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A Study on the Polarity of Generalized Neutrosophic Ideals in BCK-Algebra

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Abstract: In this study, we apply k-polar generalized neutrosophic logic to the ideal of BCK-algebra and consequently introduce the notion of a k-polar generalized neutrosophic ideal in BCK-algebra with an example. We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic ideal. We prove that every k-polar generalized neutrosophic ideal is a k-polar generalized neutrosophic subalgebra, but the converse is not true, which can be illustrated with an example. Furthermore, we prove that a k-polar generalized neutrosophic set is a k-polar generalized neutrosophic ideal if and only if its corresponding cut sets are ideals of the BCK algebra.

Keywords: k-polar Fuzzy Ideal; Cut Sets; k-polar Generalized Neutrosophic Set; k-polar Generalized Neutrosophic Subalgebra; k-polar Generalized Neutrosophic Ideal.

1. Introduction

Data with multiple dimensions, or aspects, is essential for numerous practical applications. This type of data frequently arises from sources consisting of two or more contents. To manage these types of sources, various sets, such as bipolar fuzzy sets [1], bipolar neutrosophic sets [2], m-polar fuzzy sets [3], k-polar generalized neutrosophic sets[4], etc are defined.

In 1966, Imai and Iséki [5, 6] established BCK/BCI-algebras, which enlarged the theories of set-theoretic difference and propositional calculus. The academic study of these theories has expanded substantially since then, with a healthy spotlight on the theory of their ideals. A variety of notions within these and associated algebraic structures have been explored through various research methods.

The notion of fuzzy sets (FSs) was first presented by Zadeh [7]. In 1991, Xi [8] utilized these FSs in BCK-algebras. K. T. Atanassov's [9] intuitionistic fuzzy sets (IFSs) expand FSs by gathering both truth-membership and false-membership functions. F. Smarandache [10] constructed a novel mathematical structure called the neutrosophic set (NSS) by gathering three functions (truth, indeterministic, and false) to handle uncertainty and indeterminacy.

Several modifications have been introduced into the field of NSSs to enhance our ability to better understand uncertainty and indeterminacy. Wang, H. extended the concept in 2005 by introducing interval-valued NSSs [11]. Subsequently, Jun, Y. B. et al. applied these concepts to the ideals in BCI/BCK algebras [12]. In 2018, Takalo, M. M. et al. proposed the MBJ-NSSs as an extension of the NSSs, where interval-valued fuzzy sets are used to represent indeterminacy functions [13]. Satyanarayana et al. introduced the idea of BS-NSS [14] and SB-NSS [15] in the context of BCI/BCK algebras. In BS-NSS, an interval-valued fuzzy set is utilized to represent the false membership function, while in SB-NSS, an interval-valued fuzzy set is utilized to represent the truth membership function. Following this, we apply SB-NSS to BCI/BCK algebras [16]. Smarandache, F. et al. proposed neutrosophic N-structures, where negative valued functions are utilized to represent the truth, uncertainty, and falsity membership functions [17]. Subsequently, this concept is applied to commutative ideals [18] and positive implicative ideals [19] in BCK-algebras. Recently, neutrosophic logic has been applied to rings [20]. In Figure 1, we provided a visual representation that highlights the various types of modifications of neutrosophic structures. This helps readers to understand and follow the detailed concepts discussed.

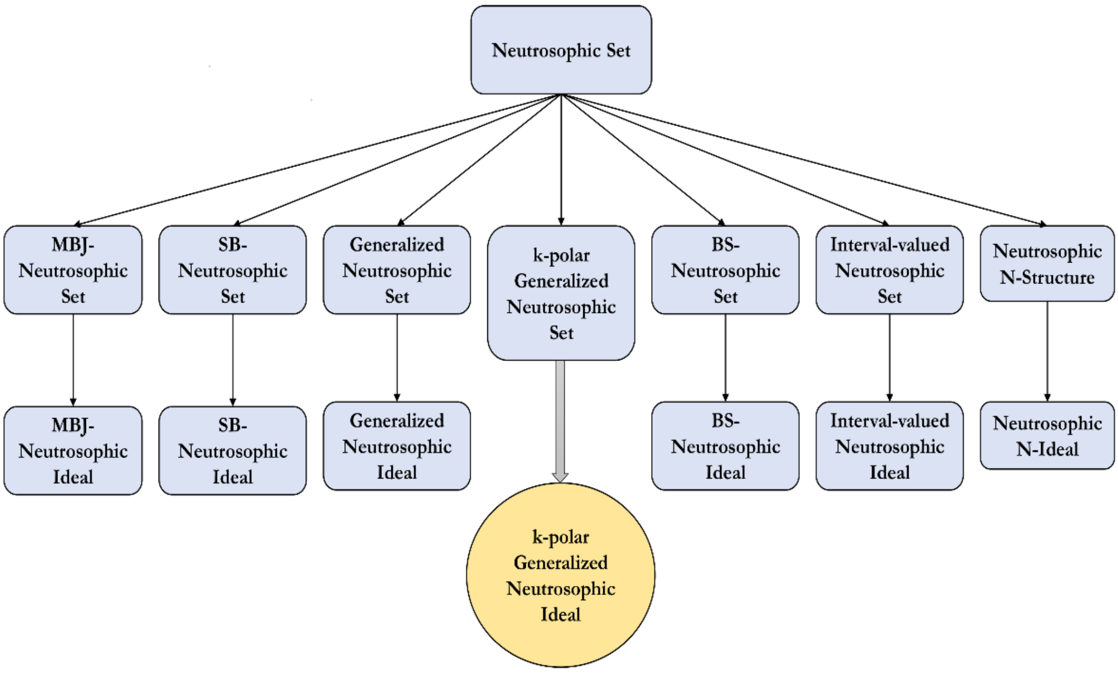

Figure 1. The various types of modifications of neutrosophic structures.

In the application of NSSs to algebraic structures, the indeterministic-membership function contributes support to either the truth-membership function or the false-membership function. To split the aspect of the indeterministic-membership function, Song et al.[21], established a generalized NSS and gave its application in BCK/BCI-algebras. Following this development, F. Smarandache et al. proposed the k-polar generalized NSS and studied its application in BCI/BCK-algebras. In this article, we apply the ϰ-polar generalized NSS (ϰpGNS-S) to the ideal of BCK-algebra and introduce the k-polar generalized neutrosophic ideal (ϰpGNS-I). We studied some of their related features.

2. Preliminaries

Definition 2.1 [5, 6] Let $\mathcal{K}$ be a non-empty set with a binary operation "$*$" and a constant "0" is called a BCK-algebra if it satisfies the following axioms, for all $\mathfrak{x}_0, \mathscr{y}_0, \mathfrak{z}_0 \in \mathcal{K}$

(I) $\big((\mathfrak{x}_0 * \mathscr{y}_0) * (\mathfrak{x}_0 * \mathfrak{z}_0)\big) * (\mathfrak{z}_0 * \mathscr{y}_0) = 0$

(II) $\big(\mathfrak{x}_0 * (\mathfrak{x}_0 * \mathscr{y}_0)\big) * \mathscr{y}_0 = 0$

(III) $\mathfrak{x}_0 * \mathfrak{x}_0 = 0$

(IV) $0 * \mathfrak{x}_0 = 0$

(V) $\mathfrak{x}_0 * \mathscr{y}_0 = 0, \mathscr{y}_0 * \mathfrak{x}_0 = 0 \Rightarrow \mathfrak{x}_0 = \mathscr{y}_0$.

The following properties are held in any BCK-algebra

(i) $\mathfrak{x}_0 * 0 = \mathfrak{x}_0$

(ii) $\mathfrak{x}_0 \le \mathscr{y}_0 \Rightarrow \mathfrak{x}_0 * \mathfrak{z}_0 \le \mathscr{y}_0 * \mathfrak{z}_0, \mathfrak{z}_0 * \mathscr{y}_0 \le \mathfrak{z}_0 * \mathfrak{x}_0$

(iii) $(\mathfrak{x}_0 * \mathscr{y}_0) * \mathfrak{z}_0 = (\mathfrak{x}_0 * \mathfrak{z}_0) * \mathscr{y}_0$

(iv) $(\mathfrak{x}_0 * \mathfrak{z}_0) * (\mathscr{y}_0 * \mathfrak{z}_0) \le \mathfrak{x}_0 * \mathscr{y}_0$ for all $\mathfrak{x}_0, \mathscr{y}_0, \mathfrak{z}_0 \in \mathcal{K}$.

Where $\mathfrak{x}_0 \le \mathscr{y}_0$ if and only if $\mathfrak{x}_0 * \mathscr{y}_0 = 0$.

Definition 2.2 [22] A sub set $\mathfrak{T}(\neq \emptyset)$ of a BCK-algebra $\mathcal{K}$ is called a sub-algebra of $\mathcal{K}$ if $\mathfrak{x}_0 * \mathscr{y}_0 \in \mathfrak{T}$ for all $\mathfrak{x}_0, \mathscr{y}_0 \in \mathfrak{T}$.

Definition 2.3 [22] A sub set $\mathfrak{I}(\neq \emptyset)$ of a BCK-algebra $Ꝃ$ is called an ideal of $Ꝃ$ if $0 \in \mathfrak{I}, and\ y_0, x_0 * y_0 \in \mathfrak{I} \Rightarrow x_0 \in \mathfrak{I}$, for all $x_0, y_0 \in Ꝃ$.
Definition 2.4 [7] Let $Ꝃ$ be a non-empty set. A FS in $Ꝃ$ is a mapping $\alpha_T: Ꝃ \to [0,1]$.
Definition 2.5 [7] The complement of FS set α_T denoted by $(\alpha_T)^c$ is also a fuzzy set defined as $(\alpha_T)^c(x_0) = 1 - \alpha_T(x_0)$ for all $x_0 \in Ꝃ$. Also $((\alpha_T)^c)^c = \alpha_T$.
Definition 2.6 [8] A FS $\alpha_T: Ꝃ \to [0,1]$ is called fuzzy sub-algebra of $Ꝃ$, if

$$\alpha_T(x_0 * y_0) \geq min\{\alpha_T(x_0), \alpha_T(y_0)\}, \text{ for all } x_0, y_0 \in Ꝃ.$$

Definition 2.7 [23] A FS $\alpha_T: Ꝃ \to [0,1]$ is called the fuzzy ideal of $Ꝃ$, if

$$\alpha_T(x_0) \geq min\{\alpha_T(x_0 * y_0), \alpha_T(y_0)\}, \text{ for all } x_0, y_0 \in Ꝃ.$$

Definition 2.8 [4] A ϰpGNS-S on a universe $Ꝃ$ is a structure of the form

$$\Upsilon = \left\{ \frac{x_0}{\left(\left(\omega_j \circ \alpha_T\right)(x_0), \left(\omega_j \circ \beta_{IT}\right)(x_0), \left(\omega_j \circ \gamma_{IF}\right)(x_0), \left(\omega_j \circ \delta_F\right)(x_0)\right)} \mid x_0 \in Ꝃ, \left(\omega_j \circ \beta_{IT}\right)(x_0) + \left(\omega_j \circ \gamma_{IF}\right)(x_0) \leq 1 \right\}$$

for $j = 1,2, \ldots, ϰ$. where $(\omega_j \circ \alpha_T)$, $(\omega_j \circ \beta_{IT})$, $(\omega_j \circ \gamma_{IF})$, and $(\omega_j \circ \delta_F)$ are mappings from $Ꝃ$ into $[0,1]^k$. The membership values of every element $x_0 \in Ꝃ$ in Υ are denoted by

$(\omega_j \circ \alpha_T)(x_0) = \left((\omega_1 \circ \alpha_T)(x_0), (\omega_2 \circ \alpha_T)(x_0), \ldots, (\omega_k \circ \alpha_T)(x_0)\right)$,
$(\omega_j \circ \beta_{IT})(x_0) = \left((\omega_1 \circ \beta_{IT})(x_0), (\omega_2 \circ \beta_{IT})(x_0), \ldots, (\omega_k \circ \beta_{IT})(x_0)\right)$,
$(\omega_j \circ \gamma_{IF})(x_0) = \left((\omega_1 \circ \gamma_{IF})(x_0), (\omega_2 \circ \gamma_{IF})(x_0), \ldots, (\omega_k \circ \gamma_{IF})(x_0)\right)$,
$(\omega_j \circ \delta_F)(x_0) = \left((\omega_1 \circ \delta_F)(x_0), (\omega_2 \circ \delta_F)(x_0), \ldots, (\omega_k \circ \delta_F)(x_0)\right)$, respectively, and satisfies the condition $(\omega_j \circ \beta_{IT})(x_0) + (\omega_j \circ \gamma_{IF})(x_0) \leq 1$ and
$0 \leq (\omega_j \circ \alpha_T)(x_0) + (\omega_j \circ \beta_{IT})(x_0) + (\omega_j \circ \gamma_{IF})(x_0) + (\omega_j \circ \delta_F)(x_0) \leq 3$ for all $j = 1,2, \ldots, ϰ$.

we shall use the ordered quadruple $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ for the ϰpGNS-S.
Definition 2.9 [4] A ϰpGNS-S $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ on $Ꝃ$ is called a ϰpGNS-SA of $Ꝃ$ if it satisfies, for all $x_0, y_0 \in Ꝃ$,

$$\begin{pmatrix} (\omega_j \circ \alpha_T)(x_0 * y_0) \geq min\{(\omega_j \circ \alpha_T)(x_0), (\omega_j \circ \alpha_T)(y_0)\} \\ (\omega_j \circ \beta_{IT})(x_0 * y_0) \geq min\{(\omega_j \circ \beta_{IT})(x_0), (\omega_j \circ \beta_{IT})(y_0)\} \\ (\omega_j \circ \gamma_{IF})(x_0 * y_0) \leq max\{(\omega_j \circ \gamma_{IF})(x_0), (\omega_j \circ \gamma_{IF})(y_0)\} \\ (\omega_j \circ \delta_F)(x_0 * y_0) \leq max\{(\omega_j \circ \delta_F)(x_0), (\omega_j \circ \delta_F)(y_0)\} \end{pmatrix}, \text{ for all } j = 1,2, \ldots, ϰ.$$

3. ϰ-polar Generalized Neutrosophic Ideal

Definition 3.1 A ϰpGNS-S $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ on $Ꝃ$ is called a ϰpGNS-I of $Ꝃ$ if it satisfies, for all $x_0, y_0 \in Ꝃ$,

(ϰ-pGNS-I I) $(\omega_j \circ \alpha_T)(0) \geq (\omega_j \circ \alpha_T)(x_0)$, $(\omega_j \circ \beta_{IT})(0) \geq (\omega_j \circ \beta_{IT})(x_0)$,
$(\omega_j \circ \gamma_{IF})(0) \leq (\omega_j \circ \gamma_{IF})(x_0)$, and $(\omega_j \circ \delta_F)(0) \leq (\omega_j \circ \delta_F)(x_0)$
(ϰ-pGNS-I II) $(\omega_j \circ \alpha_T)(x_0) \geq min\{(\omega_j \circ \alpha_T)(x_0 * y_0), (\omega_j \circ \alpha_T)(y_0)\}$
(ϰ-pGNS-I III) $(\omega_j \circ \beta_{IT})(x_0) \geq min\{(\omega_j \circ \beta_{IT})(x_0 * y_0), (\omega_j \circ \beta_{IT})(y_0)\}$
(ϰ-pGNS-I IV) $(\omega_j \circ \gamma_{IF})(x_0) \leq max\{(\omega_j \circ \gamma_{IF})(x_0 * y_0), (\omega_j \circ \gamma_{IF})(y_0)\}$
(ϰ-pGN-I V) $(\omega_j \circ \delta_F)(x_0) \leq max\{(\omega_j \circ \delta_F)(x_0 * y_0), (\omega_j \circ \delta_F)(y_0)\}$, for all $j = 1,2, \ldots, ϰ$.

Example 3.2 Consider a set $Ꝃ = \{0, æ_1, æ_2, æ_3, æ_4\}$ with the binary operation "∗" as shown in Table 1. Then, $Ꝃ$ is a BCK-algebra.

Table 1. BCK-algebra.

∗	**0**	$æ_1$	$æ_2$	$æ_3$	$æ_4$
0	0	0	0	0	0
$æ_1$	$æ_1$	0	$æ_1$	0	0
$æ_2$	$æ_2$	$æ_2$	0	0	0
$æ_3$	$æ_3$	$æ_3$	$æ_3$	0	0
$æ_4$	$æ_4$	$æ_3$	$æ_4$	$æ_1$	0

Let $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ be a 5-polar generalized neutrosophic set on Ꝁ in which $\alpha_T, \beta_{IT}, \gamma_{IF}$, and δ_F are defined as follows

$$(\omega_j \circ \alpha_T)(x_0) = \begin{cases} (0.92,0.65,0.87,0.79,0.53), & if\ x_0 = 0, æ_2, \\ (0.83, 0.54, 0.76, 0.68, 0.42), & if\ x_0 = æ_1, æ_3, æ_4 \end{cases}$$
$$(\omega_j \circ \beta_{IT})(x_0) = \begin{cases} (0.41,0.83,0.65,0.77,0.54), & if\ x_0 = 0, æ_2 \\ (0.32,0.74,0.51,0.68,0.45), & if\ x_0 = æ_1, æ_3, æ_4 \end{cases}$$
$$(\omega_j \circ \gamma_{IF})(x_0) = \begin{cases} (0.43,0.09,0.35,0.15,0.26), & if\ x_0 = 0, æ_2 \\ (0.54,0.26,0.48,0.26,0.37), & if\ x_0 = æ_1, æ_3, æ_4 \end{cases}$$
$$(\omega_j \circ \delta_F)(x_0) = \begin{cases} (0.45,0.34,0.57,0.12,0.23), & if\ x_0 = 0, æ_2 \\ (0.56,0.45,0.88,0.92,0.77), & if\ x_0 = æ_1, æ_3, æ_4 \end{cases}$$

It is routine to verify that $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ is a 5-polar generalized neutrosophic ideal of Ꝁ.

Theorem 3.3 Let $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ be a ϰpGNS-I of Ꝁ. If $x_0 \le y_0$ in Ꝁ, then $(\omega_j \circ \alpha_T)(x_0) \ge (\omega_j \circ \alpha_T)(y_0)$, $(\omega_j \circ \beta_{IT})(x_0) \ge (\omega_j \circ \beta_{IT})(y_0)$, $(\omega_j \circ \gamma_{IF})(x_0) \le (\omega_j \circ \gamma_{IF})(y_0)$, and $(\omega_j \circ \delta_F)(x_0) \le (\omega_j \circ \delta_F)(y_0)$, for all $x_0, y_0 \in$ Ꝁ and $j = 1,2,\dots,ϰ$.

Proof: Let $x_0, y_0 \in$ Ꝁ be such that $x_0 \le y_0 \Rightarrow x_0 * y_0 = 0$. By utilizing Definition 3.1, we obtain

$$\begin{aligned}(\omega_j \circ \alpha_T)(x_0) &\ge min\{(\omega_j \circ \alpha_T)(x_0 * y_0), (\omega_j \circ \alpha_T)(y_0)\} \\ &\ge min\{(\omega_j \circ \alpha_T)(0), (\omega_j \circ \alpha_T)(y_0)\} = (\omega_j \circ \alpha_T)(y_0), \\ (\omega_j \circ \beta_{IT})(x_0) &\ge min\{(\omega_j \circ \beta_{IT})(x_0 * y_0), (\omega_j \circ \beta_{IT})(y_0)\} \\ &\ge min\{(\omega_j \circ \beta_{IT})(0), (\omega_j \circ \beta_{IT})(y_0)\} = (\omega_j \circ \beta_{IT})(y_0), \\ (\omega_j \circ \gamma_{IF})(x_0) &\le max\{(\omega_j \circ \gamma_{IF})(x_0 * y_0), (\omega_j \circ \gamma_{IF})(y_0)\} \\ &\le max\{(\omega_j \circ \gamma_{IF})(0), (\omega_j \circ \gamma_{IF})(y_0)\} = (\omega_j \circ \gamma_{IF})(y_0), \\ (\omega_j \circ \delta_F)(x_0) &\le max\{(\omega_j \circ \delta_F)(x_0 * y_0), (\omega_j \circ \delta_F)(y_0)\} \\ &\le max\{(\omega_j \circ \delta_F)(0), (\omega_j \circ \delta_F)(y_0)\} = (\omega_j \circ \delta_F)(y_0), \text{ for all } j = 1,2,\dots,ϰ.\end{aligned}$$

Hence, the proof is completed.

Theorem 3.4 A ϰpGNS-I $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ of Ꝁ must be a ϰpGNS-SA of Ꝁ.

Proof: Let $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ be a ϰpGNS-I of Ꝁ. In BCK-algebra Ꝁ, we have, $x_0 * y_0 \le y_0$ for all $x_0, y_0 \in$ Ꝁ. It follows from Theorem 3.3 that $(\omega_j \circ \alpha_T)(x_0 * y_0) \ge (\omega_j \circ \alpha_T)(x_0)$, $(\omega_j \circ \beta_{IT})(x_0 * y_0) \ge (\omega_j \circ \beta_{IT})(x_0)$, $(\omega_j \circ \gamma_{IF})(x_0 * y_0) \le (\omega_j \circ \gamma_{IF})(x_0)$, and $(\omega_j \circ \delta_F)(x_0 * y_0) \le (\omega_j \circ \delta_F)(x_0)$.

By using Definition 3.1, we obtain

$$\begin{aligned}(\omega_j \circ \alpha_T)(x_0 * y_0) \ge (\omega_j \circ \alpha_T)(x_0) &\ge min\{(\omega_j \circ \alpha_T)(x_0 * y_0), (\omega_j \circ \alpha_T)(y_0)\} \\ &\ge min\{(\omega_j \circ \alpha_T)(x_0), (\omega_j \circ \alpha_T)(y_0)\}, \\ (\omega_j \circ \beta_{IT})(x_0 * y_0) \ge (\omega_j \circ \beta_{IT})(x_0) &\ge min\{(\omega_j \circ \beta_{IT})(x_0 * y_0), (\omega_j \circ \beta_{IT})(y_0)\} \\ &\ge min\{(\omega_j \circ \beta_{IT})(x_0), (\omega_j \circ \beta_{IT})(y_0)\}, \\ (\omega_j \circ \gamma_{IF})(x_0 * y_0) \le (\omega_j \circ \gamma_{IF})(x_0) &\le max\{(\omega_j \circ \gamma_{IF})(x_0 * y_0), (\omega_j \circ \gamma_{IF})(y_0)\} \\ &\le max\{(\omega_j \circ \gamma_{IF})(x_0), (\omega_j \circ \gamma_{IF})(y_0)\}, \\ (\omega_j \circ \delta_F)(x_0 * y_0) \le (\omega_j \circ \delta_F)(x_0) &\le max\{(\omega_j \circ \delta_F)(x_0 * y_0), (\omega_j \circ \delta_F)(y_0)\} \\ &\le max\{(\omega_j \circ \delta_F)(x_0), (\omega_j \circ \delta_F)(y_0)\}, \text{ for all } j = 1,2,\dots,ϰ.\end{aligned}$$

Therefore, $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ be a ϰpGNS-SA of Ꝁ.

The converse of the Theorem 3.4 may not be true, which can be shown in the following Example 3.5.

Example 3.5 Consider a set Ꝁ $= \{0, æ_1, æ_2, æ_3\}$ with the binary operation "$*$" as shown in Table 2. Then, Ꝁ is a BCK-algebra.

Table 2. BCK-algebra.

$*$	**0**	$æ_1$	$æ_2$	$æ_3$
0	0	0	0	0
$æ_1$	$æ_1$	0	0	$æ_1$
$æ_2$	$æ_2$	$æ_1$	0	$æ_2$
$æ_3$	$æ_3$	$æ_3$	$æ_3$	0

Let $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ be a 5-polar generalized neutrosophic set on K in which $\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}$, and δ_{F} are defined as follows:

$$(\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{x}_0) = \begin{cases} (0.85,0.51,0.79,0.95,0.64), & if\ \mathfrak{x}_0 = 0, æ_1, æ_3, \\ (0.66,0.32,0.51,0.73,0.46), & if\ \mathfrak{x}_0 = æ_2, \end{cases}$$
$$(\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{x}_0) = \begin{cases} (0.64,0.51,0.73,0.44,0.65), & if\ \mathfrak{x}_0 = 0, æ_1, æ_3, \\ (0.21,0.18,0.39,0.11,0.22), & if\ \mathfrak{x}_0 = æ_2, \end{cases}$$
$$(\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{x}_0) = \begin{cases} (0.25,0.01,0.13,0.42,0.31), & if\ \mathfrak{x}_0 = 0, æ_1, æ_3, \\ (0.77,0.82,0.47,0.65,0.53), & if\ \mathfrak{x}_0 = æ_2, \end{cases}$$
$$(\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{x}_0) = \begin{cases} (0.47,0.33,0.57,0.73,0.25), & if\ \mathfrak{x}_0 = 0, æ_1, æ_3, \\ (0.79,0.84,0.99,0.95,0.67), & if\ \mathfrak{x}_0 = æ_2, \end{cases}$$

It is easy to verify that $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a 5-polar generalized neutrosophic subalgebra of K. But it is not a 5-polar generalized neutrosophic ideal of K, because

$$\begin{aligned}
(\omega_j \circ \alpha_{\mathrm{T}})(2) = (0.66,0.32,0.51,0.73,0.46) &< (0.85,0.51,0.79,0.95,0.64) \\
&= (\omega_j \circ \alpha_{\mathrm{T}})(1) = min\{(\omega_j \circ \alpha_{\mathrm{T}})(2 * 1), (\omega_j \circ \alpha_{\mathrm{T}})(1)\}, \\
(\omega_j \circ \beta_{\mathrm{IT}})(2) = (0.21,0.18,0.39,0.11,0.22) &< (0.64,0.51,0.73,0.44,0.65) \\
&= (\omega_j \circ \beta_{\mathrm{IT}})(1) = min\{(\omega_j \circ \beta_{\mathrm{IT}})(2 * 1), (\omega_j \circ \beta_{\mathrm{IT}})(1)\}, \\
(\omega_j \circ \gamma_{\mathrm{IF}})(2) = (0.77,0.82,0.47,0.65,0.53) &> (0.25,0.01,0.13,0.42,0.31) \\
&= (\omega_j \circ \gamma_{\mathrm{IF}})(1) = min\{(\omega_j \circ \gamma_{\mathrm{IF}})(2 * 1), (\omega_j \circ \gamma_{\mathrm{IF}})(1)\}, \\
(\omega_j \circ \delta_{\mathrm{F}})(2) = (0.79,0.84,0.99,0.95,0.67) &> (0.47,0.33,0.57,0.73,0.25) \\
&= (\omega_j \circ \delta_{\mathrm{F}})(1) = min\{(\omega_j \circ \delta_{\mathrm{F}})(2 * 1), (\omega_j \circ \delta_{\mathrm{F}})(1)\}.
\end{aligned}$$

Theorem 3.6 A κpGNS-S $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ on K is a κpGNS-I of K if and only if for $\mathfrak{x}_0, y_0, \mathfrak{z}_0 \in \mathrm{K}$,

$$\mathfrak{x}_0 * y_0 \leq \mathfrak{z}_0 \Rightarrow \begin{pmatrix} (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{x}_0) \geq min\{(\omega_j \circ \alpha_{\mathrm{T}})(y_0), (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{z}_0)\} \\ (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{x}_0) \geq min\{(\omega_j \circ \beta_{\mathrm{IT}})(y_0), (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{z}_0)\} \\ (\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{x}_0) \leq max\{(\omega_j \circ \gamma_{\mathrm{IF}})(y_0), (\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{z}_0)\} \\ (\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{x}_0) \leq max\{(\omega_j \circ \delta_{\mathrm{F}})(y_0), (\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{z}_0)\} \end{pmatrix} \quad (1)$$

for $j = 1,2, \dots, \kappa$.

Proof: Assume that $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-I of K. Let $\mathfrak{x}_0, y_0, \mathfrak{z}_0 \in \mathrm{K}$ such that $\mathfrak{x}_0 * y_0 \leq \mathfrak{z}_0 \Rightarrow (\mathfrak{x}_0 * y_0) * \mathfrak{z}_0 = 0$. Then, we have

$$\begin{aligned}
(\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{x}_0) &\geq min\{(\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{x}_0 * y_0), (\omega_j \circ \alpha_{\mathrm{T}})(y_0)\} \\
&\geq min\{min\{(\omega_j \circ \alpha_{\mathrm{T}})((\mathfrak{x}_0 * y_0) * \mathfrak{z}_0), (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{z}_0)\}, (\omega_j \circ \alpha_{\mathrm{T}})(y_0)\} \\
&\geq min\{min\{(\omega_j \circ \alpha_{\mathrm{T}})(0), (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{z}_0)\}, (\omega_j \circ \alpha_{\mathrm{T}})(y_0)\} \\
&= min\{(\omega_j \circ \alpha_{\mathrm{T}})(y_0), (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{z}_0)\}, \\
(\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{x}_0) &\geq min\{(\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{x}_0 * y_0), (\omega_j \circ \beta_{\mathrm{IT}})(y_0)\} \\
&\geq min\{min\{(\omega_j \circ \beta_{\mathrm{IT}})((\mathfrak{x}_0 * y_0) * \mathfrak{z}_0), (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{z}_0)\}, (\omega_j \circ \beta_{\mathrm{IT}})(y_0)\} \\
&\geq min\{min\{(\omega_j \circ \beta_{\mathrm{IT}})(0), (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{z}_0)\}, (\omega_j \circ \beta_{\mathrm{IT}})(y_0)\} \\
&= min\{(\omega_j \circ \beta_{\mathrm{IT}})(y_0), (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{z}_0)\}, \\
(\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{x}_0) &\leq max\{(\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{x}_0 * y_0), (\omega_j \circ \gamma_{\mathrm{IF}})(y_0)\} \\
&\leq max\{max\{(\omega_j \circ \gamma_{\mathrm{IF}})((\mathfrak{x}_0 * y_0) * \mathfrak{z}_0), (\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{z}_0)\}, (\omega_j \circ \gamma_{\mathrm{IF}})(y_0)\} \\
&\leq max\{max\{(\omega_j \circ \gamma_{\mathrm{IF}})(0), (\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{z}_0)\}, (\omega_j \circ \gamma_{\mathrm{IF}})(y_0)\} \\
&= max\{(\omega_j \circ \gamma_{\mathrm{IF}})(y_0), (\omega_j \circ \gamma_{\mathrm{IF}})(\mathfrak{z}_0)\}, \\
(\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{x}_0) &\leq max\{(\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{x}_0 * y_0), (\omega_j \circ \delta_{\mathrm{F}})(y_0)\} \\
&\leq max\{max\{(\omega_j \circ \delta_{\mathrm{F}})((\mathfrak{x}_0 * y_0) * \mathfrak{z}_0), (\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{z}_0)\}, (\omega_j \circ \delta_{\mathrm{F}})(y_0)\} \\
&\leq max\{max\{(\omega_j \circ \delta_{\mathrm{F}})(0), (\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{z}_0)\}, (\omega_j \circ \delta_{\mathrm{F}})(y_0)\} \\
&= max\{(\omega_j \circ \delta_{\mathrm{F}})(y_0), (\omega_j \circ \delta_{\mathrm{F}})(\mathfrak{z}_0)\},
\end{aligned}$$

for all $\mathfrak{x}_0, y_0, \mathfrak{z}_0 \in \mathrm{K}$ and for $j = 1,2, \dots, \kappa$. Hence, condition (1) is valid.

Conversely, let $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-S on K that satisfies Condition (1). Since $0 * \mathfrak{x}_0 \leq \mathfrak{x}_0$, for all $\mathfrak{x}_0 \in \mathrm{K}$, we have $(\omega_j \circ \alpha_{\mathrm{T}})(0) \geq (\omega_j \circ \alpha_{\mathrm{T}})(\mathfrak{x}_0)$, $(\omega_j \circ \beta_{\mathrm{IT}})(0) \geq (\omega_j \circ \beta_{\mathrm{IT}})(\mathfrak{x}_0)$,

$(\omega_{j} \circ \gamma_{IF})(0) \leq (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$, and $(\omega_{j} \circ \delta_{F})(0) \leq (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0)$, for $j = 1,2,\ldots,\kappa$. Also, since $\mathfrak{x}_0 * (\mathfrak{x}_0 * y_0) \leq y_0$, for all $\mathfrak{x}_0, y_0 \in K$, we have
$(\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0) \geq min\{(\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \alpha_{T})(y_0)\}$
$(\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0) \geq min\{(\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \beta_{IT})(y_0)\}$
$(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$
$(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$, for all $j = 1,2,\ldots,\kappa$.
Therefore, $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ is a κpGNS-I of K.

Theorem 3.7 Let $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ be a κpGNS-S on K. Then $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ is a κpGNS-I of K if and only if the κ-polar fuzzy sets $(\omega_{j} \circ \alpha_{T})$, $(\omega_{j} \circ \beta_{IT})$, $(\omega_{j} \circ \gamma_{IF})^C$ and $(\omega_{j} \circ \delta_{F})^C$ are κ-polar fuzzy ideals of K, for $j = 1,2,\ldots,\kappa$. Where $(\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0) = 1 - (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$ and $(\omega_{j} \circ \delta_{F})^C(\mathfrak{x}_0) = 1 - (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0)$, for all $\mathfrak{x}_0 \in K$ and $j = 1,2,\ldots,\kappa$.
Proof: Assume that $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ is a κpGNS-I of K. Then, we have
$(\omega_{j} \circ \alpha_{T})(0) \geq (\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0)$, $(\omega_{j} \circ \beta_{IT})(0) \geq (\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0)$, $(\omega_{j} \circ \gamma_{IF})(0) \leq (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$, and
$(\omega_{j} \circ \delta_{F})(0) \leq (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0)$
$(\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0) \geq min\{(\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \alpha_{T})(y_0)\}$
$(\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0) \geq min\{(\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \beta_{IT})(y_0)\}$
$(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$
$(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$, for all $\mathfrak{x}_0, y_0 \in K$ and for $j = 1,2,\ldots,\kappa$.
It is clear that $(\omega_{j} \circ \alpha_{T})$ and $(\omega_{j} \circ \beta_{IT})$ are κ-polar fuzzy ideals of K. Now,

$$(\omega_{j} \circ \gamma_{IF})(0) \leq (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$$
$$\Rightarrow 1 - (\omega_{j} \circ \gamma_{IF})(0) \geq 1 - (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$$
$$\Rightarrow (\omega_{j} \circ \gamma_{IF})^C(0) \geq (\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0),$$
$$(\omega_{j} \circ \delta_{F})(0) \leq (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0)$$
$$\Rightarrow 1 - (\omega_{j} \circ \delta_{F})(0) \geq 1 - (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0)$$
$$\Rightarrow (\omega_{j} \circ \delta_{F})^C(0) \geq (\omega_{j} \circ \delta_{F})^C(\mathfrak{x}_0)$$
$$(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$$
$$\Rightarrow -(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0) \geq -max\{(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$$
$$\Rightarrow 1 - (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0) \geq 1 - max\{(\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$$
$$\Rightarrow (\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0) \geq min\{1 - (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0 * y_0), 1 - (\omega_{j} \circ \gamma_{IF})(y_0)\}$$
$$= min\left\{(\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \gamma_{IF})^C(y_0)\right\}, \text{ and}$$
$$(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0) \leq max\{(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$$
$$\Rightarrow -(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0) \geq -max\{(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$$
$$\Rightarrow 1 - (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0) \geq 1 - max\{(\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$$
$$\Rightarrow (\omega_{j} \circ \delta_{F})^C(\mathfrak{x}_0) \geq min\{1 - (\omega_{j} \circ \delta_{F})(\mathfrak{x}_0 * y_0), 1 - (\omega_{j} \circ \delta_{F})(y_0)\}$$
$$= min\left\{(\omega_{j} \circ \delta_{F})^C(\mathfrak{x}_0 * y_0), (\omega_{j} \circ \delta_{F})^C(y_0)\right\}, \text{ for all } j = 1,2,\ldots,\kappa.$$

Hence, $(\omega_{j} \circ \gamma_{IF})^C$ and $(\omega_{j} \circ \delta_{F})^C$ are κ-polar fuzzy ideals of K.

Conversely, suppose that the κ-polar fuzzy sets $(\omega_{j} \circ \alpha_{T})$, $(\omega_{j} \circ \beta_{IT})$, $(\omega_{j} \circ \gamma_{IF})^C$, and $(\omega_{j} \circ \delta_{F})^C$ are κ-polar fuzzy ideals of K for $j = 1,2,\ldots,\kappa$. Let $\mathfrak{x}_0, y_0 \in K$.
Then, $(\omega_{j} \circ \alpha_{T})(0) \geq (\omega_{j} \circ \alpha_{T})(\mathfrak{x}_0)$, $(\omega_{j} \circ \beta_{IT})(0) \geq (\omega_{j} \circ \beta_{IT})(\mathfrak{x}_0)$, $(\omega_{j} \circ \gamma_{IF})^C(0) \geq (\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0)$, and $(\omega_{j} \circ \delta_{F})^C(0) \geq (\omega_{j} \circ \delta_{F})^C(\mathfrak{x}_0)$.

$$(\omega_{j} \circ \gamma_{IF})^C(0) \geq (\omega_{j} \circ \gamma_{IF})^C(\mathfrak{x}_0)$$
$$\Rightarrow 1 - (\omega_{j} \circ \gamma_{IF})(0) \geq 1 - (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$$
$$\Rightarrow (\omega_{j} \circ \gamma_{IF})(0) \leq (\omega_{j} \circ \gamma_{IF})(\mathfrak{x}_0)$$

$$(\omega_{j} \circ \delta_{F})^{C}(0) \geq (\omega_{j} \circ \delta_{F})^{C}(x_0)$$
$$\Rightarrow 1 - (\omega_{j} \circ \delta_{F})(0) \geq 1 - (\omega_{j} \circ \delta_{F})(x_0)$$
$$\Rightarrow (\omega_{j} \circ \delta_{F})(0) \leq (\omega_{j} \circ \delta_{F})(x_0)$$
$(\omega_{j} \circ \gamma_{IF})^{C}(x_0) \geq min\left\{(\omega_{j} \circ \gamma_{IF})^{C}(x_0 * y_0), (\omega_{j} \circ \gamma_{IF})^{C}(y_0)\right\}$
$\Rightarrow -(\omega_{j} \circ \gamma_{IF})^{C}(x_0) \leq -min\left\{(\omega_{j} \circ \gamma_{IF})^{C}(x_0 * y_0), (\omega_{j} \circ \gamma_{IF})^{C}(y_0)\right\}$
$\Rightarrow 1 - (\omega_{j} \circ \gamma_{IF})^{C}(x_0) \leq 1 - min\left\{(\omega_{j} \circ \gamma_{IF})^{C}(x_0 * y_0), (\omega_{j} \circ \gamma_{IF})^{C}(y_0)\right\}$
$\Rightarrow (\omega_{j} \circ \gamma_{IF})(x_0) \leq max\left\{1 - (\omega_{j} \circ \gamma_{IF})^{C}(x_0 * y_0), 1 - (\omega_{j} \circ \gamma_{IF})^{C}(y_0)\right\}$
$\Rightarrow (\omega_{j} \circ \gamma_{IF})(x_0) \leq max\{(\omega_{j} \circ \gamma_{IF})(x_0 * y_0), (\omega_{j} \circ \gamma_{IF})(y_0)\}$, and
$(\omega_{j} \circ \delta_{F})^{C}(x_0) \geq min\left\{(\omega_{j} \circ \delta_{F})^{C}(x_0 * y_0), (\omega_{j} \circ \delta_{F})^{C}(y_0)\right\}$
$\Rightarrow -(\omega_{j} \circ \delta_{F})^{C}(x_0) \leq -min\left\{(\omega_{j} \circ \delta_{F})^{C}(x_0 * y_0), (\omega_{j} \circ \delta_{F})^{C}(y_0)\right\}$
$\Rightarrow 1 - (\omega_{j} \circ \delta_{F})^{C}(x_0) \leq 1 - min\left\{(\omega_{j} \circ \delta_{F})^{C}(x_0 * y_0), (\omega_{j} \circ \delta_{F})^{C}(y_0)\right\}$
$\Rightarrow (\omega_{j} \circ \delta_{F})(x_0) \leq max\left\{1 - (\omega_{j} \circ \delta_{F})^{C}(x_0 * y_0), 1 - (\omega_{j} \circ \delta_{F})^{C}(y_0)\right\}$
$\Rightarrow (\omega_{j} \circ \delta_{F})(x_0) \leq max\{(\omega_{j} \circ \delta_{F})(x_0 * y_0), (\omega_{j} \circ \delta_{F})(y_0)\}$, for all $x_0, y_0 \in K$ and $j = 1,2,\dots,\kappa$.
Hence, $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ is a κpGNS-I of K.

Theorem 3.8 If a $\kappa\text{pGNS} - \text{S}\ \Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ on K is a κpGNS-I of K, then so are

(i) $(\oplus_1\Upsilon) = (\alpha_T, \beta_{IT}, {\beta_{IT}}^{C}, {\alpha_T}^{C})$.
(ii) $(\oplus_2\Upsilon) = (\beta_{IT}, \alpha_T, {\alpha_T}^{C}, {\beta_{IT}}^{C})$.
(iii) $(\oplus_3\Upsilon) = ({\delta_F}^{C}, {\gamma_{IF}}^{C}, \gamma_{IF}, \delta_F)$.
(iv) $(\oplus_4\Upsilon) = ({\gamma_{IF}}^{C}, {\delta_F}^{C}, \delta_F, \gamma_{IF})$.

Note: $(\oplus_1\Upsilon)$, $(\oplus_2\Upsilon)$, $(\oplus_3\Upsilon)$, and $(\oplus_4\Upsilon)$ are κpGNS-Ss.

Theorem 3.9 For any κpGNS-I $\Upsilon = (\alpha_T, \beta_{IT}, \gamma_{IF}, \delta_F)$ of K, the following are equivalent

i) For all $x_0, y_0 \in K$ and $j = 1,2,\dots,\kappa$

$$\begin{pmatrix} (\omega_{j} \circ \alpha_T)(x_0 * y_0) \geq (\omega_{j} \circ \alpha_T)((x_0 * y_0) * y_0) \\ (\omega_{j} \circ \beta_{IT})(x_0 * y_0) \geq (\omega_{j} \circ \beta_{IT})((x_0 * y_0) * y_0) \\ (\omega_{j} \circ \gamma_{IF})(x_0 * y_0) \leq (\omega_{j} \circ \gamma_{IF})((x_0 * y_0) * y_0) \\ (\omega_{j} \circ \delta_F)(x_0 * y_0) \leq (\omega_{j} \circ \delta_F)((x_0 * y_0) * y_0) \end{pmatrix} \quad (2)$$

ii) For all $x_0, y_0, z_0 \in K$ and $j = 1,2,\dots,\kappa$

$$\begin{pmatrix} (\omega_{j} \circ \alpha_T)((x_0 * z_0) * (y_0 * z_0)) \geq (\omega_{j} \circ \alpha_T)((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \beta_{IT})((x_0 * z_0) * (y_0 * z_0)) \geq (\omega_{j} \circ \beta_{IT})((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \gamma_{IF})((x_0 * z_0) * (y_0 * z_0)) \leq (\omega_{j} \circ \gamma_{IF})((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \delta_F)((x_0 * z_0) * (y_0 * z_0)) \leq (\omega_{j} \circ \delta_F)((x_0 * y_0) * z_0) \end{pmatrix} \quad (3)$$

Proof: (i) ⇒ (ii)
For all $x_0, y_0, z_0 \in K$, we have $(x_0 * (y_0 * z_0)) * z_0 = (x_0 * z_0) * (y_0 * z_0) \leq (x_0 * y_0)$
$\Rightarrow \left((x_0 * (y_0 * z_0)) * z_0\right) * z_0 \leq (x_0 * y_0) * z_0$. It follows from Theorem 3.3 that

$$\begin{pmatrix} (\omega_{j} \circ \alpha_T)\left(\left((x_0 * (y_0 * z_0)) * z_0\right) * z_0\right) \geq (\omega_{j} \circ \alpha_T)((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \beta_{IT})\left(\left((x_0 * (y_0 * z_0)) * z_0\right) * z_0\right) \geq (\omega_{j} \circ \beta_{IT})((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \gamma_{IF})\left(\left((x_0 * (y_0 * z_0)) * z_0\right) * z_0\right) \leq (\omega_{j} \circ \gamma_{IF})((x_0 * y_0) * z_0) \\ (\omega_{j} \circ \delta_F)\left(\left((x_0 * (y_0 * z_0)) * z_0\right) * z_0\right) \leq (\omega_{j} \circ \delta_F)((x_0 * y_0) * z_0) \end{pmatrix} \quad (4)$$

Now utilizing property (iii), conditions (2) and (4), we obtain

$$\begin{aligned}(\omega_{j}\circ\alpha_{T})((x_0*z_0)*(y_0*z_0)) &= (\omega_{j}\circ\alpha_{T})\left((x_0*(y_0*z_0))*z_0\right)\\ &\geq (\omega_{j}\circ\alpha_{T})\left(\left((x_0*(y_0*z_0))*z_0\right)*z_0\right)\\ &\geq (\omega_{j}\circ\alpha_{T})((x_0*y_0)*z_0),\\ (\omega_{j}\circ\beta_{IT})((x_0*z_0)*(y_0*z_0)) &= (\omega_{j}\circ\beta_{IT})\left((x_0*(y_0*z_0))*z_0\right)\\ &\geq (\omega_{j}\circ\beta_{IT})\left(\left((x_0*(y_0*z_0))*z_0\right)*z_0\right)\\ &\geq (\omega_{j}\circ\beta_{IT})((x_0*y_0)*z_0),\\ (\omega_{j}\circ\gamma_{IF})((x_0*z_0)*(y_0*z_0)) &= (\omega_{j}\circ\gamma_{IF})\left((x_0*(y_0*z_0))*z_0\right)\\ &\leq (\omega_{j}\circ\gamma_{IF})\left(\left((x_0*(y_0*z_0))*z_0\right)*z_0\right)\\ &\leq (\omega_{j}\circ\gamma_{IF})((x_0*y_0)*z_0),\\ (\omega_{j}\circ\delta_{F})((x_0*z_0)*(y_0*z_0)) &= (\omega_{j}\circ\delta_{F})\left((x_0*(y_0*z_0))*z_0\right)\\ &\leq (\omega_{j}\circ\delta_{F})\left(\left((x_0*(y_0*z_0))*z_0\right)*z_0\right)\\ &\leq (\omega_{j}\circ\delta_{F})((x_0*y_0)*z_0), \text{ for all } j=1,2,\dots,\kappa.\end{aligned}$$

Thus (ii) holds in Ҡ.

$(ii)\Rightarrow(i)$

It follows from property (i), (III), property (iv), and condition (3) that

$$\begin{aligned}(\omega_{j}\circ\alpha_{T})(x_0*y_0) &= (\omega_{j}\circ\alpha_{T})((x_0*y_0)*0) = (\omega_{j}\circ\alpha_{T})((x_0*y_0)*(y_0*y_0))\\ &\leq (\omega_{j}\circ\alpha_{T})((x_0*y_0)*y_0),\\ (\omega_{j}\circ\beta_{IT})(x_0*y_0) &= (\omega_{j}\circ\beta_{IT})((x_0*y_0)*0) = (\omega_{j}\circ\beta_{IT})((x_0*y_0)*(y_0*y_0))\\ &\leq (\omega_{j}\circ\beta_{IT})((x_0*y_0)*y_0),\\ (\omega_{j}\circ\gamma_{IF})(x_0*y_0) &= (\omega_{j}\circ\gamma_{IF})((x_0*y_0)*0) = (\omega_{j}\circ\gamma_{IF})((x_0*y_0)*(y_0*y_0))\\ &\geq (\omega_{j}\circ\gamma_{IF})((x_0*y_0)*y_0),\\ (\omega_{j}\circ\delta_{F})(x_0*y_0) &= (\omega_{j}\circ\delta_{F})((x_0*y_0)*0) = (\omega_{j}\circ\delta_{F})((x_0*y_0)*(y_0*y_0))\\ &\geq (\omega_{j}\circ\delta_{F})((x_0*y_0)*y_0), \text{ for all } j=1,2,\dots,\kappa.\end{aligned}$$

which proves (i).

Theorem 3.10 If a ϰ pGNS-S $\Upsilon=(\alpha_T,\beta_{IT},\gamma_{IF},\delta_F)$ on Ҡ is a ϰ pGNS-I of Ҡ, then for all $x_0,a_1,a_2,\dots,a_n\in$ Ҡ, $\left(\dots\left((x_0*a_1)*a_2\right)*\dots\right)*a_n=0\Rightarrow$

$$\begin{pmatrix}(\omega_{j}\circ\alpha_{T})(x_0)\geq min\{(\omega_{j}\circ\alpha_{T})(a_1),(\omega_{j}\circ\alpha_{T})(a_2),\dots,(\omega_{j}\circ\alpha_{T})(a_n)\}\\ (\omega_{j}\circ\beta_{IT})(x_0)\geq min\{(\omega_{j}\circ\beta_{IT})(a_1),(\omega_{j}\circ\beta_{IT})(a_2),\dots,(\omega_{j}\circ\beta_{IT})(a_n)\}\\ (\omega_{j}\circ\gamma_{IF})(x_0)\leq max\{(\omega_{j}\circ\gamma_{IF})(a_1),(\omega_{j}\circ\gamma_{IF})(a_2),\dots,(\omega_{j}\circ\gamma_{IF})(a_n)\}\\ (\omega_{j}\circ\delta_{F})(x_0)\leq max\{(\omega_{j}\circ\delta_{F})(a_1),(\omega_{j}\circ\delta_{F})(a_2),\dots,(\omega_{j}\circ\delta_{F})(a_n)\}\end{pmatrix} \quad (5)$$

for $j=1,2,\dots,\kappa$.

Proof: The proof is by induction on n. Let $\Upsilon=(\alpha_T,\beta_{IT},\gamma_{IF},\delta_F)$ be a ϰpGNS-I of Ҡ. Theorem 3.3 and Theorem 3.6 show that condition (5) is valid for $n=1,2$. We assume that condition (5) is satisfied for $n=k$, that is for all $x_0,a_1,a_2,\dots,a_k\in$ Ҡ, $\left(\dots\left((x_0*a_1)*a_2\right)*\dots\right)*a_k=0\Rightarrow$

$$\begin{pmatrix}(\omega_{j}\circ\alpha_{T})(x_0)\geq min\{(\omega_{j}\circ\alpha_{T})(a_1),(\omega_{j}\circ\alpha_{T})(a_2),\dots,(\omega_{j}\circ\alpha_{T})(a_k)\}\\ (\omega_{j}\circ\beta_{IT})(x_0)\geq min\{(\omega_{j}\circ\beta_{IT})(a_1),(\omega_{j}\circ\beta_{IT})(a_2),\dots,(\omega_{j}\circ\beta_{IT})(a_k)\}\\ (\omega_{j}\circ\gamma_{IF})(x_0)\leq max\{(\omega_{j}\circ\gamma_{IF})(a_1),(\omega_{j}\circ\gamma_{IF})(a_2),\dots,(\omega_{j}\circ\gamma_{IF})(a_k)\}\\ (\omega_{j}\circ\delta_{F})(x_0)\leq max\{(\omega_{j}\circ\delta_{F})(a_1),(\omega_{j}\circ\delta_{F})(a_2),\dots,(\omega_{j}\circ\delta_{F})(a_k)\}\end{pmatrix} \quad (6)$$

for $j=1,2,\dots,\kappa$. Let $x_0,a_1,a_2,\dots,a_{k+1}\in$ Ҡ such that $\left(\dots\left((x_0*a_1)*a_2\right)*\dots\right)*a_{k+1}=0\Rightarrow$

$$\begin{pmatrix}(\omega_{j}\circ\alpha_{T})(x_0*a_1)\geq min\{(\omega_{j}\circ\alpha_{T})(a_2),(\omega_{j}\circ\alpha_{T})(a_3),\dots,(\omega_{j}\circ\alpha_{T})(a_{k+1})\}\\ (\omega_{j}\circ\beta_{IT})(x_0*a_1)\geq min\{(\omega_{j}\circ\beta_{IT})(a_2),(\omega_{j}\circ\beta_{IT})(a_3),\dots,(\omega_{j}\circ\beta_{IT})(a_{k+1})\}\\ (\omega_{j}\circ\gamma_{IF})(x_0*a_1)\leq max\{(\omega_{j}\circ\gamma_{IF})(a_2),(\omega_{j}\circ\gamma_{IF})(a_3),\dots,(\omega_{j}\circ\gamma_{IF})(a_{k+1})\}\\ (\omega_{j}\circ\delta_{F})(x_0*a_1)\leq max\{(\omega_{j}\circ\delta_{F})(a_2),(\omega_{j}\circ\delta_{F})(a_3),\dots,(\omega_{j}\circ\delta_{F})(a_{k+1})\}\end{pmatrix}$$

for $j=1,2,\dots,\kappa$. Since $\Upsilon=(\alpha_T,\beta_{IT},\gamma_{IF},\delta_F)$ is a ϰpGNS-I of Ҡ.

$$\begin{aligned}(\omega_{j} \circ \alpha_{T})(x_0) &\geq min\{(\omega_{j} \circ \alpha_{T})(x_0 * a_1), (\omega_{j} \circ \alpha_{T})(a_1)\} \\ &\geq min\{(\omega_{j} \circ \alpha_{T})(a_1), (\omega_{j} \circ \alpha_{T})(a_2), \ldots, (\omega_{j} \circ \alpha_{T})(a_{k+1})\},\\ (\omega_{j} \circ \beta_{IT})(x_0) &\geq min\{(\omega_{j} \circ \beta_{IT})(x_0 * a_1), (\omega_{j} \circ \beta_{IT})(a_1)\} \\ &\geq min\{(\omega_{j} \circ \beta_{IT})(a_1), (\omega_{j} \circ \beta_{IT})(a_2), \ldots, (\omega_{j} \circ \beta_{IT})(a_{k+1})\},\\ (\omega_{j} \circ \gamma_{IF})(x_0) &\leq max\{(\omega_{j} \circ \gamma_{IF})(x_0 * a_1), (\omega_{j} \circ \gamma_{IF})(a_1)\} \\ &\leq max\{(\omega_{j} \circ \gamma_{IF})(a_1), (\omega_{j} \circ \gamma_{IF})(a_2), \ldots, (\omega_{j} \circ \gamma_{IF})(a_{k+1})\},\\ (\omega_{j} \circ \delta_{F})(x_0) &\leq max\{(\omega_{j} \circ \delta_{F})(x_0 * a_1), (\omega_{j} \circ \delta_{F})(a_1)\} \\ &\leq max\{(\omega_{j} \circ \delta_{F})(a_1), (\omega_{j} \circ \delta_{F})(a_2), \ldots, (\omega_{j} \circ \delta_{F})(a_{k+1})\}, \text{ for } j = 1,2, \ldots, \kappa.\end{aligned}$$

Hence the proof is completed.

Theorem 3.11 Let $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ be a κpGNS-S on K satisfying condition (5). Then $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ is a κpGNS-I of K.

Proof: Suppose condition (5) is valid in $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$. Put $x_0 = 0$, $a_1 = x_0$, $a_2 = x_0$, $a_3 = x_0, \ldots, a_n = x_0$, then $(\ldots((0 * x_0) * x_0) * \ldots) * x_0 = 0 \Rightarrow$

$$\begin{pmatrix} (\omega_{j} \circ \alpha_{T})(0) \geq min\{(\omega_{j} \circ \alpha_{T})(x_0), (\omega_{j} \circ \alpha_{T})(x_0), \ldots, (\omega_{j} \circ \alpha_{T})(x_0)\} \\ (\omega_{j} \circ \beta_{IT})(0) \geq min\{(\omega_{j} \circ \beta_{IT})(x_0), (\omega_{j} \circ \beta_{IT})(x_0), \ldots, (\omega_{j} \circ \beta_{IT})(x_0)\} \\ (\omega_{j} \circ \gamma_{IF})(0) \leq max\{(\omega_{j} \circ \gamma_{IF})(x_0), (\omega_{j} \circ \gamma_{IF})(x_0), \ldots, (\omega_{j} \circ \gamma_{IF})(x_0)\} \\ (\omega_{j} \circ \delta_{F})(0) \leq max\{(\omega_{j} \circ \delta_{F})(x_0), (\omega_{j} \circ \delta_{F})(x_0), \ldots, (\omega_{j} \circ \delta_{F})(x_0)\} \end{pmatrix}$$

$\Rightarrow (\omega_{j} \circ \alpha_{T})(0) \geq (\omega_{j} \circ \alpha_{T})(x_0), (\omega_{j} \circ \beta_{IT})(0) \geq (\omega_{j} \circ \beta_{IT})(x_0), (\omega_{j} \circ \gamma_{IF})(0) \leq (\omega_{j} \circ \gamma_{IF})(x_0)$, and $(\omega_{j} \circ \delta_{F})(0) \leq (\omega_{j} \circ \delta_{F})(x_0)$, for all $x_0 \in K$ and $j = 1,2, \ldots, \kappa$.

Again take $x_0 = x_0$, $a_1 = y_0$, $a_2 = z_0$, $a_3 = 0, \ldots, a_n = 0$ in condition (5). Then,

$(\ldots((x_0 * y_0) * z_0) * \ldots) * 0 = 0 \Rightarrow$

$$\begin{aligned}(\omega_{j} \circ \alpha_{T})(x_0) &\geq min\{(\omega_{j} \circ \alpha_{T})(y_0), (\omega_{j} \circ \alpha_{T})(z_0), \ldots, (\omega_{j} \circ \alpha_{T})(0)\} \\ &= min\{(\omega_{j} \circ \alpha_{T})(y_0), (\omega_{j} \circ \alpha_{T})(z_0)\},\\ (\omega_{j} \circ \beta_{IT})(x_0) &\geq min\{(\omega_{j} \circ \beta_{IT})(y_0), (\omega_{j} \circ \beta_{IT})(z_0), \ldots, (\omega_{j} \circ \beta_{IT})(0)\} \\ &= min\{(\omega_{j} \circ \beta_{IT})(y_0), (\omega_{j} \circ \beta_{IT})(z_0)\},\\ (\omega_{j} \circ \gamma_{IF})(x_0) &\leq max\{(\omega_{j} \circ \gamma_{IF})(y_0), (\omega_{j} \circ \gamma_{IF})(z_0), \ldots, (\omega_{j} \circ \gamma_{IF})(0)\} \\ &= max\{(\omega_{j} \circ \gamma_{IF})(y_0), (\omega_{j} \circ \gamma_{IF})(z_0)\},\\ (\omega_{j} \circ \delta_{F})(x_0) &\leq max\{(\omega_{j} \circ \delta_{F})(y_0), (\omega_{j} \circ \delta_{F})(z_0), \ldots, (\omega_{j} \circ \delta_{F})(0)\} \\ &= max\{(\omega_{j} \circ \delta_{F})(y_0), (\omega_{j} \circ \delta_{F})(z_0)\}, \text{ for } j = 1,2, \ldots, \kappa.\end{aligned}$$

Hence, by using Theorem 3.6, we conclude that $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ is a κpGNS-I of K.

Theorem 3.12 If $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ is a κpGNS-I of K, then

$$\begin{aligned}(\omega_{j} \circ \alpha_{T})(x_0 * y_0) &\geq min\{(\omega_{j} \circ \alpha_{T})(x_0 * z_0), (\omega_{j} \circ \alpha_{T})(z_0 * y_0)\}\\ (\omega_{j} \circ \beta_{IT})(x_0 * y_0) &\geq min\{(\omega_{j} \circ \beta_{IT})(x_0 * z_0), (\omega_{j} \circ \beta_{IT})(z_0 * y_0)\}\\ (\omega_{j} \circ \gamma_{IF})(x_0 * y_0) &\leq max\{(\omega_{j} \circ \gamma_{IF})(x_0 * z_0), (\omega_{j} \circ \gamma_{IF})(z_0 * y_0)\}\\ (\omega_{j} \circ \delta_{F})(x_0 * y_0) &\leq max\{(\omega_{j} \circ \delta_{F})(x_0 * z_0), (\omega_{j} \circ \delta_{F})(z_0 * y_0)\}, \text{ for all } x_0, y_0, z_0 \in K \text{ and } j = 1,2, \ldots, \kappa.\end{aligned}$$

Proof: Suppose that $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ is a κpGNS-I of K. In BCK-algebra, we have $(x_0 * y_0) * (x_0 * z_0) \leq (z_0 * y_0)$. By applying Theorem 3.6, we obtain

$$\begin{aligned}(\omega_{j} \circ \alpha_{T})(x_0 * y_0) &\geq min\{(\omega_{j} \circ \alpha_{T})(x_0 * z_0), (\omega_{j} \circ \alpha_{T})(z_0 * y_0)\}\\ (\omega_{j} \circ \beta_{IT})(x_0 * y_0) &\geq min\{(\omega_{j} \circ \beta_{IT})(x_0 * z_0), (\omega_{j} \circ \beta_{IT})(z_0 * y_0)\}\\ (\omega_{j} \circ \gamma_{IF})(x_0 * y_0) &\leq max\{(\omega_{j} \circ \gamma_{IF})(x_0 * z_0), (\omega_{j} \circ \gamma_{IF})(z_0 * y_0)\}\\ (\omega_{j} \circ \delta_{F})(x_0 * y_0) &\leq max\{(\omega_{j} \circ \delta_{F})(x_0 * z_0), (\omega_{j} \circ \delta_{F})(z_0 * y_0)\},\end{aligned}$$

for all $x_0, y_0, z_0 \in K$ and $j = 1,2, \ldots, \kappa$. Hence, the proof is completed.

Definition 3.13 [4] Given a $\kappa\text{pGNS} - \text{S}\ \Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ over a universe K. Consider the following cut sets

$$\mathcal{U}_1\left((\omega_{j} \circ \alpha_{T}), t_{\alpha_{T}}\right) = \{x_0 \in K \mid (\omega_{j} \circ \alpha_{T})(x_0) \geq t_{\alpha_{T}}^{j}\ for\ all\ j = 1,2, \ldots, \kappa\},$$

$\mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right) = \left\{x_0 \in K \mid (\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0) \geq t^ȷ_{\beta_{\mathrm{IT}}} \text{ for all } ȷ = 1,2,\ldots,\kappa\right\}$,

$\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right) = \{x_0 \in K \mid (\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0) \leq t^ȷ_{\gamma_{\mathrm{IF}}} \text{ for all } ȷ = 1,2,\ldots,\kappa\}$,

$\mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right) = \left\{x_0 \in K \mid (\omega_ȷ \circ \delta_{\mathrm{F}})(x_0) \leq t^ȷ_{\delta_{\mathrm{F}}} \text{ for all } ȷ = 1,2,\ldots,\kappa\right\}$,

where $t_{\alpha_{\mathrm{T}}} = (t^1_{\alpha_{\mathrm{T}}}, t^2_{\alpha_{\mathrm{T}}}, \ldots, t^{\mathrm{k}}_{\alpha_{\mathrm{T}}})$, $t_{\beta_{\mathrm{IT}}} = (t^1_{\beta_{\mathrm{IT}}}, t^2_{\beta_{\mathrm{IT}}}, \ldots, t^{\mathrm{k}}_{\beta_{\mathrm{IT}}})$, $t_{\gamma_{\mathrm{IF}}} = (t^1_{\gamma_{\mathrm{IF}}}, t^2_{\gamma_{\mathrm{IF}}}, \ldots, t^{\mathrm{k}}_{\gamma_{\mathrm{IF}}})$, and $t_{\delta_{\mathrm{F}}} = (t^1_{\delta_{\mathrm{F}}}, t^2_{\delta_{\mathrm{F}}}, \ldots, t^{\mathrm{k}}_{\delta_{\mathrm{F}}})$. It is clear that $\mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right) = \bigcap_{ȷ=1}^{k} \mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)^ȷ$, $\mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right) = \bigcap_{ȷ=1}^{k} \mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right)^ȷ$, $\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right) = \bigcap_{ȷ=1}^{k} \mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)^ȷ$, and $\mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right) = \bigcap_{ȷ=1}^{k} \mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right)^ȷ$, where

$\mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)^ȷ = \{x_0 \in K \mid (\omega_ȷ \circ \alpha_{\mathrm{T}})(x_0) \geq t^ȷ_{\alpha_{\mathrm{T}}}\}$,

$\mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right)^ȷ = \left\{x_0 \in K \mid (\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0) \geq t^ȷ_{\beta_{\mathrm{IT}}}\right\}$,

$\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)^ȷ = \{x_0 \in K \mid (\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0) \leq t^ȷ_{\gamma_{\mathrm{IF}}}\}$,

$\mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right)^ȷ = \left\{x_0 \in K \mid (\omega_ȷ \circ \delta_{\mathrm{F}})(x_0) \leq t^ȷ_{\delta_{\mathrm{F}}}\right\}$ for $ȷ = 1,2,\ldots,\kappa$.

Theorem 3.14 Let $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-S on K. Then $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-I of K if and only if the cut sets $\mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)$, $\mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right)$, $\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)$, and $\mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right)$ are ideals of K, for all $t_{\alpha_{\mathrm{T}}}, t_{\beta_{\mathrm{IT}}}, t_{\gamma_{\mathrm{IF}}}, t_{\delta_{\mathrm{F}}} \in [0,1]^k$.

Proof: Assume that a κpGNS − S $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ on K is a κpGNS-I of K. For any $x_0 \in K$, if $x_0 \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{L}_1 \cap \mathcal{L}_2$, then $(\omega_ȷ \circ \alpha_{\mathrm{T}})(x_0) \geq t_{\alpha_{\mathrm{T}}}$, $(\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0) \geq t_{\beta_{\mathrm{IT}}}$, $(\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0) \leq t_{\gamma_{\mathrm{IF}}}$, and $(\omega_ȷ \circ \delta_{\mathrm{F}})(x_0) \leq t_{\delta_{\mathrm{F}}}$. Since $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-I of K, we get $(\omega_ȷ \circ \alpha_{\mathrm{T}})(0) \geq (\omega_ȷ \circ \alpha_{\mathrm{T}})(x_0) \geq t_{\alpha_{\mathrm{T}}}$, $(\omega_ȷ \circ \beta_{\mathrm{IT}})(0) \geq (\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0) \geq t_{\beta_{\mathrm{IT}}}$, $(\omega_ȷ \circ \gamma_{\mathrm{IF}})(0) \leq (\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0) \leq t_{\gamma_{\mathrm{IF}}}$ and $(\omega_ȷ \circ \delta_{\mathrm{F}})(0) \leq (\omega_ȷ \circ \delta_{\mathrm{F}})(x_0) \leq t_{\delta_{\mathrm{F}}}$, for all $x_0 \in K$ and $ȷ = 1,2,\ldots,\kappa$. Therefore, $0 \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{L}_1 \cap \mathcal{L}_2$.
Let for any $x_0, y_0 \in K$ and $x_0 * y_0$, $y_0 \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{L}_1 \cap \mathcal{L}_2$. Then

$(\omega_ȷ \circ \alpha_{\mathrm{T}})(x_0) \geq min\{(\omega_ȷ \circ \alpha_{\mathrm{T}})(x_0 * y_0), (\omega_ȷ \circ \alpha_{\mathrm{T}})(y_0)\} \geq min\{t_{\alpha_{\mathrm{T}}}, t_{\alpha_{\mathrm{T}}}\} = t_{\alpha_{\mathrm{T}}}$,

$(\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0) \geq min\{(\omega_ȷ \circ \beta_{\mathrm{IT}})(x_0 * y_0), (\omega_ȷ \circ \beta_{\mathrm{IT}})(y_0)\} \geq min\{t_{\beta_{\mathrm{IT}}}, t_{\beta_{\mathrm{IT}}}\} = t_{\beta_{\mathrm{IT}}}$,

$(\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0) \leq max\{(\omega_ȷ \circ \gamma_{\mathrm{IF}})(x_0 * y_0), (\omega_ȷ \circ \gamma_{\mathrm{IF}})(y_0)\} \leq max\{t_{\gamma_{\mathrm{IF}}}, t_{\gamma_{\mathrm{IF}}}\} = t_{\gamma_{\mathrm{IF}}}$,

$(\omega_ȷ \circ \delta_{\mathrm{F}})(x_0) \leq max\{(\omega_ȷ \circ \delta_{\mathrm{F}})(x_0 * y_0), (\omega_ȷ \circ \delta_{\mathrm{F}})(y_0)\} \leq max\{t_{\delta_{\mathrm{F}}}, t_{\delta_{\mathrm{F}}}\} = t_{\delta_{\mathrm{F}}}$, for $ȷ = 1,2,\ldots,\kappa$.

Hence, the cut sets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{L}_1, and\ \mathcal{L}_2$ are ideals of K, for all $t_{\alpha_{\mathrm{T}}}, t_{\beta_{\mathrm{IT}}}, t_{\gamma_{\mathrm{IF}}}, t_{\delta_{\mathrm{F}}} \in [0,1]^k$.

Conversely, let $\Upsilon = (\alpha_{\mathrm{T}}, \beta_{\mathrm{IT}}, \gamma_{\mathrm{IF}}, \delta_{\mathrm{F}})$ is a κpGNS-S on K and the cut sets $\mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)$, $\mathcal{U}_2\left((\omega_ȷ \circ \beta_{\mathrm{IT}}), t_{\beta_{\mathrm{IT}}}\right)$, $\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)$, and $\mathcal{L}_2\left((\omega_ȷ \circ \delta_{\mathrm{F}}), t_{\delta_{\mathrm{F}}}\right)$ are ideals of K, for all $t_{\alpha_{\mathrm{T}}}, t_{\beta_{\mathrm{IT}}}, t_{\gamma_{\mathrm{IF}}}, t_{\delta_{\mathrm{F}}} \in [0,1]^k$.
Suppose that $(\omega_ȷ \circ \alpha_{\mathrm{T}})(q_1) < min\{(\omega_ȷ \circ \alpha_{\mathrm{T}})(q_1 * q_2), (\omega_ȷ \circ \alpha_{\mathrm{T}})(q_2)\}$ for $ȷ = 1,2,\ldots,\kappa$. and for some $q_1, q_2 \in K$. Then, $q_1 * q_2$, $q_2 \in \mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)$ and $q_1 \notin \mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)$, where $t_{\alpha_{\mathrm{T}}} = min\{(\omega_ȷ \circ \alpha_{\mathrm{T}})(q_1 * q_2), (\omega_ȷ \circ \alpha_{\mathrm{T}})(q_2)\}$ for $ȷ = 1,2,\ldots,\kappa$. This is a contradiction to $\mathcal{U}_1\left((\omega_ȷ \circ \alpha_{\mathrm{T}}), t_{\alpha_{\mathrm{T}}}\right)$ is an ideal of K, and so

$(\omega_ȷ \circ \alpha_{\mathrm{T}})(q_1) \geq min\{(\omega_ȷ \circ \alpha_{\mathrm{T}})(q_1 * q_2), (\omega_ȷ \circ \alpha_{\mathrm{T}})(q_2)\}$, for all $q_1, q_2 \in K$ and $ȷ = 1,2,\ldots,\kappa$.

Similarly, we can check that

$(\omega_ȷ \circ \beta_{\mathrm{IT}})(q_1) \geq min\{(\omega_ȷ \circ \beta_{\mathrm{IT}})(q_1 * q_2), (\omega_ȷ \circ \beta_{\mathrm{IT}})(q_2)\}$, for all $q_1, q_2 \in K$ and $ȷ = 1,2,\ldots,\kappa$.

If there exists $q_1, q_2 \in K$ such that $(\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_1) > max\{(\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_1 * q_2), (\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_2)\}$ for $ȷ = 1,2,\ldots,\kappa$, then $q_1 * q_2$, $q_2 \in \mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)$ and $q_1 \notin \mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)$, where $t_{\alpha_{\mathrm{T}}} = max\{(\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_1 * q_2), (\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_2)\}$ for $ȷ = 1,2,\ldots,\kappa$. This is a contradiction to $\mathcal{L}_1\left((\omega_ȷ \circ \gamma_{\mathrm{IF}}), t_{\gamma_{\mathrm{IF}}}\right)$ is an ideal of K, and so

$(\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_1) \leq max\{(\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_1 * q_2), (\omega_ȷ \circ \gamma_{\mathrm{IF}})(q_2)\}$, for all $q_1, q_2 \in K$ and $ȷ = 1,2,\ldots,\kappa$.

By a similar way, we know that

$(\omega_{j} \circ \delta_{F})(q_1) \leq max\{(\omega_{j} \circ \delta_{F})(q_1 * q_2), (\omega_{j} \circ \delta_{F})(q_2)\}$, for all $q_1, q_2 \in Ҡ$ and $j = 1,2, \dots, ҡ$.
Therefore, $\Upsilon = (\alpha_{T}, \beta_{IT}, \gamma_{IF}, \delta_{F})$ is a ҡpGNS-I of Ҡ.

4. Conclusions

In this research, we applied the k-polar generalized neutrosophic set to an ideal of a BCK-algebra and introduced a novel concept k-polar generalized neutrosophic ideal of a BCK-algebra, with an example. This notion established a new framework for studying algebraic structures with indeterminacy. We investigate key properties of these ideals. The relationship between k-polar generalized neutrosophic ideals and their corresponding cut sets is a valuable approach of studying these algebraic structures. In a future study, we aim to apply the same methodology to develop the ideals in algebraic structures such as

- k-polar generalized neutrosophic dot-subalgebras of B-algebra.
- K-polar generalized neutrosophic Q-ideals of Q-algebra.
- K-polar generalized neutrosophic extended ideals in MV-Algebras, and so on.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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