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An Approach for Hybridizing N-Subalgebra with Quantified Neutrosophic Set using G-Algebra

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Abstract: Neutrosophic sets are a generalized form of fuzzy sets as well as intuitionistic fuzzy sets, as they address the uncertainty factor as an independent component along with truthfulness and falsity. However, traditional neutrosophic approaches often struggle with effectively managing and quantifying indeterminate elements in complex algebraic structures. To address this limitation, this paper employs an expanded version of the neutrosophic set, incorporating a subalgebra. The proposed research is multifaceted: firstly, the new concept of N-Subalgebra (NSU) is proposed. This is the modified setting in the family of subalgebras whose proposed name is the representation of the author's initial name. Secondly, the hybridization is called Quantified Neutrosophic N-Subalgebra. $\mathcal{Q}^t$NNSU of NSU is established with the modified version of the neutrosophic set, which is Quantified quantified neutrosophic set $\mathcal{Q}^t\mathrm{NS}$. This novel approach to hybridization is the combination of the properties related to NSU in the $\mathcal{Q}^t\mathrm{NS}$'s setting. Thirdly, the operations of P-Union, R-Union, P-Intersection, and R-Intersection are developed for the $\mathcal{Q}^t$NNSU and presented as $\mathrm{P}_{\mathcal{Q}^t}$ Union, $\mathrm{R}_{\mathcal{Q}^t}$ Union, $\mathrm{P}_{\mathcal{Q}^t}$-Intersection and, $\mathrm{R}_{\mathcal{Q}^t}$-Intersection. Furthermore, the concepts of upper bounds, lower bounds, including upper-level subalgebras, as well as lower-level subalgebras in the environment of $\mathcal{Q}^t$NNSU are developed. The proposed study is an advancement in the field of neutrosophic set theory-related algebraic structures.

Keywords: Algebraic Structures; Indeterminacy Management; Uncertainty Quantification; Hybrid Set Theory; Neutrosophic Operations.

1. Introduction

To deal with uncertainty, fuzzy sets [1] allow components to have different levels of membership in a set. Interval-valued fuzzy sets (IVFS), which expand fuzzy sets' characteristics and offer intervals for the objects, were first described by [2]. By extending the fuzzy set structure, [3] created intuitionistic fuzzy sets (IFSs), which have two components: the degree of non-existence and the degree of existence, with the requirement that their total fall between 0 and 1. Decisions are hazy and unclear due in large part to vagueness, uncertainty, or ambiguity, as is the case in every circumstance experienced in daily life. In consideration of this, Smarandache [4] devised an arrangement of neutrosophic sets, which combines the degree of indeterminacy with earlier degrees all of which are independent with sums between 0 and 3. Furthermore, an expanded version of the neutrosophic set as an interval-valued neutrosophic set [5] provides a sophisticated solution for decision analysis issues in the real world.

In the world of algebra and its extensions, the concept of BCK-algebra was first established in 1978 by Iseki et al. with its modification known as BCI-algebra which was developed in 1980 and was the most improved enhancement in the field of algebra [6, 7]. The BCK-algebra is the major subclass of BCI-algebra. In 1983, their advanced characteristics and features were explored by Hu et al. who also presented the idea of BCH -algebra which is a generalized form of BCK and BCI algebras. In 2002, J. Neggers et al. introduced a novel idea labeled as B-algebra and produced several findings. A.

Walendziak introduced the term "BFalgebra," a generalization of B-algebra, in 2007. Senapati et al. [8] implement cubic set to ideals, closed ideals, and B-algebra subalgebras. On B-algebra, Khalid et al. [9] provided the MBJ-neutrosophic T-ideal. The BMBJ and MBJ-neutrosophic subalgebraic structures and their intended applications in the BCK/BCI-algebras have also been addressed in the literature [10, 11]. [12] addressed the connections of these algebras and others including Q, BCI, BCH, BF, and B -algebras as well as the novel idea of G -algebra, which is a generalization of QS-algebras.They studied the associated features of the G-part, 0-commutative, and medial of G-algebra concepts. Saeed et al. [13] present embedded algebraic structures for soft members and soft elements. Senapati et al. [14, 15] proposed the idea of L-fuzzy G-subalgebras by implementing the FSs to G-subalgebras and Interval-valued IF BG-subalgebras. The researchers have put a lot of effort into BG-algebras [16]. Saeed et al. [17] recently provided the concept of cubic soft ideals in the settings of B-algebra. Moreover, the integrated development of neutrosophic soft cubic T-ideal in the environment of PS-algebra was developed by N.A.Khalid et al. in [18]. A collection of level subalgebras is known as the G-subalgebras. G-subalgebras are fuzzy intuitivistically along with other associated features, and the categorization of IF G-algebras is provided in [19-22]. In [23], Khalid et al. presented the idea of IF-translation to IF-subalgebra and ideals in the environment of G-algebra. They also study a couple of associated properties with IF-extension and introduce the IF-multiplication of IF-subalgebra of G-algebra. By using the concept of P-union, P-intersection, R-union, and R-intersection among other operations, Khalid et al. [24] examined the neutrosophic soft cubic G-subalgebra. Furthermore, in the context of basic logic algebras, new kinds of neutrosophic filters are developed [25]. Also, there is a new algebraic structure is available which is developed by [26] called pura vida neutrosophic algebra consisting of binary operations of addition and multiplication.

1.1 Motivation

The motivation of this proposed initiative is to establish a modified algebraic framework of a neutrosophic set named as $Q^t\mathrm{NS}$ set. This modified structure has indeterminacy as a dependent component which provides a controlled environment for indeterminate situations while dealing with real-life problems in decision-making as well as in other fields, Further, the N-subalgebra, a new substructure for the G-algebra is introduced which gives more precise conditions for a $Q^t\mathrm{NS}$ is set to be a $Q^t\mathrm{NSN}$-subalgebra.

Five sections make up this study. The introduction is in the first part, followed by fundamental definitions in the second, and theoretical analysis of our suggested structure in the third and fourth. The final part summarizes the findings and makes recommendations for the future.

1.2 Research Problem

In the realm of decision-making and mathematical modeling, uncertainty plays a critical role in complex problem-solving, particularly in environments where data is incomplete or imprecise. Existing methods, such as classical fuzzy sets and intuitionistic fuzzy sets, have made significant strides in addressing uncertainty by incorporating degrees of truth, falsity, and indeterminacy. However, these methods often fail to effectively manage the degree of indeterminacy, leading to inaccurate or imprecise outcomes in various applications, particularly in multi-criteria decision-making (MCDM) scenarios.

1.3 Importance and Necessity

The importance of handling uncertainty in decision-making cannot be overstated, especially in fields like engineering, economics, and operations research, where precise decisions are essential for optimizing systems and processes. In particular, industries such as healthcare, manufacturing, and logistics are increasingly relying on robust decision-support systems that can manage uncertainty effectively. The ability to quantify and reduce indeterminacy while maintaining flexibility in representing uncertainty is crucial for enhancing the reliability and accuracy of these systems.

1.4 Review and Critique of the Current State

Several advancements have been made in the field of neutrosophic sets, which offer a generalized framework to represent uncertainty. However, current methods primarily focus on combining truth, falsity, and indeterminacy linearly, without sufficient consideration of the algebraic structures that can enhance their application. While extensions such as neutrosophic soft sets and intuitionistic fuzzy soft sets have been proposed, they do not address the full potential of algebraic operations within the neutrosophic set framework. Furthermore, there is a lack of research on hybridizing these structures with subalgebras, leading to potential inefficiencies in decision-making processes where indeterminacy reduction is a priority.

1.5 Identification of Research Gaps

Lack of Comprehensive Algebraic Structures: While neutrosophic sets have been widely studied, there is limited research on integrating subalgebraic structures into neutrosophic sets to enhance their applicability.

1.6 Indeterminacy Control

Current methods do not offer robust solutions to control the degree of indeterminacy effectively, which is crucial for accurate decision-making in uncertain environments.

1.7 Limited Hybridization

Few studies explore the hybridization of neutrosophic sets with algebraic structures such as G-algebra, which can provide a more structured approach to uncertainty management.

1.8 Insufficient Development of Operations

There is a gap in the development of operations such as union and intersection within the context of Quantified Neutrosophic N-Subalgebra (Q^tNSSU), which could further improve the model’s flexibility and robustness.

1.9 Study’s Innovation and Contribution

This study proposes a novel hybridization of the Neutrosophic N-Subalgebra (NSU) with Quantified Neutrosophic Sets (Q^tNS), creating the Quantified Neutrosophic N-Subalgebra (Q^tNNSU). This hybrid structure allows for the precise control of indeterminacy within decision-making models, offering a new approach to managing uncertainty. Additionally, the study introduces new operations (P-Union, R-Union, P-Intersection, R-Intersection) within the Q^tNNSU framework, which significantly enhances its applicability and reliability in MCDM scenarios. The introduction of upper and lower bounds, as well as subalgebras, further strengthens the framework, offering a more nuanced approach to multi-criteria decision-making. The innovation lies in the algebraic structure and operations that support effective uncertainty management, contributing to the development of more efficient decision support systems in uncertain environments.

2. Preliminaries

Here, certain fundamental definitions are provided that are useful for the rest of the paper's presentation.

Definition 2.1. [6, 7] The conditions listed below must be satisfied for a $(2,0)$ type algebra to be a BCI-algebra for all ϑ, θ and $\lambda \in A$:
A1: $(\vartheta * \theta) * (\vartheta * \lambda) \leq (\lambda * \theta)$
A2: $(\vartheta * \vartheta) * \theta \leq \theta$
A3: $\vartheta \leq \vartheta$
A4: $\vartheta \leq \theta$ and $\theta \leq \vartheta \Rightarrow \vartheta = \theta$
A5: $\vartheta \leq 0 \Rightarrow \vartheta = 0$, where $\vartheta \leq \theta$ is defined by $\vartheta * \theta = 0$
If (A5) is replaced by (A6): $0 \leq \vartheta$, Then algebra is referred to as a BCK-algebra. Any BCK algebra is also a BCI algebra, but the reverse is not known.

Definition 2.2. [12] Let a nonempty set represented by G is said to be G-algebra $(G, *, 0)$ if it contains the constant 0 and the binary operation $' * '$ or equivalently by G with the following axioms:
G1: $\vartheta_1 * \vartheta_1 = 0$
G2: $\vartheta_1 * (\vartheta_1 * \vartheta_2) = \vartheta_2$, for all $\vartheta_1, \vartheta_2 \in G$.
For the set $U = \{0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6, \vartheta_7\}$, G-algebra can be represented with Cayley's table which is shown in Table 1:

Definition 2.3. Consider S_G a subset of G-algebra 'G'. If
- $\vartheta_1 * \vartheta_2 \in S_G \forall \vartheta_1, \vartheta_2 \in S_G$,

Then it is referred to as a G-subalgebra.

Definition 2.4. [1] A FS in G can be expressed as $F = \{\langle \vartheta_1, \psi_F(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$, where $\psi_F(\vartheta_1)$ is referred to as the existent ship value of ϑ_1 in F and $\psi_F(\vartheta_1) \in [0,1]$.

Table 1. Cayley's Table of $(G, *, 0)$.

$*$	0	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ϑ_7
0	0	ϑ_2	ϑ_1	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ϑ_7
ϑ_1	ϑ_1	0	ϑ_3	ϑ_2	ϑ_5	ϑ_4	ϑ_7	ϑ_6
ϑ_2	ϑ_2	ϑ_3	0	ϑ_1	ϑ_6	ϑ_7	ϑ_4	ϑ_5
ϑ_3	ϑ_3	ϑ_2	ϑ_1	0	ϑ_7	ϑ_6	ϑ_5	ϑ_4
ϑ_4	ϑ_4	ϑ_5	ϑ_6	ϑ_7	0	ϑ_2	ϑ_1	ϑ_3
ϑ_5	ϑ_5	ϑ_4	ϑ_7	ϑ_6	ϑ_1	0	ϑ_3	ϑ_2
ϑ_6	ϑ_6	ϑ_7	ϑ_4	ϑ_5	ϑ_2	ϑ_3	0	ϑ_1
ϑ_7	ϑ_7	ϑ_6	ϑ_5	ϑ_4	ϑ_3	ϑ_2	ϑ_1	0

- Fuzzy subalgebra is described by Ahn et al. [28] in the definition that accompanies it.

Definition 2.5. Let $F = \{\langle \vartheta_1, \psi_F(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$ be a FS in G. If
- $\psi_F(\vartheta_1 * \vartheta_2) \geq \min\{\psi_F(\vartheta_1), \psi_F(\vartheta_2)\} \forall \vartheta_1, \vartheta_2 \in G$, Then it is said to be fuzzy G-subalgebra of G.

Definition 2.6. [3] An IFS in G is a kind of structure $I = \{\langle \psi_I(\vartheta_1), \varphi_I(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$, where ψ_I is the existent ship value function of ϑ_1 and φ_I is a non-existent ship value function of ϑ_1 w.r.t I in G and $0 \leq \psi_I(\vartheta_1) + \varphi_I(\vartheta_1) \leq 1$.

Definition 2.7. [27] Let $IF = \psi_{IF}(\vartheta_1), \varphi_{IF}(\vartheta_1)$ be an IFS of any algebra K. It is said to be a t-IF subset of K represented by IF^t with t belongs to the closed interval of $[0,1]$ and is structured as

$$IF^t = \{< \vartheta_1, \psi_{IF^t}(\vartheta_1), \varphi_{IF^t}(\vartheta_1) > \mid \vartheta_1 \in K\} = < \psi_{IF^t}, \varphi_{IF^t} >$$

where,

- $\psi_{IF^t}(\vartheta_1) = \min\{\psi_{IF}(\vartheta_1), t\}$ and
- $\psi_{IF^t}(\vartheta_1) = \max\{\varphi_{IF}(\vartheta_1), 1-t\} \forall \vartheta_1 \in K$.

Definition 2.8. [27, 29] Let $IF^t =< \psi_{IF^t}, \varphi_{IF^t} >$ represents t-IFS of K along with $t \in [0,1]$ then it is to be called t-IFSU of K as it corresponds with these assumptions:

i). $\psi_{IF^t}(\vartheta_1 * \vartheta_2) \geq \min\{\psi_{IF^t}(\vartheta_1), \psi_{IF^t}(\vartheta_2)\}$ and

ii). $\varphi_{IF^t}(\vartheta_1 * \vartheta_2) \leq \max\{\varphi_{IF^t}(\vartheta_1), \varphi_{IF^t}(\vartheta_2)\} \forall \vartheta_1, \vartheta_2 \in G$.

Definition 2.9. [24]In the algebra G, the sort of structure classified as a neutrosophic set is $N_S = \{\langle \phi(\vartheta_1), \varphi(\vartheta_1), \psi(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$, Where $\phi, \varphi,$ and ψ are the existent ship value, nonexistent ship value, and indeterminate value of ϑ_1 w.r.t N_S in G and $0 \leq \phi(\vartheta_1) + \varphi(\vartheta_1) + \psi(\vartheta_1 \leq 3$.

Definition 2.10. [30] For the family of a fuzzy set $F_i = \{\langle \vartheta_1, \vartheta_{F_i}(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$ in G, where I_n stands for the index set and$i \in I_n$, The definitions of join and meet represented by $(\vee)$ and $(\wedge)$ respectively are as follows:

$$\vee_{i\in I_n} F_i = \left(\vee_{i\in I_n} \vartheta_{F_i}\right)(\vartheta_1) = \sup\{\vartheta_{F_i} \mid i \in I_n\}$$

and

$$\wedge_{i\in I_n} F_i = \left(\wedge_{i\in I_n} \vartheta_{F_i}\right)(\vartheta_1) = \inf\{\vartheta_{F_i} \mid i \in I_n\}$$

$\forall \vartheta_1 \in G$.

Definition 2.11. [31] For any cubic set $\kappa_i = (\varpi_i, \omega_i)$, where $\varpi_i = \{\langle \vartheta_1; E_{\varpi_i}(\vartheta_1), I_{\varpi_i}(\vartheta_1), F_{\varpi_i}\ (\vartheta_1)\rangle \mid \vartheta_1 \in G\}, \omega_i = \{\langle \vartheta_1; E_{\omega_i}(\vartheta_1), I_{\omega_i}(\vartheta_1), F_{\omega_i}(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$ for $i \in I_n$, the structures of P-union, P-intersection, R-union, and R-intersection are respectively given as follows:

- P-union $\rightarrow \cup_{P\, i\in I_n} \kappa_i = \left(\underset{i\in I_n}{\cup} \varpi_i, \underset{i\in I_n}{\vee} \omega_i\right)$,
- R-union $\rightarrow \cup_{i\in I_n} \kappa_i = \left(\underset{i\in I_n}{\cup} \varpi_i, \overset{\wedge}{_{i\in I_n}}\omega_i\right)$
- P-intersection $\rightarrow \cap_{i\in I_n}\kappa_i = \left(\underset{i\in I_n}{\cap_i} \varpi_i, \wedge^{\wedge}_{i\in I_n} \omega_i\right)$ and
- R-intersection $\rightarrow \cap^R_{i\in I_n}\kappa_i = \left(\underset{i\in I_n}{\cap} \varpi_i, \underset{i\in I_n}{\vee} \omega_i\right)$

Where

$$\cup_{i\in I_n} \varpi_i = \left\{\langle \vartheta_1; \left(\cup_{i\in I_n} \varpi_{iE}\right)(\vartheta_1), \left(\cup_{i\in I_n} \varpi_{iI}\right)(\vartheta_1), \left(\cup_{i\in I_n} \varpi_{iF}\right)(\vartheta_1)\rangle \mid \vartheta_1 \in G\right\}$$
$$\vee_{i\in I_n} \omega_i = \left\{\langle \vartheta_1; \left(\vee_{i\in I_n} \omega_{iE}\right)(\vartheta_1), \left(\vee_{i\in I_n} \omega_{iI}\right)(\vartheta_1), \left(\vee_{i\in I_n} \omega_{iF}\right)(\vartheta_1)\rangle \mid \vartheta_1 \in G\right\}$$
$$\cap_{i\in I_n} \varpi_i = \left\{\langle \vartheta_1; \left(\cap_{i\in I_n} \varpi_{iE}\right)(\vartheta_1), \left(\cap_{i\in I_n} \varpi_{iI}\right)(\vartheta_1), \left(\cap_{i\in I_n} \varpi_{iF}\right)(\vartheta_1)\rangle \mid \vartheta_1 \in G\right\}$$
$$\wedge_{i\in I_n} \omega_i = \left\{\langle \vartheta_1; \left(\wedge_{i\in I_n} \omega_{iE}\right)(\vartheta_1), \left(\wedge_{i\in I_n} \omega_{iI}\right)(\vartheta_1), \left(\wedge_{i\in I_n} \omega_{iF}\right)(\vartheta_1)\rangle \mid \vartheta_1 \in G\right\}$$

3. Development of Quantified Neutrosophic N-subalgebras Q^tNNSU

This section defines a novel. Q^tNS set structure and explains the requirements for a Q^tNS set to be a Quantified Neutrosophic N-subalgebra Q^tNNSU in the setting of G-algebra. By taking into consideration several additional relevant qualities, the proposed structure is investigated using a variety of concepts, including upper bounds, lower bounds, upper-level subalgebra, lower-level subalgebra, and fuzzy subalgebra.

Definition 3.1. A neutrosophic set $\mathcal{N} = \{< \vartheta_1, E(\vartheta_1), I(\vartheta_1), F(\vartheta_1) >\mid \vartheta_1 \in G\} =< E, I, F >$ is called a Q^tNS set of the form $\mathcal{N}^t = (E^t, I^t, F^t)$, where $\mathcal{N}^t = \{\langle \vartheta_1, E^t(\vartheta_1), I^t(\vartheta_1), F^t(\vartheta_1)\rangle \mid \vartheta_1 \in G\} =<$

$E^t, I^t, F^t >$ with two independent components that are $E^t(\vartheta_1)$ and $F^t(\vartheta_1)$ along one dependent component $I^t(\vartheta_1)\forall\vartheta_1 \in G$. These components are defined as:

$$E^t(\vartheta_1) = \max\{E(\vartheta_1), t\},$$
$$I^t(\vartheta_1) = \min\{I(\vartheta_1), (t + t')/2\},$$
$$F^t(\vartheta_1) = \max\{F(\vartheta_1), t'\}$$

$\forall t, t', t + t'/2 \in [0,1]$, where E^t is the existent ship value function, I^t is an indeterminate value function and F^t is a non-existent ship value function with $0 \leq E^t(\vartheta_1) + I^t(\vartheta_1) + F^t(\vartheta_1) \leq 3$. From this point forward, the symbols $\preceq$ and $\succeq$ will stand for less than and greater than, respectively, with equality.

Definition 3.2. Let $\mathcal{N}^t =< E^t(\vartheta_1), I^t(\vartheta_1), F^t(\vartheta_1) >$ be a Q^tNS set. Then $\mathcal{N}^t$ is TNNSU with a binary operation $' *'$, where $\vartheta_1, \vartheta_2, t, t', t + t'/2, \Phi, \Psi \in [0,1]$ and when it meets the three conditions listed below:

N1: $\min\{E((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)), t\} = E^t((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) \preceq \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_2 * \Psi)\}$,
N2: $\min\{I((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)), t + t'/2\} = I^t((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) \succeq \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_2 * \Psi)\}$,
N3: $\max\{F((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)), t'\} = F^t((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) \preceq \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_2 * \Psi)\}$.

To preserve everything simple, we substitute Ω for $t + t'/2'$.

Example 3.3. Let $u = \{0, \vartheta_1 * \Phi, \vartheta_2 * \Psi\}$ be a G-algebra represented by the following Cayley table.

Table 2. Cayley's Table of $(G,*,0)$.

$*$	0	$\vartheta_1 * \Phi$	$\vartheta_2 * \Psi$
0	0	$\vartheta_1 * \Phi$	$\vartheta_2 * \Psi$
$\vartheta_1 * \Phi$	$\vartheta_1 * \Phi$	0	$\vartheta_2 * \Psi$
$\vartheta_2 * \Psi$	$\vartheta_2 * \Psi$	$\vartheta_1 * \Phi$	0

A Q^tNS set $\mathcal{N}^t = \{< \vartheta_1, E^t(\vartheta_1), I^t(\vartheta_1), F^t(\vartheta_1) > \mid \vartheta_1 \in G\} =< E^t, I^t, F^t >$ of G by taking $t = 0.4$ and $t' = 1 \longrightarrow \Omega = 0.6$ is defined by

	0	$\vartheta_1 * \Phi$	$\vartheta_2 * \Psi$
E^t	0.3	0.1	0.4

	0	$\vartheta_1 * \Phi$	$\vartheta_2 * \Psi$
I^t	0.6	0.4	0.6

	0	$\vartheta_1 * \Phi$	$\vartheta_2 * \Psi$
F^t	1	1	1

To verify that a set is Quantified Neutrosophic N-subalgebra (Q^tNNSU) is standard procedure.

Proposition 3.4. Let $Q^t = \{\langle\vartheta_1, E^t(\vartheta_1), I^t(\vartheta_1), F^t(\vartheta_1)\rangle\}$ is a Q^tNNSU of G, then $\forall\vartheta_1 \in G$, $E^t(0 * \Phi) \preceq E^t(\vartheta_1 * \Phi), I^t(0 * \Phi) \succeq I^t(\vartheta_1 * \Phi)$ and $F^t(0 * \Phi) \preceq F^t(\vartheta_1 * \Phi)$. Thus, $E^t(0 * \Phi)$, $I^t(0 * \Phi)$ and $F^t(0 * \Phi)$ are the upper and lower bounds of $E^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Phi)$ and $F^t(\vartheta_1 * \Phi)$ respectively.

Proof. $\forall\vartheta_1 \in G$, we have $E^t((0 * \Phi)) = \min(E((0 * \Phi)), t) = \min(E((\vartheta_1 * \Phi) * (\vartheta_1 * \Phi)), t) \preceq \max\{\min(E((\vartheta_1 * \Phi)), t), \min(E(\vartheta_1 * \Phi), t)\} = \min(E(\vartheta_1 * \Phi), t) = E^t((\vartheta_1 * \Phi)) \Rightarrow E^t((0 * \Phi)) \preceq E^t((\vartheta_1 * \Phi)), I^t(0 * \Phi) = \min(I(0 * \Phi), t') = \min(I((\vartheta_1 * \Phi) * (\vartheta_1 * \Phi)), t') \succeq \min\{\min(I(\vartheta_1 * \Phi), t')\min(I(\vartheta_1 * \Phi), t')\} = \min(I(\vartheta_1 * \Phi), t') = I^t(\vartheta_1 * \Phi) \Rightarrow I^t(0 * \Phi) \succeq I^t(\vartheta_1 * \Phi)$ and $\max(F(0 *$

$\Phi),\Omega) = \max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_1 * \Phi)\big),\Omega\big) \preceq \max\{\max(F(\vartheta_1 * \Phi),\Omega), \max(F(\vartheta_1 * \Phi),\Omega)\} = \max(F(\vartheta_1 * \Phi), t) = F^t(\vartheta_1 * \Phi) \Rightarrow F^t(0 * \Phi) \preceq F^t(\vartheta_1 * \Phi)$.

Theorem 3.5. Let $Q^t = \{\langle(\vartheta_1), E^t(\vartheta_1), I^t(\vartheta_1), F^t(\vartheta_1)\rangle\}$ be a Q^tNNSU of G. If there exists a sequence $\{(\vartheta_1 * \Phi)_n\}$ of G such that $\lim_{n\to\infty} E^t((\vartheta_1 * \Phi)_n) = 0, \lim_{n\to\infty} I^t((\vartheta_1 * \Phi)_n) = 1$ and $\lim_{n\to\infty} F^t((\vartheta_1 * \Phi)_n) = 0$. Then $E^t(0) = 0, I^t(0) = 1$ and $F^t(0) = 0$.

Proof. Using Proposition, $E^t(0 * \Phi) \preceq E^t(\vartheta_1 * \Phi) \forall \vartheta_1 \in G$, so therefore $E^t(0 * \Phi) \preceq E^t((\vartheta_1 * \Phi)_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \preceq E^t(0 * \Phi) \preceq \lim_{n\to\infty} E^t((\vartheta_1 * \Phi)_n) = 0$. Hence, $E^t(0 * \Phi) = 0$. Using Proposition, $I^t(0 * \Phi) \succeq I^t(\vartheta_1 * \Phi) \forall \vartheta_1 \in G$, so therefore $I^t(0 * \Phi) \succeq I^t((\vartheta_1 * \Phi)_n)$ for $n \in \mathbf{Z}^+$. Consider, $1 \succeq I^t(0 * \Phi) \succeq \lim_{n\to\infty} I^t((\vartheta_1 * \Phi)_n) = 1$. Hence, $I^t(0 * \Phi) = 1$.
Again, using Proposition, $F^t(0 * \Phi) \preceq F^t(\vartheta_1 * \Phi) \forall \vartheta_1 \in G$, so therefore $F^t(0 * \Phi) \preceq F^t((\vartheta_1 * \Phi)_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \preceq F^t(0 * \Phi) \preceq \lim_{n\to\infty} F^t((\vartheta_1 * \Phi)_n) = 0$. Hence, $F^t(0 * \Phi) = 0$.

Proposition 3.6. If a Q^tNS set $\mathcal{E}^t = (E^t, I^t, F^t)$ of G is a Q^tNNSU, then $\forall \vartheta_1 \in G$, $E^t(0 * \vartheta_1) \preceq E^t(\vartheta_1 * \Phi)$ and $I^t(0 * \vartheta_1) \succeq I^t(\vartheta_1 * \Phi)$ and $F^t(0 * \vartheta_1) \preceq F^t(\vartheta_1 * \Phi)$.

Proof. $\forall \vartheta_1 \in G, E^t(0 * \vartheta_1) = \min(E(0 * \vartheta_1), t) \preceq \max\{\min(E(0), t), \min(E(\vartheta_1 * \Phi), t)\} = \max\{\min(E(\vartheta_1 * \vartheta_1), t), \min(E(\vartheta_1 * \Phi), t)\} = \max\{\min\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Phi), t)\}, \min(E(\vartheta_1 * \Phi), t)\} = \max(E(\vartheta_1 * \Phi), t) = E^t(\vartheta_1 * \Phi)$ and $I^t(0 * \vartheta_1) = \min(I(0 * \vartheta_1), t') \succeq \min\{\min(I(0), t'), \min(I(\vartheta_1 * \Phi), t')\} = \min\{\min(I(\vartheta_1 * \vartheta_1), t'), \min(I(\vartheta_1 * \Phi), t')\} = \min\{\min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Phi), t')\}, \min(I(\vartheta_1 * \Phi), t')\} = \min(I(\vartheta_1 * \Phi), t') = I^t(\vartheta_1 * \Phi)$ and $F^t(0 * \vartheta_1) = \max(F(0 * \vartheta_1), \Omega) \preceq \max\{\max(F(0), \Omega), \max(F(\vartheta_1 * \Phi), \Omega)\} = \max\{\max(F(\vartheta_1 * \vartheta_1), \Omega), \max(F(\vartheta_1 * \Phi), \Omega)\} = \max\{\max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(\vartheta_1 * \Phi), \Omega)\}, \max(F(\vartheta_1 * \Phi), \Omega)\} = \max(F(\vartheta_1 * \Phi), \Omega) = F^t(\vartheta_1 * \Phi)$▯

Lemma 3.7. If a Q^tNS set $Q^t = (E^t, I^t, F^t)$ of G is a Q^tNNSU, then $Q^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = Q^t\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big) \forall \vartheta_1, \vartheta_2 \in G$.

Proof. Let G be a G-algebra and $\vartheta_1, \vartheta_2 \in G$. Then, by lemma, we know that $\vartheta_2 = 0 * (0 * \vartheta_2)$. Hence, $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = E^t\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big)$ and $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = I^t\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big)$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = F^t\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big)$. Therefore, $Q^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = Q^t\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big)$ ▯

Proposition 3.8. If Q^tNS set $Q^t = (E^t, I^t, F^t)$ of G is a Q^tNNSU, then $\forall \vartheta_1, \vartheta_2 \in G$, $E^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \preceq \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \succeq \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \preceq \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$.

Proof. Let $\vartheta_1, \vartheta_2 \in G$. Then we have $E^t\big(\vartheta_1 * (0 * \vartheta_2)\big) = \min\big(E\big(\vartheta_1 * (0 * \vartheta_2)\big), t\big) \preceq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(0 * \vartheta_2), t)\} \preceq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Psi), t)\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big(\vartheta_1 * (0 * \vartheta_2)\big) = \min\big(I\big(\vartheta_1 * (0 * \vartheta_2)\big), t'\big) \succeq \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(0 * \vartheta_2), t')\} \succeq \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big(\vartheta_1 * (0 * \vartheta_2)\big) = \max\big(F\big(\vartheta_1 * (0 * \vartheta_2)\big), \Omega\big) \preceq \max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(0 * \vartheta_2), \Omega)\} \preceq \max\{\max(F(\vartheta_1 * \Phi), t), \max(F(\vartheta_1 * \Psi), \Omega)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$ by definition and proposition.

Proposition 3.9. If Q^tNS set $Q^t = (E^t, I^t, F^t)$ of G meets all the conditions mentioned here., then Q^t refers to a Q^tNNSU of G.
(1) $E^t(0 * \vartheta_1) \preceq E^t(\vartheta_1 * \Phi)$ and $I^t(0 * \vartheta_1) \succeq I^t(\vartheta_1 * \Phi)$ and $F^t(0 * \vartheta_1) \preceq F^t(\vartheta_1 * \Phi) \forall\ \vartheta_1 \in G$.
(2) $E^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \preceq \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \succeq \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big(\vartheta_1 * (0 * \vartheta_2)\big) \preceq \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\} \forall \vartheta_1, \vartheta_2 \in G$ and $t \in [0,1]$.

Proof. Let Q^tNS set $Q^t = (E^t, I^t, F^t)$ of G meets the statements listed above (1 and 2). then using Lemma, we have $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \big\{\min\big(E\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\big\} = \{\min(E(\vartheta_1 * (0 * (0 *$

$\vartheta_2))\big), t)\Big\} \leq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(0 * \vartheta_2), t)\} = \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(0 * \vartheta_2), t)\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\} = \Big\{\min\Big(I\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big), t'\Big)\Big\} \geq \min\{\min(I(\vartheta_1 * \Phi), t), \min(I(0 * \vartheta_2), t')\} \geq \min\{\min(I(\vartheta_1 * \Phi), t), \min(I(0 * \vartheta_2), t')\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), \Omega\big)\} = \Big\{\max\Big(F\Big(\vartheta_1 * \big(0 * (0 * \vartheta_2)\big)\Big), \Omega\Big)\Big\} \leq \max\{\max(F(\vartheta_1 * \Phi), t), \max(F(0 * \vartheta_2), \Omega)\} \leq \max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(0 * \vartheta_2), \Omega)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\} \forall \vartheta_1, \vartheta_2 \in G$. Hence, Q^t is Q^tNNSU of G .

Theorem 3.10. The Q^tNS set $Q^t = (E^t, I^t, F^t)$ of G is a Q^tNNSU of $G \Leftrightarrow E^t$ and I^t and F^t are fuzzy N-subalgebras of G.

Proof. Let E^t, I^t and J^t are fuzzy N-subalgebra of G and $\vartheta_1, \vartheta_2 \in G$ and $t, t', \Omega \in [0,1]$. Then $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(E\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\} \leq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Psi), t)\} = \min\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\} \geq \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), \Omega\big)\} \leq \max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(\vartheta_1 * \Psi), \Omega)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$.

Conversely, assume that Q^t is a Q^tNNSU of G. For any $\vartheta_1, \vartheta_2 \in G, E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(M\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\} \leq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Psi), t)\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $\big[I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\big] = I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\} \geq \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\} = \min\{[\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t'), \min(I(\vartheta_1 * \Psi), t')]\} = [\min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\}, \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\}] = [\min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}, \min\{I^t(\vartheta_1 * (\vartheta_1 * \Phi), I^t\Phi), I^t(\vartheta_1 * \Psi)\}]$. Thus, $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \geq \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}, I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \geq \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), \Omega\big)\} \leq \max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(\vartheta_1 * \Psi), \Omega)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$. Hence E^t and I^t and F^t are fuzzy N-subalgebra of G.

Theorem 3.11. Let $Q^t = (E^t, I^t, F^t)$ be a Q^tNNSU of G. Then the sets I_{E^t}, I_{I^t} and I_{F^t} which are defined as $I_{E^t} = \{\vartheta_1 \in G \mid E^t(\vartheta_1 * \Phi) = E^t(0)\}$, $I_{I^t} = \{\vartheta_1 \in G \mid I^t(\vartheta_1 * \Phi) = I^t(0)\}$ and $I_{F^t} = \{\vartheta_1 \in G \mid F^t(\vartheta_1 * \Phi) = F^t(0)\}$ are Q^t NNSU of G.

Proof. Let $\vartheta_1, \vartheta_2 \in I_{E^t}$. Then $E^t(\vartheta_1 * \Phi) = E^t(0) = E^t(\vartheta_1 * \Psi)$ and so, $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(E\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\} \leq \max\{\big(\min(E(\vartheta_1 * \Phi), t)\big), \big(\min(E(\vartheta_1 * \Psi), t)\big)\} = E^t(0)$. By using Proposition, as we know that $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = E^t(0)$ or equivalently $(\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in I_{E^t}$.

Now we let $\vartheta_1, \vartheta_2 \in I_{I^t}$. Then $I^t(\vartheta_1 * \Phi) = I^t(0) = I^t(\vartheta_1 * \Psi)$ and so, $I^t((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) = \{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\} \geq \min\{\big(\min(I(\vartheta_1 * \Phi), t')\big), \big(\min(I(\vartheta_1 * \Psi), t')\big)\} \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\} = I^t(0)$. By using Proposition, as we know that $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = I^t(0)$ or equivalently$(\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in I_{I^t}$.

Again we let $\vartheta_1, \vartheta_2 \in I_{F^t}$. Then $F^t(\vartheta_1 * \Phi) = F^t(0) = F^t(\vartheta_1 * \Psi)$ and so, $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\} \leq \max\{\big(\max(F(\vartheta_1 * \Phi), t)\big), \big(\max(F(\vartheta_1 * \Psi), t)\big)\} = F^t(0)$. Again by using Proposition, as we know that $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = F^t(0)$ or equivalently $(\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in I_{F^t}$. Hence the sets $I_{E^t} I_{I^t}$ and I_{F^t} are subalgebra of G.

Definition 3.12. Let $Q^t = (E^t, I^t, F^t)$ be a T neutrosophic set of G. For $s_1 \in [0,1]$ and $\vartheta_1, \vartheta_2 \in [0,1]$, the set $U(E^t \mid \vartheta_1) = \{\vartheta_1 \in G \mid E^t(\vartheta_1 * \Phi) \geq \vartheta_1$ is called upper ϑ_1-level of Q^t and the set $U(I^t \mid (s_1) = \{s_1 \in G \mid I^t(\vartheta_1 * \Phi) \geq s_1$ is called upper s_1-level of Q^t and $L\big(F^t \mid (t_1)\big) = \{\vartheta_1 \in G \mid F^t(\vartheta_1 * \Phi) \leq \vartheta_1$ is called lower (ϑ_1)-level of Q^t.

Theorem 3.13. If $Q^t = (E^t, I^t, F^t)$ is Q^tNNSU of G, then the upper ϑ_1-level, upper s_1-level and lower ϑ_1-level of Q^t are N-subalgebra of G.

Proof. Let $\vartheta_1, \vartheta_2 \in U(E^t \mid \vartheta_1)$. Then $E^t(\vartheta_1 * \Phi) \succeq \vartheta_1$ and $E^t(\vartheta_1 * \Psi) \succeq \vartheta_1$. It follows that $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \big\{\min\big(E\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\big\} \preceq \max\{\min(E(\vartheta_1 * \Phi), E(\vartheta_1 * \Psi), t)\} = \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Psi), t)\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\} \preceq \vartheta_1 \Rightarrow (\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in U(E^t \mid \vartheta_1)$. Hence $U(E^t \mid \vartheta_1)$ is a N-subalgebra of G. Let $\vartheta_1, \vartheta_2 \in U(I^t \mid s_1$. Then $I^t(\vartheta_1 * \Phi) \succeq s_1$ and $I^t(\vartheta_1 * \Psi) \succeq s_1$. It follows that $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \big\{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\big\} \succeq \min\{\min(I(\vartheta_1 * \Phi), I(\vartheta_1 * \Psi), t')\} = \min\{\min(I(\vartheta_1 * \Phi), t'), \min(I(\vartheta_1 * \Psi), t')\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\} \succeq s_1 \Rightarrow (\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in U(I^t \mid s_1)$. Hence, $U(I^t \mid s_1$ is a N − subalgebra of G. Let $\vartheta_1, \vartheta_2 \in L(F^t \mid \vartheta_1)$. Then $F^t(\vartheta_1 * \Phi) \preceq \vartheta_1$ and $F^t(\vartheta_1 * \Psi) \preceq \vartheta_1$. It follows that $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \big\{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), \Omega\big)\big\} \preceq \max\{\max(F(\vartheta_1 * \Phi), F(\vartheta_1 * \Psi), \Omega)\} = \max\{\max(F(\vartheta_1 * \Phi), t), \max(F(\vartheta_1 * \Psi), t)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\} \preceq \vartheta_1 \Rightarrow (\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in L(F^t \mid \vartheta_1)$. Hence, $L(F^t \mid \vartheta_1)$ is a N -subalgebra of G.

Theorem 3.14. Any G-subalgebra may be interpreted as the upper ϑ_1, upper s_1, and lower ϑ_1 levels of some Q^tNNSU.

Proof. Let $\aleph^t$ be a Q^tNNSU of G, and Q^t be a Q^tNS set on G represented by

$$E^t = \begin{cases} [v] & \text{if } \vartheta_1 \in \aleph^t \\ 1, & \text{otherwise} \end{cases}$$

$$I^t = \begin{cases} [\mu] & \text{if } \vartheta_1 \in \aleph^t \\ 0 & \text{otherwise} \end{cases}$$

$$F^t = \begin{cases} [\nu] & \text{if } \vartheta_1 \in \aleph^t \\ 0, & \text{otherwise} \end{cases}$$

$\forall v, \mu, \nu \in [0,1]$.

We discuss the following cases.

Case 1. If $\forall \vartheta_1, \vartheta_2 \in \mathcal{D}^t$ then $E^t(\vartheta_1 * \Phi) = v, I^t(\vartheta_1 * \Phi) = \mu, F^t(\vartheta_1 * \Phi) = \nu$ and $E^t(\vartheta_1 * \Psi) = v, I^t(\vartheta_1 * \Psi) = \mu, F^t(\vartheta_1 * \Psi) = \nu$. Thus $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = v = \min\{v, v\} = \min\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}$ and $I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \mu = \min\mu = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \nu = \max\{\nu, \nu\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$.

Case 2. If $\vartheta_1 \in \Psi^t$ and $\vartheta_2 \notin \Psi^t$, then $E^t(\vartheta_1 * \Phi) = v, I^t(\vartheta_1 * \Phi) = \mu, F^t(\vartheta_1 * \Phi) = \nu$ and $E^t(\vartheta_1 * \Psi) = 0, I^t(\vartheta_1 * \Psi) = 0, F^t(\vartheta_1 * \Psi) = 1$. Thus $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 0 = \max\{v, 0\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}, I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \succeq 0 = \min\{\mu, 0\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$
and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 1 = \max\{\nu, 1\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$.

Case 3. If $\vartheta_1 \notin \Psi^t$ and $\vartheta_2 \in \Psi^t$, then $E^t(\vartheta_1 * \Phi) = 0, I^t(\vartheta_1 * \Phi) = 0, F^t(\vartheta_1 * \Phi) = 1$ and $E^t(\vartheta_1 * \Psi) = v, I^t(\vartheta_1 * \Psi) = \mu, F^t(\vartheta_1 * \Psi) = \nu$. Thus $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 0 = \max\{0, v\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}, I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \succeq 0 = \min\{0, \mu\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 1 = \max\{1, \nu\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$.

Case 4. If $\vartheta_1 \notin \Psi^t$ and $\vartheta_2 \notin \Psi^t$, then $E^t(\vartheta_1 * \Phi) = 0, I^t(\vartheta_1 * \Phi) = 0, F^t(\vartheta_1 * \Phi) = 1$ and $E^t(\vartheta_1 * \Psi) = 0, I^t(\vartheta_1 * \Psi) = 0, F^t(\vartheta_1 * \Psi) = 1$. Thus $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 1 = \max\{0,0\} = \max\{E^t(\vartheta_1 * \Phi), E^t(\vartheta_1 * \Psi)\}, I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \succeq [0,0] = \min\{[0,0]\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\}$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \preceq 1 = \max 1,1 = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi)\}$. Therefore, Q^t is a Q^tNNSU of G.

Theorem 3.15. Let Q^t be a Q^tNS set on G that is presented in the proof of the aforementioned theorem and Q^t be a subset of $G.Q^t$ is a Q^tNS cubic one of G if Q^t is regarded as both the lower and upper-level subalgebra of some Q^tNNSU of G.

Proof. Let Q^t be a Q^tNNSU of G, and $\vartheta_1, \vartheta_2 \in Q^t$. Then $E^t(\vartheta_1 * \Phi) = E^t(\vartheta_1 * \Psi) = \gamma$, $I^t(\vartheta_1 * \Phi) = I^t(\vartheta_1 * \Psi) = \alpha$ and $F^t(\vartheta_1 * \Phi) = F^t(\vartheta_1 * \Psi) = \beta$. Thus $E^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(E\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t\big)\} \leq \max\{\min(E(\vartheta_1 * \Phi), t), \min(E(\vartheta_1 * \Psi), t)\} = \max\{E^t(\vartheta_1 *\Phi), E^t(\vartheta_1 * \Psi),\} = \max\{\gamma, \gamma\} = \gamma, \Rightarrow (\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in Q^t, I^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\min\big(I\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), t'\big)\} \geq \min\{\min(I(\vartheta_1 * \Phi), I(\vartheta_1 * \Psi), t')\} = \min\{\min(I(\vartheta_1 *\Phi), t'), \min(I(\vartheta_1 * \Psi), t')\} = \min\{I^t(\vartheta_1 * \Phi), I^t(\vartheta_1 * \Psi)\} = \min\alpha, \alpha = \alpha$ and $F^t\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \{\max\big(F\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big), \Omega\big)\} \leq \max\{\max(F(\vartheta_1 * \Phi), \Omega), \max(F(\vartheta_1 * \Psi), \Omega)\} = \max\{F^t(\vartheta_1 * \Phi), F^t(\vartheta_1 * \Psi),\} = \max\{\beta, \beta\} = \beta, \Rightarrow (\vartheta_1 * \Phi) * (\vartheta_2 * \Psi) \in Q^t$. Hence, the proof is completed.

4. Modified Concept of Union and Intersection for Q^tNS Se

For the suggested notion of Q^tNS N-subalgebra, the concepts of P-union, P-intersection, R-union, and R-intersection are modified in this section to P_N-Union, P_N--intersection, R_N- Union, and R_N-intersection.

Definition 4.1. For any Q^tNS set $Q_i^t = \{< \vartheta_1, E_i^t(\vartheta_1), I_i^t(\vartheta_1), F_i^t(\vartheta_1) >| \vartheta_1 \in G\} =< E_i^t, I_i^t, F_i^t >$ for $i \in I_n$, the structures of P_N--Union, P_N--Intersection, R_N-Union, and R_N- Intersection are respectively defined as follows:

- P_N-union $\rightarrow \cup_{P_N, i\in I_n} Q_i^t = \{\langle \vartheta_1; (\vee_{i\in I_n} E)(\vartheta_1), (\cup_{i\in I_n} I)(\vartheta_1), (\vee_{i\in I_n} F)(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$,
- R_N-union $\rightarrow \cup_{R_N, i\in I_n} Q_i^t = \{\langle \vartheta_1; (\wedge_{i\in I_n} E)(\vartheta_1), (\cup_{i\in I_n} I)(\vartheta_1), (\wedge_{i\in I_n} F)(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$,
- P_N-intersection $\rightarrow \cap_{P_N} Q_i^t = \{\langle \vartheta_1; (\wedge_{i\in I_n} E)(\vartheta_1), (\cap_{i\in I_n} I)(\vartheta_1), (\wedge_{i\in I_n} F)(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$,
- R_N-intersection $\rightarrow \cap_{R_N, i\in I_n} Q = \{\langle \vartheta_1; (\vee_{i\in I_n} E)(\vartheta_1), (\cap_{i\in I_n} I)(\vartheta_1), (\vee_{i\in I_n} F)(\vartheta_1)\rangle \mid \vartheta_1 \in G\}$.

Example 4.2. Suppose there are three Q^tNS sets $Q_1^t = \{< \vartheta_1, E_1^t(\vartheta_1), I_1^t(\vartheta_1), F_1^t(\vartheta_1) >| \vartheta_1 \in G\} =< E_1^t, I_1^t, F_1^t >, Q_2^t = \{< \vartheta_1, E_2^t(\vartheta_1), I_2^t(\vartheta_1), F_2^t(\vartheta_1) >| \vartheta_1 \in G\} =< E_2^t, I_2^t, F_2^t >$ and $Q_3^t = \{< \vartheta_1, E_3^t(\vartheta_1), I_3^t(\vartheta_1), F_3^t(\vartheta_1) >| \vartheta_1 \in G\} =< E_3^t, I_3^t, F_3^t >$ as:

Table 3. Q^tNS sets $Q_i^t; i = 1,2,3$.

	$E^t(\vartheta_1)$	$I^t(\vartheta_1)$	$F^t(\vartheta_1)$
Q_1^t	0.5	0.3	0.1
Q_2^t	0.9	0.2	1
Q_3^t	0.6	0.4	0.7

Then the P_N-Union, R_N-Union, P_N-intersection, and R_N-intersection of the above three sets is represented by the following Table 4:

Table 4. Tabular representation of P_N-Union, R_N-Union, P_N-intersection, and R_N-intersection.

	$E^t(\vartheta_1)$	$I^t(\vartheta_1)$	$F^t(\vartheta_1)$
P_N-union	0.9	0.4	1
R_N-union	0.5	0.4	0.1
P_N-intersection	0.5	0.2	0.1
R_N-intersection	0.9	0.2	1

Now, we examine that the P_N-Union, P_N-intersection, R_N-Union, and R_N-intersection of Q_i^t are Q^t NNSUs of G with the development of following theorems.

Theorem 4.3. Let $Q_i^t = \{\langle \vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, is a family of sets of Q^tNNSU of G .If $\inf\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Psi)\}\} = \min\{\inf(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Psi)\}$ and $\inf\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Psi)\}\} = \max\{\inf(F_i^t)(\vartheta_1 * \Phi), \inf(F_i^t)(\vartheta_1 * \Psi)\}$ and $\sup\{\min\{(I^t{}_i)(\vartheta_1 * \Phi), (I^t{}_i)(\vartheta_1 * \Psi)\}\} = \min\{\sup(I^t{}_i)(\vartheta_1 * \Phi), \sup(I^t{}_i)(\vartheta_1 * \Psi)\}\forall\vartheta_1 \in G$, and $t \in [0,1]$ then R_N-union of Q_i^t is also a Q^tNNSU of G.

Proof. Let $Q_i^t = \{\langle \vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, and $t \in [0,1]$ is a family of sets of Q^tNNSU of G such that $\inf\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Psi)\}\} = \min\{\inf(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Psi)\}$ and $\inf\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Psi)\}\} = \max\{\inf(F_i^t)(\vartheta_1 * \Phi), \inf(F_i^t)(\vartheta_1 * \Psi)\}$ and $\sup\{\min\{(I^t{}_i)(\vartheta_1 * \Phi), (I^t{}_i)(\vartheta_1 * \Psi)\}\} = \min\{\sup(I^t{}_i)(\vartheta_1 * \Phi)\sup(I^t{}_i)(\vartheta_1 * \Psi)\}\forall\vartheta_1 \in G$, and $t \in [0,1]$. Then for $\vartheta_1, \vartheta_2 \in G$ and $t \in [0,1]$.

$$\begin{aligned}
(\wedge (E_i^t))((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &= \left(\wedge \left(\min(E_i, t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\right)\right) \\
&= \inf\left\{\left(\min(E_i, t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\right)\right\} \\
&\preceq \inf\{\max\{\min(E_i, t)(\vartheta_1 * \Phi), \min(E_i, t)(\vartheta_1 * \Psi)\}\} \\
&= \max\{\inf(\min(E_i, t)(\vartheta_1 * \Phi)), \inf(\min(E_i, t)(\vartheta_1 * \Psi))\} \\
&= \max\{\inf((E_i^t)(\vartheta_1 * \Phi)), \inf((E_i^t)(\vartheta_1 * \Psi))\} \\
&= \max\{\wedge (E_i^t)(\vartheta_1 * \Phi), \wedge (E_i^t)(\vartheta_1 * \Psi)\} \\
\Rightarrow \wedge (E_i^t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &\preceq \max\{\wedge (E_i^t)(\vartheta_1 * \Phi), \wedge (E_i^t)(\vartheta_1 * \Psi)\}
\end{aligned}$$

and

$$\begin{aligned}
(\cup (I^t{}_i))((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &= (\cup (\min(I_i, t'))((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))) \\
&= \sup\{\min(I_i, t')((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\} \\
&\succeq \sup\{\min\{\min(I_i, t')(\vartheta_1 * \Phi), \min(I_i, t')(\vartheta_1 * \Psi)\}\} \\
&= \min\{\sup(\min(I_i, t')(\vartheta_1 * \Phi)), \sup(\min(I_i, t')(\vartheta_1 * \Psi))\} \\
&= \min\{\sup((I_i^t)(\vartheta_1 * \Phi)), \sup((I_i^t)(\vartheta_1 * \Psi))\} \\
&= \min\{\cup (I_i^t)(\vartheta_1 * \Phi), \cup (I_i^t)(\vartheta_1 * \Psi)\} \\
\Rightarrow \cup (I_i^t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &\succeq \min\{\cup (I_i^t)(\vartheta_1 * \Phi), \cup (I_i^t)(\vartheta_1 * \Psi)\}
\end{aligned}$$

and

$$\begin{aligned}
(\wedge (F_i^t))((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &= (\wedge (\max(F_i, \Omega))((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))) \\
&= \inf\{\max(F_i, \Omega)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\} \\
&\preceq \inf\{\max\{\max(F_i, \Omega)(\vartheta_1 * \Phi), \max(F_i, \Omega)(\vartheta_1 * \Psi)\}\} \\
&= \max\{\inf(\max(F_i, \Omega)(\vartheta_1 * \Phi)), \inf(\max(J_i, \Omega)(\vartheta_1 * \Psi))\} \\
&= \max\{\inf((F_i^t)(\vartheta_1 * \Phi)), \inf((F_i^t)(\vartheta_1 * \Psi))\} \\
&= \max\{\wedge (F_i^t)(\vartheta_1 * \Phi), \wedge (F_i^t)(\vartheta_1 * \Psi)\} \\
\Rightarrow \wedge (F_i^t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &\preceq \max\{\wedge (F_i^t)(\vartheta_1 * \Phi), \wedge (F_i^t)(\vartheta_1 * \Psi)\}
\end{aligned}$$

which show that R_N-union of Q_i^t is a Q^tNNSU of G .

Theorem 4.4. The R_N-intersection of any set of Q^tNNSU of G is also a Q^tNNSU of G.

Proof. Let $Q_i^t = \{\langle \vartheta_1, E_i^t, I^t i, F_i^t\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, be a set of Q^t NNSU of G and $\vartheta_1, \vartheta_2 \in G$ and $t, \Phi, \Psi \in [0,1]$. Then

$$\begin{aligned}
(\vee (E^t)_i)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &= \vee \left(\min(E_i, t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\right) \\
&= \sup\left(\min(E_i, t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi))\right) \\
&\preceq \sup\left\{\max\{(\min(E_i, t)(\vartheta_1 * \Phi)), (\min(E_i, t)(\vartheta_1 * \Psi))\}\right\} \\
&= \max\{\sup(\min(E_i, t)(\vartheta_1 * \Phi)), \sup(\min(E_i, t)(\vartheta_1 * \Psi))\} \\
&= \max\{\sup(E_i^t)(\vartheta_1 * \Phi), \sup(E_i^t)(\vartheta_1 * \Psi)\} \\
&= \max\{\vee (E_i^t)(\vartheta_1 * \Phi), \vee (E_i^t)(\vartheta_1 * \Psi)\} \\
\Rightarrow \vee (E_i^t)((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)) &\preceq \max\{\vee (E_i^t)(\vartheta_1 * \Phi), \vee (E_i^t)(\vartheta_1 * \Psi)\}
\end{aligned}$$

and

$$\begin{aligned}(\cap (I^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &= \cap \Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &= \inf\Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &\geq \inf\Big\{\min\big\{\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}\Big\}\\ &= \min\big\{\inf\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \inf\big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}\\ &= \min\{\inf(I_i^t)(\vartheta_1 * \Phi), \inf(I_i^t)(\vartheta_1 * \Psi)\}\\ &= \min\{\cap (I_i^t)(\vartheta_1 * \Phi), \cap (I_i^t)(\vartheta_1 * \Psi)\}\\ \Rightarrow \cap (I_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &\geq \min\{\cap (I_i^t)(\vartheta_1 * \Phi), \cap (I_i^t)(\vartheta_1 * \Psi)\}\end{aligned}$$

and

$$\begin{aligned}(\vee (F^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &= \vee \Big(\max(J_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &= \sup\Big(\max(F_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &\leq \sup\Big\{\max\big\{\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}\Big\}\\ &= \max\big\{\sup\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \sup\big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}\\ &= \max\{\sup(F_i^t)(\vartheta_1 * \Phi), \sup(F_i^t)(\vartheta_1 * \Psi)\}\\ &= \max\{\vee (F_i^t)(\vartheta_1 * \Phi), \vee (F_i^t)(\vartheta_1 * \Psi)\}\\ \Rightarrow \vee (F_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &\leq \max\{\vee (F_i^t)(\vartheta_1 * \Phi), \vee (F_i^t)(\vartheta_1 * \Psi)\}\end{aligned}$$

Which show that R_N-intersection of Q_i^t is a Q^tNNSU of G .

Theorem 4.5. Let $Q_i^t = \{\langle\vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ is a set of Q^t NNSU of G, where $i \in k$ and $t \in [0,1]$. If $\inf\big\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Phi)\}\big\} = \min\{\inf(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Phi)\}$ and $\inf\big\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Phi)\}\big\} = \max\{\inf(F_i^t)(\vartheta_1 * \Phi), \inf(F_i^t)(\vartheta_1 * \Phi)\}\forall\vartheta_1 \in G$, then P_N-intersection of Q_i^t is also a Q^tNNSU of G.

Proof. Suppose that $Q_i^t = \{\langle\vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, is a family of sets of Q^tNNSU of G such that $\inf\big\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Phi)\}\big\} = \min\{\inf(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Phi)\}$ and $\inf\big\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Phi)\}\big\} = \max\{\inf(F_i^t)(\vartheta_1 * \Phi), \inf(F_i^t)(\vartheta_1 * \Phi)\}\forall\vartheta_1, \vartheta_2 \in G$ and $t \in [0,1]$. Then

$$\begin{aligned}(\wedge (E^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &= \wedge \Big(\min(E_i, t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &= \inf\Big(\min(E_i, t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &\leq \inf\Big\{\max\big\{\big(\min(E_i, t)(\vartheta_1 * \Phi)\big), \big(\min(E_i, t)(\vartheta_1 * \Psi)\big)\big\}\Big\}\\ &= \max\big\{\inf\big(\min(E_i, t)(\vartheta_1 * \Phi)\big), \inf\big(\min(E_i, t)(\vartheta_1 * \Psi)\big)\big\}\\ &= \max\{\inf(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Psi)\}\\ &= \max\{\wedge (E_i^t)(\vartheta_1 * \Phi), \wedge (E_i^t)(\vartheta_1 * \Psi)\}\\ \Rightarrow \wedge (E_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &\leq \max\{\wedge (E_i^t)(\vartheta_1 * \Phi), \wedge (E_i^t)(\vartheta_1 * \Psi)\}\end{aligned}$$

and

$$\begin{aligned}(\cap (I^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &= \cap \Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &= \inf\Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)\\ &\geq \inf\Big\{\min\big\{\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}\Big\}\\ &= \min\big\{\inf\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \inf\big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}\\ &= \min\{\inf(I_i^t)(\vartheta_1 * \Phi), \inf(I_i^t)(\vartheta_1 * \Psi)\}\\ &= \min\{\cap (I_i^t)(\vartheta_1 * \Phi), \cap (I_i^t)(\vartheta_1 * \Psi)\}\\ \Rightarrow \cap (I_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) &\geq \min\{\cap (I_i^t)(\vartheta_1 * \Phi), \cap (I_i^t)(\vartheta_1 * \Psi)\}\end{aligned}$$

and

$$(\wedge (F^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \wedge \Big(\max(F_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$= \inf\Big(\max(F_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$\leq \inf\Big\{\max\big\{\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}\Big\}$$
$$= \max\big\{\inf\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \inf\big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}$$
$$= \max\{\inf(F_i^t)(\vartheta_1 * \Phi), \inf(F_i^t)(\vartheta_1 * \Psi)\}$$
$$= \max\{\wedge (F_i^t)(\vartheta_1 * \Phi), \wedge (F_i^t)(\vartheta_1 * \Psi)\}$$
$$\Rightarrow \wedge (F_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \leq \max\{\wedge (F_i^t)(\vartheta_1 * \Phi), \wedge (F_i^t)(\vartheta_1 * \Psi)\}$$

which show that P_N-intersection of Q_i^t is a Q^t NNSU of G.

Theorem 4.6. Let $Q^t{}_i = \{\langle \vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, is a family of sets of Q^tNNSU of G. If

$\sup\big\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Psi)\}\big\} = \min\{\sup(E_i^t)(\vartheta_1 * \Phi), \inf(E_i^t)(\vartheta_1 * \Psi$ and $\sup\big\{\min\{(I^t{}_i)(\vartheta_1 * \Phi), (I^t{}_i)(\vartheta_1 * \Psi)\}\big\} = \min\{\sup(I^t{}_i)(\vartheta_1 * \Phi), \sup(I^t{}_i)(\vartheta_1 * \Psi)\}$ and $\sup\big\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Psi)\}\big\} = \max\{\sup(F_i^t)(\vartheta_1 * \Phi), \sup(F_i^t)(\vartheta_1 * \Psi)\} \forall \vartheta_1, \vartheta_2 \in G$, then P_N-union of Q_i^t is also a Q^tNNSU of G.

Proof. Let $Q_i^t = \{\langle \vartheta_1, (E_i^t), (I^t{}_i), (F_i^t)\rangle \mid \vartheta_1 \in G\}$ where $i \in k$, is a family of sets of Q^tNNSU of G such that

$$\sup\big\{\min\{(E_i^t)(\vartheta_1 * \Phi), (E_i^t)(\vartheta_1 * \Psi)\}\big\} = \min\{\sup(E_i^t)(\vartheta_1 * \Phi), \sup(E_i^t)(\vartheta_1 * \Psi)\},$$
$$\sup\big\{\min\{(I^t{}_i)(\vartheta_1 * \Phi), (I^t{}_i)(\vartheta_1 * \Psi)\}\big\} = \min\{\sup(I^t{}_i)(\vartheta_1 * \Phi), \sup(I^t{}_i)(\vartheta_1 * \Psi)\}$$

and

$$\sup\big\{\max\{(F_i^t)(\vartheta_1 * \Phi), (F_i^t)(\vartheta_1 * \Psi)\}\big\} = \max\{\sup(F_i^t)(\vartheta_1 * \Phi), \sup(F_i^t)(\vartheta_1 * \Psi)\} \forall \vartheta_1, \vartheta_2 \in G_\circ$$

Then for $\vartheta_1, \vartheta_2 \in G$, and $t \in [0,1]$.

$$(\vee (E^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \vee \Big(\min(E_i, t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$= \sup\Big(\min(E_i, t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$\leq \sup\Big\{\max\big\{\big(\min(E_i, t)(\vartheta_1 * \Phi)\big), \big(\min(E_i, t)(\vartheta_1 * \Psi)\big)\big\}\Big\}$$
$$= \max\big\{\sup\big(\min(E_i, t)(\vartheta_1 * \Phi)\big), \sup\big(\min(E_i, t)(\vartheta_1 * \Psi)\big)\big\}$$
$$= \max\{\sup(E_i^t)(\vartheta_1 * \Phi), \sup(E_i^t)(\vartheta_1 * \Psi)\}$$
$$= \max\{\vee (E_i^t)(\vartheta_1 * \Phi), \vee (E_i^t)(\vartheta_1 * \Psi)\}$$
$$\Rightarrow \vee (E_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \leq \max\{\vee (E_i^t)(\vartheta_1 * \Phi), \vee (E_i^t)(\vartheta_1 * \Psi)\}$$

and

$$(\cup (I^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \cup \Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$= \sup\Big(\min(I_i, t')\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$\geq \sup\Big\{\min\big\{\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}\Big\}$$
$$= \min\big\{\sup\big(\min(I_i, t')(\vartheta_1 * \Phi)\big), \sup\big(\min(I_i, t')(\vartheta_1 * \Psi)\big)\big\}$$
$$= \min\{\sup(I_i^t)(\vartheta_1 * \Phi), \sup(I_i^t)(\vartheta_1 * \Psi)\}$$
$$= \text{rmin}\{\cup (I_i^t)(\vartheta_1 * \Phi), \cup (I_i^t)(\vartheta_1 * \Psi)\}$$
$$\Rightarrow \cup (I_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \geq \min\{\cup (I_i^t)(\vartheta_1 * \Phi), \cup (I_i^t)(\vartheta_1 * \Psi)\}$$

and

$$(\vee (F^t)_i)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) = \vee \Big(\max(F_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$= \sup \Big(\max(F_i, \Omega)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big)\Big)$$
$$\leq \sup \Big\{\max\big\{\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}\Big\}$$
$$= \max\big\{\sup\big(\max(F_i, \Omega)(\vartheta_1 * \Phi)\big), \sup\big(\max(F_i, \Omega)(\vartheta_1 * \Psi)\big)\big\}$$
$$= \max\{\sup(F_i^t)(\vartheta_1 * \Phi), \sup(F_i^t)(\vartheta_1 * \Psi)\}$$
$$= \max\{\vee (F_i^t)(\vartheta_1 * \Phi), \vee (F_i^t)(\vartheta_1 * \Psi)\}$$
$$\Rightarrow \vee (F_i^t)\big((\vartheta_1 * \Phi) * (\vartheta_2 * \Psi)\big) \leq \max\{\vee (F_i^t)(\vartheta_1 * \Phi), \vee (F_i^t)(\vartheta_1 * \Psi)\}$$

Which show that P_N-union of Q_i^t is a Q^t NNSU of G.

5. Conclusion

This research article introduces a modified algebraic structure for neutrosophic sets, specifically the Q^tNS N-subalgebra structure within the context of G-algebra. The proposed structure, along with its modified operations and concepts, expands the theoretical framework and offers promising avenues for applications in decision-making, computer science, and mathematics. Future research directions include exploring multi-attribute decision-making, real-world applications, efficient algorithms, integration with machine learning, formalization of mathematical properties, and comparative studies. These advancements will deepen our understanding and enhance the practicality of the Q^tNS N-subalgebra framework, driving innovation and progress in various fields.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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