017). Picture fuzzy aggregation operators and their application to multiple attribute decision making. Journal of Intelligent and Fuzzy Systems, 33, 713-724. https://doi:org/10.3233/jifs-161798
28. Guleria, A., Srivastava, S., & Bajaj, R.K. (2019). On parametric divergence measure of neutrosophic sets with its application in decision-making models, Infinite Study, 29. https://digitalrepository.unm.edu/nss_journal/vol29/iss1/9
29. Thao, N.X., & Smarandache, F. (2018). Divergence measure of neutrosophic sets and applications, Infinite Study, 21, 142-152. https://digitalrepository.unm.edu/nss_journal/vol21/iss1/15
30. Selvachandran, G., Quek, S.G., Smarandache, F., & Broumi, S. (2018). Extended technique for order preference by similarity to an ideal solution (TOPSIS) with maximizing deviation method based on integrated weight measure for single- valued neutrosophic sets. Symmetry, 10(7), 236–252. https://doi:10.3390/sym10070236
31. Broumi, S., Singh, P.K., Talea, M., Bakali, A., Smarandache, F., & Rao, V. (2019). Single-valued neutrosophic techniques for analysis of WIFI connection. Adv Intell Syst Sustain Dev., 915, 405–412. https://doi.org/10.1007/978-3-030-11928-7_36
32. Nancy, H., & Garg. (2019). A novel divergence measure and its based TOPSIS method for multi-criteria decision-making under single-valued neutrosophic environment. Journal of Intelligent and Fuzzy Systems, 36(1), 101-115. https://doi:10.3233/jifs-18040
33. Biswas, P., Pramanik, S., & Giri, B. (2016). TOPSIS method for multiattribute group decision–making under single–valued neutrosophic environment. Neural Comput. Appl., 27, 727– 737. https://doi.org/10.1007/s00521-015-1891-2
34. Vencheh, A.H., & Mirjaberi, M. (2014). Fuzzy inferior ratio method for multiple attribute decision-making problems. Information Sciences, 277, 263-272. http://dx.doi.org/10.1016/j.ins.2014.02.019
35. Ganie, A.H., & Singh, S. (2021). A picture fuzzy similarity measure based on direct operations and novel multi-attribute decision making method. Neural Computing and Applications, 33, 9199-9219. https://doi.org/10.1007/s00521-020-05682-0
36. Ruspini, E.H. (1969). A new approach to clustering. Inform Control., 15(1), 22–32. https://doi:10.1016/S0019-9958(69)90591-9
37. Zeshui, X. (2009). Intuitionistic fuzzy hierarchical clustering algorithms. Journal of Systems Engineering and Electronics, 20(1), 90-97. https://www.researchgate.net/publication/228897776_Intuitionistic_fuzzy_hierarchical_clustering_algorithms
38. Xu, Z., & Wu, J. (2010). Intuitionistic fuzzy C-means clustering algorithms. Journal of Systems Engineering and Electronics, 21(4), 580-590. https://doi.10.3969/j.issn.1004-4132.2010.04.009
39. Zhang, H.M., Xu, Z.S, & Chen, Q. (2007). On clustering approach to intuitionistic fuzzy sets. Control Decis., 22(8), 882–888. https://www.researchgate.net/publication/264960396_On_clustering_approach_to_intuitionistic_fuzzy_sets
40. Xu, Z., Chen, J., & Wu, J. (2008). Clustering algorithm for intuitionistic fuzzy sets. Information sciences, 178(19), 3775-3790. https://doi.org/10.1016/j.ins.2008.06.008

41. Ye, J. (2014). Clustering methods using distance - based similarity measures of single-valued neutrosophic sets. Journal of Intelligent Systems, 23(4), 379-389. https://doi.org/10.1515/jisys-2013-0091
42. Huang, H.L. (2016). New distance measure of single-valued neutrosophic sets and its application. International Journal of Intelligent Systems, 31(10), 1021-1032. https://doi:10.1002/int.21815
43. Chai, J.S., Selvachandran, G., Smarandache, F., Gerogiannis, V.C., Son, L.H., Bui, Q.T., & Vo, B. (2021). New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. Complex and Intelligent Systems, 7, 703-723. https://doi.org/10.1007/s40747-020-00220-w
44. Hatzimichailidis, A.G., Papakostas, G.A., & Kaburlasos, V.G. (2012). A novel distance measure of intuitionistic fuzzy sets and its application to pattern recognition problems. International Journal of Intelligent Systems, 27(4), 396-409. https://doi.org/10.1002/int.21529
45. Hwang, C. L., Yoon, K., Hwang, C. L., & Yoon, K. (1981). Methods for multiple attribute decision making. Multiple attribute decision making: methods and applications a state-of-the-art survey, 58-191. https://doi.org/10.1007/978-3-642-48318-9_3
46. Ye, J. (2014). Single valued neutrosophic cross-entropy for multicriteria decision-making problems. Applied Mathematical Modelling, 38 (3), 1170-1175. https://doi.org/10.1016/j.apm.2013.07.020
47. Shahzadi, G., Akram, M., & Saeid, A.B. (2017). An application of single valued neutrosophic sets in medical diagnosis. Neutrosophic sets and systems, 18, 80-88. https://digitalrespository.unm.edu/nss_ journal/ vol18/iss1/9
48. Ye, J. (2014). Similarity measures between interval neutrosophic sets and their applications in multicriteria decision making Journal of intelligent and fuzzy systems, 26(1), 165-172. https://doi:10.3233/ifs-120724
49. Broumi, S., & Smarandache, F. (2013). Several similarity measures of neutrosophic sets. Neutrosophic Sets Syst, 1, Article 10, 54-62. https://digitalrepository.unm.edu/nss_journal/vol1/iss1/10
50. Sahin, M., Olgun, N., Uluçay, V., Kargin, A., & Smarandache, F. (2017). A new similarity measure based on falsify value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers transformed single valued neutrosophic numbers with applications to pattern recognition. Neutrosophic Sets Syst, 15, 31–48. https://doi:10.5281/zenodo.570934

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