Neutrosophic Systems with Applications, Vol. 20, 2024
https://doi.org/10.61356/j.nswa.2024.20347

Cubic Soft Ideals on B-algebra for Solving Complex Problems: Trend Analysis, Proofs, Improvements, and Applications

Muhammad Saeed [1,*], Hafiz Inam ul Haq [1] and Mubashir Ali [1]

[1] Department of Mathematics, University of Management and Technology Lahore, 54770 Pakistan;
Emails: muhammad.saeed@umt.edu.pk; f2021109017@umt.edu.pk; v31553@umt.edu.pk

* Correspondence: muhammad.saeed@umt.edu.pk.

Abstract: In this paper, we introduce the concepts of cubic soft (CS) algebra, CS o-subalgebra, and CS ideals within the framework of B-algebra. We provide comprehensive characterizations of these new structures, elucidating their unique properties and interrelationships. Specifically, we present detailed conditions under which a CS subalgebra can be classified as a closed CS ideal. Our analysis explores the intricate relationships among closed cubic soft ideals, cubic soft subalgebras, and cubic soft o-subalgebras. By doing so, we aim to provide a deeper understanding of how these structures interact and coexist within the broader context of B-algebra The findings offer significant insights into the application and theoretical underpinnings of cubic soft sets in algebraic systems, contributing to the ongoing evolution of fuzzy set theory and its applications in various mathematical domains. Our work not only broadens the scope of B-algebra but also enhances its utility in solving complex problems where traditional algebraic approaches may fall short. Through this exploration, we seek to advance the field and open new avenues for research and practical applications in mathematical sciences.

Keywords: Soft Set, Fuzzy Set, Fuzzy Soft Set, Cubic Soft Subalgebra, Cubic Soft O-subalgebra, Cubic Soft Ideals and B-algebra.

1. Introduction

Lotfi Zadeh pioneered the notion of fuzzy sets (FS) in 1965 [1]. This pioneering innovation was quickly followed by his invention of interval-valued fuzzy sets (IVFS) [2], which broadened the scope and applications of fuzzy logic in a variety of disciplines. The early notions proposed by Zadeh sparked much study and development in the sector, resulting in several advances and modifications of his original ideas. Such as the Assessment of solid waste management strategies using an efficient complex fuzzy hypersoft algorithm based on entropy and similarity measures [3]. Following Zadeh's fundamental work, Jun et al. proposed the concept of the cubic set [4], which marked a significant advancement in the study of fuzzy sets. The cubic set idea combines the principles of FS and IVFS [1,2], resulting in a more robust and extensible framework for dealing with uncertainty and imprecision. Jun et al.'s later work expanded the concept of cubic sets to algebraic structures [5], with a special emphasis on cubic subgroups. This was a revolutionary application of cubic sets to group theory, providing a fresh perspective and tools for examining and comprehending group structures. Here theoretical framework for a decision support system for micro-enterprise supermarket investment risk assessment [6].BCK/BCI-algebras were first created by Y. Imai and K. Iseki in 1966 [7]. These algebras were primarily concerned with propositional calculus and served as the basis for several later algebraic systems [8]. Several generalizations and expansions of BCK/BCI-algebras have evolved throughout time, including B-algebra, G-algebra, BG-algebra, d-algebra, ku-algebra, and Ps-algebra. Each of these structures provided fresh insights and applications, broadening the scope of algebraic logic. Neggers and Kim introduced the basic notion of B-algebra [9], and also investigated some of its essential features. Park and Kim later established the idea of quadratic B-algebras [10], which broadened the concept of B-algebra. M Saeed introduced the theoretical development of cubic Pythagorean FSs with its application in MADM [11]. Borumand Saeed's work resulted in the development of fuzzy topological B-algebras, as well as the concept of interval-valued fuzzy (IVF) B-algebra subalgebras. Saeed's contributions included investigating several characterizations of these subalgebras, which improved our knowledge of their structure and behavior.

Walendziak's further contributions included sets of axioms defining B-algebras and establishing their independence [12]. Senapati and colleagues made substantial contributions to the study of fuzzy substructures within B-algebras. They defined and investigated fuzzy dot subalgebras, fuzzy dot ideals, interval-valued fuzzy near ideals, and fuzzy subalgebras about t-norms [13]. This research revealed novel features and prospective applications for fuzzy algebraic structures, advancing our grasp of their theoretical basis and practical relevance. Molodtsov proposed the notion of soft sets in 1999 [14], giving researchers a new mathematical tool for dealing with ambiguity and vagueness. Muhiuddin and Al-Roky expanded on this approach by merging cubic and soft-set concepts [15]. Their research entailed a thorough analysis of the characteristics and applications of cubic soft sets, particularly with B-algebras [16]. Exploration of Subalgebras Authors such as Senapati and Iqbal investigated cubic soft subalgebras and cubic soft ideals, studying their characterizations and linkages within the context of B-algebra [17]. Their findings shed light on the conditions under which cubic soft subalgebras [18] might be classified as closed cubic soft ideals [19]. In this study, we define cubic soft subalgebra, cubic soft o-subalgebra, and cubic soft ideals in the setting of B-algebra. We present thorough descriptions of these structures and investigate their interrelationships. In addition, we discuss many criteria under which a cubic soft subalgebra can be characterized as a closed cubic soft ideal. The links between closed cubic soft ideals, cubic soft subalgebras, and cubic soft o-subalgebras are carefully examined, offering a clear grasp of their roles and interactions within the larger framework of B-algebra. This study seeks to further the theoretical underpinning and practical applications of fuzzy and soft set theories in algebraic structures, contributing to the continuous growth and refinement of these fundamental mathematical tools.

Table 1. List of abbreviations.

Sr. No	Abbreviations	Meaning
1	Ss	Soft set
2	Fs	Fuzzy set
3	FSs	Fuzzy soft set
4	CSSA	Cubic Soft Sub-Algebra
5	CSoSA	Cubic Soft o-Subalgebra
6	CSI	Cubic Soft Ideals
7	BA	B-Algebra
8	FSA	Fuzzy Subalgebra
9	FTBA	Fuzzy Topological B-Algebra
10	FdSA	Fuzzy dot Subalgebra

2. Preliminaries

Some basic facts are explained in this section, which are essential to the above article. B-algebra is an important branch of logical algebra that has been expanded by much research. We can define B-algebra as follows:

An algebra $(X,*,0)$ of type $(2,0)$ is known as a B-algebra if it satisfies the following results:

$$\mathrm{B}_1.\, x_1 * x_1 = 0$$

$$\mathrm{B}_2.\, x_1 * 0 = x_1$$

$$\mathrm{B}_3.\, (x_1 * x_2) * x_3 = x_1 * \left(x_3 * (0 * x_2)\right)$$

now for all $x_1, x_2 \in X$.

A partial ordering $\leq$ is defined by $x_1 \leq x_2 \leftrightarrow x_1 * x_2 = 0$.
A B-algebra X which shows to be (S) if, for all $x_1, x_2 \in X$, the set $\{x_3 \in X \mid x_3 * x_1 \leq x_2\}$ has the greatest element, written $x_1^o x_2$.
Example 1. Let $X = \mathbb{R} - \mathbb{Z}^-$. An operation on X is defined as:

$$x_1 * x_2 = \frac{n(x_1 - x_2)}{n + x_2}$$

After that, $(X, *, 0)$ is a B-algebra.
Consider a set S which is a non-empty subset of X and where X is a B-algebra. If the relation $x_1 * x_2 \in S$ for all $x_1, x_2 \in S$, then S is called a SA of X.

Assume I is a subset of X s,t it is non-empty and it is known as ideal if it fulfills two axioms for any $x_1, x_2 \in X$:

i). $0 \in I$, and
ii). $x_1 * x_2 \in I$ and $x_2 \in I \Longrightarrow x_1 \in I$.

This ideal I is known as a closed ideal of B-algebra [20] X if it fulfills a third condition:

$$0 * x_2 \in I \text{ for all } x_1 \in I$$

Here we have some F-logical concepts:
Let X be the group of items commonly represented by x_1. Then FS A in X is defined as

$$A = \{\langle x_1, \mu_A(x_1)\rangle \mid x_1 \in X\}$$

where $\mu_A(x_1)$ is known as the membership value of x_1 in A and $\mu_A(x_1) \in [0,1]$.
An Interval-Valued Fuzzy Set (IVFS) [21], A over X, is an object having the form

$$A = \{\langle x_1, \mu_A(x_1)\rangle \mid x_1 \in X\}$$

where $\mu_A: X \to \mathcal{A}([0,1])$ is the set of whole subintervals of $[0,1]$. The interlude $\mu_A(x_1) = [\mu_A^-(x_1), \mu_A^+(x_1)]$ for all $x_1 \in X$ indicates the value of membership elements x_1 to the set A.

Also, $\mu_A^c(x_1) = [1 - \mu_A^-(x_1), 1 - \mu_A^+(x_1)]$ which shows the complement of μ_A.

Definition 1. Consider two elements $\tilde{b}_1, \tilde{b}_2 \in \tilde{B}[0,1]$. If $\tilde{b}_1 = [b_1^-, b_1^+]$and $\tilde{b}_2 = [b_2^-, b_2^+]$, then

$$\max(\tilde{b}_1, \tilde{b}_2) = [\max(b_1^-, b_2^-), \max(b_1^+, b_2^+)]$$

which is denoted by $\tilde{b}_1 \vee^r \tilde{b}_2$, and

$$\min(\tilde{b}_1, \tilde{b}_2) = [\min(b_1^-, b_2^-), \min(b_1^+, b_2^+)]$$

which is denoted by $\tilde{b}_1 \wedge^r \tilde{b}_2$.
Thus, if $\tilde{b}_i = [b_i^-, b_i^+]$belongs to $\tilde{B}[0,1]$ for $i = 1,2,3, \dots$, then we define

$$r - sup_i(\tilde{b}_i) = \left[\sup_i(b_i^-), \sup_i(b_i^+)\right]$$

i.e.,

$$\vee_i^r \tilde{b}_i = [\vee_i b_i^-, \vee_i b_i^+]$$

Similarly, we define

$$r - inf_i(\tilde{b}_i) = \left[\inf_i(b_i^-), \inf_i(b_i^+)\right]$$

i.e.,

$$\wedge_i^r \tilde{b}_i = [\wedge_i b_i^-, \wedge_i b_i^+]$$

Now we call $\tilde{b}_1 \geq \tilde{b}_2 \leftrightarrow b_1^- \geq b_2^-$and $b_1^+ \geq b_2^+$. Here we have some relations $\tilde{b}_1 \leq \tilde{b}_2$ and $\tilde{b}_1 = \tilde{b}_2$ are defined.

Definition 2. By taking U as an initial universe set, a soft set(Ss) [22] over U is a pair (F, A) such that F is a mapping which is given by

$$F: A \to P(U)$$

where A is a subset of E (the set of parameters). Also, a S s is a parametrized members of subsets of the initial universe U. For $\epsilon \in A, F(\epsilon)$ is also assume that the set of ϵ-estimated attributes of the $Ss(F, A)$.

Definition 3. By taking U as an begining universel set, a cubic set in U means a structure

$$\wp = \{\{x_1, \tilde{\mu}(x_1), v(x_1)\} \mid x_1 \in U\}$$

where $\tilde{\mu}$ (interval valued) and v are fuzzy sets in U respectively. For the sake of simplicity, a cubic set

$$\wp = \{\{x_1, \tilde{\mu}(x_1), v(x_1)\} \mid x_1 \in U\}$$

is simply denoted by

$$\wp = \{\tilde{\mu}_A, v_A\}$$

and $\mathcal{C}^U$ represents the family of all cubic sets in U.

Definition 4. By taking U as an beginning universal set, a soft cubic set [23] is a pair (ξ, A) over U such that ξ is a mapping which is given by

$$\xi: A \to \mathcal{C}^U$$

where A is a subset of E. Pay attention to the pair (ξ, A), which can also be denoted as:

$$(\xi, A) = \{(\xi(\epsilon), \epsilon) \mid \epsilon \in A\}$$

where $\xi(\epsilon) = \{\tilde{\mu}_\xi(\epsilon), v_\xi(\epsilon)\}$.

3. Cubic Soft Ideals

We take an initial universe set U. Here, from now onwards, U will be taken as B-algebra unless otherwise stated.

Definition 5. Let (ξ, A) be a cubic soft (CS) set over U. Then (ξ, A) is a CS B-subalgebra over the same universe U, if this is an existing attribute $\epsilon \in A$ s,t:

$$(S_1)\ \tilde{\mu}_{\xi_\epsilon}(x_1 * x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1), \tilde{\mu}_{\xi_\epsilon}(x_2)\}$$
$$(S_2) \tilde{v}_{\xi_\epsilon}(x_1 * x_2) \leq \max\{\tilde{v}_{\xi_\epsilon}(x_1), \tilde{v}_{\xi_\epsilon}(x_2)\}$$

for all $x_1, x_2 \in U$. We briefly denote it as an ϵ-CS subalgebra over U. If (ξ, A) is an ϵ-CS subalgebra for all $\epsilon \in A$, then (ξ, A) over U is called a CS subalgebra.

Definition 6. Consider U is a B-algebra, which satisfies condition S, and (ξ, A) over U is any CSs over U. Then (ξ, A) over the same universe U is known as CS o-subalgebra, if there exists $\epsilon \in A$ such that:

$$(S_3)\ \tilde{\mu}_{\xi_\epsilon}(x_1^o x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1), \tilde{\mu}_{\xi_\epsilon}(x_2)\}$$
$$(S_4)\ \tilde{v}_{\xi_\epsilon}(x_1^o x_2) \leq \max\{\tilde{v}_{\xi_\epsilon}(x_1), \tilde{v}_{\xi_\epsilon}(x_2)\}$$

for all $x_1, x_2 \in U$. If (ξ, A) over U is an ϵ-CS o-subalgebra (SA) for all $\epsilon \in A$, we declare that (ξ, A) over U is a CS o-SA. We briefly denote it as an $\epsilon - CSo - SA$ over U.

Definition 7. Consider that (ξ, A) over U is a CSs, and it is called an ϵ-CS ideal(CSI) over U for $\epsilon \in A$ if it holds the that results:

$$(S_5)\ \tilde{\mu}_{\xi_\epsilon}(0) \geq \tilde{\mu}_{\xi_\epsilon}(x_1), v_{\xi_\epsilon}(x_1) \leq v_{\xi_\epsilon}(x_1)$$
$$(S_6) \tilde{\mu}_{\xi_\epsilon}(x_1) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2), \tilde{\mu}_{\xi_\epsilon}(x_2)\}$$
$$(S_7) v_{\xi_\epsilon}(x_1) \leq \max\{v_{\xi_\epsilon}(x_1 * x_2), v_{\xi_\epsilon}(x_2)\}$$

for all $x_1, x_2 \in U$. If (ξ, A) is an ϵ-CS perfect for all $\epsilon \in A$ over U, we declare that (ξ, A) is a CSI.

3.1 Preposition 1

Consider that (ξ, A) over U is an ϵ-CSI, then

$$x_1 \leq x_2 \Leftrightarrow \tilde{\mu}_{\xi_\epsilon}(x_2) \preceq \tilde{\mu}_{\xi_\epsilon}(x_1),\ v_{\xi_\epsilon}(x_2) \leq v_{\xi_\epsilon}(x_1)$$

for all $x_1, x_2 \in U$.

Proof. Let $x_1, x_2 \in U$ be such that $x_1 \leq x_2$. Then $x_1 * x_2 = 0$ and therefore

$$\tilde{\mu}_{\xi_\epsilon} \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2), \tilde{\mu}_{\xi_\epsilon}(x_2)\} = \min\{\tilde{\mu}_{\xi_\epsilon}(0), \tilde{\mu}_{\xi_\epsilon}(x_2)\} = \tilde{\mu}_{\xi_\epsilon}(x_2)$$

and

$$v_{\xi_\epsilon}(x_1) \leq \max\{v_{\xi_\epsilon}(x_1 * x_2), v_{\xi_\epsilon}(x_2)\} = \max\{v_{\xi_\epsilon}(0), v_{\xi_\epsilon}(x_2)\} = v_{\xi_\epsilon}(x_2)$$

3.2 Preposition 2

Consider that (ξ, A) over U is an ϵ-CSI (for a parameter $\in A$). If the uncertainty $x_1 * x_2 \leq x_3$ satisfies in U, then

$$\tilde{\mu}_{\xi_\epsilon}(x_1) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_2), \tilde{\mu}_{\xi_\epsilon}(x_3)\}$$

and

$$v_{\xi_\epsilon}(x_1) \leq \max\{v_{\xi_\epsilon}(x_2), v_{\xi_\epsilon}(x_3)\}$$

proof.

Let us $x_1 * x_2 \leq z$ for all $x_1, x_2, x_3 \in U$. Then

$$\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2 * x_3), \tilde{\mu}_{\xi_\epsilon}(x_3)\} = \min\{\tilde{\mu}_{\xi_\epsilon}(0), \tilde{\mu}_{\xi_\epsilon}(x_3)\} = \tilde{\mu}_{\xi_\epsilon}(x_3)$$

and

$$v_{\xi_\epsilon}(x_1 * x_2) \leq \max\{v_{\xi_\epsilon}((x_1 * x_2) * x_3), v_{\xi_\epsilon}(x_3)\} = \max\{v_{\xi_\epsilon}(0), v_{\xi_\epsilon}(x_3)\} = v_{\xi_\epsilon}(x_3)$$

which implies from the above two equations that

$$\tilde{\mu}_{\xi_\epsilon}(x_1) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2), \tilde{\mu}_{\xi_\epsilon}(x_2)\} \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_2), \tilde{\mu}_{\xi_\epsilon}(x_3)\}$$

and

$$v_{\xi_\epsilon}(x_1) \leq \max\{v_{\xi_\epsilon}(x_1 * x_2), v_{\xi_\epsilon}(x_2)\} \leq \max\{v_{\xi_\epsilon}(x_2), v_{\xi_\epsilon}(x_3)\}$$

Theorem 1. In any B-algebra U, satisfying condition (S), then over U every $\epsilon - CSI(\xi, A)$ is an ϵ-CS o-SA for all $\epsilon \in$. proof.

Let $\epsilon \in A$. Because U satisfies this condition (S), we have $(x_1^o x_2) * x_1 \leq x_2$ for all $x_1, x_2 \in U$. Hence, by using (S_7), (S_8), and Proposition 3.1, this implies that

$$\tilde{\mu}_{\xi_\epsilon}(x_1^o x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}((x_1^o x_2) * x_1), \tilde{\mu}_{\xi_\epsilon}(x_1)\} \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1), \tilde{\mu}_{\xi_\epsilon}(x_2)\}$$

and

$$v_{\xi_\epsilon}(x_1^o x_2) \leq \max\{v_{\xi_\epsilon}((x_1^o x_2) * x_1), v_{\xi_\epsilon}(x_1)\} \leq \max\{v_{\xi_\epsilon}(x_1), v_{\xi_\epsilon}(x_2)\}$$

for all $x_1, x_2 \in U$. Hence, (ξ, A) is an ϵ-CS o-SA for all $\epsilon \in A$ over U.

Theorem 2. In any B-algebra U, if (ξ, A) is an $\epsilon - CSI$, then it is an ϵ-CSSA for all $\epsilon \in A$ over U. proof. Assume that over $U(\xi, A)$ is an ϵ-CSI here $\epsilon \in A$. Then for any $x_1, x_2 \in U$, Here in our hand

$$\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}((x_1 * x_2) * x_1), \tilde{\mu}_{\xi_\epsilon}(x_1)\} = \min\{\tilde{\mu}_{\xi_\epsilon}(0), \tilde{\mu}_{\xi_\epsilon}(x_1)\} = \tilde{\mu}_{\xi_\epsilon}(x_1)$$

and

$$v_{\xi_\epsilon}(x_1 * x_2) \leq \max\{v_{\xi_\epsilon}((x_1 * x_2) * x_1), v_{\xi_\epsilon}(x_1)\} = \max\{v_{\xi_\epsilon}(0), v_{\xi_\epsilon}(x_1)\} = v_{\xi_\epsilon}(x_1)$$

Therefore, (ξ, A) is an ϵ-CSSA over U.

Definition 8. Consider that U is any B-algebra and A is a subset of E (the set of parameters). For $\epsilon \in A$, over U, an $\epsilon - CSI(\xi, A)$ is said to be closed if

$$\tilde{\mu}_{\xi_\epsilon}(0 * x_1) \geq \tilde{\mu}_{\xi_\epsilon}(x_1)$$

and

$$v_{\xi_\epsilon}(0 * x_1) \leq v_{\xi_\epsilon}(x_1)$$

for all $x_1 \in U$.

Theorem 3. Every closed CSI in a B-algebra U is a CSSA over U.

Proof.

Assume that over $U(\xi, A)$ is a closed CSI. Then

$$\tilde{\mu}_{\xi_\epsilon}(0 * x_1) \geq \tilde{\mu}_{\xi_\epsilon}(x_1)$$

and

$$v_{\xi_\epsilon}(0 * x_1) \leq v_{\xi_\epsilon}(x_1)$$

for all $x_1 \in U$. It follows from this, this, and this, that

$$\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2) \geq \min\{\tilde{\mu}_{\xi_\epsilon}((x_1 * x_2) * x_1), \tilde{\mu}_{\xi_\epsilon}(x_1)\} = \min\{\tilde{\mu}_{\xi_\epsilon}(0 * x_2), \tilde{\mu}_{\xi_\epsilon}(x_1)\} \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_2), \tilde{\mu}_{\xi_\epsilon}(x_1)\}$$

and

$$v_{\xi_\epsilon}(x_1 * x_2) \geq \min\{v_{\xi_\epsilon}((x_1 * x_2) * x_1), v_{\xi_\epsilon}(x_1)\} = \max\{v_{\xi_\epsilon}(0 * x_2), v_{\xi_\epsilon}(x_1)\} \leq \max\{v_{\xi_\epsilon}(x_2), v_{\xi_\epsilon}(x_1)\}$$

for all $x_1, x_2 \in U$. Therefore, over $U, (\xi, A)$ is a CSI. For the converse of the above theorem, we will give a condition such that every CSSA is a closed CSI over U.

Theorem 4. Every CSSA is a closed CSI, in a p-semi simple B-algebra U, over U.

Proof. Assume that over a p-semi simple B-algebra $U, (\xi, A)$ is a CSSA and let $\epsilon \in A$. For each $x_1 \in U$, we have

1. $\tilde{\mu}_{\xi_\epsilon}(0) = \tilde{\mu}_{\xi_\epsilon}(x_1 * x_1) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1), \tilde{\mu}_{\xi_\epsilon}(x_1)\} = \tilde{\mu}_{\xi_\epsilon}(x_1)$,
2. $v_{\xi_\epsilon}(0) = v_{\xi_\epsilon}(x_1 * x_1) \leq \max\{v_{\xi_\epsilon}(x_1), v_{\xi_\epsilon}(x_1)\} = v_{\xi_\epsilon}(x_1)$.

Using S_1, S_2, and (1), we get

3. $\tilde{\mu}_{\xi_\epsilon}(0 * x_1) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1), \tilde{\mu}_{\xi_\epsilon}(x_1)\} = \tilde{\mu}_{\xi_\epsilon}(x_1)$,

4. $v_{\xi_\epsilon}(0 * x_1) \leq \max\{v_{\xi_\epsilon}(x_0), v_{\xi_\epsilon}(x_1)\} = v_{\xi_\epsilon}(x_1)$.

For any $x_1, x_2 \in U$, we have

$$\tilde{\mu}_{\xi_\epsilon}(x_1) = \tilde{\mu}_{\xi_\epsilon}\big(x_2 * (x_2 * x_1)\big) \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_2), \tilde{\mu}_{\xi_\epsilon}\big(0 * (x_1 * x_2)\big)\} \geq \min\{\tilde{\mu}_{\xi_\epsilon}(x_1 * x_2), v_{\xi_\epsilon}(x_2)\}$$

and

$$v_{\xi_\epsilon}(x_1) = v_{\xi_\epsilon}\big(x_2 * (x_2 * x_1)\big) \leq \max\{v_{\xi_\epsilon}(x_2), v_{\xi_\epsilon}(x_2 * x_1)\} = \max\{v_{\xi_\epsilon}(x_2), v_{\xi_\epsilon}\big(0 * (x_1 * x_2)\big)\}$$
$$\geq \max\{v_{\xi_\epsilon}(x_1 * x_2), v_{\xi_\epsilon}(x_2)\}$$

By using S_1, S_2, (3.4), and (3), we have that (ξ, A) is a closed cubic soft ideal.

3.3 Corollary

Consider that U is a B-algebra. if it satisfies as below properties.

i). $U = \{0 * x_1 \mid x_1 \in U\}$,

ii). every component of U is minimum,

iii). $(\forall x_1, x_2 \in U)\big(x_1 * (0 * x_2) = x_2 * (0 * x_1)\big)$,

iv). $(\forall x_1 \in U)(0 * x_1 = 0 \Rightarrow x_1 = 0)$,

v). $(\forall x_1, x_2 \in U)\big((x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)\big)$,

vi). $(\forall x_1, x_2 \in U)(x_1 * x_2 = x_2 * x_1)$,

vii). $(\forall x_1 \in U)(0 * x_1 = x_1)$,

viii). $(\forall x_1, x_2, x_3 \in U)\big((x_1 * x_2) * (x_1 * x_3) = x_3 * x_2\big)$,

then we conclude that every cubic CSSA is a closed CSI over the same universe U.

Theorem 5. In any B-algebra U, satisfying condition (S), then over $U(\xi, A)$ is any ϵ-cubic soft set and $\epsilon \in A$ (a subset of parameters), then alls are equal:

1. (ξ, A) over U is an ϵ-CSI.
2. For every $x_1, x_2, x_3 \in U$, if $x_1 \leq x_2^o x_3$, then

$$\mu_{\tilde{\xi}(\epsilon)}(x_1) \succeq \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}(x_2), \mu_{\tilde{\xi}(\epsilon)}(x_3)\right\}$$

and

$$v_{\xi(\epsilon)}(x_1) \leq \max\{v_{\xi(\epsilon)}(x_2), v_{\xi(\epsilon)}(x_3)\}$$

Proof. Consider that (ξ, A) over U is an $\epsilon - CSI$ and $x_1 \leq x_1^o x_2$ for all $x_1, x_2, x_3 \in U$. Then

$$\mu_{\tilde{\xi}(\epsilon)}(x_1) \succeq \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}\left(x_1 \cdot (x_2^o x_3)\right), \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3)\right\} = \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}(0), \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3)\right\} = \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3) \succeq \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}(x_2), \mu_{\tilde{\xi}(\epsilon)}(x_3)\right\}.$$

Consider that (ξ, A) over U is an ϵ-CSI and $x_1 \leq x_1^o x_2$ for all $x_1, x_2, x_3 \in U$. Then

$$\mu_{\tilde{\xi}(\epsilon)}(x_1) \succeq \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}\big(x_1 \cdot (x_2^o x_3)\big), \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3)\right\} = \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}(0), \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3)\right\} = \mu_{\tilde{\xi}(\epsilon)}(x_2^o x_3)$$
$$\succeq \text{rmin}\left\{\mu_{\tilde{\xi}(\epsilon)}(x_2), \mu_{\tilde{\xi}(\epsilon)}(x_3)\right\}$$

and

$v_{\xi(\epsilon)}(x_1) \leq \max\{v_{\xi(\epsilon)}\big(x_1 \cdot (x_2^o x_3)\big), v_{\xi(\epsilon)}(x_2^o x_3)\} = \max\{v_{\xi(\epsilon)}(0), v_{\xi(\epsilon)}(x_2^o x_3)\} = v_{\xi(\epsilon)}(x_2^o x_3) \leq \max\{v_{\xi(\epsilon)}(x_2), v_{\xi(\epsilon)}(x_3)\}$.

Conversely, assume that (ii) is valid. Because $0 \leq x_1^o x_1$ for all $x_1 \in U$, It emanates through (ii) that.

$$\mu_{\tilde{\xi}(\epsilon)}(0) \succeq \text{rmin}\{\mu_{\tilde{\xi}(\epsilon)}(x_1), \mu_{\tilde{\xi}(\epsilon)}(x_1)\} = \mu_{\tilde{\xi}(\epsilon)}(x_1)$$

and

$$v_{\xi(\epsilon)}(0) \leq \max\{v_{\xi(\epsilon)}(x_1), v_{\xi(\epsilon)}(x_1)\} = v_{\xi(\epsilon)}(x_1)$$

for all $x_1 \in U$. Since $x_1 \leq (x_1 \cdot x_2)^o x_2$ for all $x_1, x_2 \in U$, we have

$$\mu_{\tilde{\xi}(\epsilon)}(x_1) \succeq \text{rmin}\{\mu_{\xi(\epsilon)}(x_1 \cdot x_2), \mu_{\xi(\epsilon)}(x_2)\}$$

and

$$v_{\xi(\epsilon)}(x_1) \leq \max\{v_{\xi(\epsilon)}(x_1 \cdot x_2), v_{\xi(\epsilon)}(x_2)\}$$

for all $x_1, x_2 \in U$. Hence, it concludes that over $U, (\xi, A)$ is an ϵ-CSI.

Theorem 6. Consider that over U, a CSs (ξ, A) is an ε-CSI for a given parameter $\varepsilon \in A$, iff the non-empty sets

$$\mu_{\xi(\varepsilon)}[\delta_1, \delta_2] := \left\{x_1 \in U \mid \mu_{\xi(\varepsilon)}^{\tilde{x}}(x_1) \succeq [\delta_1, \delta_2]\right\}$$

and

$$v_{\xi(\varepsilon)}^{\rightarrow}(t) := \{x_1 \in U \mid v_{\xi(\varepsilon)}(x_1) \leq t\}$$

are ideals of U for all $[\delta_1, \delta_2] \in \tilde{A} \subseteq [0,1]$ and $t \in [0,1]$.

Proof. Consider that over U a CSs (ξ, A) is an ε-CSI. Suppose that $\mu_{\xi(\varepsilon)}^{\sim}[\delta_1, \delta_2] \cap_{v_{\xi(\varepsilon)}} (t) \neq \emptyset$ for all $[\delta_1, \delta_2] \in \tilde{A} \subseteq [0,1]$ and $t \in [0,1]$. evidently, $0 \in \mu_{\tilde{\xi}(\varepsilon)}^{\tilde{E}}[\delta_1, \delta_2] \cap \underset{\tilde{\xi}(\varepsilon)}{\vec{A}} (t)$. Suppose that $x_1, x_2 \in U$ such that $x_1 * x_2 \in \mu_{\tilde{\xi}(\varepsilon)}^{\sim}[\delta_1, \delta_2]$ and $x_2 \in \mu_{\tilde{\xi}(\varepsilon)}^{\sim}[\delta_1, \delta_2]$. Then $\mu_{\tilde{\xi}(\varepsilon)}^{\tilde{R}}(x_1 * x_2) \succeq [\delta_1, \delta_2]$. It follows from this equation that

$$\mu\tilde{\xi}(\varepsilon)(x_1) \succeq \text{rmin}\{\mu\tilde{\xi}(\varepsilon)(x_1 * x_2), \mu\tilde{\xi}(\varepsilon)(x_2)\} \succeq \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2]$$

Consider that over U CSs (ξ, A) is an ε-CSI. Consider that $\mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2] \cap v_{\xi(\varepsilon)}^{\rightarrow}(t) \neq \emptyset$ for all $[\delta_1, \delta_2] \in \tilde{a}[0,1]$ and $t \in [0,1]$. $0 \in \mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2] \cap v_{\xi(\varepsilon)}^{\rightarrow}(t)$. Suppose that $x_1, x_2 \in U$ such that $x_1 \cdot x_2 \in \mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$ and $x_2 \in \mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$. Then $\mu_{\xi(\varepsilon)}(x_1 \cdot x_2) \succeq [\delta_1, \delta_2]$. It follows from equation this, that $\mu_{\xi(\varepsilon)}(x_1) \succeq \min\{\mu_{\xi(\varepsilon)}(x_1 \cdot x_2), \mu_{\xi(\varepsilon)}(x_2)\} \succeq \min\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2]$.

Hence $x_1 \in \mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$. *Now if* $x_1 \cdot x_2, x_2 \in v_{\xi(\varepsilon)}^{\rightarrow}(t)$, *then* $v_{\xi(\varepsilon)}(x_1 \cdot x_2) \leq t$ *and* $v_{\xi(\varepsilon)}(x_2) \leq t$. *Using* equation this, we have $v_{\xi(\varepsilon)}(x_1) \leq \max\{v_{\xi(\varepsilon)}(x_1 \cdot x_2), v_{\xi(\varepsilon)}(x_2)\} \leq t$, and so $x_1 \in \underset{\xi(\varepsilon)}{\rightarrow} (t)$. Therefore $\mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$ and $\underset{v_{\xi(\varepsilon)}}{\rightarrow} (t)$ are ideals of U.

Contra-wise, we consider that $\mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$ and $v_{\xi(\varepsilon)}^{\rightarrow}(t)$ are ideals of U for all $[\delta_1, \delta_2] \in \tilde{b}[0,1]$ and $t \in [0,1]$. Now, we suppose As is present $b \in U$ s.t $\mu_{\xi(\varepsilon)}(0) \not\succeq \mu_{\xi(\varepsilon)}(b)$ or $\mu_{\xi(\varepsilon)}(0) > \mu_{\xi(\varepsilon)}(b)$. Let $\mu_{\xi(\varepsilon)}(0) = [0^-, 0^+]$ and $\mu_{\xi(\varepsilon)}(b) = [b^-, b^+]$. Then $0^- < b^-$and $0^+ < b^+$which gives that $0^- < \delta_1 < b^-$and $0^+ < \delta_2 < b^+$, that is $\mu_{\xi(\varepsilon)}(0) = [0^-, 0^+] < [\delta_1, \delta_2] < [b^-, b^+]$by taking $[\delta_1, \delta_2] := \left[\frac{1}{2}(0^- + b^+)\right]$. Hence 0 does not belong to $\mu_{\xi(\varepsilon)}^{\leftarrow}[\delta_1, \delta_2]$. Also, 0 does not belong to $\underset{\xi(\varepsilon)}{\rightarrow} (b_t)$ where $b_t = v_{\xi(\varepsilon)}(b)$. This is a contradiction, and so equation this is valid.

Imagine there exists. $c, d \in U$ s.t:

3. $\mu_{\xi(\varepsilon)}(c) \not\succeq \min\{\mu_{\xi(\varepsilon)}(c \cdot d), \mu_{\xi(\varepsilon)}(d)\}$,

or

4. $v_{\xi(\varepsilon)}(c) > \max\{v_{\xi(\varepsilon)}(c \cdot d), v_{\xi(\varepsilon)}(d)\}$.

For the case of 5, let $\mu_{\xi(\varepsilon)}(c) = [\delta_1, \delta_2], \mu_{\xi(\varepsilon)}(c \cdot d) = [\gamma_1, \gamma_2]$ and $\mu_{\xi(\varepsilon)}(d) = [\gamma_3, \gamma_4]$. Then $[\delta_1, \delta_2] \not\subseteq \min\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_2\}, \min\{\gamma_3, \gamma_4\}]$. Hence $\delta_1 < \min\{\gamma_1, \gamma_2\}$ and $\delta_2 < \min\{\gamma_3, \gamma_4\}$. Taking $[\tau_1, \tau_2] = \frac{1}{2}\left(\mu_{\xi(\varepsilon)}(c) + \min\{\mu_{\xi(\varepsilon)}(c \cdot d), \mu_{\xi(\varepsilon)}(d)\}\right)$ It suggests that:

$$[\tau_1, \tau_2] = \frac{1}{2}([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_2\}, \min\{\gamma_3, \gamma_4\}]) = \left[\frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_2\}), \frac{1}{2}(\delta_2 + \min\{\gamma_3, \gamma_4\})\right]$$

This entails that. $\min\{\gamma_1, \gamma_2\} > \tau_1 = \frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_2\}) > \delta_1$ and $\min\{\gamma_3, \gamma_4\} > \tau_2 = \frac{1}{2}(\delta_2 + \min\{\gamma_3, \gamma_4\}) > \delta_2$.

And so that $[\min\{\gamma_1, \gamma_2\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2] \succ [\delta_1, \delta_2] = \mu_\xi(\varepsilon)(a)$.

Therefore, $c \notin \mu_\xi \Rightarrow (\varepsilon)[\tau_1, \tau_2]$. Additionally, we note that $\mu_\xi(\varepsilon)(c \cdot d) = [\gamma_1, \gamma_2] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2]$, and $\mu_\xi(\varepsilon)(d) = [\gamma_3, \gamma_4] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2]$ which implies that $c \cdot d, d \in \mu_\xi \Rightarrow (\varepsilon)[\tau_1, \tau_2]$. Given that this is contradictory, $\mu_\xi(\varepsilon)(x_1) \succeq \min\{\mu_\xi(\varepsilon)(x_1 \cdot x_2), \mu_\xi(\varepsilon)(x_2)\}$ for all $x_1, x_2 \in U$. Now, (6) implies that there exists $t_o \in (0,1)$ such that $v_\xi(\varepsilon)(a) \geq t_o > \max\{v_\xi(\varepsilon)(c \cdot d), v_\xi(\varepsilon)(d)\}$. Hence $c \cdot d, d \in v_\xi(\varepsilon)(t_o)$ but $a \notin v_\xi(\varepsilon)(x_1)$. Given that this is contradictory, $v_\xi(\varepsilon)(x_1) \leq \max\{v_\xi(\varepsilon)(x_1 \cdot x_2), v_\xi(\varepsilon)(x_2)\}$ for all $x_1, x_2 \in U$. Therefore, we conclude that over $U(\xi, A)$ is an ϵ-CSI.

4. Conclusions and Future Work

In this paper, CSA, CSo-SA, and CSIs are introduced in B-algebra. Their characterizations are explained in the reference of B-algebra. Further, some conditions are given for CSSA to be a closed CSI. Relations between closed cubic soft ideals and cubic soft o-subalgebras are discussed. For future work, we observed that this work may be defined in some other algebraic structure such as G-algebra/ku-algebra/ps-algebra, etc and it may be the further extension in neutrosophic set theory.

Acknowledgments
The author is grateful to the editorial and reviewers, as well as the correspondent author, who offered assistance in the form of advice, assessment, and checking during the study period.

Author Contributions
All authors contributed equally to this research.

Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Funding
This research was not supported by any funding agency or institute.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

References

1. Zadeh, L. A. (2015). Fuzzy logic—a personal perspective. Fuzzy sets and systems, 281, 4-20. https://doi.org/10.1016/j.fss.2015.05.009
2. Ram Matawale, C., Datta, S., & Sankar Mahapatra, S. (2014). Leanness estimation procedural hierarchy using interval-valued fuzzy sets (IVFS). Benchmarking: An International Journal, 21(2), 150-183. https://doi.org/10.1108/BIJ-03-2012-0020
3. Saeed, M., Ahsan, M., Saeed, M. H., Mehmood, A., & El-Morsy, S. (2021). Assessment of solid waste management strategies using an efficient complex fuzzy hypersoft set algorithm based on entropy and similarity measures. IEEE Access, 9, 150700-150714. https://doi.org/10.1109/ACCESS.2021.3125727
4. Ye, J., Du, S., & Yong, R. (2021). Multifuzzy Cubic Sets and Their Correlation Coefficients for Multicriteria Group Decision-Making. Mathematical Problems in Engineering, 2021(1), 5520335. https://doi.org/10.1155/2021/5520335
5. Wang, J., Shang, X., Bai, K., & Xu, Y. (2020). A new approach to cubic q-rung orthopair fuzzy multiple attribute group decision-making based on power Muirhead mean. Neural Computing and Applications, 32, 14087-14112. https://doi.org/10.1007/s00521-020-04807-9
6. Saeed, M., & Harl, M. I. (2023). Fundamentals of picture fuzzy hypersoft set with application. Neutrosophic Sets and Systems, 53(1), 24.
7. Barbhuiya, S. R. of BCK/BCI-algebras.
8. Pudlák, P. (1999). On the complexity of the propositional calculus. London Mathematical Society Lecture Note Series, 197-218.
9. Walendziak, A. (2005). A note on normal subalgebras in B-algebras. Scientiae Mathematicae Japonicae, 62(1), 1.
10. Endam, J. C., & Mamhot, A. A. (2023). On Blo-algebras. Asian-European Journal of Mathematics, 16(1). https://doi.org/10.1142/S1793557122502345
11. Saeed, M., Ahsan, M., Saeed, M. H., & El-Morsy, S. (2022). An optimized complex fuzzy hypersoft set system based approach for the evaluation of strategic procurement techniques for fuel cell and hydrogen components. IEEE Access, 10, 71612-71631. https://doi.org/10.1109/ACCESS.2022.3188293

12. Smarandache, F. On the Classification of Bol-Moufang Type of Some Varieties of Quasi Neutrosophic Triplet Loop (Fenyves BCI-Algebras). Structures of Neutrosophic Triplets, Neutrosophic Duplets, or Neutrosophic Multisets, 202.
13. Starczewski, J. T. (2012). Advanced concepts in fuzzy logic and systems with membership uncertainty (Vol. 284). Springer.
14. Onyeozili, I. A., & Gwary, T. M. (2014). A study of the fundamentals of soft set theory. International journal of scientific and technology research, 3(4), 132-143.
15. Muhiuddin, G., & Al-roqi, A. M. (2014). Cubic soft sets with applications in BCK/BCI-algebras. Annals of Fuzzy Mathematics and Informatics, 8(2), 291-304.
16. Jun, Y. B., & Park, C. H. (2008). Applications of soft sets in ideal theory of BCK/BCI-algebras. Information sciences, 178(11), 2466-2475. https://doi.org/10.1016/j.ins.2008.01.017
17. Muhiuddin, G., Al-Kadi, D., & Balamurugan, M. (2020). Anti-intuitionistic fuzzy soft a-ideals applied to BCI-algebras. Axioms, 9(3), 79. https://doi.org/10.3390/axioms9030079
18. Muhiuddin, G., Feng, F., & Jun, Y. B. (2014). Subalgebras of BCK/BCI-Algebras Based on Cubic Soft Sets. The Scientific World Journal, 2014(1), 458638. https://doi.org/10.1155/2014/458638
19. Jun, Y. B., Muhiuddin, G., Ozturk, M. A., & Roh, E. H. (2017). Cubic soft ideals in BCK/BCI-algebras. J. Comput. Anal. Appl, 22(5), 929-940.
20. M. Fukamiya, "On b*-algebras," Proceedings of the Japan Academy, vol. 27, no. 7, pp. 321-327, 1951.
21. Bustince, H., Montero, J., Pagola, M., & Barrenechea, E. (2008). A survey of interval-valued fuzzy sets.
22. Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. Computers & mathematics with applications, 45(4-5), 555-562. https://doi.org/10.1016/S0898-1221(03)00016-6
23. Qayyum, A., Abdullah, S., & Aslam, M. (2016). Cubic soft expert sets and their application in decision making. Journal of Intelligent & Fuzzy Systems, 31(3), 1585-1596. https://doi.org/10.3233/JIFS-151652

Received: 22 Mar 2024, **Revised:** 27 Jun 2024,

Accepted: 28 Jul 2024, **Available online:** 01 Aug 2024.