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Analysis of BCK/BCI-Algebras Based on Bipolar Complex Intuitionistic Fuzzy Soft Ideals

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Abstract: In this article, we design an informative and reliable technique of bipolar complex intuitionistic fuzzy soft sets with numerous operational laws by merging the model of soft sets, complex fuzzy sets, and bipolar intuitionistic fuzzy sets to handle imprecise data. In addition, an ideal in a BCK-algebra is derived based on bipolar complex intuitionistic fuzzy soft set theory are proposed which can capture the information of hesitancy, vagueness, and non-membership information within the circumstance of BCK-algebra. Moreover, we design union, intersection, AND, and OR based on bipolar complex intuitionistic fuzzy soft ideal and simplify it with the help of numerous illustrations to justify the effectiveness of the invented theory. In the end, we also evaluate numerous results for the above-invented techniques to enhance the worth of the proposed theory.

Keywords: BCK/BCI-Algebras; Bipolar Complex Intuitionistic Fuzzy Soft Deal; Bipolar Complex Intuitionistic Fuzzy Soft Sets; Union and Intersection.

1. Introduction

In 1965, Zadeh [1] designed the system of fuzzy sets (FSs) with a perfect and reliable function, called the function of membership with a wide range of data, because the range of crisp sets is very narrow. The informative and dominant system of FSs is the modified version of the crisp set because we have just zero and one in the presence of the crisp set, but in numerous situations, we required a function with can give more space to decision-makers for making their decision like zero, one, and between of them. In addition, in 1983 as well as in 1986, Atanassov [2, 3] designed a new model, called the intuitionistic fuzzy sets (IFSs) with two major functions, defined from any fixed sets to the unit interval, where the functions are called truth function and falsity function with a condition that is the sum of the both functions must be the part of unit interval. The informative model of IFSs is very capable and flexible because in numerous real-world problems, we observed that without falsity information it is quite complex to cope with ambiguous and unreliable data, where the theory of FSs is part of IFSs.

Complex-valued contain two-dimension information, called the real one and imaginary one which is very important and dominant for coping with vague data, because, during the selection of any type of software, we provided two types of information, called the name of the software and version of the software which is represented the real and imaginary parts of complex-valued, for this, experts have noticed that to change the range of the truth function which is very exciting task for scholars. For this, in 2002, Ramot et al. [4] designed complex fuzzy sets (CFSs) with a complex version of the truth function. In addition, the complex-valued truth function is not enough for coping with vague and uncertain data, for this, the model of complex intuitionistic fuzzy sets (CIFSs) was designed by Alkouri and Salleh [5] in 2012. The informative and well-known theory of CIFSs is very advanced and famous because the existing FSs, IFSs, and CFSs have numerous specifications compared to CIFSs.

After our long assessment, we observed that the truth function and falsity function are very important because we missed the important aspects of the truth and falsity information, for instance, during purchasing any medicine, we have some problems and for those problems, we use some medicine to recover from the problems but these medicines have also some negative aspects which are very important to discuss it here, for this, Zhang [6] designed the system of bipolar fuzzy sets (BFSs) with a positive truth function and negative truth function, where the advantages of the medicine signify positive aspects and the disadvantages of the medicine signifies the negative aspects of the medicine. In 2022, Mahmood and Rehman [7] modified the BFSs to initiate the bipolar complex fuzzy sets (BCFSs) with a complex-valued positive truth function and complex-valued negative truth function.

In 2001, Lee et al. [8] initiated the bipolar intuitionistic fuzzy sets that delimited the positive and negative truth and falsity function with a condition of IFSs for both positive and negative functions. In addition, Molodtsov [9] derived the soft sets (SSs) theory with a function defined from a set of parameters to power sets. Abdullah et al. [10] derived the bipolar fuzzy soft sets with applications. Alqaraleh et al. [11] invented the bipolar complex fuzzy SSs. Jana [12] evaluated the bipolar intuitionistic fuzzy SSs with applications. Balamurugan et al. [13] designed the bipolar intuitionistic fuzzy soft ideals of BCK/BCI-algebras with applications. Gwak et al. [14] calculated the clustering analysis with bipolar complex fuzzy soft sets. In 1966, Imai and Iseki [15] presented the axiom systems of propositional calculi. In 1978, Iseki and Tanaka [16] described an introduction to the technique of BCK-algebra. From the above analysis, it is obvious that the technique of BCK/BCI-algebra was not proposed for bipolar complex intuitionistic fuzzy soft sets which is very complex and ambiguous. Inspired by the above information, we decided to propose the following models, such as:

- To design an informative and reliable technique of bipolar complex intuitionistic fuzzy soft sets (BCIFSSs) with numerous operational laws by merging the model of soft sets, complex fuzzy sets, and bipolar intuitionistic fuzzy sets to handle imprecise data.
- An ideal in a BCK-algebra is derived based on the BCIFSS theory proposed which can capture the information of hesitancy, vagueness, and non-membership information within the circumstance of BCK-algebra.
- To design union, intersection, AND, and OR based on the bipolar complex intuitionistic fuzzy soft ideal (BCIFSI) and simplify it with the help of numerous illustrations to justify the effectiveness of the invented theory.
- To evaluate numerous results for the above-invented techniques to enhance the worth of the proposed theory.

This manuscript is constructed in the following shape, In Section 2, we concisely revised the model of existing definitions, called BCK-algebra, idea, SSs, and BCFSs with basic laws. In Section 3, we designed an informative and reliable technique of BCIFSSs with numerous operational laws by merging the model of soft sets, complex fuzzy sets, and bipolar intuitionistic fuzzy sets to handle imprecise data. In addition, an ideal in a BCK-algebra is derived based on the BCIFSS theory proposed which can capture the information of hesitancy, vagueness, and non-membership information within the circumstance of BCK-algebra. In Section 4, we designed union, intersection, AND, and OR based on bipolar complex intuitionistic fuzzy soft ideal and simplified it with the help of numerous illustrations to justify the effectiveness of the invented theory. In the end, we also evaluate numerous results for the above-invented techniques to enhance the worth of the proposed theory. Numerous concluding remarks are stated in Section 5.

2. Preliminaries

In this section, we concisely revised the model of existing definitions, called BCK-algebra, idea, SSs, and BCFSs with basic laws.

Definition 1: [15] The representation of the algebra $(\ddot{U},*,0)$ of type (2,0), called the BCK-algebra when

$$\mathbb{K}_1: \big((\ddot{\varepsilon} * \ddot{\partial}) * (\ddot{\varepsilon} * \ddot{\alpha})\big) * (\ddot{\alpha} * \ddot{\partial}) = 0$$
$$\mathbb{K}_2: \big(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{\partial})\big) * \ddot{\partial} = 0$$
$$\mathbb{K}_3: \ddot{\varepsilon} * \ddot{\varepsilon} = 0$$
$$\mathbb{K}_4: 0 * \ddot{\varepsilon} = 0$$
$$\mathbb{K}_5: \ddot{\varepsilon} * \ddot{\partial} = 0 \; and \; \ddot{\partial} * \ddot{\varepsilon} = 0 \; if \; \ddot{\partial} = \ddot{\varepsilon}$$

Where for all $\ddot{\varepsilon}, \ddot{\partial}, \ddot{\alpha} \in \ddot{U}$. Using the technique of BCK-algebra $(\ddot{U}, *, 0)$, we have also some models, such as

$$\mathbb{P}_1: \ddot{\varepsilon} * 0 = \ddot{\varepsilon}$$
$$\mathbb{P}_2: (\ddot{\varepsilon} * \ddot{\partial}) * \ddot{\alpha} = (\ddot{\varepsilon} * \ddot{\alpha}) * \ddot{\partial}$$

Definition 2. [16] When $\mathbb{I}_1: 0 \in \mathbb{L}$ and $\mathbb{I}_2: \forall \ddot{\varepsilon}, \ddot{\partial} \in \mathbb{L}, \ddot{\varepsilon} * \ddot{\partial} \in \mathbb{L}, \ddot{\partial} \in \mathbb{L} \Rightarrow \ddot{\varepsilon} \in \mathbb{L}$, thus the $\mathbb{L}$ is called an ideal of $\ddot{U}$.

Definition 3. [9] A couple $(\mathbb{F}, \mathbb{L})$ is signified as an SS, where the $\mathbb{L} \subseteq \mathbb{E}$ is called the group of parameters on $\ddot{U}$ with a function $\mathbb{F}: \mathbb{L} \to P(\ddot{U})$.

Definition 4. [7] A mathematical version or model of BCFS $\check{\mathfrak{z}}^{LB}$ for a fixed set $\ddot{U}$ is deliberating by:

$$\check{\mathfrak{z}}^{LB} = \left\{\left(\Xi_{\check{\mathfrak{z}}}(\mathbb{x}) + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}}(\mathbb{x}), \Psi_{\check{\mathfrak{z}}}(\mathbb{x}) + \hat{\imath}\Phi_{\check{\mathfrak{z}}}(\mathbb{x})\right) : \mathbb{x} \in \ddot{U}\right\}$$

The terminology of truth value and falsity function in the shape of complex value is described in the following shape, such as $\Xi_{\check{\mathfrak{z}}}(\mathbb{x}) + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}}(\mathbb{x})$ and $\Psi_{\check{\mathfrak{z}}}(\mathbb{x}) + \hat{\imath}\Phi_{\check{\mathfrak{z}}}(\mathbb{x})$, where $\Xi_{\check{\mathfrak{z}}}(\mathbb{x}), \hat{\omega}_{\check{\mathfrak{z}}}(\mathbb{x}) \in [0,1]$, and $\Psi_{\check{\mathfrak{z}}}(\mathbb{x}), \Phi_{\check{\mathfrak{z}}}(\mathbb{x}) \in [-1,0], \hat{\imath} = \sqrt{-1}$. At last, the technique $\check{\mathfrak{z}}_{ь}^{LB} = \left(\Xi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}_ь}, \Psi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\Phi_{\check{\mathfrak{z}}_ь}\right), ь = 1,2,\dots,\kappa$, called BCFNs.

Definition 5. [7] Let $\check{\mathfrak{z}}_{ь}^{LB} = \left(\Xi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}_ь}, \Psi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\Phi_{\check{\mathfrak{z}}_ь}\right), ь = 1,2$, used as two BCFNs. Thus

$$\check{\mathfrak{z}}_1^{LB} \oplus \check{\mathfrak{z}}_2^{LB} = \begin{pmatrix} \left(\Xi_{\check{\mathfrak{z}}_1} + \Xi_{\check{\mathfrak{z}}_2} - \Xi_{\check{\mathfrak{z}}_1}\Xi_{\check{\mathfrak{z}}_2}\right) + \hat{\imath}\left(\hat{\omega}_{\check{\mathfrak{z}}_1} + \hat{\omega}_{\check{\mathfrak{z}}_2} - \hat{\omega}_{\check{\mathfrak{z}}_1}\hat{\omega}_{\check{\mathfrak{z}}_2}\right), \\ -\left(\Psi_{\check{\mathfrak{z}}_1}\Psi_{\check{\mathfrak{z}}_2}\right) + \hat{\imath}\left(-\left(\Phi_{\check{\mathfrak{z}}_1}\Phi_{\check{\mathfrak{z}}_2}\right)\right) \end{pmatrix}$$

$$\check{\mathfrak{z}}_1^{LB} \otimes \check{\mathfrak{z}}_2^{LB} = \begin{pmatrix} \left(\Xi_{\check{\mathfrak{z}}_1}\Xi_{\check{\mathfrak{z}}_2}\right) + \hat{\imath}\left(\hat{\omega}_{\check{\mathfrak{z}}_1}\hat{\omega}_{\check{\mathfrak{z}}_2}\right), \\ \left(\Psi_{\check{\mathfrak{z}}_1} + \Psi_{\check{\mathfrak{z}}_2} + \Psi_{\check{\mathfrak{z}}_1}\Psi_{\check{\mathfrak{z}}_2}\right) + \hat{\imath}\left(\Phi_{\check{\mathfrak{z}}_1} + \Phi_{\check{\mathfrak{z}}_2} + \Phi_{\check{\mathfrak{z}}_1}\Phi_{\check{\mathfrak{z}}_2}\right) \end{pmatrix}$$

$$q\check{\mathfrak{z}}_1^{LB} = \left(1 - \left(1 - \Xi_{\check{\mathfrak{z}}_1}\right)^{q} + \hat{\imath}\left(1 - \left(1 - \hat{\omega}_{\check{\mathfrak{z}}_1}\right)^{q}\right), -\left|\Psi_{\check{\mathfrak{z}}_1}\right|^{q} + \hat{\imath}\left(-\left|\Phi_{\check{\mathfrak{z}}_1}\right|^{q}\right)\right)$$

$$\left(\check{\mathfrak{z}}_1^{LB}\right)^{q} = \left(\left(\Xi_{\check{\mathfrak{z}}_1}\right)^{q} + \hat{\imath}\left(\left(\hat{\omega}_{\check{\mathfrak{z}}_1}\right)^{q}\right), -1 + \left(1 + \Psi_{\check{\mathfrak{z}}_1}\right)^{q} + \hat{\imath}\left(-1 + \left(1 + \Phi_{\check{\mathfrak{z}}_1}\right)^{q}\right)\right)$$

Definition 6. [7] Let $\check{\mathfrak{z}}_{ь}^{LB} = \left(\Xi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}_ь}, \Psi_{\check{\mathfrak{z}}_ь} + \hat{\imath}\Phi_{\check{\mathfrak{z}}_ь}\right), ь = 1$, used as a BCFN. Thus

$$SC\left(\check{\mathfrak{z}}_{ь}^{LB}\right) = \frac{1}{2}\left(\Xi_{\check{\mathfrak{z}}_ь} + \hat{\omega}_{\check{\mathfrak{z}}_ь} + \Psi_{\check{\mathfrak{z}}_ь} + \Phi_{\check{\mathfrak{z}}_ь}\right) \in [-1,1]$$

$$AC\left(\check{\mathfrak{z}}_{ь}^{LB}\right) = \frac{1}{2}\left(\Xi_{\check{\mathfrak{z}}_ь} + \hat{\omega}_{\check{\mathfrak{z}}_ь} - \Psi_{\check{\mathfrak{z}}_ь} - \Phi_{\check{\mathfrak{z}}_ь}\right) \in [0,1]$$

Signified the score and accuracy model, such as:

i). When $SC\left(\check{\mathfrak{z}}_1^{LB}\right) > SC\left(\check{\mathfrak{z}}_2^{LB}\right) \Rightarrow \check{\mathfrak{z}}_1^{LB} > \check{\mathfrak{z}}_2^{LB}$,
ii). When $SC\left(\check{\mathfrak{z}}_1^{LB}\right) < SC\left(\check{\mathfrak{z}}_2^{LB}\right) \Rightarrow \check{\mathfrak{z}}_1^{LB} < \check{\mathfrak{z}}_2^{LB}$,
iii). When $SC\left(\check{\mathfrak{z}}_1^{LB}\right) = SC\left(\check{\mathfrak{z}}_2^{LB}\right) \Rightarrow$, thus
iv). When $AC\left(\check{\mathfrak{z}}_1^{LB}\right) > AC\left(\check{\mathfrak{z}}_2^{LB}\right) \Rightarrow \check{\mathfrak{z}}_1^{LB} > \check{\mathfrak{z}}_2^{LB}$,
v). When $AC\left(\check{\mathfrak{z}}_1^{LB}\right) < AC\left(\check{\mathfrak{z}}_2^{LB}\right) \Rightarrow \check{\mathfrak{z}}_1^{LB} < \check{\mathfrak{z}}_2^{LB}$.

3. BCIFSI of BCK/BCI-Algebras

This section offers the BCFSSs with important properties based on the SSs and BCFSs. In addition, the bipolar complex fuzzy soft ideal of BCK/BCI-algebra is a modified form of bipolar complex fuzzy soft ideas of BCK/BCI-algebras with numerous important properties.

Definition 7. A couple $(\mathbb{F}, \mathbb{L}) = \check{\mathfrak{z}}^{LB}(\ddot{\eta})$ is signified as a BCIFSS, where the $\mathbb{L} \subseteq \mathbb{E}$ is called the group of parameters on $\ddot{U}$ with a function $\mathbb{F}: \mathbb{L} \to BCIFS(\ddot{U})$, thus the mathematical version or model of BCIFSS $\check{\mathfrak{z}}^{LB}(\ddot{\eta})$ for a fixed set $\ddot{U}$ is deliberating by:

$$\check{\mathfrak{z}}^{LB}(\ddot{\eta}) = \left\{\left(\Xi_{\check{\mathfrak{z}}}^{-}(\mathbb{x}) + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}}^{-}(\mathbb{x}), \Xi_{\check{\mathfrak{z}}}^{+}(\mathbb{x}) + \hat{\imath}\hat{\omega}_{\check{\mathfrak{z}}}^{+}(\mathbb{x}), \Psi_{\check{\mathfrak{z}}}^{-}(\mathbb{x}) + \hat{\imath}\Phi_{\check{\mathfrak{z}}}^{-}(\mathbb{x}), \Psi_{\check{\mathfrak{z}}}^{+}(\mathbb{x}) + \hat{\imath}\Phi_{\check{\mathfrak{z}}}^{+}(\mathbb{x})\right) : \mathbb{x} \in \ddot{U}\right\}$$

The terminology of positive and negative truth value and falsity function in the shape of complex value is described in the following shape, such as $\Xi^{-}_{ʒ}(𝕩) + ȋ\hat{Ѽ}^{-}_{ʒ}(𝕩), \Xi^{+}_{ʒ}(𝕩) + ȋ\hat{Ѽ}^{+}_{ʒ}(𝕩)$ and $\Psi^{-}_{ʒ}(𝕩) + ȋ\Phi^{-}_{ʒ}(𝕩), \Psi^{+}_{ʒ}(𝕩) + ȋ\Phi^{+}_{ʒ}(𝕩)$, where $\Xi^{+}_{ʒ}(𝕩), \hat{Ѽ}^{+}_{ʒ}(𝕩), \Psi^{+}_{ʒ}(𝕩), \Phi^{+}_{ʒ}(𝕩) \in [0,1]$, and $\Xi^{-}_{ʒ}(𝕩), \hat{Ѽ}^{-}_{ʒ}(𝕩), \Psi^{-}_{ʒ}(𝕩), \Phi^{-}_{ʒ}(𝕩) \in [-1,0], ȋ = \sqrt{-1}$. At last, the technique $ʒ^{LB}_{ь}(ῆ) = \left(\Xi^{-}_{ʒ}(𝕩) + ȋ\hat{Ѽ}^{-}_{ʒ}(𝕩), \Xi^{+}_{ʒ}(𝕩) + ȋ\hat{Ѽ}^{+}_{ʒ}(𝕩), \Psi^{-}_{ʒ}(𝕩) + ȋ\Phi^{-}_{ʒ}(𝕩), \Psi^{+}_{ʒ}(𝕩) + ȋ\Phi^{+}_{ʒ}(𝕩)\right)$, $ь = 1,2, \ldots, к$, called BCIFSNs.

Definition 8. A BCIFSS $ʒ^{LB}(ῆ)$ in Ü, signified a model of BCIFSI, when

$$BCIFSI_1: \Xi^{-}_{ʒ}(0) \le \Xi^{-}_{ʒ}(ȅ), \hat{Ѽ}^{-}_{ʒ}(0) \le \hat{Ѽ}^{-}_{ʒ}(ȅ), \Xi^{+}_{ʒ}(0) \ge \Xi^{+}_{ʒ}(ȅ), \hat{Ѽ}^{+}_{ʒ}(0) \ge \hat{Ѽ}^{+}_{ʒ}(ȅ), \forall\, ȅ \in Ü$$

$$BCIFSI_2: \Xi^{-}_{ʒ}(ȅ) \le \Xi^{-}_{ʒ}(ȅ * ӛ) \vee \Xi^{-}_{ʒ}(ӛ), \hat{Ѽ}^{-}_{ʒ}(ȅ) \le \hat{Ѽ}^{-}_{ʒ}(ȅ * ӛ) \vee \hat{Ѽ}^{-}_{ʒ}(ӛ), \Xi^{+}_{ʒ}(ȅ) \ge \Xi^{+}_{ʒ}(ȅ * ӛ) \wedge \Xi^{+}_{ʒ}(ӛ), \hat{Ѽ}^{+}_{ʒ}(ȅ) \ge \hat{Ѽ}^{+}_{ʒ}(ȅ * ӛ) \wedge \hat{Ѽ}^{+}_{ʒ}(ӛ), \forall\, ȅ, ӛ \in Ü$$

$$BCIFSI_3: \Psi^{-}_{ʒ}(0) \ge \Psi^{-}_{ʒ}(ȅ), \Phi^{-}_{ʒ}(0) \ge \Phi^{-}_{ʒ}(ȅ), \Psi^{+}_{ʒ}(0) \le \Psi^{+}_{ʒ}(ȅ), \Phi^{+}_{ʒ}(0) \le \Phi^{+}_{ʒ}(ȅ), \forall\, ȅ \in Ü$$

$$BCIFSI_4: \Psi^{-}_{ʒ}(ȅ) \ge \Psi^{-}_{ʒ}(ȅ * ӛ) \wedge \Psi^{-}_{ʒ}(ӛ), \Phi^{-}_{ʒ}(ȅ) \ge \Phi^{-}_{ʒ}(ȅ * ӛ) \wedge \Phi^{-}_{ʒ}(ӛ), \Psi^{+}_{ʒ}(ȅ) \le \Psi^{+}_{ʒ}(ȅ * ӛ) \vee \Psi^{+}_{ʒ}(ӛ), \Phi^{+}_{ʒ}(ȅ) \le \Phi^{+}_{ʒ}(ȅ * ӛ) \vee \Phi^{+}_{ʒ}(ӛ), \forall\, ȅ, ӛ \in Ü$$

In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(Ü, *, 0)$ with the data in Table 1.

Table 1. Representation of the Cayley information for BCK-algebra (BCIFSI).

*	**0**	ȅ	ӛ	ᾶ
0	0	0	0	0
ȅ	ȅ	0	0	ȅ
ӛ	ӛ	ȅ	0	ӛ
ᾶ	ᾶ	ᾶ	ᾶ	0

Thus, we define a function, such as

$$ʒ^{LB}(ῆ) = \begin{Bmatrix} (0, -0.2 - 0.21ȋ, 0.1 + 0.11ȋ, -0.1 - 0.11ȋ, 0.4 + 0.41ȋ), \\ (ȅ, -0.4 - 0.41ȋ, 0.3 + 0.31ȋ, -0.5 - 0.51ȋ, 0.1 + 0.11ȋ), \\ (ӛ, -0.1 - 0.11ȋ, 0.4 + 0.41ȋ, -0.2 - 0.21ȋ, 0.2 + 0.21ȋ), \\ (ᾶ, -0.0 - 0.01ȋ, 0.2 + 0.21ȋ, -0.3 - 0.31ȋ, 0.3 + 0.31ȋ) \end{Bmatrix}$$

Then, $ʒ^{LB}(ῆ)$, signified a BCIFSI.

Theorem 1. Suppose $ʒ^{LB}(ῆ)$ is a BCIFSI and $\Xi^{-}_{ʒ}(0) = \Xi^{-}_{ʒ}(ȅ), \hat{Ѽ}^{-}_{ʒ}(0) = \hat{Ѽ}^{-}_{ʒ}(ȅ), \Xi^{+}_{ʒ}(0) = \Xi^{+}_{ʒ}(ȅ), \hat{Ѽ}^{+}_{ʒ}(0) = \hat{Ѽ}^{+}_{ʒ}(ȅ)$ and $\Psi^{-}_{ʒ}(0) = \Psi^{-}_{ʒ}(ȅ), \Phi^{-}_{ʒ}(0) = \Phi^{-}_{ʒ}(ȅ), \Psi^{+}_{ʒ}(0) = \Psi^{+}_{ʒ}(ȅ), \Phi^{+}_{ʒ}(0) = \Phi^{+}_{ʒ}(ȅ), \forall\, ȅ \in Ü$, thus $\mathbb{I}(0)$ is an ideal of Ü.

Proof: Assume that $ȅ, ӛ \in Ü$ be considered that $ȅ, ӛ \in \mathbb{I}(0)$, and $ӛ \in \mathbb{I}(0)$, thus

$$\Xi^{-}_{ʒ}(ȅ * ӛ) = \Xi^{-}_{ʒ}(0), \hat{Ѽ}^{-}_{ʒ}(ȅ * ӛ) = \hat{Ѽ}^{-}_{ʒ}(0), \Xi^{-}_{ʒ}(ӛ) = \Xi^{-}_{ʒ}(0), \hat{Ѽ}^{-}_{ʒ}(ӛ) = \hat{Ѽ}^{-}_{ʒ}(0), \Xi^{+}_{ʒ}(ȅ * ӛ) = \Xi^{+}_{ʒ}(0), \hat{Ѽ}^{+}_{ʒ}(ȅ * ӛ) = \hat{Ѽ}^{+}_{ʒ}(0), \Xi^{-}_{ʒ}(ӛ) = \Xi^{+}_{ʒ}(0), \hat{Ѽ}^{+}_{ʒ}(ӛ) = \hat{Ѽ}^{+}_{ʒ}(0)$$

and

$$\Psi^{-}_{ʒ}(ȅ * ӛ) = \Psi^{-}_{ʒ}(0), \Phi^{-}_{ʒ}(ȅ * ӛ) = \Phi^{-}_{ʒ}(0), \Psi^{-}_{ʒ}(ӛ) = \Psi^{-}_{ʒ}(0), \Phi^{-}_{ʒ}(ӛ) = \Phi^{-}_{ʒ}(0), \Psi^{+}_{ʒ}(ȅ * ӛ) = \Psi^{+}_{ʒ}(0), \Phi^{+}_{ʒ}(ȅ * ӛ) = \Phi^{+}_{ʒ}(0), \Psi^{-}_{ʒ}(ӛ) = \Psi^{+}_{ʒ}(0), \Phi^{+}_{ʒ}(ӛ) = \Phi^{+}_{ʒ}(0)$$

Thus, considering the information in Def. (8), we have

$$BCIFSI_2: \Xi^{-}_{ʒ}(ȅ) \le \Xi^{-}_{ʒ}(0) \vee \Xi^{-}_{ʒ}(0) = \Xi^{-}_{ʒ}(0), \hat{Ѽ}^{-}_{ʒ}(ȅ) \le \hat{Ѽ}^{-}_{ʒ}(0) \vee \hat{Ѽ}^{-}_{ʒ}(0) = \hat{Ѽ}^{-}_{ʒ}(0), \Xi^{+}_{ʒ}(ȅ) \ge \Xi^{+}_{ʒ}(0) \wedge \Xi^{+}_{ʒ}(0) = \Xi^{+}_{ʒ}(0), \hat{Ѽ}^{+}_{ʒ}(ȅ) \ge \hat{Ѽ}^{+}_{ʒ}(0) \wedge \hat{Ѽ}^{+}_{ʒ}(0) = \hat{Ѽ}^{+}_{ʒ}(0), \forall\, ȅ, ӛ \in Ü$$

and

$$BCIFSI_4: \Psi^{-}_{ʒ}(ȅ) \ge \Psi^{-}_{ʒ}(0) \wedge \Psi^{-}_{ʒ}(0) = \Psi^{-}_{ʒ}(0), \Phi^{-}_{ʒ}(ȅ) \ge \Phi^{-}_{ʒ}(0) \wedge \Phi^{-}_{ʒ}(0) = \Phi^{-}_{ʒ}(0), \Psi^{+}_{ʒ}(ȅ) \le \Psi^{+}_{ʒ}(0) \vee \Psi^{+}_{ʒ}(0) = \Psi^{+}_{ʒ}(0), \Phi^{+}_{ʒ}(ȅ) \le \Phi^{+}_{ʒ}(0) \vee \Phi^{+}_{ʒ}(0) = \Phi^{+}_{ʒ}(0), \forall\, ȅ, ӛ \in Ü$$

Thus, considering the remaining information in Def. (8), we have

$$BCIFSI_1: \Xi^{-}_{ʒ}(0) \le \Xi^{-}_{ʒ}(ȅ), \hat{Ѽ}^{-}_{ʒ}(0) \le \hat{Ѽ}^{-}_{ʒ}(ȅ), \Xi^{+}_{ʒ}(0) \ge \Xi^{+}_{ʒ}(ȅ), \hat{Ѽ}^{+}_{ʒ}(0) \ge \hat{Ѽ}^{+}_{ʒ}(ȅ), \forall\, ȅ \in Ü$$

and

$BCIFSI_3$: $\Psi^-_{\check{\mathfrak{z}}}(0) \geq \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Phi^-_{\check{\mathfrak{z}}}(0) \geq \Phi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Psi^+_{\check{\mathfrak{z}}}(0) \leq \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Phi^+_{\check{\mathfrak{z}}}(0) \leq \Phi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \forall\, \ddot{\varepsilon} \in \ddot{U}$

So, $\ddot{\varepsilon} \in \mathbb{I}(0)$ and it is evidence that $0 \in \mathbb{I}(0)$, hence, the $\mathbb{I}(0)$ is an ideal of $\ddot{U}$.

In addition, when $\check{\mathfrak{z}}^{LB}(\ddot{\eta})$ is a BCIFSS in $\ddot{U}$ that holds data in Def. (8), $BCIFSI_1$ and $BCIFSI_3$, thus ensuring that the technique of $BCIFSI_2$ and $BCIFSI_4$ are hold or not, see the following example. This, for this, we deliberate the BCK-algebra $(\ddot{U},*,0)$ with the data in Table 2.

Table 2. Representation of the Cayley information for BCK-algebra ($\mathbb{I}(0)$ is not an ideal).

$*$	**0**	$\ddot{\varepsilon}$	$\ddot{ə}$	$\ddot{\alpha}$	$\ddot{\eta}$
0	0	0	0	0	0
$\ddot{\varepsilon}$	$\ddot{\varepsilon}$	0	$\ddot{\varepsilon}$	0	0
$\ddot{ə}$	$\ddot{ə}$	$\ddot{ə}$	0	$\ddot{ə}$	0
$\ddot{\alpha}$	$\ddot{\alpha}$	$\ddot{\alpha}$	$\ddot{\alpha}$	0	$\ddot{\alpha}$
$\ddot{\eta}$	$\ddot{\eta}$	$\ddot{\eta}$	$\ddot{\eta}$	$\ddot{\eta}$	0

Thus, we define a function, such as

$$\check{\mathfrak{z}}^{LB}(\ddot{\eta}) = \begin{Bmatrix} (0, -0.5 - 0.51\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{\varepsilon}, -0.5 - 0.51\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{ə}, -0.2 - 0.21\ddot{\imath}, 0.1 + 0.11\ddot{\imath}, -0.3 - 0.31\ddot{\imath}, 0.0 + 0.01\ddot{\imath}), \\ (\ddot{\alpha}, -0.5 - 0.51\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{\eta}, -0.5 - 0.51\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}) \end{Bmatrix}$$

Then, $\check{\mathfrak{z}}^{LB}(\ddot{\eta})$ is a BCIFSI in $\ddot{U}$ that holds data in Def. (8), $BCIFSI_1$ and $BCIFSI_3$, and $BCIFSI_2$ are hold or not $BCIFSI_4$, for this, we consider any two $\ddot{ə}, \ddot{\eta} \in \ddot{U}$, where

$$\Xi^-_{\check{\mathfrak{z}}}(\ddot{ə}) = -0.2 \nleq -0.5 = \Xi^-_{\check{\mathfrak{z}}}(\ddot{ə} * \ddot{\eta}) \vee \Xi^-_{\check{\mathfrak{z}}}(\ddot{\eta}), \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{ə}) = -0.21 \nleq -0.51 = \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{ə} * \ddot{\eta}) \vee \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\eta})$$
$$\Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) = 0.1 \ngeq 0.6 = \Xi^+_{\check{\mathfrak{z}}}(\ddot{ə} * \ddot{\eta}) \wedge \Xi^+_{\check{\mathfrak{z}}}(\ddot{\eta}), \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) = 0.11 \ngeq 0.61 = \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{ə} * \ddot{\eta}) \wedge \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\eta})$$

Hence, $\mathbb{I}(0) = \{0, \ddot{\varepsilon}, \ddot{\eta}\}$ is not an ideal of $\ddot{U}$ as $\ddot{ə} * \ddot{\eta} = 0 \in \mathbb{I}(0)$ and $\ddot{\eta} \in \mathbb{I}(0)$, but $\ddot{ə} \notin \mathbb{I}(0)$.

Theorem 2. Suppose $\check{\mathfrak{z}}^{LB}(\ddot{\eta})$ is a BCIFSI, if $\ddot{\varepsilon} \in \ddot{U}$, thus $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$.

Proof: It is clear that $0 \in \mathbb{I}(\theta)$, further, we assume that $\ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, thus $\ddot{\varepsilon} * \ddot{ə} \in \mathbb{I}(\theta)$, thus $\ddot{ə} \in \mathbb{I}(\theta)$, for example,

$$\Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}). \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Xi^-_{\check{\mathfrak{z}}}(\ddot{ə}), \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{ə}). \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Xi^+_{\check{\mathfrak{z}}}(\ddot{ə}), \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{ə})$$

and

$$\Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}). \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}), \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Psi^-_{\check{\mathfrak{z}}}(\ddot{ə}), \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Psi^-_{\check{\mathfrak{z}}}(\ddot{ə}). \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Psi^+_{\check{\mathfrak{z}}}(\ddot{ə}), \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Psi^+_{\check{\mathfrak{z}}}(\ddot{ə})$$

Thus, considering the information in Def. (8), we have

$$BCIFSI_2: \Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \vee \Xi^-_{\check{\mathfrak{z}}}(\ddot{ə}) \leq \Xi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \vee \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{ə}) \leq \hat{\mho}^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \wedge \Xi^+_{\check{\mathfrak{z}}}(\ddot{ə}) \geq \Xi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \wedge \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{ə}) \geq \hat{\mho}^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \forall\, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$$

and

$$BCIFSI_4: \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \wedge \Psi^-_{\check{\mathfrak{z}}}(\ddot{ə}) \geq \Psi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Phi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \geq \Phi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \wedge \Phi^-_{\check{\mathfrak{z}}}(\ddot{ə}) \geq \Phi^-_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \vee \Psi^+_{\check{\mathfrak{z}}}(\ddot{ə}) \leq \Psi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \Phi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}) \leq \Phi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon} * \ddot{ə}) \vee \Phi^+_{\check{\mathfrak{z}}}(\ddot{ə}) \leq \Phi^+_{\check{\mathfrak{z}}}(\ddot{\varepsilon}), \forall\, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$$

So, $\ddot{\varepsilon} \in \mathbb{I}(\theta)$, hence, the $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$. Consider a BCK-algebra $(\ddot{U},*,0)$ with the data in Table 3.

Table 3. Representation of the Cayley information for BCK-algebra ($\mathbb{I}(0)$ is an ideal).

$*$	**0**	$\ddot{ɛ}$	$\ddot{ə}$	$\ddot{\alpha}$	$\ddot{ŋ}$
0	0	0	0	0	0
$\ddot{ɛ}$	$\ddot{ɛ}$	0	$\ddot{ɛ}$	0	0
$\ddot{ə}$	$\ddot{ə}$	$\ddot{ə}$	0	$\ddot{ə}$	0
$\ddot{\alpha}$	$\ddot{\alpha}$	$\ddot{\alpha}$	$\ddot{\alpha}$	0	$\ddot{\alpha}$
$\ddot{ŋ}$	$\ddot{ŋ}$	$\ddot{ŋ}$	$\ddot{ŋ}$	$\ddot{ŋ}$	0

Thus, we define a function, such as

$$\check{ʒ}^{LB}(\ddot{ŋ}) = \begin{Bmatrix} (0, -0.5 - 0.51\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.1 - 0.11\ddot{\imath}, 0.0 + 0.01\ddot{\imath}), \\ (\ddot{ɛ}, -0.3 - 0.31\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.3 - 0.31\ddot{\imath}, 0.1 + 0.11\ddot{\imath}), \\ (\ddot{ə}, -0.1 - 0.11\ddot{\imath}, 0.2 + 0.21\ddot{\imath}, -0.5 - 0.51\ddot{\imath}, 0.4 + 0.41\ddot{\imath}), \\ (\ddot{\alpha}, -0.0 - 0.01\ddot{\imath}, 0.1 + 0.11\ddot{\imath}, -0.7 - 0.71\ddot{\imath}, 0.6 + 0.61\ddot{\imath}), \\ (\ddot{ŋ}, -0.2 - 0.21\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.2 + 0.21\ddot{\imath}) \end{Bmatrix}$$

Then, $\check{ʒ}^{LB}(\ddot{ŋ})$ is a BCIFSI in $\ddot{U}$ that holds data in Def. (8), hence, the $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$.

Theorem 3. Suppose $\check{ʒ}^{LB}(\ddot{ŋ})$ is a BCIFSS, if $\ddot{ɛ} \in \ddot{U}$, thus

1) When $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$, then

$$\Xi^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \Xi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \Xi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Xi^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \Xi^{-}_{\check{ʒ}}(\theta), \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\theta)$$

$$\Xi^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \Xi^{+}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \Xi^{+}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Xi^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \Xi^{+}_{\check{ʒ}}(\theta), \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \hat{Ѡ}^{+}_{\check{ʒ}}(\theta)$$

and

$$\Psi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \Psi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Psi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Psi^{-}_{\check{ʒ}}(\theta), \Phi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Phi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \Phi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Phi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Phi^{-}_{\check{ʒ}}(\theta)$$

$$\Psi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \Psi^{+}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Psi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Psi^{+}_{\check{ʒ}}(\theta), \Phi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Phi^{+}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \Phi^{+}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Phi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Phi^{+}_{\check{ʒ}}(\theta)$$

2) When $\check{ʒ}^{LB}(\ddot{ŋ})$ hold the data in Def. (8), then $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$.

Proof:

1) Suppose that $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$. Assume that

$$\Xi^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \Xi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \Xi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Xi^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \Xi^{-}_{\check{ʒ}}(\theta), \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \vee \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\theta)$$

and

$$\Psi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \Psi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Psi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Psi^{-}_{\check{ʒ}}(\theta), \Phi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Phi^{-}_{\check{ʒ}}(\ddot{ə} * \ddot{\alpha}) \wedge \Phi^{-}_{\check{ʒ}}(\ddot{\alpha}) \Rightarrow \Phi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Phi^{-}_{\check{ʒ}}(\theta), \forall\, \ddot{ɛ}, \ddot{ə}, \ddot{\alpha} \in \ddot{U}$$

Thus $\ddot{ə} * \ddot{\alpha} \in \mathbb{I}(\theta)$ and $\ddot{\alpha} \in \mathbb{I}(\theta)$, since, $\mathbb{I}(\theta)$ is an ideal of $\ddot{U}$, it guarantees that $\ddot{ə} \in \mathbb{I}(\theta)$ that is

$$\Xi^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \Xi^{-}_{\check{ʒ}}(\ddot{ə}), \Xi^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \Xi^{+}_{\check{ʒ}}(\ddot{ə}), \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ}) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ə}), \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ɛ}) \leq \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ə})$$

and

$$\Psi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ə}), \Psi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ə}), \Phi^{-}_{\check{ʒ}}(\ddot{ɛ}) \leq \Phi^{-}_{\check{ʒ}}(\ddot{ə}), \Phi^{+}_{\check{ʒ}}(\ddot{ɛ}) \geq \Phi^{+}_{\check{ʒ}}(\ddot{ə})$$

2) Consider that $\check{ʒ}^{LB}(\ddot{ŋ})$ hold the data in Def. (8), thus, we know that $\ddot{ɛ}, \ddot{ə} \in \ddot{U}$, thus $\ddot{ɛ} * \ddot{ə} \in \mathbb{I}(\theta)$, thus $\ddot{ə} \in \mathbb{I}(\theta)$, for example,

$$\Xi^{-}_{\check{ʒ}}(\theta) \geq \Xi^{-}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \hat{Ѡ}^{-}_{\check{ʒ}}(\theta) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}). \Xi^{+}_{\check{ʒ}}(\theta) \leq \Xi^{+}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \hat{Ѡ}^{+}_{\check{ʒ}}(\theta) \leq \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \Xi^{-}_{\check{ʒ}}(\theta) \geq \Xi^{-}_{\check{ʒ}}(\ddot{ə}), \hat{Ѡ}^{-}_{\check{ʒ}}(\theta) \geq \hat{Ѡ}^{-}_{\check{ʒ}}(\ddot{ə}). \Xi^{+}_{\check{ʒ}}(\theta) \leq \Xi^{+}_{\check{ʒ}}(\ddot{ə}), \hat{Ѡ}^{+}_{\check{ʒ}}(\theta) \leq \hat{Ѡ}^{+}_{\check{ʒ}}(\ddot{ə})$$

and

$$\Psi^{-}_{\check{ʒ}}(\theta) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \Psi^{-}_{\check{ʒ}}(\theta) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}). \Psi^{+}_{\check{ʒ}}(\theta) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \Psi^{+}_{\check{ʒ}}(\theta) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ɛ} * \ddot{ə}), \Psi^{-}_{\check{ʒ}}(\theta) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ə}), \Psi^{-}_{\check{ʒ}}(\theta) \leq \Psi^{-}_{\check{ʒ}}(\ddot{ə}). \Psi^{+}_{\check{ʒ}}(\theta) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ə}), \Psi^{+}_{\check{ʒ}}(\theta) \geq \Psi^{+}_{\check{ʒ}}(\ddot{ə})$$

Thus, considering the information in Def. (8), we have

$$BCIFSI_2: \Xi^{-}_{ǯ}(\theta) \geq \Xi^{-}_{ǯ}(ε̏ * ӛ) \vee \Xi^{-}_{ǯ}(ӛ), Ѽ^{-}_{ǯ}(\theta) \geq Ѽ^{-}_{ǯ}(ε̏ * ӛ) \vee Ѽ^{-}_{ǯ}(ӛ), \Xi^{+}_{ǯ}(\theta) \leq \Xi^{+}_{ǯ}(ε̏ * ӛ) \wedge \Xi^{+}_{ǯ}(ӛ), Ѽ^{+}_{ǯ}(\theta) \leq Ѽ^{+}_{ǯ}(ε̏ * ӛ) \wedge Ѽ^{+}_{ǯ}(ӛ), \forall\, ε̏, ӛ \in Ü$$

and

$$BCIFSI_4: \Psi^{-}_{ǯ}(\theta) \leq \Psi^{-}_{ǯ}(ε̏ * ӛ) \wedge \Psi^{-}_{ǯ}(ӛ), \Phi^{-}_{ǯ}(\theta) \leq \Phi^{-}_{ǯ}(ε̏ * ӛ) \wedge \Phi^{-}_{ǯ}(ӛ), \Psi^{+}_{ǯ}(\theta) \geq \Psi^{+}_{ǯ}(ε̏ * ӛ) \vee \Psi^{+}_{ǯ}(ӛ), \Phi^{+}_{ǯ}(\theta) \geq \Phi^{+}_{ǯ}(ε̏ * ӛ) \vee \Phi^{+}_{ǯ}(ӛ), \forall\, ε̏, ӛ \in Ü$$

thus,

$$BCIFSI_2: \Xi^{-}_{ǯ}(\theta) \geq \Xi^{-}_{ǯ}(ε̏), Ѽ^{-}_{ǯ}(\theta) \geq Ѽ^{-}_{ǯ}(ε̏), \Xi^{+}_{ǯ}(\theta) \leq \Xi^{+}_{ǯ}(ε̏), Ѽ^{+}_{ǯ}(\theta) \leq Ѽ^{+}_{ǯ}(ε̏), \forall\, ε̏, ӛ \in Ü$$

and

$$BCIFSI_4: \Psi^{-}_{ǯ}(\theta) \leq \Psi^{-}_{ǯ}(ε̏), \Phi^{-}_{ǯ}(\theta) \leq \Phi^{-}_{ǯ}(ε̏), \Psi^{+}_{ǯ}(\theta) \geq \Psi^{+}_{ǯ}(ε̏), \Phi^{+}_{ǯ}(\theta) \geq \Phi^{+}_{ǯ}(ε̏), \forall\, ε̏, ӛ \in Ü$$

So, $ε̏ \in \mathbb{I}(\theta)$, hence, $ǯ^{LB}(ῃ)$ hold the data in Def. (8) the $0 \in \mathbb{I}(\theta)$ is an ideal of $Ü$.

Theorem 4: Suppose $ǯ^{LB}(ῃ)$ is a BCIFSI, if $ε̏ \in Ü$, thus

$$\Xi^{-}_{ǯ}(ε̏ * ӛ) \leq \Xi^{-}_{ǯ}(ε̏ * α̈) \vee \Xi^{-}_{ǯ}(α̈ * ӛ), Ѽ^{-}_{ǯ}(ε̏ * ӛ) \leq Ѽ^{-}_{ǯ}(ε̏ * α̈) \vee Ѽ^{-}_{ǯ}(α̈ * ӛ)$$
$$\Xi^{+}_{ǯ}(ε̏ * ӛ) \geq \Xi^{+}_{ǯ}(ε̏ * α̈) \wedge \Xi^{-}_{ǯ}(α̈ * ӛ), Ѽ^{+}_{ǯ}(ε̏ * ӛ) \geq Ѽ^{+}_{ǯ}(ε̏ * α̈) \wedge Ѽ^{+}_{ǯ}(α̈ * ӛ)$$

and

$$\Psi^{-}_{ǯ}(ε̏ * ӛ) \geq \Psi^{-}_{ǯ}(ε̏ * α̈) \wedge \Psi^{-}_{ǯ}(α̈ * ӛ), \Phi^{-}_{ǯ}(ε̏ * ӛ) \geq \Phi^{-}_{ǯ}(ε̏ * α̈) \wedge \Phi^{-}_{ǯ}(α̈ * ӛ)$$
$$\Psi^{+}_{ǯ}(ε̏ * ӛ) \leq \Psi^{+}_{ǯ}(ε̏ * α̈) \vee \Psi^{+}_{ǯ}(α̈ * ӛ), \Phi^{+}_{ǯ}(ε̏ * ӛ) \leq \Phi^{+}_{ǯ}(ε̏ * α̈) \vee \Phi^{+}_{ǯ}(α̈ * ӛ)$$

$\forall\, α̈, ε̏, ӛ \in Ü$.

Proof: According to well-known Lemma, such as (All BCIFSI $ǯ^{LB}(ῃ)$ of $Ü$ holds the following inequality, such as $\forall\, α̈, ε̏, ӛ \in Ü$, $ε̏ \leq ӛ \Rightarrow \Xi^{-}_{ǯ}(ε̏) \leq \Xi^{-}_{ǯ}(ӛ), Ѽ^{-}_{ǯ}(ε̏) \leq Ѽ^{-}_{ǯ}(ӛ), \Xi^{+}_{ǯ}(ε̏) \geq \Xi^{+}_{ǯ}(ӛ), Ѽ^{+}_{ǯ}(ε̏) \geq Ѽ^{+}_{ǯ}(ӛ)$ and $\Psi^{-}_{ǯ}(ε̏) \geq \Psi^{-}_{ǯ}(ӛ), \Phi^{-}_{ǯ}(ε̏) \geq \Phi^{-}_{ǯ}(ӛ), \Psi^{+}_{ǯ}(ε̏) \leq \Psi^{+}_{ǯ}(ӛ), \Phi^{+}_{ǯ}(ε̏) \leq \Phi^{+}_{ǯ}(ӛ)$), thus $\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \leq (α̈ * ӛ)$, $\forall\, α̈, ε̏, ӛ \in Ü$, thus

$$\Xi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \leq \Xi^{-}_{ǯ}(α̈ * ӛ), Ѽ^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \leq Ѽ^{-}_{ǯ}(α̈ * ӛ), \Xi^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \geq \Xi^{+}_{ǯ}(α̈ * ӛ), Ѽ^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \geq Ѽ^{+}_{ǯ}(α̈ * ӛ)$$

and

$$\Psi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \geq \Psi^{-}_{ǯ}(α̈ * ӛ), \Phi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \geq \Phi^{-}_{ǯ}(α̈ * ӛ), \Psi^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \leq \Psi^{+}_{ǯ}(α̈ * ӛ), \Phi^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \leq \Phi^{+}_{ǯ}(α̈ * ӛ)$$

Thus, by using the data in Def. (8), we have

$$\Xi^{-}_{ǯ}(ε̏ * ӛ) \leq \Xi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \vee \Xi^{-}_{ǯ}(ε̏ * α̈) \leq \Xi^{-}_{ǯ}(α̈ * ӛ) \vee \Xi^{-}_{ǯ}(ε̏ * α̈), Ѽ^{-}_{ǯ}(ε̏ * ӛ) \leq Ѽ^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \vee Ѽ^{-}_{ǯ}(ε̏ * α̈) \leq Ѽ^{-}_{ǯ}(α̈ * ӛ) \vee Ѽ^{-}_{ǯ}(ε̏ * α̈)$$
$$\Xi^{+}_{ǯ}(ε̏ * ӛ) \geq \Xi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \wedge \Xi^{-}_{ǯ}(ε̏ * α̈) \geq \Xi^{-}_{ǯ}(α̈ * ӛ) \wedge \Xi^{-}_{ǯ}(ε̏ * α̈), Ѽ^{-}_{ǯ}(ε̏ * ӛ) \geq Ѽ^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \wedge Ѽ^{-}_{ǯ}(ε̏ * α̈) \geq Ѽ^{-}_{ǯ}(α̈ * ӛ) \wedge Ѽ^{-}_{ǯ}(ε̏ * α̈)$$

and

$$\Psi^{-}_{ǯ}(ε̏ * ӛ) \geq \Psi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \wedge \Psi^{-}_{ǯ}(ε̏ * α̈) \geq \Psi^{-}_{ǯ}(α̈ * ӛ) \wedge \Psi^{-}_{ǯ}(ε̏ * α̈), \Phi^{-}_{ǯ}(ε̏ * ӛ) \geq \Phi^{-}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \wedge \Phi^{-}_{ǯ}(ε̏ * α̈) \geq \Phi^{-}_{ǯ}(α̈ * ӛ) \wedge \Phi^{-}_{ǯ}(ε̏ * α̈)$$
$$\Psi^{+}_{ǯ}(ε̏ * ӛ) \leq \Psi^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \vee \Psi^{+}_{ǯ}(ε̏ * α̈) \leq \Psi^{+}_{ǯ}(α̈ * ӛ) \vee \Psi^{+}_{ǯ}(ε̏ * α̈), \Phi^{+}_{ǯ}(ε̏ * ӛ) \leq \Phi^{+}_{ǯ}\big((ε̏ * ӛ) * (ε̏ * α̈)\big) \vee \Phi^{+}_{ǯ}(ε̏ * α̈) \leq \Phi^{+}_{ǯ}(α̈ * ӛ) \vee \Phi^{+}_{ǯ}(ε̏ * α̈)$$

$\forall\, α̈, ε̏, ӛ \in Ü$.

Theorem 5. Suppose $ǯ^{LB}(ῃ)$ is a BCIFSI, thus for all $ε̏, ӛ \in Ü$, we have

i). $\Xi^{-}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \leq \Xi^{-}_{ǯ}(ӛ), Ѽ^{-}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \leq Ѽ^{-}_{ǯ}(ӛ)$.

ii). $\Xi^{+}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \geq \Xi^{+}_{ǯ}(ӛ), Ѽ^{+}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \geq Ѽ^{+}_{ǯ}(ӛ)$.

iii). $\Psi^{-}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \geq \Psi^{-}_{ǯ}(ӛ), \Phi^{-}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \geq \Phi^{-}_{ǯ}(ӛ)$.

iv). $\Psi^{+}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \leq \Psi^{+}_{ǯ}(ӛ), \Phi^{+}_{ǯ}\big(ε̏ * (ε̏ * ӛ)\big) \leq \Phi^{+}_{ǯ}(ӛ)$.

Proof: Suppose that $ǯ^{LB}(ŋ)$ is a BCIFSI, thus for all $\ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$,

$$Ξ_{ʓ}^{-}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \leq Ξ_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \vee Ξ_{ʓ}^{-}(\ddot{ə}) = Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \vee Ξ_{ʓ}^{-}(\ddot{ə}) = Ξ_{ʓ}^{-}(0) \vee Ξ_{ʓ}^{-}(\ddot{ə}) \leq Ξ_{ʓ}^{-}(\ddot{ə})$$

$$\hat{Ѡ}_{ʓ}^{-}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \leq \hat{Ѡ}_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \vee \hat{Ѡ}_{ʓ}^{-}(\ddot{ə}) = \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \vee \hat{Ѡ}_{ʓ}^{-}(\ddot{ə}) = \hat{Ѡ}_{ʓ}^{-}(0) \vee \hat{Ѡ}_{ʓ}^{-}(\ddot{ə}) \leq \hat{Ѡ}_{ʓ}^{-}(\ddot{ə})$$

The data in condition 1 is holding. In addition, we have

$$Ξ_{ʓ}^{+}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \geq Ξ_{ʓ}^{+}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \wedge Ξ_{ʓ}^{+}(\ddot{ə}) = Ξ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \wedge Ξ_{ʓ}^{+}(\ddot{ə}) = Ξ_{ʓ}^{+}(0) \wedge Ξ_{ʓ}^{+}(\ddot{ə}) \geq Ξ_{ʓ}^{+}(\ddot{ə})$$

$$\hat{Ѡ}_{ʓ}^{+}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \geq \hat{Ѡ}_{ʓ}^{+}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \wedge \hat{Ѡ}_{ʓ}^{+}(\ddot{ə}) = \hat{Ѡ}_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \wedge \hat{Ѡ}_{ʓ}^{+}(\ddot{ə}) = \hat{Ѡ}_{ʓ}^{+}(0) \wedge \hat{Ѡ}_{ʓ}^{+}(\ddot{ə}) \geq \hat{Ѡ}_{ʓ}^{+}(\ddot{ə})$$

The data in condition 2 is holding. In addition, we have

$$Ψ_{ʓ}^{-}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \geq Ψ_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \wedge Ψ_{ʓ}^{-}(\ddot{ə}) = Ψ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \wedge Ψ_{ʓ}^{-}(\ddot{ə}) = Ψ_{ʓ}^{-}(0) \wedge Ψ_{ʓ}^{-}(\ddot{ə}) \geq Ψ_{ʓ}^{-}(\ddot{ə})$$

$$Φ_{ʓ}^{-}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \geq Φ_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \wedge Φ_{ʓ}^{-}(\ddot{ə}) = Φ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \wedge Φ_{ʓ}^{-}(\ddot{ə}) = Φ_{ʓ}^{-}(0) \wedge Φ_{ʓ}^{-}(\ddot{ə}) \geq Φ_{ʓ}^{-}(\ddot{ə})$$

The data in condition 3 is holding. In addition, we have

$$Ψ_{ʓ}^{+}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \leq Ψ_{ʓ}^{+}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \vee Ψ_{ʓ}^{+}(\ddot{ə}) = Ψ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \vee Ψ_{ʓ}^{+}(\ddot{ə}) = Ψ_{ʓ}^{+}(0) \vee Ψ_{ʓ}^{+}(\ddot{ə}) \leq Ψ_{ʓ}^{+}(\ddot{ə})$$

$$Φ_{ʓ}^{+}\left(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə})\right) \leq Φ_{ʓ}^{+}(\ddot{\varepsilon} * (\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}) \vee Φ_{ʓ}^{+}(\ddot{ə}) = Φ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * (\ddot{\varepsilon} * \ddot{ə})\right) \vee Φ_{ʓ}^{+}(\ddot{ə}) = Φ_{ʓ}^{+}(0) \vee Φ_{ʓ}^{+}(\ddot{ə}) \leq Φ_{ʓ}^{+}(\ddot{ə})$$

The data in condition 4 is hold $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$.

Proposition 1. Suppose $ǯ^{LB}(ŋ)$ is a BCIFSI, thus for all $\ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, the following are equal, such as

i). $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, $Ξ_{ʓ}^{-}(\ddot{\varepsilon} * \ddot{ə}) \leq Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), \hat{Ѡ}_{ʓ}^{-}(\ddot{\varepsilon} * \ddot{ə}) \leq \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), Ξ_{ʓ}^{+}(\ddot{\varepsilon} * \ddot{ə}) \geq Ξ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), \hat{Ѡ}_{ʓ}^{+}(\ddot{\varepsilon} * \ddot{ə}) \geq \hat{Ѡ}_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right)$.

ii). $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, $Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \leq Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \leq \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), Ξ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \geq Ξ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), \hat{Ѡ}_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \geq \hat{Ѡ}_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right)$.

iii). $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, $Ψ_{ʓ}^{-}(\ddot{\varepsilon} * \ddot{ə}) \geq Ψ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), Φ_{ʓ}^{-}(\ddot{\varepsilon} * \ddot{ə}) \geq Φ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), Ψ_{ʓ}^{+}(\ddot{\varepsilon} * \ddot{ə}) \leq Ψ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right), Φ_{ʓ}^{+}(\ddot{\varepsilon} * \ddot{ə}) \leq Φ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{ə}\right)$.

iv). $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, $Ψ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \geq Ψ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), Φ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \geq Φ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), Ψ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \leq Ψ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right), Φ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) \leq Φ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right)$.

Proof: Consider that the data in condition 1 is ok, thus observed that

$$\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha} = \left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha} \leq (\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}$$

Thus, according to well-known Lemma, such as (All BCIFSI $ǯ^{LB}(ŋ)$ of $\ddot{U}$ holds the following inequality, such as $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, $\ddot{\varepsilon} \leq \ddot{ə} \Rightarrow Ξ_{ʓ}^{-}(\ddot{\varepsilon}) \leq Ξ_{ʓ}^{-}(\ddot{ə}), \hat{Ѡ}_{ʓ}^{-}(\ddot{\varepsilon}) \leq \hat{Ѡ}_{ʓ}^{-}(\ddot{ə}), Ξ_{ʓ}^{+}(\ddot{\varepsilon}) \geq Ξ_{ʓ}^{+}(\ddot{ə}), \hat{Ѡ}_{ʓ}^{+}(\ddot{\varepsilon}) \geq \hat{Ѡ}_{ʓ}^{+}(\ddot{ə})$ and $Ψ_{ʓ}^{-}(\ddot{\varepsilon}) \geq Ψ_{ʓ}^{-}(\ddot{ə}), Φ_{ʓ}^{-}(\ddot{\varepsilon}) \geq Φ_{ʓ}^{-}(\ddot{ə}), Ψ_{ʓ}^{+}(\ddot{\varepsilon}) \leq Ψ_{ʓ}^{+}(\ddot{ə}), Φ_{ʓ}^{+}(\ddot{\varepsilon}) \leq Φ_{ʓ}^{+}(\ddot{ə})$), thus $\forall \ddot{\alpha}, \ddot{\varepsilon}, \ddot{ə} \in \ddot{U}$, thus

$$Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right) \geq Ξ_{ʓ}^{-}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right), \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right) \geq \hat{Ѡ}_{ʓ}^{-}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right)$$

$$Ξ_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right) \leq Ξ_{ʓ}^{+}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right), \hat{Ѡ}_{ʓ}^{+}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right) \leq \hat{Ѡ}_{ʓ}^{+}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right)$$

thus by $\mathbb{P}_2$ and 1, we have

$$Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) = Ξ_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha}) * \ddot{\alpha}) \leq Ξ_{ʓ}^{-}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right) \leq Ξ_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right)$$

$$\hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{\alpha}) * (\ddot{ə} * \ddot{\alpha})\right) = \hat{Ѡ}_{ʓ}^{-}(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha}) * \ddot{\alpha}) \leq \hat{Ѡ}_{ʓ}^{-}\left(\left(\left(\ddot{\varepsilon} * (\ddot{ə} * \ddot{\alpha})\right) * \ddot{\alpha}\right) * \ddot{\alpha}\right) \leq \hat{Ѡ}_{ʓ}^{-}\left((\ddot{\varepsilon} * \ddot{ə}) * \ddot{\alpha}\right)$$

and

$$\Xi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\Xi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\geq\Xi^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\geq\Xi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

$$\hat{\omega}^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\hat{\omega}^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\geq\hat{\omega}^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\geq\hat{\omega}^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

Then, when the 2 is holding, we have the following data: when we change $\ddot{\alpha}$ with $\ddot{ə}$ in 2, thus

$$\Xi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Xi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Xi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\leq\Xi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

$$\hat{\omega}^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\hat{\omega}^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\hat{\omega}^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\leq\hat{\omega}^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

and

$$\Xi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Xi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Xi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\geq\Xi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

$$\hat{\omega}^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\hat{\omega}^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\hat{\omega}^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\geq\hat{\omega}^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

Hence condition 1 is holding. Further, considering that the data in condition 3 is ok, thus observed that

$$\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}=\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\leq(\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}$$

Thus, according to well-known Lemma, such as (All BCIFSI $\check{\mathfrak{z}}^{LB}(\mathring{\eta})$ of $\ddot{U}$ holds the following inequality, such as $\forall\,\ddot{\alpha},\ddot{\varepsilon},\ddot{ə}\in\ddot{U}$, $\ddot{\varepsilon}\leq\ddot{ə}\Rightarrow\Xi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\leq\Xi^{-}_{\check{\mathfrak{z}}}(\ddot{ə}),\hat{\omega}^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\leq\hat{\omega}^{-}_{\check{\mathfrak{z}}}(\ddot{ə}),\Xi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\geq\Xi^{+}_{\check{\mathfrak{z}}}(\ddot{ə}),\hat{\omega}^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\geq\hat{\omega}^{+}_{\check{\mathfrak{z}}}(\ddot{ə})$ and $\Psi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\geq\Psi^{-}_{\check{\mathfrak{z}}}(\ddot{ə}),\Phi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\geq\Phi^{-}_{\check{\mathfrak{z}}}(\ddot{ə}),\Psi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\leq\Psi^{+}_{\check{\mathfrak{z}}}(\ddot{ə}),\Phi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon})\leq\Phi^{+}_{\check{\mathfrak{z}}}(\ddot{ə})$), thus $\forall\,\ddot{\alpha},\ddot{\varepsilon},\ddot{ə}\in\ddot{U}$, thus

$$\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)\leq\Psi^{-}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big),\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)\leq\Phi^{-}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)$$

$$\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)\geq\Psi^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big),\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)\geq\Phi^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)$$

thus by $\mathbb{P}_2$ and 3, we have

$$\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\Psi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\geq\Psi^{-}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\geq\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

$$\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\Phi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\geq\Phi^{-}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\geq\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

and

$$\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\Psi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\leq\Psi^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\leq\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

$$\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{\alpha})*(\ddot{ə}*\ddot{\alpha})\big)=\Phi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})*\ddot{\alpha})\leq\Phi^{+}_{\check{\mathfrak{z}}}\Big(\Big(\big(\ddot{\varepsilon}*(\ddot{ə}*\ddot{\alpha})\big)*\ddot{\alpha}\Big)*\ddot{\alpha}\Big)\leq\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{\alpha}\big)$$

Then, when the 4 is holding, we have the following data: when we change $\ddot{\alpha}$ with $\ddot{ə}$ in 4, thus

$$\Psi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\geq\Psi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

$$\Phi^{-}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\geq\Phi^{-}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

and

$$\Psi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\leq\Psi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

$$\Phi^{+}_{\check{\mathfrak{z}}}(\ddot{\varepsilon}*\ddot{ə})=\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*0\big)=\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*(\ddot{ə}*\ddot{ə})\big)\leq\Phi^{+}_{\check{\mathfrak{z}}}\big((\ddot{\varepsilon}*\ddot{ə})*\ddot{ə}\big)$$

Hence condition 1 is holding.

4. Operations on BCIFSIs of BCK/BCI-Algebras

This section explains numerous operational laws for bipolar complex intuitionistic fuzzy soft deals of BCK/BCI-algebra and also describes numerous examples.

Definition 9. Consider any two BCIFSS $\check{\mathfrak{z}}^{LB}(\mathring{\eta}),\Xi^{LB}(\mathring{\eta})\in BCIF(\ddot{U})$ in $\ddot{U}$, thus the intersection of $\check{\mathfrak{z}}^{LB}(\mathring{\eta})$ and $\Xi^{LB}(\mathring{\eta})$ is mentioned and designed below:

$$\check{\mathfrak{z}}^{LB}(\mathring{\eta})\cap\Xi^{LB}(\mathring{\eta})=\mathbb{H}^{LB}(\mathring{\eta})$$

Noticed that $\mathcal{D}=\mathcal{A}\cap\mathcal{B}$ for all $\mathring{\eta}\in\mathcal{D}$. In addition, the intersection of $\check{\mathfrak{z}}^{LB}(\mathring{\eta})$ and $\Xi^{LB}(\mathring{\eta})$ is a BCIFSI. In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(\ddot{U},*,0)$ with the data in Table 4.

Table 4. Representation of the Cayley information for BCK-algebra ($ǯ^{LB}(ῆ) \cap \Xi^{LB}(ῆ)$ is an BCIFSI).

*	**0**	ε̏	ə̈	α̏
0	0	ε̏	ə̈	α̏
ε̏	ε̏	0	α̏	ə̈
ə̈	ə̈	α̏	0	ε̏
α̏	α̏	ə̈	ε̏	0

Consider $\mathbb{E} = \{ῆ_1, ῆ_2, ῆ_3, ῆ_4\}, \mathcal{A} = \{ῆ_1, ῆ_2\}$ and $\mathcal{B} = \{ῆ_1, ῆ_2, ῆ_3\}$, thus, we define a function, such as

$$ǯ^{LB}(ῆ_1) = \begin{Bmatrix} (0, -0.8 - 0.81ι̏, 0.7 + 0.71ι̏, -0.1 - 0.11ι̏, 0.2 + 0.21ι̏), \\ (ε̏, -0.3 - 0.31ι̏, 0.6 + 0.61ι̏, -0.6 - 0.61ι̏, 0.3 + 0.31ι̏), \\ (ə̈, -0.7 - 0.71ι̏, 0.4 + 0.41ι̏, -0.2 - 0.21ι̏, 0.5 + 0.51ι̏), \\ (α̏, -0.3 - 0.31ι̏, 0.4 + 0.41ι̏, -0.6 - 0.61ι̏, 0.5 + 0.51ι̏) \end{Bmatrix}$$

$$ǯ^{LB}(ῆ_2) = \begin{Bmatrix} (0, -0.7 - 0.71ι̏, 0.6 + 0.61ι̏, -0.2 - 0.21ι̏, 0.3 + 0.31ι̏), \\ (ε̏, -0.2 - 0.21ι̏, 0.5 + 0.51ι̏, -0.7 - 0.71ι̏, 0.4 + 0.41ι̏), \\ (ə̈, -0.6 - 0.61ι̏, 0.3 + 0.31ι̏, -0.3 - 0.31ι̏, 0.6 + 0.61ι̏), \\ (α̏, -0.2 - 0.21ι̏, 0.3 + 0.31ι̏, -0.7 - 0.71ι̏, 0.6 + 0.61ι̏) \end{Bmatrix}$$

and

$$\Xi^{LB}(ῆ_1) = \begin{Bmatrix} (0, -0.6 - 0.61ι̏, 0.5 + 0.51ι̏, -0.3 - 0.31ι̏, 0.4 + 0.41ι̏), \\ (ε̏, -0.1 - 0.11ι̏, 0.4 + 0.41ι̏, -0.8 - 0.81ι̏, 0.5 + 0.51ι̏), \\ (ə̈, -0.5 - 0.51ι̏, 0.2 + 0.21ι̏, -0.4 - 0.41ι̏, 0.7 + 0.71ι̏), \\ (α̏, -0.1 - 0.11ι̏, 0.2 + 0.21ι̏, -0.8 - 0.81ι̏, 0.7 + 0.71ι̏) \end{Bmatrix}$$

$$\Xi^{LB}(ῆ_2) = \begin{Bmatrix} (0, -0.5 - 0.51ι̏, 0.4 + 0.41ι̏, -0.4 - 0.41ι̏, 0.5 + 0.51ι̏), \\ (ε̏, -0.0 - 0.01ι̏, 0.3 + 0.31ι̏, -0.9 - 0.91ι̏, 0.6 + 0.61ι̏), \\ (ə̈, -0.4 - 0.41ι̏, 0.1 + 0.11ι̏, -0.5 - 0.51ι̏, 0.8 + 0.81ι̏), \\ (α̏, -0.0 - 0.01ι̏, 0.1 + 0.11ι̏, -0.9 - 0.91ι̏, 0.8 + 0.81ι̏) \end{Bmatrix}$$

$$\Xi^{LB}(ῆ_3) = \begin{Bmatrix} (0, -0.9 - 0.91ι̏, 0.8 + 0.81ι̏, -0.0 - 0.01ι̏, 0.1 + 0.11ι̏), \\ (ε̏, -0.4 - 0.41ι̏, 0.6 + 0.61ι̏, -0.5 - 0.51ι̏, 0.2 + 0.21ι̏), \\ (ə̈, -0.8 - 0.81ι̏, 0.5 + 0.51ι̏, -0.1 - 0.11ι̏, 0.4 + 0.41ι̏), \\ (α̏, -0.4 - 0.41ι̏, 0.5 + 0.51ι̏, -0.5 - 0.51ι̏, 0.4 + 0.41ι̏) \end{Bmatrix}$$

thus

$$ǯ^{LB}(ῆ) \cap \Xi^{LB}(ῆ) = \mathbb{H}^{LB}(ῆ)$$

Noticed that $\mathcal{D} = \mathcal{A} \cap \mathcal{B} = \{ῆ_1, ῆ_2\}$ is described by:

$$\mathbb{H}^{LB}(ῆ_1) = \begin{Bmatrix} (0, -0.6 - 0.61ι̏, 0.5 + 0.51ι̏, -0.3 - 0.31ι̏, 0.4 + 0.41ι̏), \\ (ε̏, -0.1 - 0.11ι̏, 0.4 + 0.41ι̏, -0.8 - 0.81ι̏, 0.5 + 0.51ι̏), \\ (ə̈, -0.5 - 0.51ι̏, 0.2 + 0.21ι̏, -0.4 - 0.41ι̏, 0.7 + 0.71ι̏), \\ (α̏, -0.1 - 0.11ι̏, 0.2 + 0.21ι̏, -0.8 - 0.81ι̏, 0.7 + 0.71ι̏) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(ῆ_2) = \begin{Bmatrix} (0, -0.5 - 0.51ι̏, 0.4 + 0.41ι̏, -0.4 - 0.41ι̏, 0.5 + 0.51ι̏), \\ (ε̏, -0.0 - 0.01ι̏, 0.3 + 0.31ι̏, -0.9 - 0.91ι̏, 0.6 + 0.61ι̏), \\ (ə̈, -0.4 - 0.41ι̏, 0.1 + 0.11ι̏, -0.5 - 0.51ι̏, 0.8 + 0.81ι̏), \\ (α̏, -0.0 - 0.01ι̏, 0.1 + 0.11ι̏, -0.9 - 0.91ι̏, 0.8 + 0.81ι̏) \end{Bmatrix}$$

Finally, we concluded that $\mathbb{H}^{LB}(ῆ)$ is also BCIFSI.

Definition 10. Consider any two BCIFSS $ǯ^{LB}(ῆ), \Xi^{LB}(ῆ) \in BCIF(\ddot{U})$ in $\ddot{U}$, thus the union of $ǯ^{LB}(ῆ)$ and $\Xi^{LB}(ῆ)$ is mentioned and designed below:

$$ǯ^{LB}(ῆ) \cup \Xi^{LB}(ῆ) = \mathbb{H}^{LB}(ῆ) = \begin{cases} ǯ^{LB}(ῆ) & ῆ \in \mathcal{A} - \mathcal{B} \\ \Xi^{LB}(ῆ) & ῆ \in \mathcal{B} - \mathcal{A} \\ ǯ^{LB}(ῆ) \cup \Xi^{LB}(ῆ) & ῆ \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

Noticed that $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ for all $ῆ \in \mathcal{D}$. In addition, the union of $ǯ^{LB}(ῆ)$ and $\Xi^{LB}(ῆ)$ is a BCIFSI. In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(\ddot{U}, *, 0)$ with the data in Table 4. Consider $\mathbb{E} = \{ῆ_1, ῆ_2, ῆ_3, ῆ_4\}, \mathcal{A} = \{ῆ_1, ῆ_2, ῆ_3\} \subseteq \mathbb{E}$ and $\mathcal{B} = \{ῆ_1, ῆ_2, ῆ_3, ῆ_4\} \subseteq \mathbb{E}$, where $\ddot{U} = \{0, ε̏, ə̈, α̏\}$, thus, we define a function, such as

$$\breve{\mathfrak{Z}}^{LB}(\ddot{\eta}_1)=\left\{\begin{matrix}(0,-0.4-0.41\ddot{\iota},0.0+0.01\ddot{\iota},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.4-0.41\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\alpha},-0.2-0.21\ddot{\iota},0.2+0.21\ddot{\iota},-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

$$\breve{\mathfrak{Z}}^{LB}(\ddot{\eta}_2)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota},-0.6-0.61\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.5-0.51\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota},-0.4-0.41\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.5-0.51\ddot{\iota},0.2+0.21\ddot{\iota})\end{matrix}\right\}$$

$$\breve{\mathfrak{Z}}^{LB}(\ddot{\eta}_3)=\left\{\begin{matrix}(0,-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.4+0.41\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.2-0.21\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{ə},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\alpha},-0.3-0.31\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

and

$$\Xi^{LB}(\ddot{\eta}_1)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.4+0.41\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.1-0.11\ddot{\iota},0.5+0.51\ddot{\iota}),\\(\ddot{ə},-0.0-0.01\ddot{\iota},0.3+0.31\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.2-0.21\ddot{\iota},0.5+0.51\ddot{\iota},-0.0-0.01\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

$$\Xi^{LB}(\ddot{\eta}_2)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.3-0.31\ddot{\iota},0.5+0.51\ddot{\iota}),\\(\ddot{\varepsilon},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{ə},-0.2-0.21\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota},-0.3-0.31\ddot{\iota},0.4+0.41\ddot{\iota})\end{matrix}\right\}$$

$$\Xi^{LB}(\ddot{\eta}_3)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.5+0.51\ddot{\iota},-0.2-0.21\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.2-0.21\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.2-0.21\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

$$\Xi^{LB}(\ddot{\eta}_4)=\left\{\begin{matrix}(0,-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{\varepsilon},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{ə},-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.1-0.11\ddot{\iota},0.4+0.41\ddot{\iota},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota})\end{matrix}\right\}$$

thus

$$\breve{\mathfrak{Z}}^{LB}(\ddot{\eta})\cup\Xi^{LB}(\ddot{\eta})=\mathbb{H}^{LB}(\ddot{\eta})$$

Noticed that $\mathcal{D}=\mathcal{A}\cup\mathcal{B}=\{\ddot{\eta}_1,\ddot{\eta}_2,\ddot{\eta}_3,\ddot{\eta}_4\}$ is described by:

$$\mathbb{H}^{LB}(\ddot{\eta}_1)=\left\{\begin{matrix}(0,-0.4-0.41\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.1-0.11\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.4-0.41\ddot{\iota},0.3+0.31\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\alpha},-0.2-0.21\ddot{\iota},0.5+0.51\ddot{\iota},-0.1-0.11\ddot{\iota},0.0+0.01\ddot{\iota})\end{matrix}\right\}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.3+0.31\ddot{\iota},-0.1-0.11\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.3-0.31\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.3-0.31\ddot{\iota},0.3+0.31\ddot{\iota},-0.0-0.01\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_3)=\left\{\begin{matrix}(0,-0.3-0.31\ddot{\iota},0.5+0.51\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{ə},-0.2-0.21\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{\alpha},-0.3-0.31\ddot{\iota},0.4+0.41\ddot{\iota},-0.1-0.11\ddot{\iota},0.1+0.11\ddot{\iota})\end{matrix}\right\}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_4)=\left\{\begin{matrix}(0,-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota}),\\(\ddot{\varepsilon},-0.2-0.21\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.2+0.21\ddot{\iota}),\\(\ddot{ə},-0.3-0.31\ddot{\iota},0.1+0.11\ddot{\iota},-0.1-0.11\ddot{\iota},0.3+0.31\ddot{\iota}),\\(\ddot{\alpha},-0.1-0.11\ddot{\iota},0.4+0.41\ddot{\iota},-0.3-0.31\ddot{\iota},0.2+0.21\ddot{\iota})\end{matrix}\right\}$$

Finally, $\breve{\mathfrak{Z}}^{LB}(\ddot{\eta})\cup\Xi^{LB}(\ddot{\eta})=\mathbb{H}^{LB}(\ddot{\eta})$, where $\mathcal{D}=\mathcal{A}\cup\mathcal{B}=\{\ddot{\eta}_1,\ddot{\eta}_2,\ddot{\eta}_3,\ddot{\eta}_4\}$.

Theorem 6. The union of any two BCIFSI is again a BCIFSI.

Proof: Consider the data in Def. (10), $\mathbb{H}^{LB}(\ddot{\eta})$ is computed based on $\mathcal{D}=\mathcal{A}\cup\mathcal{B}$ for all $\ddot{\eta}\in\mathcal{D}$, thus

$$\breve{\mathfrak{Z}}^{LB}(\ddot{\eta})\cup\Xi^{LB}(\ddot{\eta})=$$

We have the above cases, such as

$$\begin{cases} \mathbb{H}^{LB}(\ddot{\eta}) = \check{ʒ}^{LB}(\ddot{\eta}) & \ddot{\eta} \in \mathcal{A} - \mathcal{B} \\ \mathbb{H}^{LB}(\ddot{\eta}) = \Xi^{LB}(\ddot{\eta}) & \ddot{\eta} \in \mathcal{B} - \mathcal{A} \\ \mathbb{H}^{LB}(\ddot{\eta}) = \check{ʒ}^{LB}(\ddot{\eta}) \cup \Xi^{LB}(\ddot{\eta}) & \ddot{\eta} \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

Thus,

Case 1: When $\ddot{\eta} \in \mathcal{A} - \mathcal{B}$, thus

$$\begin{aligned} \mathbb{H}^{LB}(\ddot{\eta}) &= \check{ʒ}^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{A} - \mathcal{B} \\ &= \check{ʒ}^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{A} \\ &= \check{ʒ}^{LB}(\ddot{\eta}) \; from \; conditon \; 6. \end{aligned}$$

Case 2: When $\ddot{\eta} \in \mathcal{B} - \mathcal{A}$, thus

$$\begin{aligned} \mathbb{H}^{LB}(\ddot{\eta}) &= \Xi^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{B} - \mathcal{A} \\ &= \Xi^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{B} \\ &= \Xi^{LB}(\ddot{\eta}) \; from \; conditon \; 7. \end{aligned}$$

Case 3: When $\ddot{\eta} \in \mathcal{A} \cap \mathcal{B}$, thus

$$\begin{aligned} \mathbb{H}^{LB}(\ddot{\eta}) &= \check{ʒ}^{LB}(\ddot{\eta}) \cup \Xi^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{A} \cap \mathcal{B} \\ &= \Xi^{LB}(\ddot{\eta}) \cup \check{ʒ}^{LB}(\ddot{\eta}) \; when \; \ddot{\eta} \in \mathcal{B} \cap \mathcal{A} \; from \; conditon \; 8. \end{aligned}$$

Collecting the above cases, we can easily obtain that the $\mathbb{H}^{LB}(\ddot{\eta})$ is a BCIFSI. In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(\ddot{U}, *, 0)$ with the data in Table 4. Consider $\mathbb{E} = \{\ddot{\eta}_1, \ddot{\eta}_2, \ddot{\eta}_3, \ddot{\eta}_4\}, \mathcal{A} = \{\ddot{\eta}_1, \ddot{\eta}_2\} \subseteq \mathbb{E}$ and $\mathcal{B} = \{\ddot{\eta}_1, \ddot{\eta}_2, \ddot{\eta}_3\} \subseteq \mathbb{E}$, where $\ddot{U} = \{0, \ddot{ε}, \ddot{ə}, \ddot{\alpha}\}$, thus, we define a function, such as

$$\check{ʒ}^{LB}(\ddot{\eta}_1) = \begin{Bmatrix} (0, -0.8 - 0.81\ddot{\imath}, 0.7 + 0.71\ddot{\imath}, -0.1 - 0.11\ddot{\imath}, 0.2 + 0.21\ddot{\imath}), \\ (\ddot{ε}, -0.3 - 0.31\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.6 - 0.61\ddot{\imath}, 0.3 + 0.31\ddot{\imath}), \\ (\ddot{ə}, -0.7 - 0.71\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.2 - 0.21\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.6 - 0.61\ddot{\imath}, 0.5 + 0.51\ddot{\imath}) \end{Bmatrix}$$

$$\check{ʒ}^{LB}(\ddot{\eta}_2) = \begin{Bmatrix} (0, -0.7 - 0.71\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.2 - 0.21\ddot{\imath}, 0.3 + 0.31\ddot{\imath}), \\ (\ddot{ε}, -0.2 - 0.21\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.7 - 0.71\ddot{\imath}, 0.4 + 0.41\ddot{\imath}), \\ (\ddot{ə}, -0.6 - 0.61\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.3 - 0.31\ddot{\imath}, 0.6 + 0.61\ddot{\imath}), \\ (\ddot{\alpha}, -0.2 - 0.21\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.7 - 0.71\ddot{\imath}, 0.6 + 0.61\ddot{\imath}) \end{Bmatrix}$$

and

$$\Xi^{LB}(\ddot{\eta}_1) = \begin{Bmatrix} (0, -0.6 - 0.61\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.3 - 0.31\ddot{\imath}, 0.4 + 0.41\ddot{\imath}), \\ (\ddot{ε}, -0.1 - 0.11\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.8 - 0.81\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{ə}, -0.5 - 0.51\ddot{\imath}, 0.2 + 0.21\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.7 + 0.71\ddot{\imath}), \\ (\ddot{\alpha}, -0.1 - 0.11\ddot{\imath}, 0.2 + 0.21\ddot{\imath}, -0.8 - 0.81\ddot{\imath}, 0.7 + 0.71\ddot{\imath}) \end{Bmatrix}$$

$$\Xi^{LB}(\ddot{\eta}_2) = \begin{Bmatrix} (0, -0.5 - 0.51\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{ε}, -0.0 - 0.01\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.9 - 0.91\ddot{\imath}, 0.6 + 0.61\ddot{\imath}), \\ (\ddot{ə}, -0.4 - 0.41\ddot{\imath}, 0.1 + 0.11\ddot{\imath}, -0.5 - 0.51\ddot{\imath}, 0.8 + 0.81\ddot{\imath}), \\ (\ddot{\alpha}, -0.0 - 0.01\ddot{\imath}, 0.1 + 0.11\ddot{\imath}, -0.9 - 0.91\ddot{\imath}, 0.8 + 0.81\ddot{\imath}) \end{Bmatrix}$$

$$\Xi^{LB}(\ddot{\eta}_3) = \begin{Bmatrix} (0, -0.9 - 0.91\ddot{\imath}, 0.8 + 0.81\ddot{\imath}, -0.0 - 0.01\ddot{\imath}, 0.1 + 0.11\ddot{\imath}), \\ (\ddot{ε}, -0.4 - 0.41\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.5 - 0.51\ddot{\imath}, 0.2 + 0.21\ddot{\imath}), \\ (\ddot{ə}, -0.8 - 0.81\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.1 - 0.11\ddot{\imath}, 0.4 + 0.41\ddot{\imath}), \\ (\ddot{\alpha}, -0.4 - 0.41\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.5 - 0.51\ddot{\imath}, 0.4 + 0.41\ddot{\imath}) \end{Bmatrix}$$

thus

$$\check{ʒ}^{LB}(\ddot{\eta}) \cup \Xi^{LB}(\ddot{\eta}) = \mathbb{H}^{LB}(\ddot{\eta})$$

Noticed that $\mathcal{D} = \mathcal{A} \cup \mathcal{B} = \{\ddot{\eta}_1, \ddot{\eta}_2, \ddot{\eta}_3\}$ is described by:

$$\mathbb{H}^{LB}(\ddot{\eta}_1) = \begin{Bmatrix} (0, -0.8 - 0.81\ddot{\imath}, 0.7 + 0.71\ddot{\imath}, -0.1 - 0.11\ddot{\imath}, 0.2 + 0.21\ddot{\imath}), \\ (\ddot{ε}, -0.3 - 0.31\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.6 - 0.61\ddot{\imath}, 0.3 + 0.31\ddot{\imath}), \\ (\ddot{ə}, -0.7 - 0.71\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.2 - 0.21\ddot{\imath}, 0.5 + 0.51\ddot{\imath}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\imath}, 0.4 + 0.41\ddot{\imath}, -0.6 - 0.61\ddot{\imath}, 0.5 + 0.51\ddot{\imath}) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2) = \begin{Bmatrix} (0, -0.7 - 0.71\ddot{\imath}, 0.6 + 0.61\ddot{\imath}, -0.2 - 0.21\ddot{\imath}, 0.3 + 0.31\ddot{\imath}), \\ (\ddot{ε}, -0.2 - 0.21\ddot{\imath}, 0.5 + 0.51\ddot{\imath}, -0.7 - 0.71\ddot{\imath}, 0.4 + 0.41\ddot{\imath}), \\ (\ddot{ə}, -0.6 - 0.61\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.3 - 0.31\ddot{\imath}, 0.6 + 0.61\ddot{\imath}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\imath}, 0.3 + 0.31\ddot{\imath}, -0.0 - 0.01\ddot{\imath}, 0.1 + 0.11\ddot{\imath}) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_3)=\begin{Bmatrix}(0,-0.9-0.91\ddot{\imath},0.8+0.81\ddot{\imath},-0.0-0.01\ddot{\imath},0.1+0.11\ddot{\imath}),\\(\ddot{\varepsilon},-0.4-0.41\ddot{\imath},0.6+0.61\ddot{\imath},-0.5-0.51\ddot{\imath},0.2+0.21\ddot{\imath}),\\(\ddot{ə},-0.8-0.81\ddot{\imath},0.5+0.51\ddot{\imath},-0.1-0.11\ddot{\imath},0.4+0.41\ddot{\imath}),\\(\ddot{\alpha},-0.4-0.41\ddot{\imath},0.5+0.51\ddot{\imath},-0.5-0.51\ddot{\imath},0.4+0.41\ddot{\imath})\end{Bmatrix}$$

Finally, $\check{Ʒ}^{LB}(\ddot{\eta})\cup\Xi^{LB}(\ddot{\eta})=\mathbb{H}^{LB}(\ddot{\eta})$ is a BCIFSI.

Definition 11. The AND of $\check{Ʒ}^{LB}(\ddot{\eta}),\Xi^{LB}(\ddot{\eta})\in BCIF(\ddot{U})$ is a BCIFSS on $\ddot{U}$, defined by:

$$\check{Ʒ}^{LB}(\ddot{\eta})\wedge\Xi^{LB}(\delta)=\mathbb{H}^{LB}(\ddot{\eta},\delta)$$

and $(\ddot{\eta},\delta)\in\mathcal{D}=\mathcal{A}\times\mathcal{B}$ for all $\ddot{\eta}\in\mathcal{D}$. In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(\ddot{U},*,0)$ with the data in Table 4. Consider $\mathbb{E}=\{\ddot{\eta}_1,\ddot{\eta}_2,\ddot{\eta}_3,\ddot{\eta}_4\},\mathcal{A}=\{\ddot{\eta}_1,\ddot{\eta}_2\}\subseteq\mathbb{E}$ and $\mathcal{B}=\{\ddot{\eta}_3,\ddot{\eta}_4\}\subseteq\mathbb{E}$, where $\ddot{U}=\{0,\ddot{\varepsilon},\ddot{ə},\ddot{\alpha}\}$, thus, we define a function, such as

$$\check{Ʒ}^{LB}(\ddot{\eta}_1)=\begin{Bmatrix}(0,-0.8-0.81\ddot{\imath},0.7+0.71\ddot{\imath},-0.1-0.11\ddot{\imath},0.2+0.21\ddot{\imath}),\\(\ddot{\varepsilon},-0.3-0.31\ddot{\imath},0.6+0.61\ddot{\imath},-0.6-0.61\ddot{\imath},0.3+0.31\ddot{\imath}),\\(\ddot{ə},-0.7-0.71\ddot{\imath},0.4+0.41\ddot{\imath},-0.2-0.21\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{\alpha},-0.3-0.31\ddot{\imath},0.4+0.41\ddot{\imath},-0.6-0.61\ddot{\imath},0.5+0.51\ddot{\imath})\end{Bmatrix}$$

$$\check{Ʒ}^{LB}(\ddot{\eta}_2)=\begin{Bmatrix}(0,-0.7-0.71\ddot{\imath},0.6+0.61\ddot{\imath},-0.2-0.21\ddot{\imath},0.3+0.31\ddot{\imath}),\\(\ddot{\varepsilon},-0.2-0.21\ddot{\imath},0.5+0.51\ddot{\imath},-0.7-0.71\ddot{\imath},0.4+0.41\ddot{\imath}),\\(\ddot{ə},-0.6-0.61\ddot{\imath},0.3+0.31\ddot{\imath},-0.3-0.31\ddot{\imath},0.6+0.61\ddot{\imath}),\\(\ddot{\alpha},-0.2-0.21\ddot{\imath},0.3+0.31\ddot{\imath},-0.7-0.71\ddot{\imath},0.6+0.61\ddot{\imath})\end{Bmatrix}$$

and

$$\Xi^{LB}(\ddot{\eta}_3)=\begin{Bmatrix}(0,-0.6-0.61\ddot{\imath},0.5+0.51\ddot{\imath},-0.3-0.31\ddot{\imath},0.4+0.41\ddot{\imath}),\\(\ddot{\varepsilon},-0.1-0.11\ddot{\imath},0.4+0.41\ddot{\imath},-0.8-0.81\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{ə},-0.5-0.51\ddot{\imath},0.2+0.21\ddot{\imath},-0.4-0.41\ddot{\imath},0.7+0.71\ddot{\imath}),\\(\ddot{\alpha},-0.1-0.11\ddot{\imath},0.2+0.21\ddot{\imath},-0.8-0.81\ddot{\imath},0.7+0.71\ddot{\imath})\end{Bmatrix}$$

$$\Xi^{LB}(\ddot{\eta}_4)=\begin{Bmatrix}(0,-0.5-0.51\ddot{\imath},0.4+0.41\ddot{\imath},-0.4-0.41\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{\varepsilon},-0.0-0.01\ddot{\imath},0.3+0.31\ddot{\imath},-0.9-0.91\ddot{\imath},0.6+0.61\ddot{\imath}),\\(\ddot{ə},-0.4-0.41\ddot{\imath},0.1+0.11\ddot{\imath},-0.5-0.51\ddot{\imath},0.8+0.81\ddot{\imath}),\\(\ddot{\alpha},-0.0-0.01\ddot{\imath},0.1+0.11\ddot{\imath},-0.9-0.91\ddot{\imath},0.8+0.81\ddot{\imath})\end{Bmatrix}$$

thus

$$\check{Ʒ}^{LB}(\ddot{\eta})\wedge\Xi^{LB}(\ddot{\eta})=\mathbb{H}^{LB}(\ddot{\eta})$$

Noticed that $\mathcal{D}=\mathcal{A}\times\mathcal{B}=\{\ddot{\eta}_1,\ddot{\eta}_2\}\times\{\ddot{\eta}_3,\ddot{\eta}_4\}=\{(\ddot{\eta}_1,\ddot{\eta}_3),(\ddot{\eta}_1,\ddot{\eta}_4),(\ddot{\eta}_2,\ddot{\eta}_3),(\ddot{\eta}_2,\ddot{\eta}_4)\}$ is described by:

$$\mathbb{H}^{LB}(\ddot{\eta}_1,\ddot{\eta}_3)=\begin{Bmatrix}(0,-0.6-0.61\ddot{\imath},0.5+0.51\ddot{\imath},-0.3-0.31\ddot{\imath},0.4+0.41\ddot{\imath}),\\(\ddot{\varepsilon},-0.1-0.11\ddot{\imath},0.4+0.41\ddot{\imath},-0.8-0.81\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{ə},-0.5-0.51\ddot{\imath},0.2+0.21\ddot{\imath},-0.4-0.41\ddot{\imath},0.7+0.71\ddot{\imath}),\\(\ddot{\alpha},-0.1-0.11\ddot{\imath},0.2+0.21\ddot{\imath},-0.8-0.81\ddot{\imath},0.7+0.71\ddot{\imath})\end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_1,\ddot{\eta}_4)=\begin{Bmatrix}(0,-0.5-0.51\ddot{\imath},0.4+0.41\ddot{\imath},-0.4-0.41\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{\varepsilon},-0.0-0.01\ddot{\imath},0.3+0.31\ddot{\imath},-0.9-0.91\ddot{\imath},0.6+0.61\ddot{\imath}),\\(\ddot{ə},-0.4-0.41\ddot{\imath},0.1+0.11\ddot{\imath},-0.5-0.51\ddot{\imath},0.8+0.81\ddot{\imath}),\\(\ddot{\alpha},-0.0-0.01\ddot{\imath},0.1+0.11\ddot{\imath},-0.9-0.91\ddot{\imath},0.8+0.81\ddot{\imath})\end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2,\ddot{\eta}_3)=\begin{Bmatrix}(0,-0.6-0.61\ddot{\imath},0.5+0.51\ddot{\imath},-0.3-0.31\ddot{\imath},0.4+0.41\ddot{\imath}),\\(\ddot{\varepsilon},-0.1-0.11\ddot{\imath},0.4+0.41\ddot{\imath},-0.8-0.81\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{ə},-0.5-0.51\ddot{\imath},0.2+0.21\ddot{\imath},-0.4-0.41\ddot{\imath},0.7+0.71\ddot{\imath}),\\(\ddot{\alpha},-0.1-0.11\ddot{\imath},0.2+0.21\ddot{\imath},-0.8-0.81\ddot{\imath},0.7+0.71\ddot{\imath})\end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2,\ddot{\eta}_4)=\begin{Bmatrix}(0,-0.5-0.51\ddot{\imath},0.4+0.41\ddot{\imath},-0.4-0.41\ddot{\imath},0.5+0.51\ddot{\imath}),\\(\ddot{\varepsilon},-0.0-0.01\ddot{\imath},0.3+0.31\ddot{\imath},-0.9-0.91\ddot{\imath},0.6+0.61\ddot{\imath}),\\(\ddot{ə},-0.4-0.41\ddot{\imath},0.1+0.11\ddot{\imath},-0.5-0.51\ddot{\imath},0.8+0.81\ddot{\imath}),\\(\ddot{\alpha},-0.0-0.01\ddot{\imath},0.1+0.11\ddot{\imath},-0.9-0.91\ddot{\imath},0.8+0.81\ddot{\imath})\end{Bmatrix}$$

Finally, $\mathbb{H}^{LB}(\ddot{\eta})$ is a BCIFSI.

Definition 12. The OR of $\check{Ʒ}^{LB}(\ddot{\eta}),\Xi^{LB}(\ddot{\eta})\in BCIF(\ddot{U})$ is a BCIFSS on $\ddot{U}$, defined by:

$$\check{Ʒ}^{LB}(\ddot{\eta})\vee\Xi^{LB}(\delta)=\mathbb{H}^{LB}(\ddot{\eta},\delta)$$

and $(\ddot{\eta}, \delta) \in \mathcal{D} = \mathcal{A} \times \mathcal{B}$ for all $\ddot{\eta} \in \mathcal{D}$. In addition, we simplify the above model with the help of a suitable example, for this, we deliberate the BCK-algebra $(\ddot{U}, *, 0)$ with the data in Table 4. Consider $\mathbb{E} = \{\ddot{\eta}_1, \ddot{\eta}_2, \ddot{\eta}_3, \ddot{\eta}_4\}, \mathcal{A} = \{\ddot{\eta}_1, \ddot{\eta}_2\} \subseteq \mathbb{E}$ and $\mathcal{B} = \{\ddot{\eta}_3, \ddot{\eta}_4\} \subseteq \mathbb{E}$, where $\ddot{U} = \{0, \ddot{\varepsilon}, \ddot{\partial}, \ddot{\alpha}\}$, thus, we define a function, such as

$$\check{\mathfrak{z}}^{LB}(\ddot{\eta}_1) = \begin{Bmatrix} (0, -0.8 - 0.81\ddot{\iota}, 0.7 + 0.71\ddot{\iota}, -0.1 - 0.11\ddot{\iota}, 0.2 + 0.21\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\partial}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.5 + 0.51\ddot{\iota}) \end{Bmatrix}$$

$$\check{\mathfrak{z}}^{LB}(\ddot{\eta}_2) = \begin{Bmatrix} (0, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}), \\ (\ddot{\partial}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}), \\ (\ddot{\alpha}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}) \end{Bmatrix}$$

and

$$\Xi^{LB}(\ddot{\eta}_3) = \begin{Bmatrix} (0, -0.6 - 0.61\ddot{\iota}, 0.5 + 0.51\ddot{\iota}, -0.3 - 0.31\ddot{\iota}, 0.4 + 0.41\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.1 - 0.11\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.8 - 0.81\ddot{\iota}, 0.5 + 0.51\ddot{\iota}), \\ (\ddot{\partial}, -0.5 - 0.51\ddot{\iota}, 0.2 + 0.21\ddot{\iota}, -0.4 - 0.41\ddot{\iota}, 0.7 + 0.71\ddot{\iota}), \\ (\ddot{\alpha}, -0.1 - 0.11\ddot{\iota}, 0.2 + 0.21\ddot{\iota}, -0.8 - 0.81\ddot{\iota}, 0.7 + 0.71\ddot{\iota}) \end{Bmatrix}$$

$$\Xi^{LB}(\ddot{\eta}_4) = \begin{Bmatrix} (0, -0.5 - 0.51\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.4 - 0.41\ddot{\iota}, 0.5 + 0.51\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.0 - 0.01\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.9 - 0.91\ddot{\iota}, 0.6 + 0.61\ddot{\iota}), \\ (\ddot{\partial}, -0.4 - 0.41\ddot{\iota}, 0.1 + 0.11\ddot{\iota}, -0.5 - 0.51\ddot{\iota}, 0.8 + 0.81\ddot{\iota}), \\ (\ddot{\alpha}, -0.0 - 0.01\ddot{\iota}, 0.1 + 0.11\ddot{\iota}, -0.9 - 0.91\ddot{\iota}, 0.8 + 0.81\ddot{\iota}) \end{Bmatrix}$$

thus

$$\check{\mathfrak{z}}^{LB}(\ddot{\eta}) \vee \Xi^{LB}(\ddot{\eta}) = \mathbb{H}^{LB}(\ddot{\eta})$$

Noticed that $\mathcal{D} = \mathcal{A} \times \mathcal{B} = \{\ddot{\eta}_1, \ddot{\eta}_2\} \times \{\ddot{\eta}_3, \ddot{\eta}_4\} = \{(\ddot{\eta}_1, \ddot{\eta}_3), (\ddot{\eta}_1, \ddot{\eta}_4), (\ddot{\eta}_2, \ddot{\eta}_3), (\ddot{\eta}_2, \ddot{\eta}_4)\}$ is described by:

$$\mathbb{H}^{LB}(\ddot{\eta}_1, \ddot{\eta}_3) = \begin{Bmatrix} (0, -0.8 - 0.81\ddot{\iota}, 0.7 + 0.71\ddot{\iota}, -0.1 - 0.11\ddot{\iota}, 0.2 + 0.21\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\partial}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.5 + 0.51\ddot{\iota}) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_1, \ddot{\eta}_4) = \begin{Bmatrix} (0, -0.8 - 0.81\ddot{\iota}, 0.7 + 0.71\ddot{\iota}, -0.1 - 0.11\ddot{\iota}, 0.2 + 0.21\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\partial}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}), \\ (\ddot{\alpha}, -0.3 - 0.31\ddot{\iota}, 0.4 + 0.41\ddot{\iota}, -0.6 - 0.61\ddot{\iota}, 0.5 + 0.51\ddot{\iota}) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2, \ddot{\eta}_3) = \begin{Bmatrix} (0, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}), \\ (\ddot{\partial}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}), \\ (\ddot{\alpha}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}) \end{Bmatrix}$$

$$\mathbb{H}^{LB}(\ddot{\eta}_2, \ddot{\eta}_4) = \begin{Bmatrix} (0, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}), \\ (\ddot{\varepsilon}, -0.2 - 0.21\ddot{\iota}, 0.5 + 0.51\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.4 + 0.41\ddot{\iota}), \\ (\ddot{\partial}, -0.6 - 0.61\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.3 - 0.31\ddot{\iota}, 0.6 + 0.61\ddot{\iota}), \\ (\ddot{\alpha}, -0.2 - 0.21\ddot{\iota}, 0.3 + 0.31\ddot{\iota}, -0.7 - 0.71\ddot{\iota}, 0.6 + 0.61\ddot{\iota}) \end{Bmatrix}$$

Finally, $\mathbb{H}^{LB}(\ddot{\eta})$ is a BCIFSI. In the above examples, when we remove the imaginary parts, then the proposed theory with examples will be reduced to the proposed theory of Balamurugan et al. [13] designed the bipolar intuitionistic fuzzy soft ideals of BCK/BCI-algebras with applications.

Future advancements can be made in the following directions as well to enhance the literature broadness especially in neutrosophic set structure [17-19], other type-2 fuzzy sets [20-21], and T-S fuzzy sets [22] to define the soft ideals and other axioms. It can be extended to explore Multi-Polarity Fuzziness Subalgebras of BCK/BCI-Algebras [23], fuzzy fixed-point results of fuzzy differential equations [24] in complex scenario of uncertainty, fuzzy sampled-data stabilization for chaotic nonlinear systems [25-26]. Also, It can be extended to cubic multi-polar structures in BCK/BCI-algebras, highlighting their algebraic properties and significance [27] and generalized m-polar fuzzy positive implicative ideals, contributing to fuzzy algebra research [28].

5. Conclusions

The key goals of this article are described below:

i). We design an informative and reliable technique of BCIFSSs with numerous operational laws by merging the model of soft sets, complex fuzzy sets, and bipolar intuitionistic fuzzy sets to handle imprecise data.

ii). An ideal in a BCK-algebra is derived based on the BCIFSS theory proposed which can capture the information of hesitancy, vagueness, and non-membership information within the circumstance of BCK-algebra.

iii). We designed union, intersection, AND, and OR based on BCIFSI and simplified it with the help of numerous illustrations to justify the effectiveness of the invented theory.

iv). We evaluate numerous results for the above-invented techniques to enhance the worth of the proposed theory.

In the future, we will propose the bipolar complex Pythagorean fuzzy soft ideal with their operational laws, called the union, intersection, AND, and NO. Further, we will discuss their application in decision-making, artificial intelligence, and data mining to improve the worth of the proposed theory.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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