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Application of the Neutrosophic Poisson Distribution Series on the Harmonic Subclass of Analytic Functions using the Salagean Derivative Operator

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Abstract: The authors explore the innovative application of the Neutrosophic series, particularly the Neutrosophic Poisson Distribution Series (NPDS), to investigate various indeterminacy or uncertainties inherent in the classical univalent harmonic function class. The Neutrosophic Poisson Distribution Series is equipped with a Salangean derivative operator and convoluted with analytic univalent harmonic function class to derive new properties, such as inclusion relation, and coefficient inequalities for star-likeness. The results obtained demonstrate the effectiveness of this approach in capturing the inherent uncertainties and complexities associated with harmonic functions. There are several other areas of importance of our results that can be unlocked by computer engineers, scientists, and other experts. In this investigation, some of these indeterminacy and complexities are revealed using graphs by employing Python software tools. This novel integration enhances the analytical techniques available and opens a new stairway for future research in neutrosophic series and geometric function theory.

Keywords: Neutrosophic; Harmonic Function; Analytical Function; Starlikeness; Univalent Functions; Salagean Operator.

1. Introduction

Classical harmonic analysis in geometric function theory has been a center of attraction to the researcher in the field of geometric function theory, and it has been extensively studied. This is likely to be associated with its broad areas of applications, such as signal processing (filtering, modulation, demodulation), vibrations and oscillation, waves, antennae image and video processing, audio processing, financial analysis, geophysics, and medical imaging to mention but a few. It is worth saying that some life situations and their concurrences are often not properly captured by classical modeling or analysis because not every situation or occurrence behaved as assumed by the classical conditions, and such may cause serious damage if overlooked or neglected. This observation gave rise to the concepts known as fuzzy and neutrosophic sets as new areas of study in mathematics to cater for indeterminacy value or situations not accommodated or captured in classical situations. The present investigation is designed to address both the classical and neutrosophic harmonic analysis using neutrosophic Poisson distribution polynomials. The investigation will assist in a long way to address some missing value in classical harmonic analysis.

The authors in [8] open a staircase for a deeper study of harmonic functions by defining a subclass of harmonic functions to obtain the geometric properties of class *SH* such as coefficient bounds and many more. Thereafter, authors in [2] investigated a new subclass of harmonic univalent functions, and the geometric properties of the defined class were extensively discussed. Also, some connections between various subclasses of planar harmonic mappings involving classical Poisson distribution series were considered in [22].

A continuous complex-valued function $f = u + iv$ is defined to be harmonic in a simply connected domain $D \subset C$ if both u and v are real harmonic in D. For example, $f = \omega + i\overline{\varphi}$, can be written as follows, where ω is the analytic part and φ the co-analytic part of f

$$f(z) = \omega(z) + \overline{\varphi(z)}.$$

The function $f = \omega + \overline{\varphi}$ is defined as harmonic univalent in D if the mapping $z \to f(z)$ is orientation preserving harmonic, and one-to-one in D (see [8]). The class of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving is denoted by H in the open unit disk $D = \{z: |z| < 1\}$ for which $f(0) = 0$ and $f(0) = 1$, for $f = \omega + \overline{\varphi} \in H$, the analytic functions f and g can be expressed as

$$\omega(z) = z + \sum_{k=2}^{\infty} a_k z^k, \varphi(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1}$$

If the co-analytic part reduces to zero, then H reduces to the class of normalized analytic univalent functions denoted by S and it is expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2}$$

Denote by T_H the class of function belonging to S_H which is expressed as:

$$\omega(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \varphi(z) = \sum_{k=1}^{\infty} |b_k| z^k, |b_1| < 1 \tag{3}$$

If $0 \le \beta < 1$, then

$$N_H(\beta) = \left\{ f \in H : \mathrm{Re}\left(\frac{f'(z)}{z'} \right) \ge \beta, z = re^{i\theta} \in D \right\},$$

and

$$R_H(\beta) = \left\{ f \in H : \mathrm{Re}\left(\frac{f''(z)}{z''} \right) \ge \beta, z = re^{i\theta} \in D \right\},$$

where

$$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right), z'' = \frac{\partial}{\partial\theta}(z'), f'(z) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right), f'' = \frac{\partial}{\partial\theta}(f''(z)).,$$

and the following equalities hold.

$TN_H(\beta) = N_H(\beta) \cap T_H$ *and* $TR_H(\beta) = R_H(\beta) \cap T_H$.

Recently, a subclass of complex-valued harmonic univalent functions defined by a generalized linear operator was introduced in [1], the authors presented some interesting results such as coefficient bounds and compactness. In addition, authors in [9] established some results involving coefficient conditions, distortion bounds, extreme points using convolution, and convex combinations for a new class of harmonic univalent functions class in the open unit disc, associated with the Salagean operator. Various authors have investigated various sub-classes of harmonic functions which can be found in [10, 12-14, 20, 25, 28]. Furthermore, the authors in [2, 24] defined and investigated the following classes:

T_H, $N_H(\beta)$, $TN_H(\beta)$, $R_H(\beta)$, $TR_H(\beta)$.

Let $D^n f = D^n\omega + \overline{D^n\varphi}$ with h and g be given by (1), then Darwish *et al.* [9] defined and investigated the class $S^*_{H,n}(\alpha,\beta)$ of the function (1) that satisfies the geometrical condition

$$R\left\{\frac{\alpha D^{n+2}\omega(z)+(1-\alpha)D^{n+1}\omega(z)+\alpha\overline{D^{n+2}\varphi(z)}+(\alpha-1)\overline{D^{n+1}\varphi(z)}}{D^n\omega(z)+\overline{D^n\varphi(z)}}\right\}\geq\beta$$

$$= f(z)\in H : R\left\{\frac{\alpha z^2\omega''(z)+z\omega'(z)+\overline{\alpha z^2\varphi''(z)+(2\alpha-1)z\varphi'(z)}}{\omega(z)+\varphi(z)}\right\}\geq\beta$$

where

$$D^n\omega(z)=z+\sum_{k=2}^{\infty}k^n a_k z^k,\quad \overline{D^n\varphi(z)}=\sum_{k=1}^{\infty}k^n\overline{b_k z^k},\quad |b_1|<1$$

for β $(0\leq\beta<1)$ and $\alpha\geq 0$. It is obvious that if the co-analytic part $g(z)$ is zero, then the class $S_{H,n}(\alpha,\beta)$ reduces to the class $P_0(\alpha,\beta)$ of function $f\in S$ satisfying.

$$R\left(\frac{\alpha z^2\left(D^n f\right)''+z\left(D^n f\right)'}{D^n f(z)}\right)\geq\beta$$

for some β $(0\leq\beta<1)$, $\alpha\geq 0$ and $z\in D$. The classes of $P_0(\alpha,\beta)$ and $P_0(\alpha,0)$ were explored by several researchers, (see [15-17, 19, 22, 23,30]).

In 2014, Porwal [21], rigorously discussed the application of a Poisson distribution series on certain analytic univalent functions and obtained necessary and sufficient conditions for this series belonging to the classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$, and studied the integral operator related to this series. Thereafter, other researchers in the field of geometric function theory engaged in poison probability distribution series (PPDS) for different types of analytic functions. For example, the author in [29] introduced and investigated some new subclasses of analytic multivalent functions $S(p,\mu,\delta)$ and $C(p,\mu,\delta)$ to determine the necessary and sufficient conditions for the generalized Poisson distribution series to be in the said subclasses by finding connections between various sub-classes of analytic univalent functions, (see also [22]).

The Neutrosophic Poisson probability distribution series (NPPDS) is a generalization of the classical Poisson distribution, which incorporates neutrosophic logic to handle indeterminacy and uncertainty. Applications of NPPDS include reliability engineering, quality control, medical research, financial modeling, and data analysis. The authors in [3, 26, 27], studied the concept of neutrosophic theory to model disease outbreaks, and environmental phenomena and evaluate the risks in situations where there is uncertainty about the occurrence of events. A variable y is said to be a neutrosophic Poisson distribution if it takes values 0, 1, 2, ... the probability $e^{-m_N}, \frac{me^{-m_N}}{1!}, m^2\frac{e^{-m_N}}{2!}, \ldots$ respectively and m_N is called distribution parameters which are equal to the expected values and the variance. Hence,

$$P(y=v)=\frac{m_N^v e^{-m_N}}{y!},\quad k=0,1,2,\ldots \tag{4}$$

That is

$$NE_{(y)} = NV_{(y)} = m_N \tag{5}$$

Where $N = d + I$ is a neutrosophic number [4]. Neutrosophic Poisson Distribution was expressed in [18] in the form of a power series as follows

$$\tau(m_N, z) = z + \sum_{v=2}^{\infty} \frac{m_N^{v-1}}{(v-1)!} e^{-m_N} z^v, \qquad z \in D, \tag{6}$$

and $m \in [1, \infty]$, and by ratio test, the radius of convergence of the above series was shown to be infinity. In [18], the author defined certain analytic function classes and obtained the first few coefficients bound and the Fekete-Szego function with some practical examples for justification. Subsequently, Awolere and Oladipo [6] derived necessary and sufficient conditions for neutrosophic Poisson distribution series to be in the classes $M_\gamma^n(\theta)$ through coefficient inequality. Recently, authors in [5] introduced a class analogous to the one defined in [18] by generalizing the neutrosophic q-Poisson distribution series to investigate a new subclass of analytic and bi-univalent functions in the open unit disk associated with the q-Gegenbauer polynomials, obtain estimates for the Taylor coefficients and Fekete-Szeg type inequalities for the defined functions class. Awolere et al. [7] investigated the possibility of finding a connection between harmonic analytic functions and the neutrosophic Poisson distribution using q-derivative and the sigmoid function. The authors derived necessary and sufficient conditions for the neutrosophic Poisson distribution series to be in the defined harmonic analytic function class. The Neutrosophic Poisson distribution series is gradually attracting the attention of researchers in geometric function theory, which is due to its application in other areas of study and its relationship to the harmonic analytic function class. Therefore, there is a need for further research to explore the relationship between neutrosophic Poisson distribution series to derive neutrosophic harmonic subclasses of analytic functions to expose the likely dangers inherent in classical models. Connections between neutrosophic Poisson distribution series and harmonic analytic functions require advanced mathematical concepts, by integrating elements from probability theory, neutrosophic sets, and complex analysis. The Poisson kernel used in constructing harmonic analytic functions class can be extended to the neutrosophic concept. In the analysis of neutrosophic Poisson processes, harmonic functions can be employed to study the underlying structure of the process to reveal deeper insights into the interplay between determinacy and indeterminacy through potential theory, which heavily relies on harmonic functions.

Now for $m_{N1}, m_{N2} \in [0,\infty]$,we defined that $\gamma(m_{N1},m_{N2}) \in f(z) \in S^*_{H,n}$ as

$\gamma(f) = \gamma(m_{N1}, m_{N2})f(z) = \tau(m_{N1},z) * \omega(z) + \tau(m_{N2},z) * \varphi(z)$

$= \epsilon(z) + \mu(z)$,

where

$$\epsilon(z) = z + \sum_{v=2}^{\infty} \frac{m_{N_1}^{v-1}}{(v-1)!} e^{-m_{N_1}} z^v, \qquad \mu(z) = z + \sum_{v=2}^{\infty} \frac{m_{N_2}^{v-1}}{(v-1)!} e^{-m_{N_2}} z^v \tag{7}$$

Motivated by the works of [7] and [11] the authors wish to investigate the indeterminacy inherent in classical Poisson and harmonic functions by employing a neutrosophic set and convolution operator.

2. Preliminaries and Lemmas

If $u(x, y)$ is harmonic then the NPDS representation $u(x, y)$ converges to $u(x, y)$ uniformly on compact subsets. The statement is established using the mean-value property of harmonic functions and the definition of NPDS. Furthermore, we also know that the NPDS representation of $u(x, y)$ converges to $u(x, y)$ as $n \to \infty$, this is possible by employing the uniform convergence of NPDS and the continuity of $u(x, y)$. In addition, the harmonic function attains its maximum value at the boundary. Therefore, the authors wish to employ NPDS to investigate harmonic because any harmonic function can be represented as an NPDS, and also because of the flexibility, accuracy, and efficiency of NPDS.

For our results, the following lemmas shall be employed

Lemma 1 [9]. Consider $f = \omega + v$, where ω and v are given by (1), and suppose that $\alpha \geq 0, 0 \leq \beta < 1$ and

$$\sum_{K=2}^{\infty} k^n \left[\alpha k(k-1) + k - \beta\right] a_k \Big| + \sum_{k=1}^{\infty} k^n \left[\alpha k(k-1) + k + \beta\right] b_k \Big| \leq 1 - \beta.$$

Then $f \in H(\alpha, \beta)$. and

$$|a_k| \leq \frac{1-\beta}{k^n\left[\alpha k(k-1) + k - \beta\right]}, k \geq 2, \tag{8}$$

and

$$|d_k| \leq \frac{1-\beta}{k^n\left[\alpha k(k-1) + k + \beta\right]}, k \geq 1. \tag{9}$$

Lemma2 [3]: Consider $f = \omega + \bar{v}$ where ω and v are given by (2) and $0 \leq \delta < 1$, then $f \in T_{NH}(\delta)$ if and only if

$$\sum_{k=2}^{\infty} k\,|a_k| + \sum_{k=1}^{\infty} k\,|b_k| \leq 1 - \delta$$

when $f \in TN_H(\delta)$, then

$$|a_k| \leq \frac{1-\delta}{k}, \qquad k \geq 2 \tag{10}$$

$$|b_k| \leq \frac{1-\delta}{k}, \qquad k \geq 1 \tag{11}$$

Lemma 3 [1]: Consider $f = \omega + \bar{v}$ where ω and v are given by (2) and $0 \leq \delta < 1$, then $f \in T_{RH}(\delta)$ if and only if

$$\sum_{k=2}^{\infty} k^2\,|a_k| + \sum_{k=1}^{\infty} k^2\,|b_k| \leq 1 - \delta$$

when $f \in TR_H(\delta)$, then

$$|a_k| \leq \frac{1-\delta}{k^2}, \qquad k \geq 2 \tag{12}$$

$$|b_k| \leq \frac{1-\delta}{k^2}, \qquad k \geq 1$$

(13)

Lemma 4 [8]: If $f = \omega + \bar{v} \in S_H^*$ where ω and v are given by (1) with $b_1 = 0$, then

$$|a_k| \leq \frac{(2k+1)(k+1)}{6} \quad and \quad |b_k| \leq \frac{(2k-1)(k-1)}{6} \tag{14}$$

Lemma 5 [2]: If $f = \omega + v \in \bar{K}_H$, where ω and v are given by (1) with $b_1 = 0$, then

$$|a_k| \leq \frac{(k+1)}{2} \quad and \quad |b_k| \leq \frac{(k-1)}{2} \tag{15}$$

For convergence throughout, except otherwise stated we use the following notations

$$\sum_{k=2}^{\infty} \frac{m_N^{k-1}}{(k-1)!} = e^{m_N} - 1 \quad and \quad \sum_{k=j}^{\infty} \frac{m_N^{k-1}}{(k-j)!} = m_N^{j-1} e^{m_N}, \quad j \geq 2$$

3. Main Result

Theorem 1: Let $m_{N1}, m_{N2} \in [0, \infty], 0 \leq \beta < 1$ and $\alpha \geq 0$. If $2\alpha[m^4_{N1}+m^4_{N2}] + [21\alpha+2]\,[m^3_{N1}+m^3_{N2}] + [54\alpha-2\beta+15]m^2_{N1} + [42\alpha+2\beta+9]m^2_{N2} + [30\alpha - 9\beta + 24]m_{N1} + [19\alpha + 7]m_{N2} \leq 6(1 - \beta)$, (16)

Then,

$$\gamma\left(S_H^*\right) \subset S_{H,0}^*(\alpha, \beta)$$

Proof: Let $f=\omega+\overline{\varphi}\in S_H^*$ so, ω and φ are given by (1) with $d_1 = 0$. We need to prove that $\gamma(f)=\varepsilon(z)+\overline{\mu(z)}\in S_{H,0}^*(\alpha,\beta)$ where ϵ and μ are analytic functions in *D* defined by (7) with $d_1 = 0$. As a result of Lemma 1, we need to establish that

$W_0(m_{N1}, m_{N2}, \alpha, 0) \leq 1-\beta$

where

$$W_0(m_{N1},m_{N2},\alpha,0)=\sum_{k=2}^{\infty}[\alpha k(k-1)+k-\beta\ ]\left|\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}a_k\right|+\sum_{k=2}^{\infty}[\alpha k(k+1)+k+\beta\ ]\left|\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}d_k\right| \quad (17)$$

Applying relation (14) of lemma 4, we have

$$W_0(m_{N1},m_{N2},\alpha,o)=\frac{1}{6}\left\{\sum_{k=2}^{\infty}(\alpha k(k-1)+k-\beta)(2k+1)(k+1)\left|\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}\right|\right\}$$

$$+\frac{1}{6}\left\{\sum_{k=2}^{\infty}(\alpha k(k+1)+k+\beta)(2k-1)(k-1)\left|\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}\right|\right\}$$

$$=\frac{1}{6}\left\{\sum_{k=2}^{\infty}\left[2\alpha k^4+(2+\alpha)k^3+(3-2\alpha-2\beta)k^2+(1-3\beta-\alpha)k-\beta\right]\left|\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}\right|\right\}$$
$$+\frac{1}{6}\left\{\sum_{k=2}^{\infty}\left[2\alpha k^4+(2-\alpha)k^3+(2\alpha-2\beta-3)k^2+(\alpha-3\beta+1)k-\beta\right]\left|\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}\right|\right\} \quad (18)$$

Setting

$k = (k-1) + 1$ (19)

$K^2 = (k-1)(k-2) + 3(k-1) + 1$ (20)

$k^3 = (k-i)(k-2)(k-3) + 6(k-1)(k-2) + 7(k-1) + 1$ (21)

$k^4 = (k-1)(k-2)(k-3)(k-4) + 10(k-1)(k-2)(k-3) + 25(k-1)(k-2) + 15(k-1) + 1$ (22)

In (18), we write

$$W_0(m_{N1,}m_{N2,}\alpha)\leq\frac{1}{6}\left\{\begin{array}{l}\sum_{k=2}^{\infty}\begin{bmatrix}2\alpha(k-1)(k-2)(k-3)(k-4)+(21\alpha+2)(k-1)(k-2)(k-3)+\\(54\alpha-2\beta+15)(k-1)(k-2)+(30\alpha-9\beta+24)(k-1)\end{bmatrix}\frac{m_{N1}^{k-1}e^{-mN1}}{(K-1)!}\\+\sum_{k=2}^{\infty}\begin{bmatrix}2\alpha(k-1)(k-2)(k-3)(k-4)+(21\alpha+2)(k-1)(k-2)(k-3)+\\(42\alpha-2\beta+9)(k-1)(k-2)+(19\alpha+7)(k-1)\end{bmatrix}\frac{m_{N1}^{k-1}e^{-mN2}}{(K-1)!}\end{array}\right\}$$

$$=\frac{1}{6}\left\{2\alpha\sum_{k=5}^{\infty}\frac{e^{-m_{N1}m_{N1}^{k-1}}}{(k-5)!}+(21\alpha+2)\sum_{k=4}^{\infty}\frac{e^{-m_{N1}m_{N1}^{k-1}}}{(k-4)!}+(54\alpha-2\beta+15)\sum_{k=3}^{\infty}\frac{e^{-m_{N1}m_{N1}^{k-1}}}{(k-3)!}+(30\alpha-9\beta+24)\sum_{k=2}^{\infty}\frac{e^{-m_{N1}m_{N1}^{k-1}}}{(k-2)!}\right\}$$
$$+\frac{1}{6}\left\{2\alpha\sum_{k=5}^{\infty}\frac{e^{-m_{N1}m_{N2}^{k-1}}}{(k-5)!}+(21\alpha+2)\sum_{k=4}^{\infty}\frac{e^{-m_{N1}m_{N2}^{k-1}}}{(k-4)!}+(42\alpha-2\beta+9)\sum_{k=3}^{\infty}\frac{e^{-m_{N1}m_{N2}^{k-1}}}{(k-3)!}+(19\alpha+7)\sum_{k=3}^{\infty}\frac{e^{-m_{N1}m_{N2}^{k-1}}}{(k-2)!}\right\}$$

$$=\frac{1}{6}\left[2\alpha m_{N1}^{4}+(21\alpha+2)m_{N1}^{3}+(54\alpha-2\beta+15)m_{N1}^{2}+(30\alpha-9\beta+24)m_{N1}\right]$$
$$+\frac{1}{6}\left[2\alpha m_{N2}^{4}+(21\alpha+2)m_{N2}^{3}+(54\alpha-2\beta+15)m_{N2}^{2}+(30\alpha-9\beta+24)m_{N2}\right].$$

The last expression is bounded above by $1-\beta$ if the condition (16) holds.

Corollary 1.1: Let $m_{N1}, m_{N2}\in[0,\infty], 0\le\beta<1 and \alpha=0$. If

$2[m^3_{N1}+m^3_{N2}]+[15-2\beta]m^2_{N1}+[9+2\beta]m^2_{N2}+[24-9\beta]\ m_{N1}+7m_{N2}\le 6(1-\beta)$,

then

$\gamma(S_H^*)\subset S_{H,0}^*(\alpha)$

Corollary 1.2: Let $m_{N1}, m_{N2}\in[0,\infty], \beta=0$ and $\alpha\ge 0$. If

$2\alpha[m^4_{N1}+m^4_{N2}]+[21\alpha+2][m^3_{N1}+m_N{}^3{}_2]+[54\alpha+15]m_N{}^2{}_1+[42\alpha+9]m^2_{N2}+[30\alpha+24]m_{N1}+[19\alpha+7]m_{N2}\le 6$,

then

$\gamma(SH*)\subset SH,\alpha*\quad(0)$.

Theorem 2: Let $m_{N1}, m_{N2}\in[1,\infty]$, $\alpha\ge 0$, $0\le\beta<1$. If

$\alpha(m^3_{N1}+m^3_{N2})+(6\alpha+1)(m^2_{N1}+m^2_{N2})+(6\alpha+\beta+2)m_{N2}+2(1-\beta)(1-e^{-mN1})\le 2(1-\beta)$, (23)

then

$Y(K_H)\subset S_{H,}{}^*{}_0(\alpha,\beta)$.

Proof. Let $f=\omega+\varphi\in \overline{K}_H$, so that ω and φ are given by (1) with $d_1=0$. We need to prove that $\gamma(f)=\varepsilon(z)+\mu(z)\in S_H^*{}_0(\alpha,\beta)$ where ϵ and μ are analytic functions in D defined by (7) with $d_1=0$. As a result of lemma 1, we need to establish that $\Delta_n(m_{N1}, m_{N2},\alpha)\le 1-\beta$,

$$\Delta_n(m_{N1},m_{N2},\alpha)\le$$
$$\frac{1}{2}\left[\sum_{k=2}^{\infty}[k+1][\alpha k(k-1)+k-\beta]\frac{m_{N1}^{k-1}e^{-mN1}}{(k-1)!}+\sum_{k=2}^{\infty}[k-1][\alpha k(k+1)+k+\beta]\frac{m_{N1}^{k-1}e^{-mN1}}{(k-1)!}\right]$$
$$=\frac{1}{2}\left[\sum_{k=2}^{\infty}[\alpha(k-1)(k-2)(k-3)+(6\alpha+1)(k-1)(k-2)+(6\alpha+4-\beta)(k-1)+2(1-\beta)]\frac{m_{N1}^{k-1}e^{-mN1}}{(k-1)!}\right]$$
$$+\frac{1}{2}\left[\sum_{k=2}^{\infty}[\alpha(k-1)(k-2)(k-3)+(6\alpha+1)(k-1)(k-2)+(6\alpha+\beta+2)(k-1)+2(1-\beta)]\frac{m_{N2}^{k-1}e^{-mN2}}{(k-1)!}\right]$$

$$=\frac{1}{2}\left[\alpha\sum_{k=4}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-4)!}+(6\alpha+1)\sum_{k=3}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-3)!}+(6\alpha-\beta+4)\sum_{k=2}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-2)!}+2(1-\beta)\sum_{k=1}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-1)!}\right]$$
$$+\frac{1}{2}\left[\alpha\sum_{k=4}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-4)!}+(6\alpha+1)\sum_{k=3}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-3)!}+(6\alpha+\beta+2)\sum_{k=2}^{\infty}\frac{m_{N1}^{k-1}e^{-mN1}}{(k-2)!}\right]$$
$$=\alpha\left(m^3{}_{N1}+m^3{}_{N2}\right)+(6\alpha+1)\left(m^2{}_{N1}+m^2{}_{N2}\right)+(6\alpha+\beta+2)m_{N2}+2(1-\beta)\left(1-e^{-m_{N1}}\right)$$

The last expression is bounded above by $1-\beta$ if the condition (10) holds.

Theorem 3: Let $m_{N1}, m_{N2}\in[1,\infty]$, $\alpha\ge 0$, $0\le\beta<1$. If

$(1-\delta)[\alpha(m^2_{N1}+m^2_{N2})+(2\alpha+1)(m^2_{N1}+(4\alpha+1)m_{N2}+(1-\alpha-\beta)(1-e^{-mN1})+(2\alpha+\beta+1)(1-e^{-mN2})]$

$\le 1-\beta-|d_1|$, (24)

then

$Y(TN_H(\delta)) \subset S^*_{H,1}(\alpha\beta\ )$.

Proof. Let $f = \omega + \bar{\varphi} \in TN_H(\delta)$ so that u and v are given by (2) with $d_1 = 0$. From Lemma 1, We need to establish that $\Omega_n(m_{N1},m_{N2},\alpha) \leq 1-\beta$ and moreover when $n = 1$ in Lemma 1, we have

$$\Omega_1(m_{N1},m_{N1},\alpha)=\sum_{K=2}^{\infty}k[\alpha k(k-1)+k\beta]\left|\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}ak\right|+|d_1|\sum_{K=2}^{\infty}k[\alpha k(k+1)+k\beta]\left|\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}ak\right|$$

(25)

Application of the inequalities (10) and (11) of Lemma 2, it follows that.

$$\Omega_1(m_{N1},m_{N2},\alpha)\leq$$
$$(1-\delta)\left[\sum_{K=2}^{\infty}k[\alpha k(k-1)+k-\beta]\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}ak+\sum_{K=2}^{\infty}k[\alpha k(k+1)+k+\beta]\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}ak\right]+|d_1|$$
$$=(1-\delta)\left[\alpha\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-3)!}+(2\alpha+1)\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-2)!}+(1-\alpha-\beta)\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}\right]$$
$$=(1-\delta)\left[\alpha\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-3)!}+(4\alpha+1)\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-2)!}+(2\alpha+\beta+1)\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-1)!}\right]+|d_1|$$
$$=(1-\delta)\left[\alpha(m^2{}_{N1},m^2{}_{N2})+(2\alpha+1)(m^2{}_{N1}+(4\alpha+1)m_{N2}+(1-\alpha-\beta)(1-e^{-mN1})+(2\alpha+\beta+1)(1-e^{-mN1}))\right]$$
$$\leq 1-\beta-|d_1|$$

By the given assertion, which completes the proof of Theorem 3.

Theorem 4: Let $m_{N1},m_{N2} \in [1,\infty]$, $\alpha \geq 0$, δ, $\beta \in [0,1)$. If

$$(1-\delta)\left[\alpha(m_{N1}+m_{N2})+(1-e^{-m_{N1}})+(1+\alpha)(1-e^{-m_{N2}})-\frac{\beta}{m_{N1}}(1-e^{-m_{N1}}-m_{N1}e^{-m_{N1}})\right]$$
$$+\frac{\beta}{m_{N1}}(1-e^{-m_{N2}}-m_{N2}e^{-m_{N2}}\leq 1-\beta\ -|d_1| \quad (26)$$

Then

$Y\ (TR_H(\delta)) \subset S_{H,^*1}(\beta,\alpha)$.

Proof: By Lemma 1, we need to show that $\Omega_1(m_{N1}, m_{N2},\alpha) \leq 1-\beta$, where $\Omega_1(m_{N1},m_{N2},\alpha)$ as given by (25). Using the inequalities (12) and (13) of Lemma 3, it follows that

$$\Omega_1(m_{N1},m_{N2},\alpha)=\sum_{K=2}^{\infty}k[\alpha k(k-1)+k-\beta]\left|\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}ak\right|+|d_1|+\sum_{K=2}^{\infty}k[\alpha k(k+1)+k+\beta]\left|\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}ak\right|$$

$$\leq(1-\delta)\left[\sum_{K=2}^{\infty}\left[\alpha k+(1-\alpha)-\frac{\beta}{k}\right]\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}+\sum_{K=2}^{\infty}\left[\alpha k+(1-\alpha)+\frac{\beta}{k}\right]\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}\right]+|d_1|$$
$$=\leq(1-\delta)\left[\alpha\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-2)!}+\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}-\sum_{K=2}^{\infty}\left(\frac{\beta}{k+2}\right)\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}\right]$$

$$=\leq (1-\delta)\left[\alpha\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-2)!}+\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}-\sum_{K=2}^{\infty}\left(\frac{\beta}{k+2}\right)\frac{m_{N1}^{k-1}e^{-m_{N1}}}{(k-1)!}\right]$$

$$+\leq (1-\delta)\left[\alpha\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-2)!}+\sum_{K=2}^{\infty}(1+\alpha)\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-1)!}-\sum_{K=2}^{\infty}\left(\frac{\beta}{k+2}\right)\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-1)!}\right]+|d_1|$$

$$=(1-\delta)\left[\alpha(m_{N1}+m_{N2})-\left(1-e^{-mN1}\right)+(1+\alpha)\left(1-e^{-mN2}\right)-\frac{\beta}{m_{N1}}\left(1-e^{-mN1}-m_{N1}e^{-m_{N1}}\right)\right]$$

$$+\frac{\beta}{m_{N1}}\left(1-e^{-mN2}-m_{N2}e^{-m_{N2}}\leq 1-\beta-|d_1|\right)$$

By the given assertion.

Theorem 5: Let m_{N1}, $m_{N2} \in [1,\infty]$, $\alpha \geq 0$, δ, $\beta \in [0,1)$. If

$$e^{-m_{N1}}+e^{-m_{N2}}\leq 1+\frac{|d_1|}{1-\beta} \quad (27)$$

Then,

Proof: Using the inequalities (8) and (9) of Lemma 1, we inferred that

$$\Omega_1(m_{N1},m_{N2},\alpha)=(1-\beta)\left[\sum_{K=2}^{\infty}\frac{m_{N1}^{k-1}e^{-m_{N2}}}{(k-1)!}+\sum_{K=2}^{\infty}\frac{m_{N2}^{k-1}e^{-m_{N2}}}{(k-1)!}\right]+|d_1|$$

$$=(1-\beta)\left[1-e^{-mN1}+1-e^{-m_{N2}}\right]+|d_1|$$

$$=(1-\beta)\left[2-e^{-mN1}-e^{-m_{N2}}\right]+|d_1|\leq 1-\beta$$

By the given relation (27).

4. Graphical Representation of Some of the Results

By employing the use of Python software, and with various choices of α and β describing M_{N_2} as a function of M_{N_1}, the output from Python gave the listed graphs from Figures 1-7. These graphs revealed various inherent indeterminacy nature of harmonic structure.

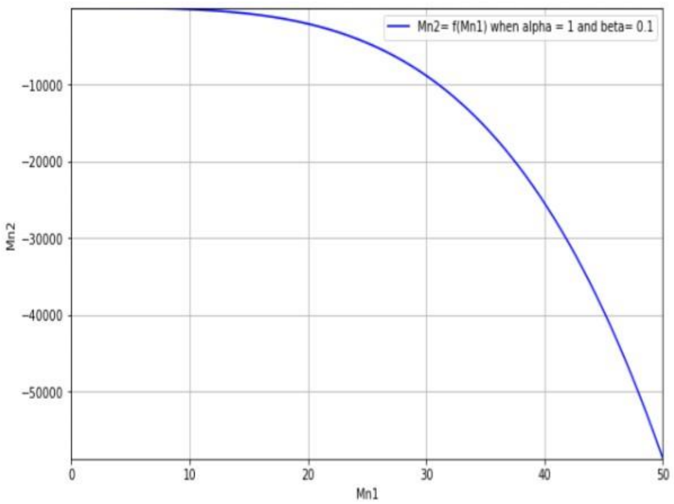

Figure 1. Plot of M_{n2} as a function of M_{n1}.

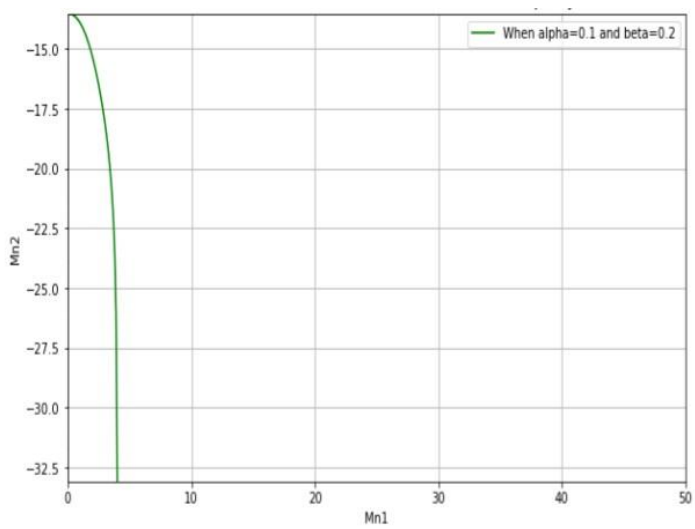

Figure 2. M_{n2} as a function of M_{n1} (for the first two inequality).

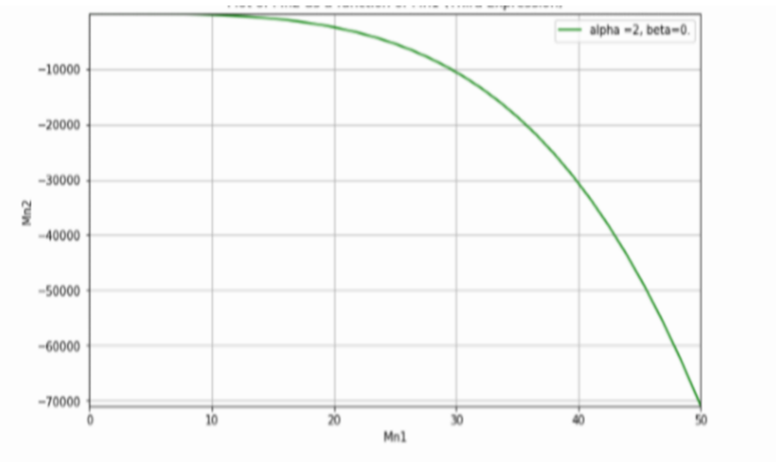

Figure 3. M_{n2} as a function of M_{n1} (Corollary).

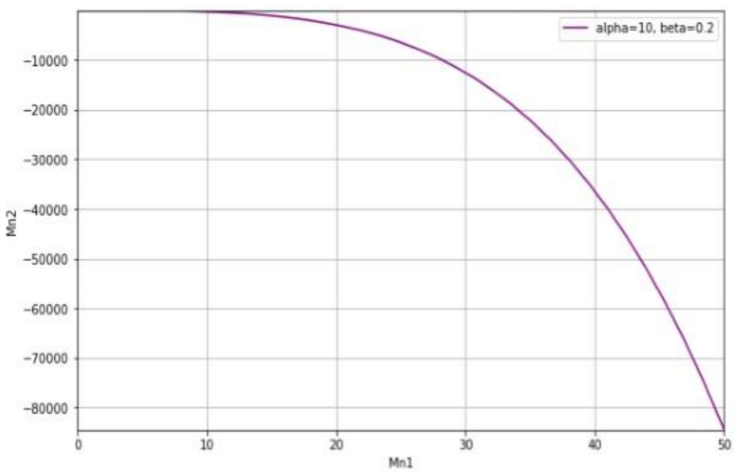

Figure 4. M_{n2} as a function of M_{n1}.

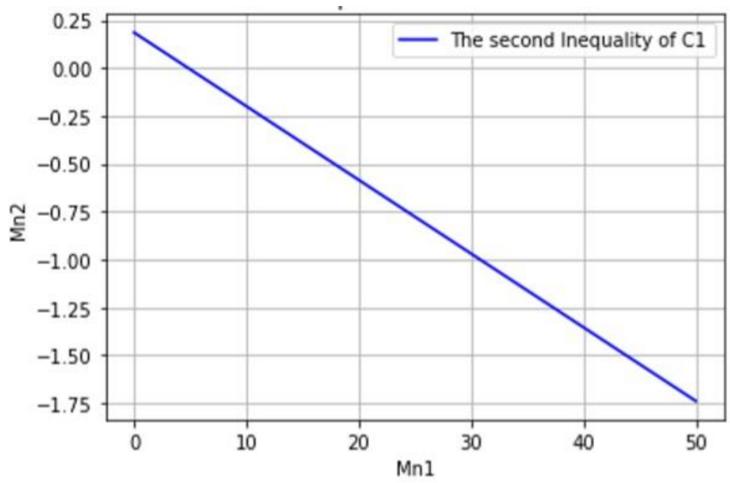

Figure 5. M_{n2} as a function of M_{n1}.

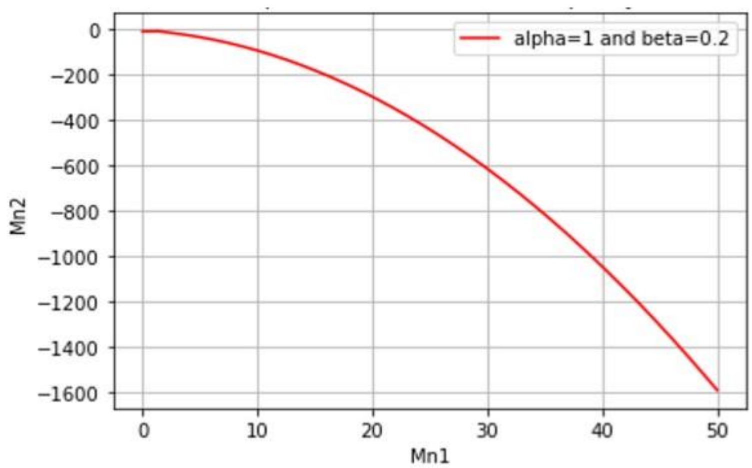

Figure 6. M_{n2} as a function of M_{n1}.

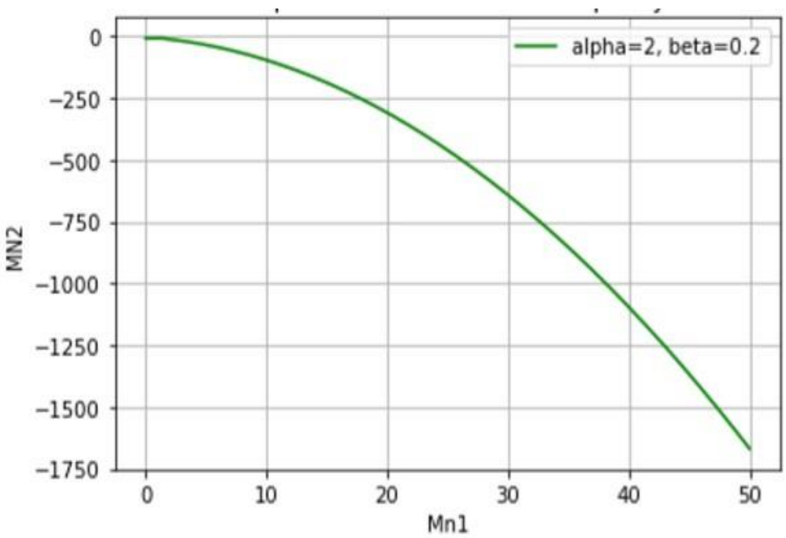

Figure 7. Graph of the first and last inequality.

5. Conclusion

The authors employed a neutrosophic Poisson distribution series to understudy classical harmonic analytic functions to expose some of the inherent indeterminacy values in classical Poisson and harmonic models, which are represented in the form of graphs presented in Figures 1- 7 or an improvement on the classical value. From the graphs, Figures 1- 7, it is observed that in between the various stop-points, many inherent indeterminate situations can be discovered, some of these indeterminacy values may likely produce invalidation and some consequences as an improvement. Therefore, the authors wish that signal processing, computer engineers or scientists, and medical imagers should explore our model further and do a comparative study with their results and with some of the existing results.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

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Author Contribution

Writing original draft, AMG., ITA., OA., and ATO., writing-review and editing, AMG., ITA., OA., and ATO.,. All authors contributed equally to the manuscript and read and approved the final manuscript.

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