Abstract. As concerning the views of $T$ norms and $T$ conorms, the intent of article is to define and probe the fuzzy semigroups, fuzzy ideals, fuzzy bi-ideals, bifuzzy subsemigroups, bifuzzy ideals, bifuzzy bi-ideals, fuzzy $(1, 2)$-ideals and bifuzzy $(1, 2)$-ideals in any given semigroup. Also, we indicate and study their basic properties of them in completely regular semigroups. Finally, we extend these concepts and so characterise (pre)image of them in semigroup homomorphisms.

Keywords: Regular semigroups, Fuzzy subsemigroups, Fuzzy bi-ideals, Bifuzzy subsemigroup, Bifuzzy ideal, Bifuzzy bi-ideals.

1. Introduction

Theory of semigroup with one operation in universal algebra, initiated in the 20th century [7]. In the real world, a purely mathematical set alone is not of much use, and having a weight for each element in this set is a necessity. Combining algebraic structures as systematic systems in the form of sets with labeled and weighted elements can be used as precise complex networks with many applications in the real world. Therefore, in addition to algebraic structures, it is necessary to have collections that can create indexes or weights in the elements of these structures. Fuzzy set theory which is inserted (in this regard) by Zadeh [28] is a generalization of crisp sets. Based on this concepts, Kuroki [12, 13], presented the fuzzy semigroup and fuzzy ideals in semigroups and delineated them and later was extended by Mordeson et al. [18]. In [25], the substructures prime, strongly prime, semiprime and irreducible fuzzy bi-ideals of a semigroup were expressed by Shabir, Jun and Bano. The related notions of fuzzy bi-ideals [11, 14, 27], intuitionistic fuzzy sets [1], intuitionistic fuzzy generalized bi-ideal of a semigroup [9] and intuitionistic fuzzy bi-ideals and intuitionistic F. I [10], are mentioned in the bibliography. Today, some research are investigated in these scoups such as hesitant bifuzzy set (an introduction): a new approach to assess the reliability of
the systems [3], B. F. I of d-algebras [3], singlevalued neutrosophic filters in EQ-algebras [4], EQ-algebras based on fuzzy hyper EQ-filters [5], F. I and F. F on topologies generated by fuzzy relations [24] and rough bipolar F. I in semigroups [17]. In this work, we inspected some assets of fuzzy algebraic structures, by using norms, defined fuzzy subsemigroups as ideals of [6], B. F. I of d-algebras [4], singlevalued neutrosophic filters in EQ-algebras [6], B. F. I as fuzzy relations [24] and rough bipolar F. I in semigroups [17]. In addition, we by using norms, define the novel concept fuzzy (1,2)-ideal of $S$ as $F(1,2)IT(S)$ and bifuzzy (1,2)-ideal of $S$ as $BF(1,2)IN(S)$ and we prove that $\overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BF(1,2)IN(S)$ if and only if $\xi_\Theta \in F(1,2)IT(S)$ and $\partial_\Theta \in F(1,2)IT(S)$. Also we show that $\bigtriangledown \overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BF(1,2)IN(S)$ if and only if $\bigtriangleup \overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BF(1,2)IN(S)$ and $\bigtriangledown \overline{\delta} = (\partial_\Theta, \partial_\Theta) \in BF(1,2)IN(S)$. Also we show that for any given $\overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BF(1,2)IN(S)$ and $B = (\xi_B, \partial_B) \in BF(1,2)IN(S)$, $\overline{\delta} \cap B \in BF(1,2)IN(S)$. Finally we prove that under some conditions $\overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BF(1,2)IN(S) \iff \overline{\delta} = (\xi_\Theta, \partial_\Theta) \in BFBIN(S)$. In final, we investigate image and pre-image of $FIT(S), FBIT(S), BFSN(S), BFIN(S), BFBIN(S), F(1,2)IT(S), BF(1,2)IN(S)$ under homomorphisms.

2. Preliminaries

**Lemma 2.1.** [13, 19] As $S = (S, \ast)$ be a semigroup so for all $a \in S$, $S$ is completely regular iff $a \in a^2Sa^2$.

**Definition 2.2.** [3, 17] Let $O \neq \emptyset$ be a set. Define

(i) $\overline{\delta} = \{ (x, \overline{\delta}(x)) : x \in O \}$ is a fuzzy subset of $O$, which $\overline{\delta} : O \to [0,1]$ ($\overline{\delta} \in [0,1]^O$).

For any $k \in [0,1]$, $U(\overline{\delta}; k) = \{ x \in O : \overline{\delta}(x) \geq T_{nor} \}$ is an upper level cut set and $L(\overline{\delta}; k) = \{ x \in O : \overline{\delta}(x) \leq T_{nor} \}$ is a lower level cut set.

(ii) $\overline{\delta} = \{ (x, \overline{\delta}_{\Theta}(x), \overline{\delta}_{\Theta}(x)) : x \in O \}$ is a bifuzzy subset of $O$, which $\overline{\delta}_{\Theta}, \overline{\delta}_{\Theta} \in [0,1]^O$ and for all $x \in O$ we get $0 \leq \overline{\delta}_{\Theta}(x) + \overline{\delta}_{\Theta}(x) \leq 1$ ($\overline{\delta} \in BF(O)$).

**Definition 2.3.** [2] Let $l, m, n \in \mathcal{I} = [0,1]$.

(i) triangular norm is a map $T_{nor} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $T_{nor}(l, 1) = l$, $T_{nor}(l, m) \leq T_{nor}(l, n)$ if $m \leq n$, $T_{nor}(l, m) = T_{nor}(m, n)$ and $T_{nor}(l, T_{nor}(m, n)) = T_{nor}(T_{nor}(l, m), n)$.

(ii) triangular conorm is a function $C_{con} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $C_{con}(l, 0) = l$, $C_{con}(l, m) \leq C_{con}(l, n)$ if $m \leq n$, $C_{con}(l, m) = C_{con}(m, l)$ and $C_{con}(l, C_{con}(m, n)) = C_{con}(C_{con}(l, m), n)$.

**Definition 2.4.** [21, 21] Let $\overline{\delta} \in [0,1]^S$ and $x, y, w \in S$.

(i) $\overline{\delta}$ is a fuzzy subsemigroup of $S$ regarding $T_{nor}$, if $\overline{\delta}(xy) \geq T_{nor}(\overline{\delta}(x), \overline{\delta}(y)) (\overline{\delta} \in FST(S))$. 

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(ii) $\delta$ is a F. I of $S$ regarding $T_{nor}$ if, (1) $\delta(xy) \geq T_{nor}(\delta(x), \delta(y)), (2) \delta(xy) \geq \delta(y), (3)$ $\delta(xy) \geq \delta(x)$ \text{(} $\delta \in FIT(S))$.

(iii) $\delta$ is a fuzzy bi-ideal of $S$ regarding $T_{nor}$-norm $T_{nor}$ if, (1) $\delta(xy) \geq T_{nor}(\delta(x), \delta(y)), (2) \delta(xwy) \geq T_{nor}(\delta(x), \delta(y))$ \text{(} $\delta \in FBIT(S))$.

**Definition 2.5.** [20, 21] Let $\bar{\delta} = (\delta_B, \delta_I) \in BF(S)$. Then $\bar{\delta}$ is a

(i) bifuzzy subsemigroup of $S$ regarding $T_{nor}$ and a $C^{con}$, if, (1) $\delta_B(x) \geq T_{nor}(\delta_B(x), \delta_I(y)), (2) \delta_B(x) \leq C^{con}(\delta_B(x), \delta_I(y)) \text{(} \delta \in BFSN(S))$.

(ii) B. F. I of $S$ regarding $T_{nor}$ and $C^{con}$, if (1) $\delta_B(x) \geq T_{nor}(\delta_B(x), \delta_I(y)), (2) \delta_B(x) \geq \delta_I(x), (3) \delta_I(xy) \geq \delta_B(y), (4) \delta_B(xy) \leq C^{con}(\delta_B(x), \delta_I(y)), (5) \delta_I(x) \leq \delta_I(y), (6) \delta_I(xy) \leq \delta_I(x)$ \text{(} $\bar{\delta} \in BFIN(S))$.

(iii) bifuzzy bi-ideal of $S$ regarding $T_{nor}$ and $C^{con}$, if it satisfies: (1) $\delta_B(x) \geq T_{nor}(\delta_B(x), \delta_I(y)), (2) \delta_B(xwy) \geq T_{nor}(\delta_B(x), \delta_I(y)), (3)$ $\delta_B(xy) \leq C^{con}(\delta_B(x), \delta_I(y)), (4) \delta_B(xwy) \leq C^{con}(\delta_B(x), \delta_I(y)) \text{(} \bar{\delta} \in BFIN(S))$.

3. Results On $BFBIN(S)$

In this section, we investigate some properties of $BFBIN(S), BFIN(S)$ and obtain the relation beween them. Let $\bar{\mathcal{U}} = (\delta_B, \delta_I), B = (\delta_B, \delta_B) \in BF(O).$ Then $\bar{\mathcal{U}} \cap B = (\delta_B, \delta_I) \cap (\delta_B, \delta_B) = (\delta_B, \delta_B) \in BF(O)$ is a bifuzzy subset, which $\bar{\mathcal{U}} \cap B : S \rightarrow [0, 1]$ will be defined by $(\bar{\mathcal{U}} \cap B)(s) = (\delta_B(s), \delta_B(s)) = (T_{nor}(\delta_B(s), \delta_B(s)), C^{con}(\delta_B(s), \delta_B(s)))$ with $s \in S$.

**Theorem 3.1.** Let $\bar{\mathcal{U}} = (\delta_B, \delta_I) \in BFBIN(S)$ and $B = (\delta_B, \delta_B) \in BFIN(S)$. Thus $\bar{\mathcal{U}} \cap B \in BFBIN(S)$.

**Proof.** Let $p, q, r \in O$. Then

\[
\delta_{\bar{\mathcal{U}} \cap B}(pq) = T_{nor}(\delta_B(pq), \delta_B(pq)) \geq T_{nor}(T_{nor}(\delta_B(p), \delta_B(q)), T_{nor}(\delta_B(p), \delta_B(q)))
\]

\[
= T_{nor}(T_{nor}(\delta_B(p), \delta_B(p)), T_{nor}(\delta_B(q), \delta_B(q))) = T_{nor}(\delta_{\bar{\mathcal{U}} \cap B}(p), \delta_{\bar{\mathcal{U}} \cap B}(q)).
\]

In a similar way, one can see that

\[
\delta_{\bar{\mathcal{U}} \cap B}(prq) \geq T_{nor}(\delta_{\bar{\mathcal{U}} \cap B}(p), \delta_{\bar{\mathcal{U}} \cap B}(q)), \delta_{\bar{\mathcal{U}} \cap B}(pq) \leq C^{con}(\delta_{\bar{\mathcal{U}} \cap B}(p), \delta_{\bar{\mathcal{U}} \cap B}(q)) \text{ and} \delta_{\bar{\mathcal{U}} \cap B}(prq) \leq C^{con}(\delta_{\bar{\mathcal{U}} \cap B}(p), \delta_{\bar{\mathcal{U}} \cap B}(q)).
\]

Therefore, we get that $\bar{\mathcal{U}} \cap B \in BFBIN(S. \Box$

**Example 3.2.** Let $S$ has a zero and

\[
x \ast y = \begin{cases} 
x & \text{if } x = y \\
0 & \text{otherwise}
\end{cases}
\]
and assume that $|S| > 2$, where $|S|$ denotes the cardinality of $S$, then $(S, *)$ is a semigroup.

Define $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BF(S)$ and $B = (\mathcal{B}_1, \mathcal{B}_2) \in BF(S)$ as

$$
\mathcal{U}_1(x) = \begin{cases} 
0.55 & \text{if } x = 0 \\
0.2 & \text{otherwise}
\end{cases}
, \quad
\mathcal{U}_2(x) = \begin{cases} 
0.35 & \text{if } x = 0 \\
0.15 & \text{otherwise}
\end{cases}
$$

$$
\mathcal{B}_1(x) = \begin{cases} 
0.45 & \text{if } x = 0 \\
0.1 & \text{otherwise}
\end{cases}
, \quad
\mathcal{B}_2(x) = \begin{cases} 
0.25 & \text{if } x = 0 \\
0.05 & \text{otherwise}
\end{cases}
$$

$$
T_{nor}(u, z) = T_{m}^{nor}(u, z) = \min\{u, z\} \text{ and } C_{con}(u, z) = C_{m}^{con}(u, z) = \max\{u, z\}, \text{ for all } u, z \in \mathcal{T}. \text{ Then } \mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BFBIN(S) \text{ and } B = (\mathcal{B}_1, \mathcal{B}_2) \in BFBIN(S) \text{ and } \mathcal{U} \cap B \in BFBIN(S).
$$

**Corollary 3.3.** (1) If $\{\mathcal{U}_i\}_{i \in I} \subseteq BFBIN(S)$, then $\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i \in BFBIN(S)$.

(2) If $\{\mathcal{U}_i\}_{i \in I} \subseteq BFSN(S)$, then $\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i \in BFSN(S)$.

**Theorem 3.4.** Let $\mathcal{U}_1 \in FBIT(S)$. Then $\square \mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BFBIN(S)$, which $\mathcal{U}_1 = 1 - \mathcal{U}_2$.

**Proof.** Let $r, s, t \in \mathcal{O}$. Since $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BFBIN(S)$, we get $\mathcal{U}_1(rs) \geq T_{nor}(\mathcal{U}_1(r), \mathcal{U}_1(s))$ and $\mathcal{U}_2(rts) \geq T_{nor}(\mathcal{U}_2(r), \mathcal{U}_2(s))$. Now

$$
\mathcal{U}_1(rs) \geq T_{nor}(\mathcal{U}_1(r), \mathcal{U}_1(s)) \Rightarrow -\mathcal{U}_1(rs) \leq -T_{nor}(\mathcal{U}_1(r), \mathcal{U}_1(s))
$$

$$
\Rightarrow 1 - \mathcal{U}_1(rs) \leq 1 - T_{nor}(\mathcal{U}_1(r), \mathcal{U}_1(s)) \Rightarrow \mathcal{U}_1(rs) \leq C_{con}(1 - \mathcal{U}_1(r), 1 - \mathcal{U}_1(s))
$$

and

$$
\mathcal{U}_2(rts) \geq T_{nor}(\mathcal{U}_2(r), \mathcal{U}_2(s)) \Rightarrow -\mathcal{U}_2(rts) \leq -T_{nor}(\mathcal{U}_2(r), \mathcal{U}_2(s))
$$

$$
\Rightarrow 1 - \mathcal{U}_2(rts) \leq 1 - T_{nor}(\mathcal{U}_2(r), \mathcal{U}_2(s)) \Rightarrow \mathcal{U}_2(rts) \leq C_{con}(1 - \mathcal{U}_2(r), 1 - \mathcal{U}_2(s))
$$

Therefore, $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BFBIN(S)$.

We recall that $T_{nor}$ and $C_{con}$ are idempotent, if for any $t \in \mathcal{T}, T_{nor}(t, t) = t$ and $S(t, t) = t$.

**Theorem 3.5.** Let $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in BFBIN(S)$. If $T_{nor}$ and $C_{con}$ are idempotent which $S$ is completely regular, then $\mathcal{U}(s) = \mathcal{U}(s^2)$ with $s \in S$.

**Proof.** Assume $s \in S$. Since $S$ is completely regular, using of Lemma 3.1, there exists $x \in S$ so $s = s^2 x s^2$. Now

$$
\mathcal{U}_1(s) = \mathcal{U}_1(s^2 x s^2) \geq T_{nor}(\mathcal{U}_1(s^2), \mathcal{U}_1(s^2)) = \mathcal{U}_1(s^2) = \mathcal{U}_1(s s) \geq T_{nor}(\mathcal{U}_1(s), \mathcal{U}_1(s)) = \mathcal{U}_1(s)
$$
and so \( \partial_3(s) = \partial_3(s^2) \). Also \( \partial_3(s) = \partial_3(s^2xs^2) \leq C^{con}(\partial_3(s^2), \partial_3(s^2)) = \partial_3(s^2) = \partial_3(ss) \leq C^{con}(\partial_3(s), \partial_3(s)) = \partial_3(s) \) and so \( \partial_3(s) = \partial_3(s^2) \). Thus, we get that \( \bar{U}(s) = (\partial_3(s), \partial_3(s)) = (\partial_3(s^2), \partial_3(s^2)) = \bar{U}(s^2) \).

**Theorem 3.6.** Let \( \bar{U} = (\partial_1, \partial_1) \in BF(S) \). If \( T^{nor} \) and \( C^{con} \) are idempotent and \( S \) is an intra-regular, then for all \( a \in S \), \( \bar{U}(a) = \bar{U}(a^2) \).

**Proof.** Suppose \( a \in S \). Since \( S \) is an intra-regular, find \( x, y \in S \) so \( a = xa^2y \). Then \( \partial_1(a) = \partial_1(xa^2y) \geq \partial_1(a^2y) \geq \partial_1(a^2) \geq T^{nor}(\partial_1(a), \partial_1(a)) = \partial_1(a) \) thus \( \partial_1(a) = \partial_1(a^2) \). Also, \( \partial_1(a) = \partial_1(xa^2y) \leq \partial_1(a^2y) \leq \partial_1(a^2) \leq C^{con}(\partial_1(a), \partial_1(a)) = \partial_1(a) \), so \( \partial_1(a) = \partial_1(a^2) \).

Therefore, we get that \( \bar{U}(a) = (\partial_1(a), \partial_1(a)) = (\partial_1(a^2), \partial_1(a^2)) = \bar{U}(a) \).

**Theorem 3.7.** Let \( \bar{U} = (\partial_1, \partial_1) \in BF(S) \). If \( T^{nor} \) and \( C^{con} \) are idempotent and \( S \) is an intra-regular, then for all \( a, b \in S \), \( \bar{U}(ab) = \bar{U}(ba) \).

**Proof.** Let \( a, b \in S \). Using Theorem 3.6, we have that \( \partial_1(a) = \partial_1(a^2) \) and \( \partial_1(a) = \partial_1(a^2) \). It follows that \( \partial_1(ab) = \partial_1((ab)^2) \) and \( \partial_1(ab) = \partial_1((ab)^2) \). Thus

\[
\partial_1(ab) = \partial_1((ab)^2) = \partial_1(abab) \geq \partial_1(bab) \geq \partial_1(ba)
\]

\[
= \partial_1((ba)^2) = \partial_1(baba) \geq \partial_1(aba) \geq \partial_1(ab)
\]

then \( \partial_1(ab) = \partial_1(ba) \). In addition,

\[
\partial_1(ab) = \partial_1((ab)^2) = \partial_1(abab) \leq \partial_1(bab) \leq \partial_1(ba)
\]

\[
= \partial_1((ba)^2) = \partial_1(baba) \leq \partial_1(aba) \leq \partial_1(ab),
\]

then \( \partial_1(ab) = \partial_1(ba) \). Thus, we get that \( \bar{U}(ab) = (\partial_1(ab), \partial_1(ab)) = (\partial_1(ba), \partial_1(ba)) = \bar{U}(ba) \).

**Theorem 3.8.** Let \( \bar{U} = (\partial_1, \partial_1) \in BF(S) \). Then \( \bar{U} = (\partial_1, \partial_1) \in BF(S) \) if and only if \( \partial_1 \in FBIT(S) \) and \( \partial_1 \in FBIT(S) \).

**Proof.** Let \( \bar{U} = (\partial_1, \partial_1) \in BF(S) \). Then for all \( f, g, h \in S \), we get that \( \partial_1(fg) \geq T^{nor}(\partial_1(f), \partial_1(g)) \) and \( \partial_1(fg) \geq T^{nor}(\partial_1(f), \partial_1(g)) \) which mean that \( \partial_1 \in FBIT(S) \). Also

\[
\partial_1(fg) \leq C^{con}(\partial_1(f), \partial_1(g)) \iff -\partial_1(fg) \geq -C^{con}(\partial_1(f), \partial_1(g))
\]

\[
\iff 1 - \partial_1(fg) \geq 1 - C^{con}(\partial_1(f), \partial_1(g)) \iff \partial_1(fg) \geq T^{nor}(1 - \partial_1(f), 1 - \partial_1(g))
\]

\[
\iff \partial_1(fg) \geq T^{nor}(\partial_1(f), \partial_1(g)).
\]
thus $\delta_3(fg) \geq T_{nor}(\delta_3(f), \delta_3(g))$. Also

$$\delta_3(fhg) \leq C_{con}(\delta_3(f), \delta_3(g)) \iff -\delta_3(fhg) \geq -C_{con}(\delta_3(f), \delta_3(g))$$

$$\iff 1 - \delta_3(fhg) \geq 1 - C_{con}(\delta_3(f), \delta_3(g)) \iff \delta_3(fhg) \geq T_{nor}(1 - \delta_3(f), 1 - \delta_3(g))$$

$$\iff \delta_3(fhg) \geq T_{nor}(\delta_3(f), \delta_3(g)),$$

then $\delta_3(fhg) \geq T_{nor}(\delta_3(f), \delta_3(g))$ and so $\delta_3 \in FBIT(S)$.

Conversely, let $\delta_3 \in FBIT(S)$, $\delta_3 \in FBIT(S)$ and $f, g, h \in S$. As $\delta_3 \in FBIT(S)$ so $\delta_3(fg) \geq T_{nor}(\delta_3(f), \delta_3(g))$ and $\delta_3(fhg) \geq T_{nor}(\delta_3(f), \delta_3(g))$. Since $\delta_3 \in FBIT(S)$,

$$\delta_3(fg) \geq T_{nor}(\delta_3(f), \delta_3(g)) \iff -\delta_3(fg) \leq -T_{nor}(\delta_3(f), \delta_3(g))$$

$$\iff 1 - \delta_3(fg) \leq 1 - T_{nor}(\delta_3(f), \delta_3(g)) \iff \delta_3(fg) \leq C_{con}(1 - \delta_3(f), 1 - \delta_3(g))$$

$$\iff \delta_3(fg) \leq C_{con}(\delta_3(f), \delta_3(g)),$$

thus $\delta_3(fg) \leq C_{con}(\delta_3(f), \delta_3(g))$ and

$$\delta_3(fhg) \geq T_{nor}(\delta_3(f), \delta_3(g)) \iff -\delta_3(fhg) \leq -T_{nor}(\delta_3(f), \delta_3(g))$$

$$\iff 1 - \delta_3(fhg) \leq 1 - T_{nor}(\delta_3(f), \delta_3(g)) \iff \delta_3(fhg) \leq C_{con}(1 - \delta_3(f), 1 - \delta_3(g))$$

$$\iff \delta_3(fhg) \leq C_{con}(\delta_3(f), \delta_3(g)).$$

Therefore, we conclude that $\delta = (\delta_3, \delta_3) \in BFIN(S)$. \(\square\)

4. **Bifuzzy (1,2)-ideals of subsemigroups and norms**

In this section, we define the notion of bifuzzy (1,2)-ideals of subsemigroups regarding norms and study their properties. In [20], F. Wang, introduced the concepts of fuzzy subsemigroups and fuzzy (1,2)-ideal in semigroups an in special case. In what follows, we introduce the fuzzy (1,2)-ideal and bifuzzy (1,2)-ideal of semigroups regarding any arbitrary triangular norms and any arbitrary triangular conorms.

**Definition 4.1.** Let $\delta \in T^S$ and $x, y, z, w \in S$. Then

(i) $\delta$ is a fuzzy (1,2)-ideal of $S$ regarding $T_{nor}$ if, (1) $\delta(xy) \geq T_{nor}(\delta(x), \delta(y))$, (2) $\delta(xwyz) \geq T_{nor}(\delta(x), T_{nor}(\delta(y), \delta(z))) (\delta \in F(1,2)IT(S))$.

(ii) A bifuzzy set $\delta = (\delta_3, \delta_3) \in BF(S)$ is a S regarding $T_{nor}$ and $C_{con}$, if (1) $\delta_3(xy) \geq T_{nor}(\delta_3(x), \delta_3(y))$ (2) $\delta_3(xwyz) \geq T_{nor}(\delta_3(x), T_{nor}(\delta_3(y), \delta_3(z)))$ (3) $\delta_3(xy) \leq C_{con}(\delta_3(x), \delta_3(y))$ (4) $\delta_3(xwyz) \leq C_{con}(\delta_3(x), C_{con}(\delta_3(y), \delta_3(z))) (\delta \in BF(1,2)IN(S))$. 

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Example 4.2. Let $S = \{-2, -4, -6, -8\}$. Then $(S, \ast)$ is a semigroup and $\tilde{0} \in I^S$ as follows:

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$\tilde{0} = \begin{cases} 0.1 \text{ if } b = -2 \\ 0.2 \text{ if } b = -4 \\ 0.3 \text{ if } b = -6 \\ 0.4 \text{ if } b = -8 \end{cases}$

and for all $u, z \in I$, $T^{nor}(u, z) = T^{nor}_p(u, z) = uz$. Clearly, $\tilde{0} \in F(1, 2)IT(S)$. Also define $\tilde{\tilde{0}} = (\tilde{0}_1, \tilde{0}_3) \in BF(S)$, which

$T^{nor}(u, z) = T^{nor}_p(u, z) = uz$ and $C^{con}(u, z) = C^{con}_p(u, z) = u + z - uz$, for all $u, z \in I$. One can see that $\tilde{\tilde{0}} = (\tilde{0}_1, \tilde{0}_3) \in BF(1, 2)IN(S)$.

Theorem 4.3. Assume $\tilde{0} = (\tilde{0}_1, \tilde{0}_3) \in BF(S)$. Then $\tilde{0} = (\tilde{0}_1, \tilde{0}_3) \in BF(1, 2)IN(S)$ if and only if $\tilde{0}_3 \in F(1, 2)IT(S)$ and $\tilde{0}_1 \in F(1, 2)IT(S)$.

Proof. Let $i, j, k, \in S$ and $\tilde{0} = (\tilde{0}_1, \tilde{0}_3) \in BF(1, 2)IN(S)$. Then $\tilde{0}_1(ij) \geq T^{nor}(\tilde{0}_1(i), \tilde{0}_1(j))$ and $\tilde{0}_3(i(kjm)) \geq T^{nor}(\tilde{0}_3(i), T^{nor}(\tilde{0}_3(j), \tilde{0}_3(m)))$ and so $\tilde{0}_3 \in F(1, 2)IT(S)$. Since $\tilde{0} = (\tilde{0}_1, \tilde{0}_3) \in BF(1, 2)IN(S)$,

$\tilde{0}_1(i(kjm)) \leq C^{con}(\tilde{0}_1(i), \tilde{0}_1(j)) \Rightarrow -\tilde{0}_1(ij) \geq -C^{con}(\tilde{0}_1(i), \tilde{0}_1(j))$

$\Rightarrow 1 - \tilde{0}_1(ij) \geq 1 - C^{con}(\tilde{0}_1(i), \tilde{0}_1(j)) = T^{nor}(1 - \tilde{0}_1(i), 1 - \tilde{0}_1(j))$

$\Rightarrow \tilde{0}_1(ij) \geq T^{nor}(\tilde{0}_1(i), \tilde{0}_1(j))$

and

$\tilde{0}_1(ij) \leq C^{con}(\tilde{0}_1(i), C^{con}(\tilde{0}_1(j), \tilde{0}_1(m))) \Rightarrow -\tilde{0}_1(i(kjm)) \geq -C^{con}(\tilde{0}_1(i), C^{con}(\tilde{0}_1(j), \tilde{0}_1(m)))$

$\Rightarrow 1 - \tilde{0}_1(i(kjm)) \geq 1 - C^{con}(\tilde{0}_1(i), C^{con}(\tilde{0}_1(j), \tilde{0}_1(m)))$

$= T^{nor}(1 - \tilde{0}_1(i), 1 - C^{con}(\tilde{0}_1(j), \tilde{0}_1(m))) = T^{nor}(1 - \tilde{0}_1(i), T^{nor}(1 - \tilde{0}_1(j), 1 - \tilde{0}_1(m)))$

$\Rightarrow \tilde{0}_1(ikjm) \geq T^{nor}(\tilde{0}_1(i), T^{nor}(\tilde{0}_1(j), \tilde{0}_1(m)))$.

Therefore, $\tilde{\tilde{0}} \in F(1, 2)IT(S)$.

Conversely, let $\tilde{0}_3 \in F(1, 2)IT(S)$ and $\tilde{0}_3 \in F(1, 2)IT(S)$ then $\tilde{0}_1(ij) \geq T^{nor}(\tilde{0}_1(i), \tilde{0}_1(j))$ and $\tilde{0}_3(ikjm) \geq T^{nor}(\tilde{0}_3(i), T^{nor}(\tilde{0}_3(j), \tilde{0}_3(m)))$. Also

$\tilde{0}_1(ij) \geq T^{nor}(\tilde{0}_1(i), \tilde{0}_1(j)) = T^{nor}(1 - \tilde{0}_1(i), 1 - \tilde{0}_1(j)) \Rightarrow 1 - \tilde{0}_1(ij) \geq 1 - C^{con}(\tilde{0}_1(i), \tilde{0}_1(j))$

$\Rightarrow -\tilde{0}_1(ij) \leq -C^{con}(\tilde{0}_1(i), \tilde{0}_1(j)) \Rightarrow \tilde{0}_1(ij) \leq C^{con}(\tilde{0}_1(i), \tilde{0}_1(j))$.

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Thus, $\partial_0(ij) \leq C^\con(\partial_0(i), \partial_0(j))$ and

$$
\tilde{\partial}_3(ik jm) \geq T^n_\nor(\tilde{\partial}_3(i), T^n_\nor(\tilde{\partial}_3(j), \tilde{\partial}_3(m)))
= T^n_\nor(1 - \partial_3(i), T^n_\nor(1 - \partial_3(j), 1 - \partial_3(m)))
= T^n_\nor(1 - \partial_3(i), 1 - C^\con(\partial_3(j), \partial_3(m)))
\Rightarrow 1 - \partial_3(ik jm) \geq 1 - C^\con(\partial_3(i), C^\con(\partial_3(j), \partial_3(m)))
\Rightarrow -\partial_3(ik jm) \geq -C^\con(\partial_3(i), C^\con(\partial_3(j), \partial_3(m)))
\Rightarrow \partial_3(ik jm) \leq C^\con(\partial_3(i), C^\con(\partial_3(j), \partial_3(m))).
$$

Hence, $\partial_3(ik jm) \leq C^\con(\partial_3(i), C^\con(\partial_3(j), \partial_3(m)))$ and so $\mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$.

\[\square\]

**Theorem 4.4.** Let $\mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$. Then $\mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$ if and only if $\Delta \mathcal{U} = (\partial_3, \tilde{\partial}_3) \in BF(1, 2)IN(S)$ and $\nabla \mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$.

**Proof.** Let $u, v, z, w \in S$. If $\mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$, then $\partial_3(uv) \geq T^n_\nor(\partial_3(u), \partial_3(v))$ and $\partial_3(uw(vz)) \geq T^n_\nor(\partial_3(u), T^n_\nor(\partial_3(v), \partial_3(z)))$ and

$$
\partial_3(uw(vz)) \geq T^n_\nor(\partial_3(u), \partial_3(v)) \Rightarrow -\partial_3(uv) \leq -T^n_\nor(\partial_3(u), \partial_3(v))
\Rightarrow 1 - \partial_3(uv) \leq 1 - T^n_\nor(\partial_3(u), \partial_3(v)) = C^\con(1 - \partial_3(u), 1 - \partial_3(v))
\Rightarrow \tilde{\partial}_3(uv) \leq C^\con(\partial_3(u), \partial_3(v))
$$

and so $\partial_3(uv) \leq C^\con(\partial_3(u), \partial_3(v))$. Moreover,

$$
\partial_3(uw(vz)) \geq T^n_\nor(\partial_3(u), T^n_\nor(\partial_3(v), \partial_3(z)))
\Rightarrow -\partial_3(uw(vz)) \leq -T^n_\nor(\partial_3(u), T^n_\nor(\partial_3(v), \partial_3(z)))
\Rightarrow 1 - \partial_3(uw(vz)) \leq 1 - T^n_\nor(\partial_3(u), T^n_\nor(\partial_3(v), \partial_3(z)))
= C^\con(1 - \partial_3(u), 1 - T^n_\nor(\partial_3(v), \partial_3(z))) = C^\con(1 - \partial_3(u), C^\con(1 - \partial_3(v), 1 - \partial_3(z)))
\Rightarrow \tilde{\partial}_3(uw(vz)) \leq C^\con(\partial_3(u), C^\con(\partial_3(v), \partial_3(z))),
$$

thus $\partial_3(uw(vz)) \leq C^\con(\partial_3(u), C^\con(\partial_3(v), \partial_3(z)))$. Hence, we give that $\Delta \mathcal{U} = (\partial_3, \tilde{\partial}_3) \in BF(1, 2)IN(S)$. Now, we prove that $\nabla \mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$. As $\mathcal{U} = (\partial_0, \partial_0) \in BF(1, 2)IN(S)$, then $\partial_0(uv) \leq C^\con(\partial_0(u), \partial_0(v))$ and $\partial_0(xw(vz)) \leq C^\con(\partial_0(u), C^\con(\partial_0(v), \partial_0(z)))$. Hence, we give that $\Delta \mathcal{U} = (\partial_3, \tilde{\partial}_3) \in BF(1, 2)IN(S)$. Now, we prove that $\nabla \mathcal{U} = (\tilde{\partial}_3, \partial_3) \in BF(1, 2)IN(S)$. As $\mathcal{U} = (\partial_0, \partial_0) \in BF(1, 2)IN(S)$, then $\partial_0(uv) \leq C^\con(\partial_0(u), \partial_0(v))$ and $\partial_0(xw(vz)) \leq C^\con(\partial_0(u), C^\con(\partial_0(v), \partial_0(z)))$. Hence, we give that $\Delta \mathcal{U} = (\partial_3, \tilde{\partial}_3) \in BF(1, 2)IN(S)$.
Let \( C^{con}(\partial_{13}(u), C^{con}(\partial_{13}(v), \partial_{13}(z))) \) and then
\[
\partial_{13}(uv) \leq C^{con}(\partial_{13}(u), \partial_{13}(v))
\]
\[
\Rightarrow -\partial_{13}(uv) \geq -C^{con}(\partial_{13}(u), \partial_{13}(v))
\]
\[
\Rightarrow 1 - \partial_{13}(uv) \geq 1 - C^{con}(\partial_{13}(u), \partial_{13}(v)) = T^{nor}(1 - \partial_{13}(u), 1 - \partial_{13}(v))
\]
\[
\Rightarrow \partial_{13}(uv) \geq T^{nor}(\partial_{13}(u), \partial_{13}(v)) \text{ and so } \partial_{13}(uv) \geq T^{nor}(\partial_{13}(u), \partial_{13}(v))
\]

Also
\[
\partial_{13}(uw(vz)) \leq C^{con}(\partial_{13}(u), C^{con}(\partial_{13}(v), \partial_{13}(z)))
\]
\[
\Rightarrow -\partial_{13}(uw(vz)) \geq -C^{con}(\partial_{13}(u), C^{con}(\partial_{13}(v), \partial_{13}(z)))
\]
\[
\Rightarrow 1 - \partial_{13}(uw(vz)) \geq 1 - C^{con}(\partial_{13}(u), C^{con}(\partial_{13}(v), \partial_{13}(z))) = T^{nor}(1 - \partial_{13}(u), 1 - C^{con}(\partial_{13}(v), \partial_{13}(z)))
\]
\[
= T^{nor}(1 - \partial_{13}(u), T^{nor}(1 - \partial_{13}(v), 1 - \partial_{13}(z)))
\]
\[
\Rightarrow \partial_{13}(uw(vz)) \geq T^{nor}(\partial_{13}(u), T^{nor}(\partial_{13}(v), \partial_{13}(z))).
\]

Thus, \( \partial_{13}(uw(vz)) \geq T^{nor}(\partial_{13}(u), T^{nor}(\partial_{13}(v), \partial_{13}(z))). \) Hence, we get that \( \forall \bar{U} = (\partial_{13}, \bar{U}) \in BF(1,2)IN(S). \)

Assume \( S \) be a semigroup and \( \emptyset \neq B \subseteq S \). We recall that \( B \) is a \((1,2)\)-ideal of \( S \), if for every \( x, y, z \in B \) and for every \( w \in S \), \( xw(yz) \in B \).

**Theorem 4.5.** Let \( \bar{U} = (\partial_{13}, \partial_{13}) \in BF(1,2)IN(S) \) and \( T^{nor} \) and \( C^{con} \) be idempotent. Then for all \( T^{nor} \in [0,1] \), \( U(\partial_{13}; t) \) and \( L(\partial_{13}; t) \) are \((1,2)\)-ideal of \( S \).

**Proof.** Let \( x, y \in U(\partial_{13}; t) \). Then, \( \partial_{13}(xy) \geq T^{nor}(\partial_{13}(x), \partial_{13}(y)) \geq T^{nor}(t, t) = t \) and so \( xy \in U(\partial_{13}; t) \) and \( U(\partial_{13}; t) \neq \emptyset \). Let \( x, y, z \in U(\partial_{13}; t) \) and \( w \in S \). Then
\[
\partial_{13}(xw(yz)) \geq T^{nor}(\partial_{13}(x), T^{nor}(\partial_{13}(y), \partial_{13}(z))) \geq T^{nor}(t, T^{nor}(t, t)) = T^{nor}(t, t) = t
\]
and so \( xw(yz) \in U(\partial_{13}; t) \). It follows that \( U(\partial_{13}; t) \) is a \((1,2)\)-ideal of \( S \) for all \( T^{nor} \in [0,1] \).

Similarly, if \( x, y \in L(\partial_{13}; t) \), then \( \partial_{13}(xy) \leq C^{con}(\partial_{13}(x), \partial_{13}(y)) \leq C^{con}(t, t) = t \). Hence, \( xy \in L(\partial_{13}; t) \) and \( L(\partial_{13}; t) \neq \emptyset \). Let \( x, y, z \in L(\partial_{13}; t) \) and \( w \in S \). Then
\[
\partial_{13}(xw(yz)) \leq C^{con}(\partial_{13}(x), C^{con}(\partial_{13}(y), \partial_{13}(z))) \leq C^{con}(t, C^{con}(t, t)) = C^{con}(t, t) = t.
\]
Thus, \( xw(yz) \in L(\partial_{13}; t) \) and so \( L(\partial_{13}; t) \) is a \((1,2)\)-ideal of \( S \) for all \( T^{nor} \in [0,1] \).

**Corollary 4.6.** Let \( \bar{U} = (\partial_{13}, \partial_{13}) \in BF(1,2)IN(S) \) and \( a \in S \) be a fixed element. Then \( M = \{ x \in O : \partial_{13}(x) \geq \partial_{13}(a) \} \) and \( N = \{ x \in O : \partial_{13}(x) \leq \partial_{13}(a) \} \) are \((1,2)\)-ideal of \( S \).
Theorem 4.7. Let $J \subseteq S$ and $\mathfrak{U} = (\partial_3, \partial_3) \in BF(S)$ defined by

$$
\partial_3(a) = \begin{cases} 
    c_0 & \text{if } a \in J \\
    c_1 & \text{if } a \notin J 
\end{cases}, \quad \partial_3(a) = \begin{cases} 
    c_0 & \text{if } a \notin J \\
    c_1 & \text{if } a \in J 
\end{cases}
$$

for all $a \in S$ and $c_0, c_1 \in [0, 1]$ so $c_0 > c_1$ and $T^{nor}$ and $C^{con}$ be idempotent. Then $\mathfrak{U} = (\partial_3, \partial_3) \in BF(1, 2)IN(S)$ if and only if $J = U(\partial_3; c_0) = L(\partial_3; c_0)$ be a $(1, 2)$-ideal of $S$.

Proof. Let $J = U(\partial_3; c_0) = L(\partial_3; c_0)$ be a $(1, 2)$-ideal of $S$. Since $c_0 > c_1$ so $c_1 = T^{nor}(c_1, c_0)$ and $c_0 = C^{con}(c_1, c_0)$ for $x, y \in S$, we have the following conditions:

(a) Assume $x \in J$ with $y \notin J$, so $xy \notin J$ and $\partial_3(xy) = c_1 \geq c_1 = T^{nor}(c_0, c_1) = T^{nor}(\partial_3(x), \partial_3(y))$ and $\partial_3(xy) = c_0 \leq c_0 = C^{con}(c_1, c_0) = C^{con}(\partial_3(x), \partial_3(y))$.

(b) As $x \notin J$ that $y \in J$, thus $xy \notin J$ and so $\partial_3(xy) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(\partial_3(x), \partial_3(y))$ and $\partial_3(xy) = c_0 \leq c_0 = C^{con}(c_0, c_0) = C^{con}(\partial_3(x), \partial_3(y))$.

(c) Assume $x \notin J$ which $y \notin J$, hence $xy \notin J$ then $\partial_3(xy) = c_1 \geq c_1 = T^{nor}(c_1, c_1) = T^{nor}(\partial_3(x), \partial_3(y))$ and $\partial_3(xy) = c_0 \leq c_0 = C^{con}(c_0, c_0) = C^{con}(\partial_3(x), \partial_3(y))$.

(d) Suppose $x \in J$ with $y \in J$, hence $xy \in J$ and so $\partial_3(xy) = c_0 \geq c_0 = T^{nor}(c_0, c_0) = T^{nor}(\partial_3(x), \partial_3(y))$ and $\partial_3(xy) = c_1 \leq c_1 = C^{con}(c_1, c_1) = C^{con}(\partial_3(x), \partial_3(y))$. Thus from (a)-(d) we get that $\partial_3(xy) \geq T^{nor}(\partial_3(x), \partial_3(y))$ and $\partial_3(xy) \leq C^{con}(\partial_3(x), \partial_3(y))$ for all $x, y \in S$.

Now, let $x, y, z, w \in S$ and we investigate the following conditions:

(a) As $x \in J$ and $y, z \notin J$, thus, $xw(yz) \notin J$. Hence,

$$
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_0, c_1) = T^{nor}(c_0, T^{nor}(c_1, c_1))
$$

(b) If $y \in J$ and $x, z \notin J$, then, $xw(yz) \notin J$. Hence

$$
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(T^{nor}(c_1, c_1), c_0) = T^{nor}(T^{nor}(\partial_3(x), \partial_3(y)), \partial_3(z))
$$

(c) Assume $z \in J$ and $x, y \notin J$, so, $xw(yz) \notin J$. Thus,

$$
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(T^{nor}(c_1, c_1), c_0)
$$

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(d) Suppose $x, y \in J$ such that $z \notin J$, then, $xw(yz) \notin J$. Hence,
\[
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(c_1, T^{nor}(c_0, c_0))
\]
\[
= T^{nor}(\partial_3(z), T^{nor}(\partial_3(x), \partial_3(y))) = T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), \partial_3(z))) \text{ and}
\]
\[
\partial_3(xw(yz)) = c_0 \leq c_0 = C^{con}(c_0, c_1) = C^{con}(c_0, C^{con}(c_0, c_1))
\]
\[
= C^{con}(\partial_3(z), C^{con}(\partial_3(x), \partial_3(y))) = C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]

(e) As $x, z \in J$ and $y \notin J$, hence, $xw(yz) \notin J$. It follows that
\[
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(c_1, T^{nor}(c_0, c_0))
\]
\[
= T^{nor}(\partial_3(y), T^{nor}(\partial_3(x), \partial_3(z))) = T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), \partial_3(z))) \text{ and}
\]
\[
\partial_3(xw(yz)) = c_0 \leq c_0 = C^{con}(c_0, c_1) = C^{con}(c_0, C^{con}(c_0, c_1))
\]
\[
= C^{con}(\partial_3(y), C^{con}(\partial_3(x), \partial_3(z))) = C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]

(f) Assume $y, z \in J$ with $x \notin J$, then, $xw(yz) \notin J$. Thus,
\[
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_0) = T^{nor}(c_1, T^{nor}(c_0, c_0))
\]
\[
= T^{nor}(\partial_3(y), T^{nor}(\partial_3(x), \partial_3(z))) = T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), \partial_3(z))) \text{ and}
\]
\[
\partial_3(xw(yz)) = c_0 \leq c_0 = C^{con}(c_0, c_1) = C^{con}(c_0, C^{con}(c_0, c_1))
\]
\[
= C^{con}(\partial_3(y), C^{con}(\partial_3(x), \partial_3(z))) = C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]

(j) Suppose $x, y, z \in J$ hence, $xw(yz) \in J$. So
\[
\partial_3(xw(yz)) = c_0 \geq c_0 = T^{nor}(c_0, c_0)
\]
\[
= T^{nor}(c_0, T^{nor}(c_0, c_0)) = T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), c_0)) \text{ and}
\]
\[
\partial_3(xw(yz)) = c_1 \leq c_1 = C^{con}(c_1, c_1)
\]
\[
= C^{con}(c_1, C^{con}(c_1, c_1)) = C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]

(h) If $x, y, z \notin J$ then, $xw(yz) \notin J$. Hence,
\[
\partial_3(xw(yz)) = c_1 \geq c_1 = T^{nor}(c_1, c_1)
\]
\[
= T^{nor}(c_1, T^{nor}(c_1, c_1)) = T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), \partial_3(z))) \text{ and}
\]
\[
\partial_3(xw(yz)) = c_0 \leq c_0 = C^{con}(c_0, c_0)
\]
\[
= C^{con}(c_0, C^{con}(c_0, c_0)) = C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]

Therefore, from (a)-(h), we get that $\partial_3(xw(yz)) \geq T^{nor}(\partial_3(x), T^{nor}(\partial_3(y), \partial_3(z))) \text{ and}$
\[
\partial_3(xw(yz)) \leq C^{con}(\partial_3(x), C^{con}(\partial_3(y), \partial_3(z))).
\]
Thus, $U = (\partial_0, \partial_3) \in BF(1,2)IN(S)$. □

**Theorem 4.8.** Let $U = (\partial_0, \partial_3) \in BF(1,2)IN(S)$ and $B = (\partial_B, \partial_B) \in BF(1,2)IN(S)$. Then $U \cap B \in BF(1,2)IN(S)$. 

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Proof. Let \( e, f, w \in \mathcal{O} \). Then
\[
\partial_{\mathcal{U}\cap B}(ef) = T^{nor}(\partial_{\mathcal{U}}(ef), \partial_B(ef)) \geq T^{nor}(T^{nor}(\partial_{\mathcal{U}}(e), \partial_B(f)), T^{nor}(\partial_{\mathcal{U}}(e), \partial_B(f)))
\]
\[
= T^{nor}(T^{nor}(\partial_{\mathcal{U}}(e), \partial_B(e)), T^{nor}(\partial_B(f), \partial_B(f))) = T^{nor}(\partial_{\mathcal{U}\cap B}(e), \partial_{\mathcal{U}\cap B}(f)),
\]
thus \( \partial_{\mathcal{U}\cap B}(ef) \geq T^{nor}(\partial_{\mathcal{U}\cap B}(e), \partial_{\mathcal{U}\cap B}(f)) \). Also
\[
\partial_{\mathcal{U}\cap B}(ew(fz)) = T^{nor}(\partial_{\mathcal{U}}(ew(fz)), \partial_B(ew(fz)))
\]
\[
\geq T^{nor}(T^{nor}(\partial_{\mathcal{U}}(e), T^{nor}(\partial_{\mathcal{U}}(f), \partial_{\mathcal{U}}(z))), T^{nor}(\partial_B(e), T^{nor}(\partial_B(f), \partial_B(z))))
\]
\[
= T^{nor}(T^{nor}(\partial_{\mathcal{U}}(e), \partial_B(e)), T^{nor}(\partial_B(f), \partial_B(f))) = T^{nor}(\partial_{\mathcal{U}\cap B}(e), T^{nor}(\partial_{\mathcal{U}\cap B}(f), \partial_{\mathcal{U}\cap B}(z))),
\]
so \( \partial_{\mathcal{U}\cap B}(ew(fz)) \geq T^{nor}(\partial_{\mathcal{U}\cap B}(e), T^{nor}(\partial_{\mathcal{U}\cap B}(f), \partial_{\mathcal{U}\cap B}(z))) \). Since
\[
\partial_{\mathcal{U}\cap B}(ef) = C^{con}(\partial_{\mathcal{U}}(ef), \partial_B(ef)) \leq C^{con}(C^{con}(\partial_{\mathcal{U}}(e), \partial_B(f)), C^{con}(\partial_B(e), \partial_B(f)))
\]
\[
= C^{con}(C^{con}(\partial_{\mathcal{U}}(e), \partial_B(e)), C^{con}(\partial_B(f), \partial_B(f))) = C^{con}(\partial_{\mathcal{U}\cap B}(e), \partial_{\mathcal{U}\cap B}(f)),
\]
so \( \partial_{\mathcal{U}\cap B}(ef) \leq C^{con}(\partial_{\mathcal{U}\cap B}(e), \partial_{\mathcal{U}\cap B}(f)) \). As
\[
\partial_{\mathcal{U}\cap B}(ew(fz)) = C^{con}(\partial_{\mathcal{U}}(ew(fz)), \partial_B(ew(fz)))
\]
\[
\leq C^{con}(C^{con}(\partial_{\mathcal{U}}(e), C^{con}(\partial_{\mathcal{U}}(f), \partial_{\mathcal{U}}(z))), C^{con}(\partial_B(e), C^{con}(\partial_B(f), \partial_B(z))))
\]
\[
= C^{con}(C^{con}(\partial_{\mathcal{U}}(e), \partial_B(e)), C^{con}(C^{con}(\partial_{\mathcal{U}}(f), \partial_{\mathcal{U}}(z)), C^{con}(\partial_B(f), \partial_B(z))))
\]
\[
= C^{con}(C^{con}(\partial_{\mathcal{U}}(e), \partial_B(e)), C^{con}(C^{con}(\partial_{\mathcal{U}}(f), \partial_B(y)), C^{con}(\partial_{\mathcal{U}}(z)), \partial_B(z))))
\]
\[
= C^{con}(\partial_{\mathcal{U}\cap B}(e), C^{con}(\partial_{\mathcal{U}\cap B}(f), \partial_{\mathcal{U}\cap B}(z)))
\]
so \( \partial_{\mathcal{U}\cap B}(ew(fz)) \leq C^{con}(\partial_{\mathcal{U}\cap B}(x), C^{con}(\partial_{\mathcal{U}\cap B}(f), \partial_{\mathcal{U}\cap B}(z))) \). Therefore, we get that \( \mathcal{U} \cap B \in BF(1, 2)IN(S) \). \( \square \)

Example 4.9. Let \( S = \{-10, -20, -30, -40, -50, -60\} \). Then \( (S, *) \) is a semigroup and \( \bar{\delta} \in T^S \) as follows:

<table>
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<td>-30</td>
<td>-40</td>
<td>-50</td>
<td>-60</td>
</tr>
</tbody>
</table>

\[
\bar{\delta}(p) = \begin{cases} 
0.1 & \text{if } p = -10 \\
0.2 & \text{if } p = -20 \\
0.3 & \text{if } p = -30 \\
0.4 & \text{if } p = -40 \\
0.5 & \text{if } p = -50 \\
0.6 & \text{if } p = -60 
\end{cases}
\]
and for all \( u, z \in I \), \( T^{\text{nor}}(u, z) = T^{\text{nor}}_p(u, z) = uz \). Clearly, \( \overline{U} \in F(1,2)IT(S) \). Define \( \overline{U} = (\overline{U}_3, \partial U) \in BF(S) \) and \( B = (\overline{B}_3, \partial B) \in BF(S) \) with

\[
\partial U_3(p) = \begin{cases} 
0.3 & \text{if } p = -10 \\
0.5 & \text{if } p = -20 \\
0.2 & \text{if } p = -30 \\
0.6 & \text{if } p = -40 \\
0.4 & \text{if } p = -50 \\
0.1 & \text{if } p = -60 
\end{cases},
\partial U(p) = \begin{cases} 
0.4 & \text{if } p = -10 \\
0.2 & \text{if } p = -20 \\
0.6 & \text{if } p = -30 \\
0.1 & \text{if } p = -40 \\
0.3 & \text{if } p = -50 \\
0.7 & \text{if } p = -60 
\end{cases},
\]

\( T^{\text{nor}}(u, z) = T^{\text{nor}}_p(u, z) = uz \) and \( C^{\text{con}}(u, z) = C^{\text{con}}_p(u, z) = u + z - uz \), for all \( u, z \in I \). Thus \( \overline{U} = (\overline{U}_3, \partial U) \in BF(1,2)IN(S) \) and \( B = (\overline{B}_3, \partial B) \in BF(1,2)IN(S) \) and \( \overline{U} \cap B \in BF(1,2)IN(S) \).

**Corollary 4.10.** (i) If \( \{ \overline{U}_i \}_{i \in I} \subseteq BF(1,2)IN(S) \), then \( \overline{U} = \bigcap_{i \in I} \overline{U}_i \in BF(1,2)IN(S) \).

(ii) If \( \{ \overline{U}_i \}_{i \in I} \subseteq F(1,2)IT(S) \), then \( \overline{U} = \bigcap_{i \in I} \overline{U}_i \in F(1,2)IT(S) \).

**Theorem 4.11.** Every \( BF BIN(S) \) is a \( BF(1,2)IN(S) \).

**Proof.** Let \( \overline{U} = (\overline{U}_3, \partial U) \in BF BIN(S) \) and \( m, n, z, w \in S \). Since \( \overline{U} = (\overline{U}_3, \partial U) \in FBIT(S) \), we get that \( \overline{U}_3(mn) \geq T^{\text{nor}}(\overline{U}_3(m), \overline{U}_3(n)) \). Also

\[
\overline{U}_3(mw(nz)) = \overline{U}_3((mwn)z) \geq T^{\text{nor}}(\overline{U}_3(mwn), \overline{U}_3(z)) \\
\geq T^{\text{nor}}(T^{\text{nor}}(\overline{U}_3(m), \overline{U}_3(n)), \overline{U}_3(z)) = T^{\text{nor}}(\overline{U}_3(m), T^{\text{nor}}(\overline{U}_3(n), \overline{U}_3(z))),
\]

thus \( \overline{U}_3(mw(nz)) \geq T^{\text{nor}}(\overline{U}_3(m), T^{\text{nor}}(\overline{U}_3(n), \overline{U}_3(z))) \). Moreover \( \overline{U}_3(mn) \leq C^{\text{con}}(\overline{U}_3(m), \overline{U}_3(n)) \) and

\[
\overline{U}_3(mw(nz)) = \overline{U}_3((mwn)z) \leq C^{\text{con}}(\overline{U}_3(mwn), \overline{U}_3(z)) \\
\leq C^{\text{con}}(C^{\text{con}}(\overline{U}_3(m), \overline{U}_3(n)), \overline{U}_3(z)) = C^{\text{con}}(\overline{U}_3(m), C^{\text{con}}(\overline{U}_3(n), \overline{U}_3(z))),
\]

then \( \overline{U}_3(mw(nz)) \leq C^{\text{con}}(\overline{U}_3(m), C^{\text{con}}(\overline{U}_3(n), \overline{U}_3(z))) \). Therefore, we give that \( \overline{U} = (\overline{U}_3, \partial U) \in BF(1,2)IN(S) \).
Theorem 4.12. Let $S$ be a regular semigroup and $T^\text{nor}$ and $C^\text{con}$ be idempotent. Then every $BF(1,2)IN(S)$ is a $BFBIN(S)$.

Proof. Let $\bar{\varnothing} = (\bar{\mathfrak{o}}, \bar{\mathfrak{m}}) \in BF(1,2)IN(S)$ and $c, d, w, s \in S$. Because $S$ is a regular semigroup for all $c \in S$, there exists $s \in S$ so $c = \text{csc}$. Then $cw \in (cSc)S \subseteq cSc$ and $cw = \text{csc}$. As $\bar{\varnothing} = (\bar{\mathfrak{o}}, \bar{\mathfrak{m}}) \in BF(1,2)IN(S)$, then $\bar{\mathfrak{o}}_\Theta(cd) \geq T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d))$. Also

$$\bar{\mathfrak{o}}_\Theta(cwd) = \bar{\mathfrak{o}}_\Theta(csxd) \geq T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d)))$$

$$= T^\text{nor}(T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(c)), \bar{\mathfrak{m}}_\Theta(d)) = T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d))$$

and so $\bar{\mathfrak{o}}_\Theta(cwd) \geq T^\text{nor}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d))$. Also $\bar{\mathfrak{o}}_\Theta(cd) \leq C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d))$. In addition,

$$\bar{\mathfrak{o}}_\Theta(cwd) = \bar{\mathfrak{o}}_\Theta(cs(cd))$$

$$\leq C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d)))$$

$$= C^\text{con}(C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(c)), \bar{\mathfrak{m}}_\Theta(d)) = C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d)),$$

and so $\bar{\mathfrak{o}}_\Theta(cwd) \leq C^\text{con}(\bar{\mathfrak{o}}_\Theta(c), \bar{\mathfrak{m}}_\Theta(d))$. Hence, $\bar{\varnothing} = (\bar{\mathfrak{o}}, \bar{\mathfrak{m}}) \in BFBIN(S)$. \qed

5. Homomorphisms on $F(1,2)IT(S)$ and $BF(1,2)IN(S)$.

In this section, we apply the concept of homomorphism over $FIT(S)$, $FBIT(S)$, $BFISN(S)$, $BFIN(S)$, $BFBIN(S)$, $F(1,2)IT(S)$, $BF(1,2)IN(S)$ and extend the bifuzzy bi-ideal on semirings. Throughout this section we let that $\hat{S} = (\hat{S}, \hat{\varnothing})$ be a semigroup.

Theorem 5.1. Assume $\bar{\Theta} \in FIT(S)$ and $\bar{\varnothing} \in FIT(\hat{S})$ and $\varnothing : S \rightarrow \hat{S}$ be an onto homomorphism. Hence

(1) $\varnothing(\bar{\Theta}) \in FIT(\hat{S})$,

(2) $\varnothing^{-1}(\bar{\varnothing}) \in FIT(S)$.

Proof. (1) Let $u, v \in \hat{S}$ and $b, d \in S$ so $u = \varnothing(b)$ and $v = \varnothing(d)$. Then

$$\varnothing(\bar{\Theta}(uv)) = \sqrt{\{\bar{\Theta}(bd) \mid u = \varnothing(b), v = \varnothing(d)\} \geq \sqrt{\{T^\text{nor}(\bar{\Theta}(b), \bar{\Theta}(d)) \mid u = \varnothing(b), v = \varnothing(d)\}}$$

$$= T^\text{nor}(\sqrt{\{\bar{\Theta}(b) \mid u = \varnothing(b), v = \varnothing(d)\}}, \sqrt{\{\bar{\Theta}(d) \mid u = \varnothing(d)\}}) = T^\text{nor}(\varnothing(\bar{\Theta}(u)), \varnothing(\bar{\Theta}(v)))$$

and

$$\varnothing(\bar{\Theta}(uv)) = \sqrt{\{\bar{\Theta}(bd) \mid u = \varnothing(b), v = \varnothing(d)\} \geq \sqrt{\{\bar{\Theta}(b) \mid u = \varnothing(b)\}} = \varnothing(\bar{\Theta}(u))$$

and

$$\varnothing(\bar{\Theta}(uv)) = \sqrt{\{\bar{\Theta}(bd) \mid u = \varnothing(b), v = \varnothing(d)\} \geq \sqrt{\{\bar{\Theta}(d) \mid v = \varnothing(d)\}} = \varnothing(\bar{\Theta}(v)).$$

Then, $\varnothing(\bar{\Theta}) \in FIT(\hat{S})$.

(2) assume $b, d \in S$. Therefore

$$\varnothing^{-1}(\bar{\Theta})(bd) = \bar{\Theta}(\varnothing(bd)) = \bar{\Theta}(\varnothing(b)\varnothing(d)) \geq T^\text{nor}(\bar{\Theta}(\varnothing(b)), \bar{\Theta}(\varnothing(d))) = T^\text{nor}(\varnothing^{-1}(\bar{\Theta})(b), \varnothing^{-1}(\bar{\Theta})(d))$$
Theorem 5.2. Let $\partial \in FBIT(S)$ and $\partial \in FBIT(\hat{S})$ and $\varrho : S \to \hat{S}$ be an onto homomorphism. Then $\varrho(\partial) \in FBIT(\hat{S})$ and $\varrho^{-1}(\partial) \in FBIT(S)$.

Proof. Suppose $u, v, z \in \hat{S}$ that $f, g, w \in S$ so $u = \varrho(f)$ and $v = \varrho(g)$ and $z = \varrho(w)$. Then

$$\varrho(\partial)(uv) = \bigvee \{\partial(fg) | u = \varrho(f), v = \varrho(g)\}$$

$$\geq \bigvee \{T_{nor}(\partial(f), \partial(g)) | u = \varrho(f), v = \varrho(g)\}$$

$$= T_{nor}(\bigvee \{\partial(f) | u = \varrho(f)\}, \bigvee \{\partial(g) | v = \varrho(g)\})$$

$$= T_{nor}(\varrho(\partial)(u), \varrho(\partial)(v))$$

Then, $\varrho(\partial) \in FBIT(\hat{S})$. Assume $f, g, w \in S$. Since

$$\varrho^{-1}(\partial)(fg) = \partial(\varrho(fg)) = \partial(\varrho(f)\varrho(g))$$

$$\geq T_{nor}(\partial(\varrho(f)), \partial(\varrho(g))) = T_{nor}(\varrho^{-1}(\partial)(f), \varrho^{-1}(\partial)(g))$$

we get $\varrho^{-1}(\partial) \in FBIT(S)$. \qed

Theorem 5.3. Let $\partial = (\partial_1, \partial_3) \in BFSN(S)$ and $B = (\partial_B, \partial_B) \in BFSN(\hat{S})$ and $\varrho : S \to \hat{S}$ be a homomorphism. Then $\varrho(\partial) \in BFSN(\hat{S})$ and $\varrho^{-1}(B) \in BFSN(S)$.  

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Theorem 5.4. Let $\mathcal{U} = (\mathcal{H}, \mathcal{V}) \in BFIN(S)$ and $B = (\mathcal{B}, \mathcal{B}) \in BFIN(\mathcal{H})$ and $\varphi : S \to \mathcal{H}$ be a homomorphism. Then, $\varphi(\mathcal{U}) \in BFIN(\mathcal{H})$ and $\varphi^{-1}(B) \in BFIN(S)$. 

Proof. Let $u, v \in \mathcal{H}$ and $x, y \in S$ so $u = \varphi(x)$ and $v = \varphi(y)$. Thus

$\varphi(\partial_{\mathcal{U}}(uv)) = \bigvee \{\partial_{\mathcal{U}}(xy) : u = \varphi(x), v = \varphi(y)\}$

$\geq \bigvee \{T^{nor}(\partial_{\mathcal{U}}(x), \partial_{\mathcal{U}}(y)) : u = \varphi(x), v = \varphi(y)\}$

$= T^{nor}(\bigvee \{\partial_{\mathcal{U}}(x) : u = \varphi(x)\}, \bigvee \{\partial_{\mathcal{U}}(y) : v = \varphi(y)\}) = T^{nor}(\varphi(\mathcal{U}(x)), \varphi(\mathcal{U}(y))) = T^{nor}(\varphi(\partial_{\mathcal{U}}(x)), \varphi(\partial_{\mathcal{U}}(y)))$.

Also $\varphi(\partial_{\mathcal{U}}(uv)) = \bigvee \{\partial_{\mathcal{U}}(xy) : u = \varphi(x), v = \varphi(y)\} \geq \bigvee \{\partial_{\mathcal{U}}(x) : u = \varphi(x)\} = \varphi(\mathcal{U}(x)) = \varphi(\partial_{\mathcal{U}}(x))$ and $\varphi(\partial_{\mathcal{U}}(uv)) = \bigvee \{\partial_{\mathcal{U}}(xy) : u = \varphi(x), v = \varphi(y)\} \geq \bigvee \{\partial_{\mathcal{U}}(y) : v = \varphi(y)\} = \varphi(\partial_{\mathcal{U}}(y))$. Moreover

$\varphi(\partial_{\mathcal{U}}(uv)) = \bigwedge \{\partial_{\mathcal{U}}(xy) : u = \varphi(x), v = \varphi(y)\}$

$\leq \bigwedge \{C^{con}(\partial_{\mathcal{U}}(x), \partial_{\mathcal{U}}(y)) : u = \varphi(x), v = \varphi(y)\}$

$= C^{con}(\bigwedge \{\partial_{\mathcal{U}}(x) : u = \varphi(x)\}, \bigwedge \{\partial_{\mathcal{U}}(y) : v = \varphi(y)\}) = C^{con}(\varphi(\partial_{\mathcal{U}}(x)), \varphi(\partial_{\mathcal{U}}(y)))$. 

Thus, $\varphi^{-1}(B) = (\varphi^{-1}(\partial_{\mathcal{B}}), \varphi^{-1}(\partial_{\mathcal{B}})) \in BFIN(S)$. □
Also
\[ \varrho(\partial_3)(uv) = \bigwedge \{ \partial_3(xy) : u = \varrho(x), v = \varrho(y) \} \]
\[ \leq \bigwedge \{ \partial_3(x) : u = \varrho(x) \} = \varrho(\partial_3)(u) \]
\[ \varrho(\partial_3)(uv) = \bigwedge \{ \partial_3(xy) | u = \varrho(x), v = \varrho(y) \} \]
\[ \leq \bigwedge \{ \partial_3(y) : v = \varrho(y) \} = \varrho(\partial_3)(v). \]

Thus \( \varrho(\mathcal{U}) = (\varrho(\partial_3), \varrho(\partial_3)) \in \text{BFIN}(\hat{S}) \). Let \( x, y \in S \). Then
\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \]
\[ \geq T^{\text{nor}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) = T^{\text{nor}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)), \]
\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \geq \partial_B(\varrho(x)) = \varrho^{-1}(\partial_B)(x) \]
\[ \text{and} \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \geq \partial_B(\varrho(y)) = \varrho^{-1}(\partial_B)(y). \]

Also
\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \leq C^{\text{con}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) \]
\[ = C^{\text{con}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)), \]
\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \leq \partial_B(\varrho(x)) = \varrho^{-1}(\partial_B)(x) \] and
\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(x)) = \partial_B(\varrho(x) \varrho(y)) \leq \partial_B(\varrho(y)) = \varrho^{-1}(\partial_B)(y). \]

Therefore, \( \varrho^{-1}(B) = (\varrho^{-1}(\partial_B), \varrho^{-1}(\partial_B)) \in \text{BFIN}(S) \). \( \square \)

**Theorem 5.5.** Let \( \mathcal{U} = (\partial_3, \partial_3) \in \text{BFBIN}(S) \), \( B = (\partial_B, \partial_B) \in \text{BFBIN}(\hat{S}) \) and \( \varrho : S \to \hat{S} \) be a homomorphism. Then \( \varrho(\mathcal{U}) \in \text{BFBIN}(\hat{S}) \) and \( \varrho^{-1}(B) \in \text{BFBIN}(S) \).

**Proof.** Let \( u, v, z \in \hat{S} \) and \( x, y, w \in S \) so \( u = \varrho(x) \) and \( v = \varrho(y) \) and \( z = \varrho(w) \). Then
\[ \varrho(\partial_3)(uvz) = \bigvee \{ \partial_3(xwy) | u = \varrho(x), v = \varrho(y), z = \varrho(w) \} \]
\[ \geq \bigvee \{ T^{\text{nor}}(\partial_3(x), \partial_3(y)) | u = \varrho(x), v = \varrho(y) \} \]
\[ = T^{\text{nor}}(\bigvee \{ \partial_3(x) | u = f(x) \}, \bigvee \{ \partial_3(y) | v = \varrho(y) \}) = T^{\text{nor}}(\varrho(\partial_3)(u), \varrho(\partial_3)(v)) \] and
\[ \varrho(\partial_3)(uvz) = \bigvee \{ \partial_3(xwy) | u = \varrho(x), v = \varrho(y), z = \varrho(w) \} \]
\[ \geq \bigvee \{ T^{\text{nor}}(\partial_3(x), \partial_3(y)) | u = \varrho(x), v = \varrho(y) \} \]
\[ = T^{\text{nor}}(\bigvee \{ \partial_3(x) | u = f(x) \}, \bigvee \{ \partial_3(y) | v = \varrho(y) \}) = T^{\text{nor}}(\varrho(\partial_3)(u), \varrho(\partial_3)(v)). \]
Also

\[ \varrho(\partial_3)(uv) = \bigwedge \{ \partial_3(xy) \mid u = \varrho(x), v = \varrho(y) \} \]

\[ \leq \bigwedge \{ C^{\text{con}}(\partial_3(x), \partial_3(y)) \mid u = \varrho(x), v = \varrho(y) \} \]

\[ = C^{\text{con}}\left( \bigwedge \{ \partial_3(x) \mid u = f(x) \}, \bigwedge \{ \partial_3(y) \mid v = \varrho(y) \} \right) = C^{\text{con}}(\varrho(\partial_3)(u), \varrho(\partial_3)(v)) \quad \text{and} \]

\[ \varrho(\partial_3)(uzv) = \bigwedge \{ \partial_3(xwv) \mid u = \varrho(x), v = \varrho(y), z = \varrho(w) \} \]

\[ \leq \bigwedge \{ C^{\text{con}}(\partial_3(x), \partial_3(y)) \mid u = \varrho(x), v = \varrho(y) \} \]

\[ = C^{\text{con}}\left( \bigwedge \{ \partial_3(x) \mid u = f(x) \}, \bigwedge \{ \partial_3(y) \mid v = \varrho(y) \} \right) = C^{\text{con}}(\varrho(\partial_3)(u), \varrho(\partial_3)(v)). \]

Thus \( \varrho(\mathcal{U}) = (\varrho(\partial_3), \varrho(\partial_3)) \in BFBIN(\hat{S}) \). Let \( x, y, w \in S \). So

\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(xy)) = \partial_B(\varrho(x)\varrho(y)) \geq T^{\text{nor}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) \]

\[ = T^{\text{nor}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)) \quad \text{and} \]

\[ \varrho^{-1}(\partial_B)(xwy) = \partial_B(\varrho(xwy)) = \partial_B(\varrho(x)\varrho(w)\varrho(y)) \]

\[ \geq T^{\text{nor}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) = T^{\text{nor}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)). \]

Also

\[ \varrho^{-1}(\partial_B)(xy) = \partial_B(\varrho(xy)) = \partial_B(\varrho(x)\varrho(y)) \]

\[ \leq C^{\text{con}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) = C^{\text{con}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)) \quad \text{and} \]

\[ \varrho^{-1}(\partial_B)(xwy) = \partial_B(\varrho(xwy)) = \partial_B(\varrho(x)\varrho(w)\varrho(y)) \]

\[ \leq C^{\text{con}}(\partial_B(\varrho(x)), \partial_B(\varrho(y))) = C^{\text{con}}(\varrho^{-1}(\partial_B)(x), \varrho^{-1}(\partial_B)(y)). \]

Then, \( \varrho^{-1}(B) = (\varrho^{-1}(\partial_B), \varrho^{-1}(\partial_B)) \in BFBIN(S) \).

**Theorem 5.6.** Let \( \partial \in F(1, 2)IT(S) \), \( \vartheta \in F(1, 2)IT(\hat{S}) \) and \( \varrho : S \rightarrow \hat{S} \) be a homomorphism. Then, \( \varrho(\partial) \in F(1, 2)IT \) and \( \varrho^{-1}(\vartheta) \in F(1, 2)IT(S) \).

**Proof.** Let \( m, n, p, q \in \hat{S} \) and \( x, w, y, z \in S \) so \( m = \varrho(x) \) and \( n = \varrho(w) \) and \( p = \varrho(y) \) and \( q = \varrho(z) \). Now

\[ \varrho(\partial)(mn) = \bigvee \{ \partial(xw) \mid m = \varrho(x), n = \varrho(y) \} \]

\[ \geq \bigvee \{ T^{\text{nor}}(\partial(x), \partial(w)) \mid m = \varrho(x), n = \varrho(y) \} \]

\[ = T^{\text{nor}}(\bigvee \{ \partial(x) \mid u = \varrho(x) \}, \bigvee \{ \partial(y) \mid v = \varrho(y) \}) = T^{\text{nor}}(\varrho(\partial)(m), \varrho(\partial)(n)) \quad \text{and} \]

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Theorem 5.7. Let $\varrho(\emptyset) = (\varrho(1), \varrho(2))$. Let

$$
\varrho(\emptyset)(m, n, p, q) = \sqrt{\{x(w(yz)) \mid m = \varrho(x), n = \varrho(w), p = \varrho(y), q = \varrho(z)\}
\geq \sqrt{\{\varrho(x), T^\varrho(\varrho(y), \varrho(z)) \mid m = \varrho(x), p = \varrho(y), q = \varrho(z)\}
= T^\varrho(\sqrt{\{\varrho(x) \mid m = f(x)\}, \varrho(y), \varrho(z) \mid p = \varrho(y), q = \varrho(z)\})
= T^\varrho(\sqrt{\{\varrho(x) \mid f(x)\}, \varrho(y) \varrho(z) \mid p = \varrho(y), q = \varrho(z)\})
= T^\varrho(\varrho(\emptyset)(m), T^\varrho(\varrho(\emptyset)(p), \varrho(\emptyset)(q))).
$$

Then $\varrho(\emptyset) \in F(1, 2)IT(\hat{S})$. Let $x, w, y, z \in S$. Then

$$
\varrho^{-1}(\emptyset)(x, w, y, z, S) = \emptyset(\varrho(x)) \emptyset(\varrho(y)) \emptyset(\varrho(z)) \emptyset(\varrho(\emptyset)(x), \varrho(\emptyset)(y), \varrho(\emptyset)(z))
$$
and

$$
\varrho^{-1}(\emptyset)(xwyz) = \emptyset(\varrho(xwyz))
= \emptyset(\varrho(x) \emptyset(w) \emptyset(y) \emptyset(z)) = \emptyset(\varrho(x) \emptyset(w) \emptyset(y) \emptyset(z))
\geq T^\varrho(\emptyset(\varrho(x)), T^\varrho(\emptyset(\varrho(y)), \varrho(\emptyset(x), \varrho(\emptyset(y), \varrho(\emptyset(z))))
= T^\varrho(\varrho^{-1}(\emptyset)(x), \varrho^{-1}(\emptyset)(y), \varrho^{-1}(\emptyset)(z))).
$$
Thus, $\varrho^{-1}(\emptyset) \in F(1, 2)IT(S)$. □

**Theorem 5.7.** Let $\mathcal{U} = (\varrho(1), \varrho(2)) \in BF(1, 2)IN(S)$ and $B = (\varrho(1), \varrho(2)) \in BF(1, 2)IN(\hat{S})$. Suppose that $\varrho : S \to \hat{S}$ be a homomorphism. Then

1. $\varrho(\mathcal{U}) \in BF(1, 2)IN(\hat{S})$.
2. $\varrho^{-1}(B) \in BF(1, 2)IN(S)$.

**Proof.** (1) Let $m, n, p, q \in S$ and $x, w, y, z \in S$ so $m = \varrho(x)$ and $n = \varrho(w)$ and $p = \varrho(y)$ and $q = \varrho(z)$. Hence

$$
\varrho(\emptyset)(\emptyset)(mn) = \sqrt{\{\varrho(\emptyset)(xw) \mid m = \varrho(x), n = \varrho(y)\}
\geq \sqrt{\{\varrho(\emptyset)(x), T^\varrho(\varrho(\emptyset)(w)) \mid m = \varrho(x), n = \varrho(y)\}
= T^\varrho(\sqrt{\{\varrho(\emptyset)(x) \mid u = f(x)\}, \varrho(w) \varrho(y) \varrho(z) \mid p = \varrho(y), q = \varrho(z)\})
$$
and

$$
\varrho(\emptyset)(\emptyset)(mn(pq)) = \sqrt{\{\varrho(\emptyset)(xwyz) \mid m = \varrho(x), n = \varrho(w), p = \varrho(y), q = \varrho(z)\}
\geq \sqrt{\{T^\varrho(\varrho(\emptyset)(x), T^\varrho(\varrho(\emptyset)(y), \varrho(\emptyset)(z)) \mid m = \varrho(x), p = \varrho(y), q = \varrho(z)\}
= T^\varrho(\sqrt{\{\varrho(\emptyset)(x) \mid m = f(x)\}, \varrho(y) \varrho(z) \mid p = \varrho(y), q = \varrho(z)\})
= T^\varrho(\sqrt{\{\varrho(\emptyset)(x) \mid m = f(x)\}, \varrho(y) \varrho(z) \mid p = \varrho(y), q = \varrho(z)\})
= T^\varrho(\varrho(\emptyset)(m), T^\varrho(\varrho(\emptyset)(p), \varrho(\emptyset)(q))).
$$
Also

$$\varrho(\partial_3)(mn) = \bigwedge \{ \partial_3(xw) | m = \varrho(x), n = \varrho(y) \}$$

$$\leq \bigwedge \{ C^\text{con}(\partial_3(x), \partial_3(w)) | m = \varrho(x), n = \varrho(y) \}$$

$$= C^\text{con}(\bigwedge \{ \partial_3(x) | u = f(x) \}, \bigwedge \{ \partial_3(y) | v = \varrho(y) \}) = C^\text{con}(\varrho(\partial_3(m), \varrho(\partial_3(n)))$$

and

$$\varrho(\partial_3)(mnpq) = \bigwedge \{ \partial_3(xwyz) | m = \varrho(x), n = \varrho(w), p = \varrho(y), q = \varrho(z) \}$$

$$\leq \bigwedge \{ C^\text{con}(\partial_3(x), C^\text{con}(\partial_3(y), \partial_3(z))) | m = \varrho(x), p = \varrho(y), q = \varrho(z) \}$$

$$= C^\text{con}(\bigwedge \{ \partial_3(x) | m = f(x) \}, C^\text{con}(\partial_3(y), \partial_3(z)) | p = \varrho(y), q = \varrho(z) \})$$

$$= C^\text{con}(\bigwedge \{ \partial_3(x) | m = f(x) \}, C^\text{con}(\bigwedge \{ \partial_3(y) | p = \varrho(y) \}, \bigwedge \{ \partial_3(z) | q = \varrho(z) \})$$

$$= C^\text{con}(\varrho(\partial_3(m), C^\text{con}(\varrho(\partial_3(p), \varrho(\partial_3(q)))).$$

Then, \( \varrho(\tilde{S}) = (\varrho(\tilde{B}_3), \varrho(\tilde{\partial}_3)) \in BF(1, 2)IN(\tilde{S}). \) Let \( x, w, y, z \in S. \) Then

$$\varrho^{-1}(\tilde{B}_3)(xy) = \tilde{B}_3(\varrho(xy)) = \tilde{B}_3(\varrho(x)\varrho(y))$$

$$\geq T^\text{nor}(\tilde{B}_3(\varrho(x)), \tilde{B}_3(\varrho(y))) = T^\text{nor}(\varrho^{-1}(\tilde{B}_3)(x), \varrho^{-1}(\tilde{B}_3)(y))$$

and

$$\varrho^{-1}(\tilde{B}_3)(xwyz) = \tilde{B}_3(\varrho(xwyz)) = \tilde{B}_3(\varrho(x)\varrho(w)\varrho(y)\varrho(z)) = \tilde{B}_3(\varrho(x)\varrho(w)(\varrho(y)\varrho(z)))$$

$$\geq T^\text{nor}(\tilde{B}_3(\varrho(x)), T^\text{nor}(\tilde{B}_3(\varrho(y)), \varrho(z))))$$

$$= T^\text{nor}(\varrho^{-1}(\tilde{B}_3)(x), T^\text{nor}(\varrho^{-1}(\tilde{B}_3)(y), \varrho^{-1}(\tilde{B}_3)(z))).$$

Also

$$\varrho^{-1}(\tilde{B}_3)(xy) = \tilde{B}_3(\varrho(xy)) = \tilde{B}_3(\varrho(x)\varrho(y))$$

$$\leq C^\text{con}(\tilde{B}_3(\varrho(x)), \tilde{B}_3(\varrho(y))) = C^\text{con}(\varrho^{-1}(\tilde{B}_3)(x), \varrho^{-1}(\tilde{B}_3)(y))$$

and

$$\varrho^{-1}(\tilde{B}_3)(xwyz) = \tilde{B}_3(\varrho(xwyz)) = \tilde{B}_3(\varrho(x)\varrho(w)(\varrho(y)\varrho(z)))$$

$$= \tilde{B}_3(\varrho(x)\varrho(w)(\varrho(y)\varrho(z))) \leq C^\text{con}(\tilde{B}_3(\varrho(x)), C^\text{con}(\tilde{B}_3(\varrho(y)), \varrho(z))))$$

$$= C^\text{con}(\varrho^{-1}(\tilde{B}_3)(x), C^\text{con}(\varrho^{-1}(\tilde{B}_3)(y), \varrho^{-1}(\tilde{B}_3)(z))).$$

Therefore, \( \varrho^{-1}(B) = (\varrho^{-1}(\tilde{B}_3), \varrho^{-1}(\tilde{\partial}_3)) \in BF(1, 2)IN(S). \) \( \square \)
6. Applications, discussion and conclusions

In this study, as using the notions of triangular norms and triangular conorms, the fuzzy semigroups, fuzzy ideals, fuzzy bi-ideals, bifuzzy subsemigroups, bifuzzy ideals, bifuzzy bi-ideals, fuzzy (1, 2)-ideals and bifuzzy (1, 2)-ideals in any given semigroup will be defined and investigated and obtained some basic properties of them. Now one can study fuzzy semirings, fuzzy ideals, fuzzy bi-ideals, bifuzzy subsemirings, bifuzzy ideals, bifuzzy bi-ideals, fuzzy (1, 2)-ideals and bifuzzy (1, 2)-ideals in any given semiring and this can be an open problem for future research directions.

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