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Neutrosophic Automata and Its Algebraic Properties

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Abstract: This research endeavors to elucidate the interrelationships among various classes of operators, including neutrosophic successor/neutrosophic source/neutrosophic core operators of neutrosophic automata based on neutrosophic resituated lattices. Furthermore, it delves into the characterization of algebraic properties such as neutrosophic subsystem, neutrosophic connectivity, and neutrosophic separability of a neutrosophic automaton using these operators. Finally, we study the concepts of neutrosophic primaries of the given neutrosophic automata using neutrosophic core operators.

Keywords: Neutrosophic Automata; Neutrosophic Point; Neutrosophic Source Operator; Neutrosophic Successor Operator; Neutrosophic Core Operator.

1. Introduction

In 1965, Zadeh [1] pioneered fuzzy set theory, fundamentally transforming the methodologies employed to address uncertainty and ambiguity in both mathematical frameworks and practical problems. By permitting elements to possess varying degrees of membership, as opposed to rigid binary classifications, fuzzy set theory unlocked a spectrum of possibilities. This innovation subsequently led to the fuzzification of numerous mathematical concepts and other disciplines. Among these advancements was the introduction of the intuitionistic fuzzy set by Atanassov [2], which captures not only the degree of membership but also the inherent hesitancy or indeterminacy prevalent in real-world situations. Building upon these foundations, Smarandache [3, 4] proposed the neutrosophic set, which offers a triadic framework for managing uncertainty by incorporating degrees of truth, indeterminacy, and falsity for each element. This comprehensive representation is especially useful in contexts characterized by incomplete, inconsistent, or conflicting data.

The algebraic analysis of automata theory holds critical significance [5, 6, 7]. Numerous scholars have dedicated efforts to the algebraic exploration of fuzzy automata across various forms (cf., [8, 9, 10, 11, 12-18, 19-22]). In [6], concepts such as separateness, connectedness, and retrievability of an automaton were introduced, along with their operator-driven applications. Following Bavel's influential work [6] on algebraic automata theory, the intersection of topology and automata was first established in [23, 24], where it was demonstrated that several established topological concepts could be seamlessly applied to automata theory, including their connectivity and separation properties. The notion of a fuzzy automaton was initially conceptualized by Wee [25] and Santos [26] to address challenges posed by vagueness and imprecision, which are frequently encountered in the analysis of natural languages. Subsequently, Malik et al. [27] introduced a significantly simplified version of the fuzzy finite state machine, thereby making notable contributions to the algebraic study of fuzzy automata and fuzzy languages. Neutrosophic automata, an extension of classical automata theory grounded in neutrosophic logic, present a novel and adaptive paradigm in computational theory. Neutrosophic logic, formulated by Florentin Smarandache [4], generalizes classical, fuzzy, and intuitionistic logic by addressing uncertainty, imprecision, vagueness, incompleteness, and inconsistency through the integration of three degrees: truth (T), indeterminacy (I), and falsity (F).

This multifaceted approach facilitates a more nuanced representation of real-world problems, particularly those marked by ambiguity and incomplete information. Wang et al. [28, 29] have introduced the concepts of single-valued neutrosophic sets and interval neutrosophic sets as a more manageable instance of neutrosophic sets. However, the algebraic and operator-centric study of neutrosophic automata remains relatively underdeveloped. The primary objective of this work is to introduce and analyze the properties of various operators, such as neutrosophic source, neutrosophic successor, neutrosophic core, and neutrosophic primary, and to explore the diverse properties these operators induce within the framework of neutrosophic automata.

1.1 *Motivation*

There are many algebraic structures of neutrosophic automata theory that have not been studied yet, especially automatons with different neutrosophic operators based on neutrosophic residuated lattice. The neutrosophic source, neutrosophic successor, and neutrosophic core operators proposed in my paper demonstrate superior precision and efficacy compared to their fuzzy and intuitionistic fuzzy analogs. Because it provides a more flexible and nuanced way of modeling uncertainty, ambiguity, and vagueness in automata based on the neutrosophic residuated lattice.

1.2 *Application*

Neutrosophic automata, an advanced construct derived from the theoretical foundations of fuzzy automata, has garnered attention from numerous researchers who have extensively explored automata theory within a neutrosophic framework [30-32, 33, 34, 35]. These automata establish a sophisticated framework for handling computational uncertainties, making them invaluable in tackling intricate problems across various disciplines. Their applications span diverse fields, including learning management systems [36], control systems [37, 38], medical diagnostics [39, 40, 41], algebraic structures [30-32, 33, 34, 35], financial decision-making [42], design of intelligent product-service ecosystems [43] and the studies in topological and metric spaces [44-49,50,51,52]. The operators used in these papers are used to diversify and refine earlier algebraic approaches to automata theory, along with enriching concepts in topology, financial analysis, and other interdisciplinary fields.

In this paper, we delve into the theory of neutrosophic automata through the concepts of operator theory, which provides a robust framework for the algebraic exploration of neutrosophic automata. The paper is organized as follows: Section 2 presents the foundational concepts pertinent to the study. In Section 3, we introduce and rigorously analyze two key operators, namely the neutrosophic source and neutrosophic successor operators, for a given neutrosophic automaton, exploring their algebraic properties in detail. Section 4 introduces the neutrosophic core operator, examining its interrelations with the operators discussed in Section 3. Finally, Section 5 offers the concluding remarks.

2. Preliminaries

In this section, we revisit the essential notations and foundational concepts about neutrosophic sets, encompassing implication operators, neutrosophic residuated lattices, neutrosophic points, and the various properties and operations governing neutrosophic sets. The theoretical underpinnings of neutrosophic sets and their properties are derived from the seminal works of [3, 22, 28, 31] while the distinct properties of neutrosophic automata, implication operators, and fundamental neutrosophic set operations are explored in [35]. The discourse begins by addressing the following key points.

Definition 2.1. A neutrosophic set (NS, in short) N on a non-empty set X is defined as an entity of the form $N = \{< u, F_N(u), G_N(u), H_N(u) > : u \in X\}$, where the functions F_N, G_N, H_N: $X \rightarrow]0-, 1+[$ define respectively the degree of membership (or truth), the degree of indeterminacy, and the degree of non-membership (or false) of each element $u \in X$ to the set N. As, the sum of $F_N(u)$, $G_N(u)$, $H_N(u)$, have unrestricted. Consequently, for each $u \in X, 0^- \leq F_N(u) + G_N(u) + H_N(u) \leq 3^+$.

Remark 2.1. A neutrosophic set $N = \{< u, F_N(u), G_N(u), H_N(u) >: u \in X\}$ is conventionally represented as an ordered triple $<F_N, G_N, H_N>$ in the non-standard unit interval] 0−, 1 + [on$X$. The neutrosophic sets (NSs, in short) 0_N and 1_N which signify constant NSs in X and are defined as $0_N =< 0, 1, 1 >$ *and* $1_N =< 1, 0, 0 >$, *where* $0, 1: X \to]0^-, 1^+[$ are defined respectively by $0(u) = 0$ and $1(u) = 1$. The NS $\eta = (\sigma, \beta, \gamma)$ such that $(\hat{\eta}) = \left(\widehat{\alpha, \beta, \gamma}\right)$ is expressed as $\hat{\eta}(u) = \eta$ for all $u \in X$, where σ, β, and γ are the σ-valued, β-valued, and γ-valued constant neutrosophic sets in X respectively, with the condition $0^- \leq \sigma + \beta + \gamma \leq 3^+$.

This paper adopts the interval $[0, 1]$ rather than the notation $]\mathbf{0}^-, 1^+[$for practical purposes, as the latter may present obstacles in real-world applications. Moreover, $NS(X)$ will signify the collection of all neutrosophic sets within X and I^* denotes the set $\{(u, v, w): (u, v, w) \in [0, 1] \times [0, 1] \times [0, 1], 0 \leq u + v + w \leq 3\}$. A neutrosophic set $N =< F_N, G_N, H_N >$ in X will frequently be viewed as a function $N: X \to I^*$, given by $N(u) = \{F_N(u), G_N(u), H_N(u): u \in X\}$.

To begin, we revisit several foundational properties of neutrosophic sets in$X$.

Definition 2.2. For NSs $N_1 =< F_{N_1}, G_{N_1}, H_{N_1} >$, $N_2 =< F_{N_2}, G_{N_2}, H_{N_2} >$ and $N_i =< F_{N_i}, G_{N_i}, H_{N_i} >$, $i \in J$ & $u \in X$. We have

1) $N_1 \leq N_2$, if $F_{N_1}(u) \leq F_{N_2}(u)$, $G_{N_1}(u) \geq G_{N_2}(u)$ and $H_{N_1}(u) \geq H_{N_2}(u)$;
2) $\vee_{i\in J} N_i(u) = (\vee_{i\in J} F_{N_i}(u), \wedge_{i\in J} G_{N_i}(u), \wedge_{i\in J} H_{N_i}(u))$;
3) $\wedge_{i\in J} N_i(u) = (\wedge_{i\in J} F_{N_i}(u) \vee_{i\in J} G_{N_i}(u), \vee_{i\in J} H_{N_i}(u))$;
4) $C(N) = (1 - F_N, 1 - G_N, 1 - H_N)$;
5) $0_N \subseteq N \subseteq 1_N$; $C(0_N) = 1_N$ *and* $C(1_N) = 0_N$;
6) $N \cup 0_N = N$, $N \cup 1_N = 1_N$ *and* $N \cap 0_N = 0_N$, $N \cap 1_N = N$.

Definition 2.3. A neutrosophic residuated lattice is an algebra $I_N = \{I^*, \wedge, \vee, \otimes, \to, 0_N, 1_N\}$, where

1) $(I^*, \wedge, \vee, 0_N, 1_N)$ is a lattice with the least element 0_N and greatest element 1_N,
2) $(I^*, \otimes, 1_N)$ is a commutative monoid with unit 1_N, and
3) $\otimes$ and $\to$ form an adjoint pair, i.e., for all $\sigma_N = (\sigma_1, \sigma_2, \sigma_3)$, $\beta_N = (\beta_1, \beta_2, \beta_3)$, $\gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*$, $\sigma_N \otimes \beta_N \leq \gamma_N \Leftrightarrow \sigma_N \leq \beta_N \to \gamma_N$, *i.e.*, $\sigma_1 \otimes \beta_1 \leq \gamma_1 \Leftrightarrow \sigma_1 \leq \beta_1 \to \gamma_1$, $\sigma_2 \otimes \beta_2 \geq \gamma_2 \Leftrightarrow \sigma_2 \geq \beta_2 \leftarrow \gamma_2$ and $\sigma_3 \otimes \beta_3 \geq \gamma_3 \Leftrightarrow \sigma_3 \geq \beta_3 \leftarrow \gamma_3$.

Definition 2.4. The neutrosophic point (NP, in short) $x_{(\alpha,\beta,\gamma)}$ on X is a NS in X such that for all $x, y \in X$ and for all $(\alpha, \beta, \gamma) \in I^*$,

$$x_{(\alpha,\beta,\gamma)}(y) = \begin{cases} (\alpha, \beta, \gamma) & \text{if } x = y \\ 0_N & \text{if } x \neq y \end{cases}, \text{ i.e.,}$$

$$F_{x_\alpha}(y) = \begin{cases} F_\alpha & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$G_{x_\beta}(y) = \begin{cases} G_\beta & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \&$$

$$H_{x_\gamma}(y) = \begin{cases} H_\gamma & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 2.1. Let $I_N = \{I^*, \wedge, \vee, \otimes, \to, 0_N, 1_N\}$ be a neutrosophic residuated lattice and $N = (F_N, G_N, H_N) \in NS(X)$. Then for all $\sigma_N = (\sigma_1, \sigma_2, \sigma_3)$, $\beta_N = (\beta_1, \beta_2, \beta_3)$, *and* $\gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*$ following properties hold:

i). $1_N \to N = (1, 0, 0) \to (F_N, G_N, H_N) = (F_N, G_N, H_N) = N$;

ii). $\sigma_N \leq \beta_N \Rightarrow \sigma_N \rightarrow \beta_N = 1_N$;
iii). $\beta_N \leq \sigma_N \rightarrow (\sigma_N \otimes \beta_N)$;
iv). $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = \sigma_N \rightarrow (\beta_N \rightarrow \gamma_N)$;
v). $\sigma_N \otimes (\wedge_{i\in J} \beta_{N_i} \leq \wedge_{i\in J} (\sigma_N \otimes \beta_{N_i})$;
vi). $\sigma_N \otimes (V_{i\in J} \beta_{N_i}) = V_{i\in J} (\sigma_N \otimes \beta_{N_i})$;
vii). $(\sigma_N \rightarrow \beta_N) \otimes \sigma_N \leq \beta_N$;
viii). $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = (\beta_N \otimes \sigma_N) \rightarrow \gamma_N$; and
ix). $\vee_{i\in J} \sigma_{Ni} \rightarrow \beta_N = \wedge_{i\in J} (\sigma_{Ni} \rightarrow \beta_N)$.

3. Neutrosophic Source and Successor Operator

In this section, we have to study the concepts of neutrosophic source and neutrosophic successor operators for a given neutrosophic automaton and discuss some algebraic concepts of a neutrosophic automaton based on the neutrosophic residuated lattice.

Definition 3.1. [35] A Neutrosophic Automaton, ($NAut$, in short) is a triple $A = (S, X, \delta^n)$, where S and X are non-empty sets, called the set of states and the set of inputs respectively. The neutrosophic transition function $\delta^N = (F_\delta, G_\delta, H_\delta)$ be a neutrosophic subset of $S \times X \times S$, i.e a map δ_N:$S \times X \times S \rightarrow I^*$.

Remark 3.1. Let X^* denote the free monoid generated by the set X, with e being its identity. The extension of δ is denoted as $\delta^{N*} = (F_{\delta*}, G_{\delta*}, H_{\delta*})$: $S \times X^* \times S \rightarrow I^*$. This extension is characterized by the property that for any $s_1, s_2 \in S$, $u \in X^*$, and $x \in X$, the following holds:

$$F_{\delta^*}(s_1, e, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 \neq s_2 \end{cases}$$

$$G_{\delta^*}(s_1, e, s_2) = H_{\delta^*}(s_1, e, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2 \\ 1 & \text{if } s_1 \neq s_2 \end{cases}$$

$F_{\delta^*}(s_1, ux, s_2) = \vee\{F_{\delta^*}(s_1, u, s_3) \otimes F_{\delta^*}(s_3, x, s_2): s_3 \in S\}$, $G_{\delta^*}(s_1, ux, s_2) = \wedge\{G_{\delta^*}(s_1, u, s_3) \otimes G_{\delta^*}(s_3, x, s_2) : s_3 \in S\}$ and $H_{\delta^*}(s_1, ux, s_3) = \wedge\{H_{\delta^*}(s_1, u, s_3) \otimes H_{\delta^*}(s_3, x, s_2): s_3 \in S\}$.

Lemma 3.1. Let $A = (S, X, \delta^n)$ be a $NAut$. Then $\forall\ s_1, s_2 \in S$ and $\forall\ u, v \in X^*$, $F_{\delta^*}(s_1, uv, s_2) = \vee \{F_{\delta^*}(s_1, u, s_3) \otimes F_{\delta^*}(s_3, v, s_2) : s_3 \in S\}$, $G_{\delta^*}(s_1, uv, s_2) = \wedge\{G_{\delta^*}(s_1, u, s_3) \otimes G_{\delta^*}(s_3, v, s_2) : s_3 \in S\}$ and $H_{\delta^*}(s_1, uv, s_2) = \wedge\{H_{\delta^*}(s_1, u, s_3) \otimes H_{\delta^*}(s_3, v, s_2): s_3 \in S\}$.

Proof: - Let $s_1, s_2 \in S$, and $u, v \in X^*$. We establish the result by induction on the length of v, denoted by $|v|$. For which, let $|v| = K$. Then for $K = 0$, the result is obvious by the definition of δ^{N*}. Now, assume that the result holds for all $u, v \in X^*$ with $|v| \leq K - 1$. Also, let $= v_1, v_2, \ldots\ldots v_K \in X^*$. Then from the definition of $\delta^{N*} = (F_{\delta*}, G_{\delta*}, H_{\delta*})$ and the distributive property of $\otimes$ over v,

$F_{\delta*}(s_1, uv, s_2) = \vee\{F_{\delta*}(s_1, u\, v_1, v_2, \ldots\ldots\ldots v_{K-1}, s\_3) \otimes F_{\delta*}(s_3, v_K, s_2) : s_3 \in S\} = \vee\{\vee\{F_{\delta*}(s_1, u, s_4) \otimes F_{\delta*}(s_4, v_1, v_2, \ldots\ldots\ldots v_{K-1}, s_3) : s_4 \in S\} \otimes F_{\delta*}(s_3, v_K, s_2) : s_3 \in S\} = \vee\{F_{\delta*}(s_1, u, s_4) \otimes F_{\delta*}(s_4, v_1, v_2, \ldots\ldots\ldots v_{K-1}, s_3) \otimes F_{\delta*}(s_3, v_K, s_2) : s_3 \in S\} = \vee\{\vee\{F_{\delta*}(s_1, u, s_4) \otimes F_{\delta*}(s_4, v_1, v_2, \ldots\ldots\ldots v_{K-1}, s_3) : s_4 \in S\} \otimes F_{\delta*}(s_3, v_K, s_2) : s_3 \in S\} = \vee\{\vee\{F_{\delta*}(s_4, v_1, \ldots, v_{k-1}, s_3) \otimes F_{\delta*}(s_3, v_K, s_2) : s_3 \in S\} \otimes F_{\delta*}(s_1, u, s_4) : s_4 \in S\} = \vee\{F_{\delta*}(s_4, v_1, v_2, \ldots\ldots v_K, s_2) \otimes F_{\delta*}(s_1, u, s_4,) : s_4 \in S\} = \vee\{F_{\delta*}(s_1, u, s_4) \otimes F_{\delta*}(s_4, v_1, v_2, \ldots.. v_K, s_2) : s_4 \in S\} = \vee\{F_{\delta*}(s_1, u, s_4) \otimes F_{\delta*}(s_4, v, s_2) : s_4 \in S\}$. Similarly, $G_{\delta^*}(s_1, uv, s_2) = \wedge\{G_{\delta^*}(s_1, u, s_4,) \otimes G_{\delta^*}(s_4, v, s_2) : s_4 \in S\}$ and $H_{\delta^*}(s_1, uv, s_2) = \wedge\{H_{\delta^*}(s_1, u, s_4,) \otimes H_{\delta^*}(s_4, v, s_2) : s_4 \in S\}$.

The existing studies of neutrosophic automata [30] is differs from this neutrosophic automata, as the set of states may be infinite and the input set may be a monoid rather than just a set.

Definition 3.2. A *NAut* is a triple $A = (S, X, \delta^n)$, where S is a nonempty set of states of A, X is a monoid (the input monoid of A), whose identity shall be denoted as e. The neutrosophic transition function $\delta^N = (F_\delta, G_\delta, H_\delta)$ is a neutrosophic subset of $S \times X \times S$, i. e, a map δ^N :S × X × S → I*, such that for all $s_1, s_2 \in$ S and $x, y \in X$,

$$F_{\delta^*}(s_1, e, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 \neq s_2 \end{cases}$$

$$G_{\delta^*}(s_1, e, s_2) = H_{\delta^*}(s_1, e, s_2) = \text{-} \begin{cases} 0 & \text{if } s_1 = s_2 \\ 1 & \text{if } s_1 \neq s_2 \end{cases}$$

$F_{\delta^*}(s_2, xy, s_1) = \vee\{F_{\delta^*}(s_2, u, s_3) \otimes F_{\delta^*}(s_3, y, s_2): s_3 \in S\}$, $G_{\delta^*}(s_2, xy, s_1) = \wedge\{G_{\delta^*}(s_1, u, s_3) \otimes G_{\delta^*}(s_3, x, s_2): s_3 \in S\}$ and $H_{\delta^*}(s_2, ux, s_1) = \wedge\{H_{\delta^*}(s_2, u, s_3) \otimes H_{\delta^*}(s_3, x, s_2): s_3 \in S\}$.

In the subsequent sections of this paper, we delve into the framework of these *NAut* concepts.

Definition 3.3. Let $A = (S, X, \delta^n)$ be a *NAut* and $N \in NS(S)$. The Neutrosophic source (*NSo*, in short) and the Neutrosophic successor (*NSu*, in short) of N are defined as:

$$So(F_N)(s_2) = \vee\{F_N(s_1) \otimes F_\delta(s_2, u, s_1): s_1 \in S, u \in X\},$$
$$So(G_N)(s_2) = \wedge\{G_N(s_1) \otimes G_\delta(s_2, u, s_1): s_1 \in S, u \in X\},$$
$$So(H_N)(s_1) = \wedge\{H_N(s_1) \otimes H_\delta(s_2, u, s_1): s_1 \in S, u \in X\}, \text{ and}$$
$$Su(F_N)(s_2) = \vee\{F_N(s_1) \otimes F_\delta(s_1, u, s_2): s_1 \in S, u \in X\}$$
$$Su(G_N)(s_2) = \wedge\{G_N(s_1) \otimes G_\delta(s_1, u, s_2): s_1 \in S, u \in X\}$$
$$Su(H_N)(s_2) = \wedge\{H_N(s_1) \otimes H_\delta(s_1, u, s_2): s_1 \in S, u \in X\}, \forall\ s_2 \in S.$$

Remark 3.2. In this paper, we can use the *NSo* and *NSu* operators as $So^n(N)(s_2) = \vee\{N(s_1) \otimes \delta^n(s_2, u, s_1): s_1 \in S, u \in X\}$ and $Su^n(N)(s_2) = \vee\{N(s_1) \otimes \delta^n(s_1, u, s_2): s_1 \in S, u \in X\}$.

Proposition 3.1. Let $A = (S, X, \delta^n)$ be a *NAut* and So^n, $Su^n : NS(S) \rightarrow NS(S)$ be the induced *NSo* and *NSu* operators, respectively. Then $\forall\ N_1, N_2, (N_j)j \in J \in NS(S)$,

- i). $Su^n\left(\overbrace{\alpha, \beta, \gamma}\right) = \left(\overbrace{\alpha, \beta, \gamma}\right)$ *and* $So^n\left(\overbrace{\alpha, \beta, \gamma}\right) = \left(\overbrace{\alpha, \beta, \gamma}\right), \forall\ (\alpha, \beta, \gamma) \in I^*$;
- **ii).** *if* $N_1 \leq N_2$ then $Su^n(N_1) \leq Su^n(N_2)$ and $So^n(N_1) \leq So^n(N_2)$;
- **iii).** $N_1 \leq Su^n(N_1)$ and $N_1 \leq So^n(N_1)$;
- **iv).** $Su^n(\cup\{N_j : j \in J\}) = \cup\{Su^n(N_j : j \in J)\}$ and $So^n(\cup\{N_j : j \in J\}) = \cup\{(So^n(N_j : j \in J)\}$;
- **v).** $Su^n(Su^n(N_1)) = Su^n(N_1)$ and $So^n(So^n(N_1)) = So^n(N_1)$;
- **vi).** $Su^n(\cap\{N_j : j \in J\}) \leq \cap\{Su^n(N_j : j \in J)\}$ and $So^n(\cap\{N_j : j \in J\}) \leq \cap\{So^n(N_j : j \in J)\}$.

Proof: We have to prove here only for *NSu* operators. The *NSo* operators can be proved in a similar method: -

- **i).** Let $s_2 \in$ S. Then $Su^n\left(\overbrace{\alpha, \beta, \gamma}\right)(s_2) = \vee\{\left(\overbrace{\alpha, \beta, \gamma}\right)(s_1) \otimes \delta^n(s_1, u, s_2): s_1 \in S, u \in X\}$ $= \vee\{(\alpha, \beta, \gamma) \otimes \delta^n(s_1, u, s_2): s_1 \in S, u \in X\} = (\alpha, \beta, \gamma) \otimes \vee\{\delta^n(s_1, u, s_2): s_1 \in S, u \in X\} = (\alpha, \beta, \gamma) \otimes 1_N = \left(\overbrace{\alpha, \beta, \gamma}\right)(s_2)$.
- **ii).** The definition of *NSu* operators makes this immediately apparent.
- iii). Let $s_2 \in$ S. Then $Su(F_{N_1})(s_2) = \vee\{F_{N_1}(s_1) \otimes F_\delta(s_1 u, s_2): s_1 \in S, u \in X\} \geq F_{N_1}(s_2) \otimes F_\delta(s_2, e, s_2) = F_{N_1}(s_2) \otimes 1 = F_{N_1}(s_2)$ and $Su(G_{N_1})(s_2) = \wedge\{G_{N_1}(s_1) \otimes G_\delta(s_1, u, s_2): s_1 \in S, u \in X\} \leq G_{N_1}(s_2) \otimes G_\delta(s_2, e, s_2) = G_{N_1}(s_2) \otimes 0 = G_{N_1}(s_2)$. Similarly, $Su(H_{N_1})(s_2) \leq H_{N_1}(s_2)$. Thus, $Su^n(N_1) \geq N_1$.
- **iv).** Let $s_2 \in$ S. Then $Su^n(\cup\{N_j : j \in J\})(s_2) = \left\{Su\left(\cup\left\{F_{N_j} : j \in J\right\}\right), Su\left(\cap\left\{G_{N_j} : j \in J\right\}\right), Su\left(\cap\left\{H_{N_j} : j \in J\right\}\right)\right\}$, where $Su\left(\cup\left\{F_{N_j} : j \in J\right\}\right)(s_2) = \vee\{(\vee(F_{N_J}(s_1) : j \in J) \otimes F_\delta(s_1, u, s_2): s_1 \in S, u \in X\} = \vee\{(F_{N_J}(s_1) : j \in J) \otimes F_\delta(s_1, u, s_2): s_1 \in$

$S, u \in X, j \in J\} = \cup\{Su(F_{N_j}: j \in J)\}(s_2)$. Similarly, $Su\left(\cap\left\{G_{N_j}: j \in J\right\}\right)(s_2) = \cap\{Su(G_{N_j}: j \in J)\}(s_2)$ and $Su\left(\cap\left\{H_{N_j}: j \in J\right\}\right)(s_2) = \cap\{Su(H_{N_j}: j \in J)\}(s_2)$. Thus, $Su^n(\cup\{N_j: j \in J\}) = \cup\{Su^n(N_j: j \in J)\}$.

v). Let $s_2 \in$ S. Then $Su^n(Su^n(N_1))(s_2) = \{Su(Su(F_{N_1})), Su(Su(G_{N_1})), Su(Su(H_{N_1}))\}(s_1)$ where, $Su(Su(F_{N_1}))(s_2) = \vee\{Su(F_{N_1})(s_1) \otimes F_\delta(s_1, u, s_2): s_1 \in S, u \in X\} = \vee\{\vee\{F_{N_1}(s_3) \otimes F_\delta(s_3, v, s_1): s_3 \in S, v \in X\} \otimes F_\delta(s_1, u, s_2): s_1 \in S, u \in X\} \leq \vee\{F_{N_1}(s_3) \otimes F_\delta(s_3, vu, s_2): s_3 \in S, u, v \in X\} = \vee\{F_{N_1}(s_3) \otimes F_\delta(s_3, w, s_2): s_3 \in S, w \in X\} = Su(F_{N_1})(s_2)$. Similarly, $Su(Su(G_{N_1}))(s_2) \geq Su(G_{N_1})(s_2)$ and $Su(Su(H_{N_1}))(s_2) \geq Su(H_{N_1})(s_2)$. Thus $Su^n(Su^n)(N_1) \leq Su^n(N_1)$. Also, the converse inequality follows from the fact that $Su^n(Su^n)(N_1) \geq Su^n(N_1)$ (cf., Property (iii)). Hence, $Su^n(Su^n)(N_1) = Su^n(N_1)$.

vi). This follows the fact that $N_1 \cap N_2 \leq N_1$ and $N_1 \cap N_2 \leq N_2$ and from the Property (ii).

Definition 3.4. Let $A = (S, X, \delta^n)$ be a $NAut$. Then $N = (F_N, G_N, H_N) \in NS(S)$ is called a Neutrosophic Subsystem ($NSSy$, in short) of A if $Su^n(N) \leq N$. Also, this $NSSy$ is called Neutrosophic Separated ($NSep$, in short), if $Su^n(C(N)) \leq C(N)$.

Proposition 3.2. For a $NSSy$ $N = (F_N, G_N, H_N)$ of a $NAut$ A, the following conditions are equivalent.

i). N is $NSep$;

ii). $Su^n(C(N)) = C(N)$.

Proof: This follows directly from Proposition (3.1) and the definition of neutrosophic separateness in $NSSy$.

Proposition 3.3. Let $A = (S, X, \delta^n)$ be a $NAut$ and $N \in NS(S)$. Then the following assertions are equivalent.

i). N is a $NSSy$ of A;

ii). $Su^n(s_{2(\alpha,\beta,\gamma)}) \leq N$, $\forall (s_{2(\alpha,\beta,\gamma)}) \leq N$.

Proof: $(i) \Rightarrow (ii)$. Let N be a $NSSy$ of a $NAutA$. Then $F_{Su(N)} \leq F_N$, $G_{Su(N)} \geq G_N$ and $H_{Su(N)} \geq H_N$,now let $s_{2(\alpha,\beta,\gamma)}(s_2) \leq N(s_2), \forall s_2 \in S$. Then $\forall s_1 \in$ S, $F_{Su(\dot{s}_{2\alpha})}(s_1) = \vee\{(s_{2\alpha}(s_2) \otimes F_\delta(s_2, u, s_1): s_2 \in S, u \in X\} \leq \vee\{F_N(s_2) \otimes F_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = F_{Su(N)}(s_1) \leq F_N(s_1)$, and $G_{Su(\dot{s}_{2\beta})}(s_1) = \wedge\{s_{2\beta}(s_2) \otimes G_\delta(s_2, u, s_1): s_2 \in S, u \in X\} \geq \wedge\{G_N(s_2) \otimes G_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = G_{Su(N)}(s_1) \geq G_N(s_1)$. Similarly, $H_{Su(N)}(s_1) \geq H_N(s_1)$. Hence $Su^n(s_{2(\alpha,\beta,\gamma)}) \leq N$.

$(ii) \Rightarrow (i)$. Let s_1, $s_2 \in S$ and $u \in X$, if $N(s_2) = 0_N$ or $\delta^n(s_2, u, s_1) = 0_N$. Then $N(s_1) \geq 0_N = \vee\{N(s_2) \otimes \delta^n(s_2, u, s_1): s_2 \in S, u \in X\} = Su^n(N)(s_1)$. Now, let $N(s_2) \neq 0_N$, $\delta^n(s_2, u, s_1) \neq 0_N$ and $N(s_2) = s_{2(\alpha,\beta,\gamma)}(s_2)$. Then $s_{2(\alpha,\beta,\gamma)} \leq N$, and $F_{Su(\dot{s}_{2\alpha})} \leq F_N$, $G_{Su(\dot{s}_{2\beta})} \geq G_N$, $H_{Su(\dot{s}_{2\gamma})} \geq H_N$,$\forall s_1 \in$ S, which implies that $F_N(s_1) \geq F_{Su(\dot{s}_{2\alpha})}(s_1) = \vee\{s_{2\alpha}(s_2) \otimes F_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = \vee\{\alpha \otimes F_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = \vee\{F_N(s_2) \otimes F_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = F_{Su(N)}(s_1)$, and $G_N(s_1) \leq G_{Su(\dot{s}_{2\beta})}(s_1) = \wedge\{s_{2\beta}(s_2) \otimes G_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = \wedge\{\beta \otimes G_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = \wedge\{G_N(s_2) \otimes G_\delta(s_2, u, s_1): s_2 \in S, u \in X\} = G_{Su(N)}(s_1)$. Similarly, $H_N(s_1) \leq H_{Su(N)}(s_1)$. Thus N is a $NSSy$ of A.

Proposition 3.4. Let N be a $NSSy$ of a $NAutA = (S, X, \delta^n)$. Also, let there exists $s_2 \in S$ such that $N = Su^n(s_{2(\alpha,\beta,\gamma)})$. Then:

i). $N(s_2) = (\alpha, \beta, \gamma)$;

ii). $\forall s_1 \in S$, $F_N(s_2) \geq F_N(s_1)$, $G_N(s_2) \leq G_N(s_1)$ and $H_N(s_2) \leq H_N(s_1)$;

iii). For all $NSSy$ N' of N such that $N' \leq N$, if $F_{N'}(s_2) \geq F_{N'}(s_1)$, $G_{N'}(s_2) \leq G_{N'}(s_1)$ and $H_{N'}(s_2) \leq H_{N'}(s_1)$ $\forall s_1 \in S$, then $N' = Su^n(s_{2N'(s_2)})$.

Proof:

i). Let there exist $s_2 \in S$ such that $N(s_2) = Su^n(s_{2(\alpha,\beta,\gamma)})(s_2) = \vee\{s_{2(\alpha,\beta,\gamma)}(s_1) \otimes \delta^n(s_1, u, s_2) : s_1 \in S, u \in X\} = (\alpha,\beta,\gamma) \otimes (\vee\{\delta^n(s_2, u, s_2) : u \in X\}) = (\alpha,\beta,\gamma) \otimes 1_N = (\alpha,\beta,\gamma)$.

ii). Let $s_1 \in S$. Then $N(s_1) = Su^n(s_{2(\alpha,\beta,\gamma)})(s_1) = \vee\{s_{2(\alpha,\beta,\gamma)}(s_3) \otimes \delta^n(s_3, u, s_1) : s_3 \in S, u \in X\} = (\alpha,\beta,\gamma) \otimes \delta^n(s_2, u, s_1) = (\alpha,\beta,\gamma) \otimes (\vee\{\delta^n(s_2, u, s_1) : u \in X\}) = N(s_2) \otimes (\vee\{\delta^n(s_2,u,s_1) : u \in X\}) \le N(s_2)$.

iii). Let $s_1 \in S$. Then $N'(s_1) = N'(s_1) \otimes N(s_1) = N'(s_1) \otimes Su^n(s_{2(\alpha,\beta,\gamma)})(s_1) = \vee\{N'(s_1) \otimes s_{2(\alpha,\beta,\gamma)}(s_2) \otimes \delta^n(s_2, u, s_1): u \in X\} = \vee\{N'(s_1) \otimes (\alpha,\beta,\gamma) \otimes \delta^n(s_2,u,s_1) : u \in X\} = \vee\{N'(s_1) \otimes N(s_1) \otimes \delta^n(s_2,u,s_1) : u \in X\}$ (from (i)) $= \vee\{N'(s_1) \otimes \delta^n(s_2,u,s_1) : u \in X\} \le \vee\{N'(s_2) \otimes \delta^n(s_2, u, s_1): u \in X\} = Su^n(s_{2N'(s_2)})(s_1)$. Hence $N' \le Su^n(s_{2N'(s_2)})$. Similarly, from Proposition 3.3 (ii), $Su^n(s_{2N'(s_2)}) \le N'$. Thus $N' = Su^n(s_{2N'(s_2)})$.

4. Neutrosophic Core Operator

In this section, we present an additional operator, termed the neutrosophic core operator. Utilizing this operator, we establish its interrelation with the operators defined in the prior section and further delineate the neutrosophic primary concepts of the given *NAut*.

Definition 4.1. Let $A = (S, X, \delta^n)$ be a *NAut* and $N \in NS(S)$. The Neutrosophic core (NCOR, in short) of N is given as: -

$$\mu^n(N)(s_2) = (F_{\mu(N)}, G_{\mu(N)}, H_{\mu(N)})(s_2), \text{ Where}$$
$$F_{\mu(N)}(s_2) = \wedge\{F_\delta(s_1,u,s_2) \rightarrow F_N(s_1) : s_1 \in S, u \in X\};$$
$$G_{\mu(N)}(s_2) = \vee\{G_\delta(s_1,u,s_2) \leftarrow G_N(s_1) : s_1 \in S, u \in X\}, \text{ and}$$
$$H_{\mu(N)}(s_2) = \vee\{H_\delta(s_1,u,s_2) \leftarrow H_N(s_1) : s_1 \in S, u \in X\}, \forall s_2 \in S.$$

Remark 4.1. The NCOR can also be written as $\mu^n(N)(s_2) = \wedge\{\delta^n(s_1,u,s_2) \rightarrow N(s_1) : s_1 \in S, u \in X\}$.

Proposition 4.1. Let $A = (S, X, \delta^n)$ be a *NAut* and $\mu^n : NS(S) \rightarrow NS(S)$ be the induced NCOR operator. Then for all $N_1, N_2, \{N_j : j \in J\} \in NS(S)$,

i). $\mu^n \widehat{(\alpha,\beta,\gamma)} = \widehat{(\alpha,\beta,\gamma)}, \forall(\alpha,\beta,\gamma) \in I^*$;
ii). if $N_1 \le N_2$ then $\mu^n(N_1) \le \mu^n(N_2)$;
iii). $\mu^n(N_1) \le N_1$;
iv). $\mu^n(\cap\{N_j : j \in J\}) = \cap\{\mu^n(N_j : j \in J)\}$;
v). $\mu^n(\mu^n(N_1)) = \mu^n(N_1)$;
vi). $\cup\{\mu^n(N_j : j \in J)\} \le \mu^n(\cup\{N_j : j \in J\}$.

Proof:

i). Let $s_2 \in S$. Then $\mu^n(\widehat{\alpha,\beta,\gamma})(s_2) = \wedge\{\delta^n(s_1,u,s_2) \rightarrow (\widehat{\alpha,\beta,\gamma})(s_1): s_1 \in S, u \in X\} = \wedge\{\delta^n(s_1, u, s_2) \rightarrow (\alpha,\beta,\gamma) : s_1 \in S, u \in X\} = \vee\{\delta^n(s_1,u,s_2) : s_1 \in S, u \in X\} \rightarrow (\alpha,\beta,\gamma) = 1_N \rightarrow (\alpha,\beta,\gamma) = (\alpha,\beta,\gamma) = (\widehat{\alpha,\beta,\gamma})(s_2)$.

ii). It is obvious from the definition of NCOR operator.

iii). Let $s_2 \in S$. Then $\mu^n(N_1)(s_2) = \wedge\{\delta^n(s_1,u,s_2) \rightarrow N_1(s_1): s_1 \in S, u \in X\}$, i.e., $F_{\mu(N_1)}(s_2) = \wedge\{F_\delta(s_1,u,s_2) \rightarrow F_{N_1}(s_1): s_1 \in S, u \in X\} \le F_\delta(s_2,e,s_2) \rightarrow F_{N_1}(s_2) = 1 \rightarrow F_{N_1}(s_2) = F_{N_1}(s_2)$, and $G_{\mu(N_1)}(s_2) = \vee\{G_\delta(s_1,u,s_2) \leftarrow G_{N_1}(s_1): s_1 \in S, u \in X\} \ge G_\delta(s_2,e,s_2) \leftarrow G_{N_1}(s_2) = 0 \leftarrow G_{N_1}(s_2) = G_{N_1}(s_2)$. Similarly, $H_{\mu(N_1)}(s_2) = \vee\{H_\delta(s_1,u,s_2) \leftarrow H_{N_1}(s_1): s_1 \in S, u \in X\} \ge H_\delta(s_2,e,s_2) \leftarrow H_{N_1}(s_2) = 0 \leftarrow H_{N_1}(s_2)$. Hence $\mu^n(N_1) \le N_1$.

iv). Let $s_2 \in S$. Then $\mu^n(\cap\{N_j : j \in J\})(s_2) = \wedge\{\delta^n(s_1,u,s_2) \to \{\wedge(N_j(s_1) : j \in J)\} : s_1 \in S, u \in X\}$, i.e., $F_{\mu(\cup\{N_j: j\in J\})}(s_2) = \wedge\{F_\delta(s_2,e,s_2) \to \{\wedge F_{N_j}(s_1) : j \in J)\} : s_1 \in S, u \in X\} = \wedge\{\wedge\{F_\delta(s_2,u,s_2) \to (F_{N_j}(s_1) : j \in J)\} : s_1 \in S, u \in X\} = \wedge\{F_\delta(s_1,u,s_2) \to F_{N_j}(s_1)\} : s_1 \in S, u \in X, j \in J\} = \cap\{F_{\mu(N_j)} : j \in J\}(s_2)$, and $G_{\mu(\cup\{N_j: j\in J\})}(s_2) = \vee\{G_\delta(s_1,u,s_2) \leftarrow \{\vee\{G_{N_j}(s_1) : j \in J)\} : s_1 \in S, u \in X\} = \vee\{\vee\{G_\delta(s_2,u,s_2) \leftarrow (G_{N_j}(s_1) : j \in J)\} : s_1 \in S, u \in X\} = \vee\{G_\delta(s_1,u,s_2) \leftarrow G_{N_j}(s_1)\} : s_1 \in S, u \in X, j \in J\} = \cup\{G_{\mu(N_j)} : j \in J\}(s_2)$. Similarly, $H_{\mu(\cup\{N_j: j\in J\})}(s_2) = \cup\{H_{\mu(N_j)} : j \in J\}(s_2)$. Hence $\mu^n(\cap\{N_j : j \in J\}) = \cap\{\mu^n(N_j : j \in J)\}$.

v). Let $s_2 \in S$. Then $F_{\mu(\mu(N_1)}(s_2) = \wedge\{F_\delta(s_1,u,s_2) \to F_{\mu(N_1)}(s_1) : s_1 \in S, u \in X\} = \wedge\{F_\delta(s_1,u,s_2) \to [\wedge\{F_\delta(s_3,\nu,s_1) \to F_{N_1}(s_3) : s_3 \in S, \nu \in X\}] : s_1 \in S, u \in X\} = \wedge\{\wedge\{F_\delta(s_1,u,s_2) \to (F_\delta(s_3,\nu,s_1) \to F_{N_1}(s_3)) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\} = \wedge\{\wedge\{(F_\delta(s_1,u,s_2) \otimes F_\delta(s_3,\nu,s_1)) \to F_{N_1}(s_3) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\}$ (c.f., Proposition 2.2 (ix)) $= \wedge\{\wedge\{(F_\delta(s_3,\nu,s_1) \otimes F_\delta(s_1,u,s_2)) \to F_{N_1}(s_1) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\} \geq \wedge\{F_\delta(s_3,w,s_2) \to F_{N_1}(s_3) : s_3 \in S, w \in X\} = F_{\mu(N_1)}(s_2)$ and $G_{\mu(\mu(N_1)}(s_2) = \vee\{G_\delta(s_1,u,s_2) \leftarrow G_{\mu(N_1)}(s_1) : s_1 \in S, u \in X\} = \vee\{G_\delta(s_1,u,s_2) \leftarrow [\vee\{G_\delta(s_3,\nu,s_1) \leftarrow G_{N_1}(s_3) : s_3 \in S, \nu \in X\}] : s_1 \in S, u \in X\} = \vee\{\vee\{G_\delta(s_1,u,s_2) \leftarrow (G_\delta(s_3,\nu,s_1) \leftarrow G_{N_1}(s_3)) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\} = \vee\{\vee\{(G_\delta(s_1,u,s_2) \otimes G_\delta(s_3,\nu,s_1)) \leftarrow G_{N_1}(s_3) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\} = \vee\{\vee\{(G_\delta(s_3,\nu,s_1) \otimes G_\delta(s_1,u,s_2)) \leftarrow G_{N_1}(s_3) : s_3 \in S, \nu \in X\} : s_1 \in S, u \in X\} \leq \vee\{G_\delta(s_3,w,s_2) \leftarrow G_{N_1}(s_3) : s_3 \in S, w \in X\} = G_{\mu(N_1)}(s_2)$. Similarly, we have $H_{\mu(\mu(N_1))}(s_2) \leq H_{\mu(N_1)}(s_2)$. Thus $\mu^n(\mu^n(N_1))(s_2) \geq \mu^n(N_1)(s_2)$; also, from the property (iii), converse inequality holds. Hence, $\mu^n(\mu^n(N_1)) = \mu^n(N_1)$.

vi). Follows from the fact that $N_j \leq \cup\{N_j : j \in J\}$ and property (ii).

Definition 4.2. Let $A = (S, X, \delta^n)$ be a $NAut$. Then $N \in NS(S)$ is called
Neutrosophic genetic (*NG*, in short) if $So^n(N) \leq Su^n(N)$;
Neutrosophic genetically closed (*NGC*, in short) if $\exists$ P = $(F_P, G_P, H_P) \leq N = (F_N, G_N, H_N)$, i.e., $F_P \leq F_N$, $G_P \geq G_N$ and $H_P \geq H_N$ such that $So^n(P) \leq Su^n(P)$ and $Su^n(P) = N$;
A Neutrosophic primary if N is a minimal neutrosophic genetically closed subsystem of A.

Proposition 4.2. Let $A = (S, X, \delta^n)$ be a $NAut$. Then for each $N \in NS(S)$, $Su^n(So^n(N))$ is NGC.
Proof: For all $N \in NS(S)$, $So^n(So^n(N)) = So^n(N) \leq Su^n(So^n(N))$. Hence, $Su^n(So^n(N))$ is NGC.

We shall now, demonstrate the interrelationships among the *NSo, NSu,* and *NCo* operators.

Proposition 4.3. Let $A = (S, X, \delta^n)$ be a $NAut$ and $N \in NS(S)$ is a neutrosophic primary. Then $Su^n(So^n(s_{2(\alpha,\beta,\gamma)})) = N$ iff $s_{2(\alpha,\beta,\gamma)} \leq \mu^n(N)$.

Proof: - Let $N \in NS(S)$ be a neutrosophic primary and $Su^n(So^n(s_{2(\alpha,\beta,\gamma)})) = N$. Then

$\forall\ s_2 \in S$, $F_{Su(So(s_{2\alpha}))}(s_2) = F_N(s_2) \Rightarrow F_{So(s_{2\alpha})}(s_2) \leq F_{Su(So(s_{2\alpha}))}(s_2) = F_N(s_2)$

$\Rightarrow\ Fs_{2\alpha} \otimes F_\delta(s_2, u, s_1) \leq \vee\{Fs_{2\alpha}(s_3) \otimes F_\delta(s_2, u, s_3) : s_3 \in S, u \in X\} \leq F_N(s_2)$

$\Rightarrow\ Fs_{2\alpha} \otimes F_\delta(s_2, u, s_1) \leq F_N(s_2), \forall\ s_2 \in S, u \in X$

$\Rightarrow\ Fs_{2\alpha} \leq F_\delta(s_2, u, s_1) \to F_N(s_2), \forall\ s_2 \in S, u \in X$

$\Rightarrow Fs_{2\alpha}(s_1) \leq F_{\mu(N)}(s_1) \Rightarrow Fs_{2\alpha} \leq F_{\mu(N)}$ and,

$G_{Su(So(s_{2\beta}))}(s_2) = G_N(s_2) \Rightarrow G_{So(s_{2\beta})}(s_2) \geq G_{Su(So(s_{2\beta}))}(s_2) = G_N(s_2)$

$\Rightarrow G_{s_{2\beta}} \otimes G_\delta(s_2, u, s_1) \geq \wedge\{G_{s_{2\beta}}(s_3) \otimes G_\delta(s_2, u, s_3) : s_3 \in S, u \in X\} \geq G_N(s_2)$

$\Rightarrow\ G_{s_{2\beta}} \otimes G_\delta(s_2, u, s_1) \geq G_N(s_2), \forall s_2 \in S, u \in X$

$\Rightarrow\ G_{s_{2\beta}} \geq G_\delta(s_2, u, s_1) \leftarrow G_N(s_2), \forall\ s_2 \in S, u \in X$

$\Rightarrow G_{s_{2\beta}}(s_1) \leq G_{\mu(N)}(s_1) \Rightarrow\ G_{s_{2\beta}} \geq G_{\mu(N)}$. Similarly, $H_{s_{2\gamma}} \geq H_{\mu(N)}$. Thus, $s_{2(\alpha,\beta,\gamma)} \leq \mu^n(N)$.

Conversely, Let $s_{2(\alpha,\beta,\gamma)} \leq \mu^n(N)$. Then $F s_{2\alpha}(s_2) \leq F_{\mu(N)}(s_2) \Rightarrow F s_{2\alpha}(s_2) \leq \wedge\{F_\delta(s_3, u, s_2) \rightarrow F_N(s_3) : s_3 \in S,\ u \in X\} \Rightarrow F s_{2\alpha}(s_2) \leq F_\delta(s_3, u, s_2) \rightarrow F_N(s_3),\ \forall\ s_3 \in S, u \in X \Rightarrow F s_{2\alpha}(s_2) \otimes F_\delta(s_3, u, s_2) \leq F_N(s_3) \Rightarrow \vee\{F s_{2\alpha}(s_2) \otimes F_\delta(s_3, u, s_2) : s_3 \in S, u \in X\} \leq F_N(s_3) \Rightarrow F_{So(s_{2\alpha})}(s_3) \leq F_N(s_3) \Rightarrow F_{Su(So(s_{2\alpha}))}(s_3) \leq F_{Su(N)}(s_3) = F_N(s_3)$ and, $G_{s_{2\beta}}(s_2) \geq G_{\mu(N)}(s_2) \Rightarrow G_{s_{2\beta}}(s_2) \geq \vee\{G_\delta(s_3, u, s_2) \leftarrow G_N(s_3) : s_3 \in S,\ u \in X\} \Rightarrow G_{s_{2\beta}}(s_2)(s_2) \geq G_\delta(s_3, u, s_2) \leftarrow G_N(s_3),\ \forall\ s_3 \in S, u \in X \Rightarrow G_{s_{2\beta}}(s_2) \otimes G_\delta(s_3, u, s_2) \geq G_N(s_3) \Rightarrow \wedge\{G_{s_{2\beta}}(s_2) \otimes G_\delta(s_3, u, s_2) : s_3 \in S, u \in X\} \geq G_N(s_3) \Rightarrow G_{So(s_{2\beta})}(s_3) \geq G_N(s_3) \Rightarrow G_{Su(So(s_{2\beta}))}(s_3) \geq G_{Su(N)}(s_3) = G_N(s_3)$. Similarly, $H_{Su(So(s_{2\gamma}))}(s_3) \geq H_{Su(N)}(s_3) = H_N(s_3)$. Also, from Proposition 4.2, $Su^n(So^n(s_{2(\alpha,\beta,\gamma)}))$ is NGC, while N being a neutrosophic primary, it is minimal neutrosophic genetically closed subset, i.e., $N \leq Su^n(So^n(s_{2(\alpha,\beta,\gamma)}))$. Hence $= Su^n(So^n(s_{2(\alpha,\beta,\gamma)}))$.

Proposition 4.4. Let (S, X, δ^n) be a NAut and $N \in NS(S)$. Then $N \leq Su^n(\mu^n(s_{2(\alpha,\beta,\gamma)})) \Rightarrow So^n(N) \leq s_{2(\alpha,\beta,\gamma)}$.

Proof: - Let $s_1 \in S$ with $N(s_1) \leq Su^n(\mu^n(s_{2(\alpha,\beta,\gamma)}))(s_1)$. Then $F_N(s_1) \leq F_{Su(\mu(s_{2\alpha}))}(s_1) = \vee\{F_\delta(s_3, u, s_1) \otimes F_{\mu(s_{2\alpha})}(s_3) : s_3 \in S, u \in X\} = \vee\{F_\delta(s_3, u, s_1) \otimes \wedge\{F_\delta(s_4, v, s_3) \rightarrow F s_{2\alpha}(s_4) : s_4 \in S, v \in X\} : s_3 \in S, u \in X\} \leq \vee\{F_\delta(s_3, u, s_1) \otimes \wedge\{F_\delta(s_2, v, s_3) \rightarrow F_\alpha : v \in X\} : s_3 \in S, u \in X\} \leq \vee\{\wedge\{F_\delta(s_3, u, s_1) \otimes F_\delta(s_2, v, s_3) \rightarrow F_\alpha : v \in X\} : s_3 \in S, u \in X\} \leq \wedge\{\vee\{F_\delta(s_2, v, s_3) \otimes F_\delta(s_3, u, s_1) \rightarrow F\alpha : s_3 \in S, u \in X\} : v \in X\} \leq \wedge\{F_\delta(s_2, w, s_1) \rightarrow F\alpha : w \in X\}$, which implies that $F_N(s_1) \leq F_\delta(s_2, w, s_1) \rightarrow F\alpha \Rightarrow F_N(s_1) \otimes F_\delta(s_2, w, s_1) \leq F\alpha \Rightarrow \vee\{F_N(s_1) \otimes F_\delta(s_2, w, s_1) : s_1 \in S, w \in X\} \leq F s_{2\alpha}(s_2) \Rightarrow F_{So(N)}(s_2) \leq F s_{2\alpha}(s_2) \Rightarrow F_{So(N)} \leq F s_{2\alpha}$ and $G_N(s_1) \geq G_{Su(\mu(s_{2\beta}))}(s_1) = \wedge\{G_\delta(s_3, u, s_1) \otimes G_{\mu(s_{2\beta})}(s_3) : s_3 \in S, u \in X\} = \wedge\{G_\delta(s_3, u, s_1) \otimes \vee\{G_\delta(s_4, v, s_3) \leftarrow G s_{2\beta}(s_4) : s_4 \in S, v \in X\} : s_3 \in S, u \in X\} \geq \wedge\{G_\delta(s_3, u, s_1) \otimes \vee\{G_\delta(s_2, v, s_3) \leftarrow G_\beta : v \in X\} : s_3 \in S, u \in X\} \geq \wedge\{\vee\{G_\delta(s_3, u, s_1) \otimes G_\delta(s_2, v, s_3) \leftarrow G_\beta : v \in X\} : s_3 \in S, u \in X\} \geq \vee\{\wedge\{G_\delta(s_2, v, s_3) \otimes G_\delta(s_3, u, s_1) \leftarrow G_\beta : s_3 \in S, u \in X\} : v \in X\} \geq \vee\{G_\delta(s_2, w, s_1) \leftarrow G_\beta : w \in X\}$, which implies that $G_N(s_1) \geq G_\delta(s_2, w, s_1) \leftarrow G_\beta \Rightarrow G_N(s_1) \otimes G_\delta(s_2, w, s_1) \geq G_\beta \Rightarrow \wedge\{G_N(s_1) \otimes G_\delta(s_2, w, s_1) : s_1 \in S, w \in X\} \geq G s_{2\beta}(s_2) \Rightarrow G_{So(N)}(s_2) \geq G s_{2\beta}(s_2) \Rightarrow G_{So(N)} \geq G s_{2\beta}$. Similarly, $H_{So(N)} \geq H s_{2\gamma}$. Thus, $So^n(N) \leq s_{2(\alpha,\beta,\gamma)}$.

Proposition 4.5. Let (S, X, δ^n) be a NAut and $N \in NS(S)$. Then $Su^n(\mu^n(N)) = \mu^n(N)$.

Proof: - Let $s_2 \in S$. Then $F_{SU(\mu(N))}(s_2) = \vee\{F_\delta(s_1, u, s_2) \otimes F_{\mu(N)}(s_1) : s_1 \in S, u \in X\} = \vee\{F_\delta(s_1, u, s_2) \otimes (\wedge\{F_\delta(s_3, v, s_1) \rightarrow F_N(s_3) : s_3 \in S, v \in X\}) : s_1 \in S, u \in X\} \leq \vee\{\wedge\{F_\delta(s_1, u, s_2) \otimes (F_\delta(s_3, v, s_1) \rightarrow F_N(s_3)) : s_3 \in S, v \in X\} : s_1 \in S, u \in X\} \leq \vee\{\wedge\{(F_\delta(s_1, u, s_2) \otimes F_\delta(s_3, v, s_1)) \rightarrow F_N(s_3) : s_3 \in S, v \in X\} : s_1 \in S, u \in X\} \leq \wedge\{\vee\{(F_\delta(s_3, v, s_1) \otimes F_\delta(s_1, u, s_2)) \rightarrow F_N(s_3) : s_1 \in S, u \in X\}) : s_3 \in S, v \in X\} \leq \wedge\{F_\delta(s_3, w, s_2) \rightarrow F_N(s_3) : s_3 \in S, w \in X\} = F_{\mu(N)}(s_2)$ and $G_{SU(\mu(N))}(s_2) = \wedge\{G_\delta(s_1, u, s_2) \otimes G_{\mu(N)}(s_1) : s_1 \in S, u \in X\} = \wedge\{G_\delta(s_1, u, s_2) \otimes (\vee\{G_\delta(s_3, v, s_1) \leftarrow G_N(s_3) : s_3 \in S, v \in X\}) : s_1 \in S, u \in X\} \geq \wedge\{\vee\{G_\delta(s_1, u, s_2) \otimes (G_\delta(s_3, v, s_1) \leftarrow G_N(s_3)) : s_3 \in S, v \in X\} : s_1 \in S, u \in X\} \geq \wedge\{\vee\{(G_\delta(s_1, u, s_2) \otimes G_\delta(s_3, v, s_1)) \leftarrow G_N(s_3) : s_3 \in S, v \in X\} : s_1 \in S, u \in X\} \geq \vee\{\wedge\{(G_\delta(s_3, v, s_1) \otimes G_\delta(s_1, u, s_2)) \leftarrow G_N(s_3) : s_1 \in S, u \in X\}) : s_3 \in S, v \in X\} \geq \vee\{G_\delta(s_3, w, s_2) \leftarrow G_N(s_3) : s_3 \in S, w \in X\} = G_{\mu(N)}(s_2)$. Similarly, $H_{Su(\mu(N))}(s_2) \geq H_{\mu(N)}(s_2)$. Hence, $Su^n(\mu^n(N)) = \mu^n(N)$.

5. Conclusion

In this paper, we have examined the algebraic properties of neutrosophic automata through the application of various operators. We have demonstrated the interconnections between the neutrosophic source, neutrosophic successor, and neutrosophic core operators under specific conditions. These operators are pivotal in analyzing the neutrosophic primary structure of the given neutrosophic automata.

5.1 Future Plan

The discussions presented here offer hope for some new insights into:

i). The study of neutrosophic topology/neutrosophic co-topology/neutrosophic homomorphisms.

ii). The decompositions of a neutrosophic automaton can be proposed.

iii). The decomposition of a neutrosophic automaton in terms of its layers and propose a framework for the construction of a neutrosophic automaton corresponding to a given finite partially ordered set (poset).

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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Author Contribution

All authors contributed equally to this research.

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