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Improved Estimator for Population Mean Utilizing Known Medians of Two Auxiliary Variables under Neutrosophic Framework

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Abstract: In the context of classical statistics, the estimation of the population mean is done with determinate, precise, and crisp data when auxiliary information is available. However, there are instances where dealing with uncertain, indeterminate, and imprecise data in interval form is required. To overcome this issue, Florentin Smarandache introduced neutrosophic statistics as a novel approach. This paper introduces a neutrosophic modified ratio-cum-product log-type estimator for the estimation of the population mean using known medians of two auxiliary variables in the neutrosophic context. The bias and mean squared error (MSE) for the proposed estimators are computed to the first-order approximation. The proposed estimator shows better results than existing ones in terms of MSE and percent relative efficiency (PRE). It is recommended to use the estimator with the highest PRE or lowest MSE for practical applications across different fields. The suggested estimator's effectiveness is validated through empirical studies, and its real-world applicability is illustrated using agricultural data.

Keywords: Classical Statistics; Neutrosophic Statistics; Ratio-cum-product Log-type Estimator; Median; Auxiliary Variables; Study Variable; Bias; Population Mean; Mean Squared Error; Percent Relative Efficiency.

1. Introduction

There has been much discussion in the literature on finite population sampling regarding the issue of estimating the population mean when the auxiliary variable is present. The practice of using auxiliary data in the estimation of parameters like the population mean, ratio, and product of two means, and the coefficient of variation has been well established. Ratio, product, and regression estimators serve as key examples in this regard. In areas like economics, finance, environmental science, and health, data analysis often involves nonlinear and complex relationships between variables. The traditional ratio estimators may not be sufficient, but the modified ratio-cum-product estimator better addresses these challenges. For instance, researchers can use the modified ratio-cum-product type estimator to improve estimates of crop yield or pollutant concentration by incorporating the median of the auxiliary variables (e.g., rainfall, fertilizer usage, irrigation days) or (wind speed, temperature, water flow) respectively, which is often known. This method enhances accuracy, as these auxiliary variables are closely related to the study variable, such as crop production or pollutant levels.

The use of auxiliary data at the estimation stage was first introduced by [1], who proposed the ratio estimator for population mean estimation. [2] proposed the product estimator, which performs better than the simple mean estimator when there is a negative correlation between the study variable and auxiliary variables. [3] employed the ratio estimator to estimate the population median by utilizing the auxiliary variable's median information. [4] took the initiative by suggesting a modified ratio estimator that uses the population median of the auxiliary variable to estimate the population mean of the study variable. Several authors, including [5,6,7,8,9] have utilized auxiliary information to enhance the estimation of the population mean of the study variable.

However, the primary goal of this study is to introduce an improved estimator for the population mean of the study variable, employing the known median of two auxiliary variables within a neutrosophic framework.

Classical statistics works with data represented by precise and crisp numbers, where there is no uncertainty in the observations. However, in real life, data is often imprecise, uncertain, and vague. To resolve this vagueness, L.A. Zadeh made his first effort by proposing the concept of fuzzy logic, [10]. However, fuzzy logic overlooks indeterminacy. To overcome this limitation, Florentin Smarandache introduced the concept of neutrosophy, a generalization of fuzzy logic that accounts for both randomness and indeterminacy, [11]. Neutrosophic statistics are applied when there is indeterminacy in the data, whereas classical statistics is employed when the data is fully precise or deterministic, [12]. In sampling theory, [13] tried to estimate the population mean under conditions of uncertainty and indeterminacy in the data. [8] proposed an almost unbiased estimator for population mean utilizing neutrosophic information. [14] introduced a generalized neutrosophic technique aimed at providing an elevated estimation of the population mean.

In the classical framework, numerous estimators have been developed using single and multiple auxiliary variables. Under optimal conditions, the ratio, product, and regression estimators show similar performance. However, the regression estimator shows better performance than the ratio and product estimators when the regression line between the study and auxiliary variables shifts from the origin. The ratio-cum-product estimator proposed by [15] was shown to perform more effectively than the usual estimator and the ratio or product estimators under specific conditions. [6] proposed a new estimator incorporating two auxiliary variables. [9] proposed to combine two auxiliary variables to achieve a more accurate estimation of the finite population mean under a neutrosophic framework. [16] suggested a new log-product-type estimator incorporating auxiliary information.

Designed to address uncertainty in both decision-making and mathematical modeling, neutrosophic theory has seen widespread use in fields like earthquake management, smart grid development, and internet-based streaming. Recent work highlights its significant influence in these areas. [23] studied the use of neutrosophic graphs in Japan's earthquake response facilities, illustrating their potential in disaster management through the effective handling of unclear and uncertain information. [24, 25] demonstrated various neutrosophic fixed point results and generalized theorems concerning neutrosophic cone metric spaces. In addition, [26] focused on identifying internet streaming services via the max product of complements in neutrosophic graphs, illustrating their application in optimizing decisions within digital environments. [27] work on fuzzy irresolute mappings demonstrated their key role in decision-making for electric vehicle systems, showing the impact of fuzzy logic on enhancing energy-efficient transportation. The study in [28] presents neutrosophic g°-closed and g°-open sets, proving several generalized theorems in the context of neutrosophic theory. [29] presents a new structure for fuzzy neutrosophic topological spaces, referred to as fuzzy neutrosophic pre-$\tau 0,1$ and pre-$\tau 0,2$ spaces, and explores more general relationships between the defined functions, using counter-examples for comparison.

Our paper is structured in the following sequences: Section 1 covers the Introduction, while Section 2 focuses on the terminology of neutrosophic statistics. Section 3 presents a review study on various existing neutrosophic estimators for estimating the finite population mean in the context of the neutrosophic framework. Section 4 covers our proposed estimator, including the derivations of the bias and MSE for the neutrosophic modified ratio-cum-product log-type estimator at the first-order approximation. Section 5 discusses the comparison of theoretical efficiency. Section 6 includes an empirical study employing agricultural data [17], related to rice production. Section 7 involves the discussion, and Section 8 outlines the conclusion.

2. Neutrosophic Terminology

Different forms of neutrosophic observations were given, including quantitative data that implied a number could be situated within an unknown range (a,b), [12]. Multiple approaches can be used to express the interval value of a neutrosophic number. [13] and [14] describe neutrosophic interval values as $Z_N = Z_L + Z_U I_N$, where, $I_N \in [I_L, I_U]$. The lower and upper values of the neutrosophic variable Z_N are denoted by $'Z_L'$ and $'Z_U'$ respectively, while I_N reflecting the indeterminacy level in Z_N, with values from 0 to 1.

Several of the notations used here are adopted from [13]. Let us consider a neutrosophic finite population of N_{NN} units $P_N = (P_{1N}, P_{2N}, ..., P_{NN})$. Each unit $P_{iN} \in (i = 1,2,...,N)$ has two neutrosophic auxiliary variables $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$ are recorded, and the study variable is $Y_N \in [Y_L, Y_U]$. The neutrosophic sample means $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $\bar{x}_{1N} \in [\bar{x}_{1L}, \bar{x}_{1U}]$, $\bar{x}_{2N} \in [\bar{x}_{2L}, \bar{x}_{2U}]$ are derived from the i^{th} observation of the study variable $Y_N \in [Y_L, Y_U]$ and the auxiliary variables $X_{1N} \in [X_{1L}, X_{1U}]$, $X_{2N} \in [X_{2L}, X_{2U}]$, respectively. Where $y_N(i) \in [y_L(i), y_U(i)]$, $x_{1N}(i) \in [x_{1L}(i), x_{1U}(i)]$, $x_{2N}(i) \in [x_{2L}(i), x_{2U}(i)]$ represent the i^{th} sample observation of our neutrosophic data. The overall means of the neutrosophic data are indicated by $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{X}_{1N} \in [\bar{X}_{1L}, \bar{X}_{1U}]$ and $\bar{X}_{2N} \in [\bar{X}_{2L}, \bar{X}_{2U}]$, respectively.

Moreover, the coefficients of variation in neutrosophic form for the variables Y_N, X_{1N}, X_{2N} are given as $C_{yN} \in [C_{yL}, C_{yU}]$, $C_{x1N} \in [C_{x1L}, C_{x1U}]$, and $C_{x2N} \in [C_{x2L}, C_{x2U}]$, respectively. Additionally, The neutrosophic correlation coefficients between the study variable $Y_N \in [Y_L, Y_U]$ and the auxiliary variables $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$ are given by $\rho_{yx1N} \in [\rho_{yx1L}, \rho_{yx1U}]$ and $\rho_{yx2N} \in [\rho_{yx2L}, \rho_{yx2U}]$, respectively.

The expressions $e_{0N} = \dfrac{\bar{y}_N - \bar{Y}_N}{\bar{Y}_N}$, $e_{1N} = \dfrac{\bar{x}_{1N} - \bar{X}_{1N}}{\bar{X}_{1N}}$, and $e_{2N} = \dfrac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{X}_{2N}}$ represent the neutrosophic relative errors for the sample means of the study variable Y_N and the auxiliary variables X_{1N} and X_{2N}, respectively. To determine the estimator's bias and MSE, we proceed by writing,

$$\bar{y}_N = \bar{Y}_N(1+e_{0N}),\ \bar{x}_{1N} = \bar{X}_{1N}(1+e_{1N}),\ \bar{x}_{2N} = \bar{X}_{2N}(1+e_{2N}).$$

$$E(e_{0N}) = E(e_{1N}) = E(e_{2N}) = 0;\ E(e_{0N}^2) = \theta_N C_{yN}^2,\ E(e_{1N}^2) = \theta_N C_{x1N}^2,\ E(e_{2N}^2) = \theta_N C_{x2N}^2,$$

$$E(e_{0N}e_{1N}) = \theta_N C_{x1N} C_{yN} \rho_{yx1N},\ E(e_{0N}e_{2N}) = \theta_N C_{x2N} C_{yN} \rho_{yx2N},\ E(e_{1N}e_{2N}) = \theta_N C_{x1N} C_{x2N} \rho_{x1x2N}.$$

Where, $e_{0N} \in [e_{0L}, e_{0U}]$, $e_{1N} \in [e_{1L}, e_{1U}]$, $e_{2N} \in [e_{2L}, e_{2U}]$.

$$C_{yN}^2 = \frac{\sigma_{yN}^2}{\bar{Y}_N^2},\ C_{x1N}^2 = \frac{\sigma_{x1N}^2}{\bar{X}_{1N}^2},\ C_{x2N}^2 = \frac{\sigma_{x2N}^2}{\bar{X}_{2N}^2},$$ where, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{x1N}^2 \in [C_{x1L}^2, C_{x1U}^2]$, $C_{x2N}^2 \in [C_{x2L}^2, C_{x2U}^2]$.

$\theta_N = \dfrac{(1 - f_N)}{n_N}$; where, $\theta_N \in [\theta_L, \theta_U]$, $n_N \in [n_L, n_U]$, $\sigma^2_{x1N} \in [\sigma^2_{x1L}, \sigma^2_{x1U}]$, $\sigma^2_{x2N} \in [\sigma^2_{x2L}, \sigma^2_{x2U}]$, $\sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$.

Similarly, in neutrosophic terms, MSE and percentage relative efficiency (PRE) are given by $MSE_N(\bar{y}_N) \in [MSE_L(\bar{y}_N), MSE_U(\bar{y}_N)]$ and $PRE_N(\bar{y}_N) \in [PRE_L(\bar{y}_N), PRE_U(\bar{y}_N)]$, respectively.

The flowchart of the proposed study using neutrosophic numbers is illustrated in Figure 1.

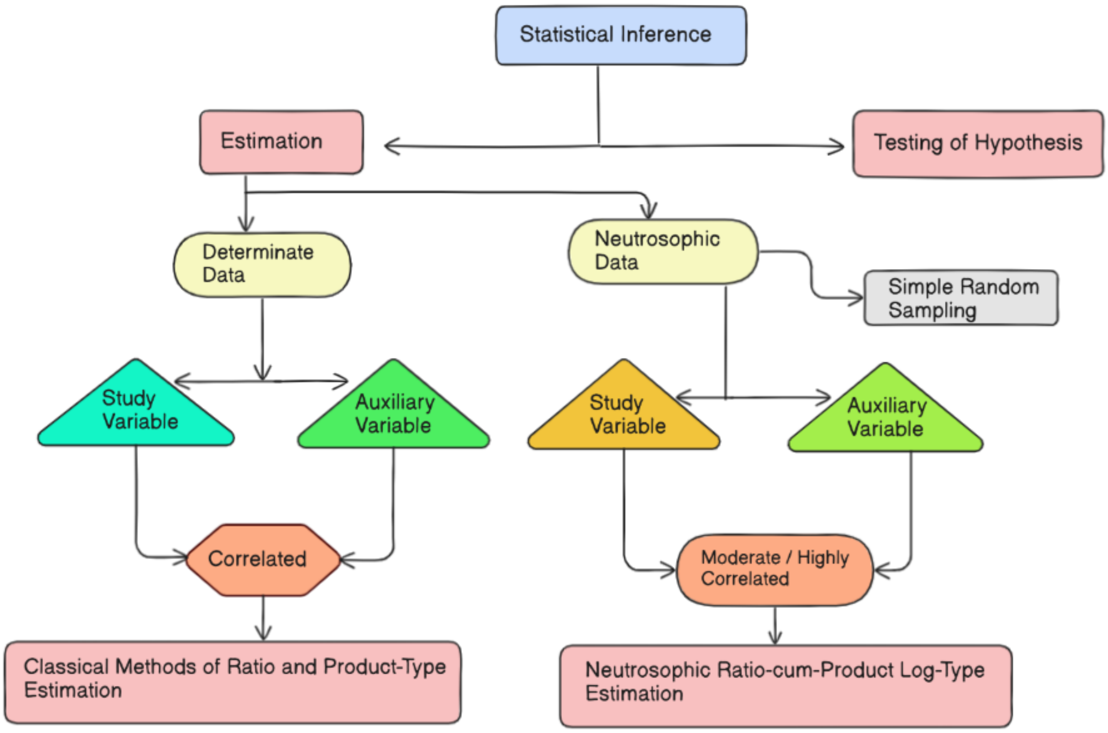

Figure 1. Process for parameter estimation.

3. Existing Neutrosophic Estimators

The neutrosophic population mean $(\bar{Y}_N)$ is estimated through the neutrosophic sample mean, which is provided as follows,

$$t_{0N} = \bar{y}_N \tag{3.1}$$

The variance expression for the estimator (t_{0N}) is provided as follows,

$$V(t_{0N}) = \theta_N \bar{Y}_N^2 C^2_{yN} \quad \text{, where, } t_{0N} \in [t_{0L}, t_{0U}] . \tag{3.2}$$

[13] proposed the usual neutrosophic ratio estimator for estimating the neutrosophic population mean with the known population mean of X_{1N} as follows,

$$t_{1N} = \bar{y}_N \left(\frac{\bar{X}_{1N}}{\bar{x}_{1N}} \right) \tag{3.3}$$

The bias and MSE of the estimator t_{1N} are provided respectively as follows,

$$Bias(t_{1N}) = \theta_N \bar{Y}_N (C^2_{x1N} - C_{yx1N}) \tag{3.4}$$

$$MSE(t_{1N}) = \theta_N \bar{Y}_N^2 \left(C^2_{yN} + C^2_{x1N} - 2C_{yx1N} \right) \tag{3.5}$$

Where, $C_{yx1N} = \rho_{yx1N} C_{x1N} C_{yN}$ and $t_{1N} \in [t_{1L}, t_{1U}]$.

Inspired by [18], the neutrosophic product estimators are presented as follows.

$$t_{2N} = \bar{y}_N \left(\frac{\bar{x}_{1N}}{\bar{X}_{1N}} \right) \tag{3.6}$$

The bias and MSE of the estimator (t_{2N}) are expressed by,

$$Bias(t_{2N}) = \theta_N \bar{Y}_N C_{yx1N} \tag{3.7}$$

$$MSE(t_{2N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{x1N}^2 + 2C_{yx1N} \right) \text{, where, } t_{2N} \in [t_{2L}, t_{2U}]. \tag{3.8}$$

Inspired by [19], [13] presented the neutrosophic exponential ratio-type estimator as outlined below,

$$t_{3N} = \bar{y}_N exp\left(\frac{\bar{X}_{1N} - \bar{x}_{1N}}{\bar{x}_{1N} + \bar{X}_{1N}} \right) \tag{3.9}$$

The bias and MSE for the estimator (t_{3N}) are expressed by,

$$Bias(t_{3N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{x1N}^2 - \frac{1}{2} C_{yx1N} \right) \tag{3.10}$$

$$MSE(t_{3N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{x1N}^2}{4} - C_{yx1N} \right) \text{, where, } t_{3N} \in [t_{3L}, t_{3U}]. \tag{3.11}$$

Motivated by the work of [19, 20] presented the neutrosophic product-type exponential estimator as,

$$t_{4N} = \bar{y}_N exp\left(\frac{\bar{x}_{1N} - \bar{X}_{1N}}{\bar{x}_{1N} + \bar{X}_{1N}} \right) \tag{3.12}$$

Its Bias and MSE expressions, are respectively provided by,

$$Bias(t_{4N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{x1N}^2 + \frac{1}{2} C_{yx1N} \right) \tag{3.13}$$

$$MSE(t_{4N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{x1N}^2}{4} + C_{yx1N} \right) \text{, where, } t_{4N} \in [t_{4L}, t_{4U}]. \tag{3.14}$$

A neutrosophic ratio estimator utilizing the correlation coefficient was introduced by [14].

$$t_{5N} = \bar{y}_N \left(\frac{\bar{X}_{1N} + \rho_{yx1N}}{\bar{x}_{1N} + \rho_{yx1N}} \right) \tag{3.15}$$

The estimator's (t_{5N}) bias and MSE are respectively stated as,

$$Bias(t_{5N}) = \theta_N \bar{Y}_N (\delta_N^2 C_{x1N}^2 - \delta_N C_{yx1N}) \tag{3.16}$$

$$MSE(t_{5N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \delta_N^2 C_{x1N}^2 - 2\delta_N C_{yx1N} \right) \tag{3.17}$$

where, $\delta_N = \dfrac{\bar{X}_{1N}}{\bar{X}_{1N} + \rho_{yx1N}}$ and $t_{5N} \in [t_{5L}, t_{5U}]$.

Inspired by [16], the neutrosophic modified log-product estimator is formulated as,

$$t_{6N} = \bar{y}_N + \alpha_N log\left(\frac{\bar{x}_{1N}}{\bar{X}_{1N}} \right) \tag{3.18}$$

where, the constant $'\alpha_N'$ is obtained by minimizing the MSE of the estimator (t_{6N}).

The bias along with the MSE of the estimator (t_{6N}) are respectively given by,

$$Bias(t_{6N}) = -\frac{\alpha}{2}\theta_N C_{x1N}^2 \tag{3.19}$$

$$MSE(t_{6N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 \left(1 - \rho_{yx1N}^2\right), \text{ where, } t_{6N} \in [t_{6L}, t_{6U}]. \tag{3.20}$$

Inspired by [21], the neutrosophic ratio-cum-product type exponential estimator, along with its bias and MSE, is expressed as follows,

$$t_{7N} = \bar{y}_N exp\left(\frac{\bar{X}_{1N} - \bar{x}_{1N}}{\bar{x}_{1N} + \bar{X}_{1N}}\right) exp\left(\frac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{x}_{2N} + \bar{X}_{2N}}\right) \tag{3.21}$$

$$Bias(t_{7N}) = \left(\frac{\theta_N}{2}\right)\bar{Y}_N\left[\frac{3}{4}C_{x1N}^2 - C_{x1N}C_{yN}\rho_{yx1N} + C_{x2N}C_{yN}\rho_{yx2N} - C_{x1N}C_{x2N}\rho_{x1x2N} - \frac{1}{4}C_{x2N}^2\right] \tag{3.22}$$

$$MSE(t_{7N}) = \theta_N \bar{Y}_N^2\left[C_{yN}^2 + \left\{C_{x1N}^2\frac{(1-4K_{01N})}{4}\right\} + \left\{C_{x2N}^2\frac{(1+4K_{02N}-2K_{12N})}{4}\right\}\right] \tag{3.23}$$

where, $t_{7N} \in [t_{7L}, t_{7U}]$ and $K_{01N} = \rho_{yx1N}\frac{C_{yN}}{C_{x1N}}$, $K_{01N} \in [K_{01L}, K_{01U}]$, $K_{02N} = \rho_{yx2N}\frac{C_{yN}}{C_{x2N}}$, $K_{02N} \in [K_{02L}, K_{02U}]$, $K_{12N} = \rho_{x1x2N}\frac{C_{x1N}}{C_{x2N}}$, $K_{12N} \in [K_{12L}, K_{12U}]$.

Motivated by [6], when two auxiliary variables are present, the regression estimate of $\bar{Y}_N$ is given by,

$$t_{8N} = \bar{y}_N + b_{1N}\left(\bar{X}_{1N} - \bar{x}_{1N}\right) + b_{2N}\left(\bar{X}_{2N} - \bar{x}_{2N}\right) \tag{3.24}$$

The MSE of the estimator (t_{8N}) can be determined as,

$$MSE(t_{8N}) = \theta_N S_{yN}^2(1 - \rho_{yx1N}^2 - \rho_{yx2N}^2 + 2\rho_{yx1N}\rho_{yx2N}\rho_{x1x2N}), \text{ where, } t_{8N} \in [t_{8L}, t_{8U}]. \tag{3.25}$$

Where, S_{yN}^2 = The population variance of $y_N(i)$.

4. The Proposed Neutrosophic Estimator

This section highlights improved neutrosophic estimators designed to predict the finite neutrosophic population mean with precision. Inspired by the work in [16] and [22], we proposed an improved estimator that incorporates the known medians of two auxiliary variables within the neutrosophic framework. The estimator being proposed is defined as,

$$t_N^* = \bar{y}_N + K_1 log\left[\frac{\bar{X}_{1N} + Md_{x1N}}{\bar{x}_{1N} + Md_{x1N}}\right] + K_2 log\left[\frac{\bar{x}_{2N} + Md_{x2N}}{\bar{X}_{2N} + Md_{x2N}}\right] \tag{4.1}$$

Where,

Md_{x1N} refers to the population's median for the auxiliary variable X_{1N}.

Md_{x2N} refers to the population's median for the auxiliary variable X_{2N}.

The constants K_1 and K_2 are derived by minimizing the MSE of the estimator t_N^*.

By expressing the equation (4.1) in terms of errors up to the first-order approximation gives us the following result,

$$t_N^* = (1+e_{0N})\bar{Y}_N + K_1 log\left(1+e_{1N}\lambda_{1N}\right)^{-1} + K_2 log\left(1+e_{2N}\lambda_{2N}\right) \quad (4.2)$$

Where, $\lambda_{1N} = \dfrac{\bar{X}_{1N}}{\bar{X}_{1N} + Md_{x1N}}$ and $\lambda_{2N} = \dfrac{\bar{X}_{2N}}{\bar{X}_{2N} + Md_{x2N}}$.

After expanding and simplifying the right-hand side, while retaining terms for the first-order approximation, we obtain,

$$t_N^* = (1+e_{0N})\bar{Y}_N + K_1 log\left(1 - e_{1N}\lambda_{1N} + e_{1N}^2\lambda_{1N}^2\right) + K_2 log\left(1+e_{2N}\lambda_{2N}\right) \quad (4.3)$$

$$t_N^* = (1+e_{0N})\bar{Y}_N + K_1\left(\frac{e_{1N}^2\lambda_{1N}^2}{2} - e_{1N}\lambda_{1N}\right) + K_2\left(e_{2N}\lambda_{2N} - \frac{(e_{2N}\lambda_{2N})^2}{2}\right) \quad (4.4)$$

By subtracting $\bar{Y}_N$ from both sides of the above Eq. (4.4), we obtain,

$$t_N^* - \bar{Y}_N = \bar{Y}_N e_{0N} + K_1\left(\frac{e_{1N}^2\lambda_{1N}^2}{2} - e_{1N}\lambda_{1N}\right) + K_2\left(e_{2N}\lambda_{2N} - \frac{(e_{2N}\lambda_{2N})^2}{2}\right) \quad (4.5)$$

By applying expectations to both sides of Eq. (4.5) and substituting the respective expectation values, the bias of (t_N^*) is obtained as,

$$E(t_N^* - \bar{Y}_N) = \bar{Y}_N E(e_{0N}) + K_1\left(\frac{E(e_{1N}^2)\lambda_{1N}^2}{2} - E(e_{1N})\lambda_{1N}\right) + K_2\left(E(e_{2N})\lambda_{2N} - \frac{E(e_{2N}^2)\lambda_{2N}^2}{2}\right) \quad (4.6)$$

$$Bias(t_N^*) = K_1\left(\frac{\lambda_{1N}^2}{2}\theta_N C_{x1N}^2\right) - K_2\left(\frac{\lambda_{2N}^2}{2}\theta_N C_{x2N}^2\right) \quad (4.7)$$

Using Eq. (4.5), we obtain,

$$t_N^* - \bar{Y}_N \cong \bar{Y}_N e_{0N} - K_1\lambda_{1N}e_{1N} + K_2\lambda_{2N}e_{2N} \quad (4.8)$$

By squaring both sides of Eq. (4.8) and computing the expected values, we derive the MSE of the estimator t_N^* to the first-order approximation as,

$$MSE(t_N^*) = \begin{bmatrix} \bar{Y}_N^2\theta_N C_{yN}^2 + K_1^2\lambda_{1N}^2\theta_N C_{x1N}^2 + K_2^2\lambda_{2N}^2\theta_N C_{x2N}^2 - 2K_1\bar{Y}_N\lambda_{1N}\theta_N\rho_{yx1N}C_{yN}C_{x1N} + \\ 2K_2\bar{Y}_N\lambda_{2N}\theta_N\rho_{yx2N}C_{yN}C_{x2N} - 2K_1K_2\lambda_{1N}\lambda_{2N}\theta_N\rho_{x1x2N}C_{x1N}C_{x2N} \end{bmatrix} \quad (4.9)$$

$$MSE(t_N^*) = A + BK_1^2 + CK_2^2 - 2K_1D + 2K_2E - 2K_1K_2F . \quad (4.10)$$

where,

$A = \bar{Y}_N^2 \theta_N C_{yN}^2$.

$B = \lambda_{1N}^2 \theta_N C_{x1N}^2$.

$C = \lambda_{2N}^2 \theta_N C_{x2N}^2$.

$D = \bar{Y}_N \lambda_{1N} \theta_N \rho_{yx1N} C_{yN} C_{x1N}$.

$E = \bar{Y}_N \lambda_{2N} \theta_N \rho_{yx2N} C_{yN} C_{x2N}$.

$F = \lambda_{1N} \lambda_{2N} \theta_N \rho_{x1x2N} C_{x1N} C_{x2N}$.

The minimum MSE of the proposed estimator (t_N^*) is achieved when the constants take the following optimal values, respectively,

$$K_{1(opt)} = \frac{(FE - DC)}{(F^2 - BC)} \quad \text{and} \quad K_{2(opt)} = \frac{(BE - FD)}{(F^2 - BC)} .$$

Using the optimal values for $K_{1(opt)}$ and $K_{2(opt)}$, the estimator's minimum MSE is defined as,

$$MSE_{min}(t_N^*) = \begin{bmatrix} A + B\left(\frac{FE - DC}{F^2 - BC}\right)^2 + C\left(\frac{BE - FD}{F^2 - BC}\right)^2 - 2\left(\frac{FE - DC}{F^2 - BC}\right)D + \\ 2\left(\frac{BE - FD}{F^2 - BC}\right)E - 2\left(\frac{FE - DC}{F^2 - BC}\right)\left(\frac{BE - FD}{F^2 - BC}\right)F \end{bmatrix} \tag{4.11}$$

$$MSE_{min}(t_N^*) = \left[A + \left(\frac{BE^2 + CD^2 - 2DEF}{F^2 - BC} \right) \right] \tag{4.12}$$

$$MSE_{min}(t_N^*) = \left[A + \frac{P}{Q} \right] . \tag{4.13}$$

Where,

$P = BE^2 + CD^2 - 2DEF$.

$Q = F^2 - BC$.

Particular case $(K_1 + K_2 = 1)$.

By substituting $(K_1 + K_2 = 1)$ into Eq. (4.1), the proposed estimator t_N^* transforms into a new class of estimators as,

$$t_N^@ = \bar{y}_N + K_1 log\left[\frac{\bar{X}_{1N} + Md_{x1N}}{\bar{x}_{1N} + Md_{x1N}}\right] + (1 - K_1) log\left[\frac{\bar{x}_{2N} + Md_{x2N}}{\bar{X}_{2N} + Md_{x2N}}\right] \tag{4.14}$$

The bias and MSE of the estimator $(t_N^@)$ can easily be derived to the first degree of approximation by applying the constraints $(K_1 + K_2 = 1)$ in equations (4.7) and (4.10), respectively.

$$Bias(t_N^{@}) = K_1\left(\frac{\lambda_{1N}^2}{2} f_N C_{x1N}^2\right) - (1-K_1)\left(\frac{\lambda_{2N}^2}{2} f_N C_{x2N}^2\right) \tag{4.15}$$

$$MSE(t_N^{@}) = A + BK_1^2 + C(1-K_1)^2 - 2K_1D + 2(1-K_1)E - 2K_1(1-K_1)F \tag{4.16}$$

$$MSE(t_N^{@}) = (A + C + 2E) + K_1^2(B + C + 2F) - 2K_1(C + D + E + F) \tag{4.17}$$

The MSE of the estimator $(t_N^{@})$ is minimized when,

$$K_{1(opt)} = \frac{C + D + E + F}{B + C + 2F} = K_1^*.$$

Consequently, the derived minimum MSE of the estimator $(t_N^{@})$ is,

$$MSE_{min}(t_N^{@}) = (A + C + 2E) - \left[\frac{(C + D + E + F)^2}{(B + C + 2F)}\right]. \tag{4.18}$$

5. Theoretical Efficiency Comparison

This part compares the suggested neutrosophic estimator with competing estimators of $\bar{Y}_N$ using neutrosophic auxiliary parameters. The efficiency of the proposed estimator is analyzed in terms of MSEs, and the conditions for its superior efficiency over the competing estimator are established.

The proposed estimator (t_N^*) is more efficient than the estimator (t_{0N}) under the condition that,

$V(t_{0N}) - MSE(t_N^*) > 0$ or,

$$\theta_N \bar{Y}_N^2 C_{yN}^2 - \left[A + \frac{P}{Q}\right] > 0.$$

The estimator (t_N^*) has a lower MSE than (t_{1N}) under the following condition,

$MSE(t_{1N}) - MSE(t_N^*) > 0$ or,

$$\theta_N \bar{Y}_N^2\left(C_{yN}^2 + C_{x1N}^2 - 2C_{yx1N}\right) - \left[A + \frac{P}{Q}\right] > 0.$$

The proposed estimator (t_N^*) outperforms the estimator (t_{2N}) under the condition that,

$MSE(t_{2N}) - MSE(t_N^*) > 0$ or,

$$\theta_N \bar{Y}_N^2\left(C_{yN}^2 + C_{x1N}^2 + 2C_{yx1N}\right) - \left[A + \frac{P}{Q}\right] > 0.$$

The proposed estimator (t_N^*) outperforms the exponential ratio estimator (t_{3N}) from [19], given the condition that,

$$MSE(t_{3N}) - MSE(t_N^*) > 0 \text{ or,}$$

$$\theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{x1N}^2}{4} - C_{yx1N} \right) - \left[A + \frac{P}{Q} \right] > 0 .$$

The introduced estimator (t_N^*) is more effective than (t_{4N}) under the condition that.

$$MSE(t_{4N}) - MSE(t_N^*) > 0 \text{ or,}$$

$$\theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{x1N}^2}{4} + C_{yx1N} \right) - \left[A + \frac{P}{Q} \right] > 0 .$$

The estimator (t_N^*) outperforms the estimator (t_{5N}) if,

$$MSE(t_{5N}) - MSE(t_N^*) > 0 \text{ or,}$$

$$\theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \delta_N^2 C_{x1N}^2 - 2\delta_N C_{yx1N} \right) - \left[A + \frac{P}{Q} \right] > 0 .$$

The estimator (t_N^*) has a lower MSE than (t_{6N}) if the condition is satisfied,

$$MSE(t_{6N}) - MSE(t_N^*) > 0 \text{ or,}$$

$$\theta_N \bar{Y}_N^2 C_{yN}^2 \left(1 - \rho_{yx1N}^2 \right) - \left[A + \frac{P}{Q} \right] > 0 .$$

The proposed estimator (t_N^*) outperforms the estimator (t_{7N}) if,

$$MSE(t_{7N}) - MSE(t_N^*) > 0$$

$$\theta_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{x1N}^2 \frac{(1 - 4K_{01N})}{4} + C_{x2N}^2 \frac{(1 + 4K_{02N} - 2K_{12N})}{4} \right] - \left[A + \frac{P}{Q} \right] > 0 .$$

The estimator (t_N^*) performs better than (t_{8N}), provided that,

$$MSE(t_{8N}) - MSE(t_N^*) > 0$$

$$\theta_N S_{yN}^2 (1 - \rho_{yx1N}^2 - \rho_{yx2N}^2 + 2\rho_{yx1N}\rho_{yx2N}\rho_{x1x2N}) - \left[A + \frac{P}{Q} \right] > 0 .$$

6. Empirical Study

Population 1: In this analysis, we use the data from [17], where rice yield is our neutrosophic study variable (Y_N), and rain sowing (X_{1N}) and rain ripening (X_{2N}) are the two neutrosophic auxiliary variables. This data is applied to assess the performance of different estimators for the neutrosophic study variable $Y_N \in [Y_L, Y_U]$.

The PRE equation is expressed as,

$$PRE = \frac{MSE(\bar{y}_N)}{MSE(estimator)} \times 100$$
.

Table 1. The descriptive statistics of the population being studied.

Parameter	Neutrosophic Value	Parameter	Neutrosophic Value
μ_{yN}	[3.9222, 18.4333]	σ^2_{x2N}	[1035.0988, 5643.2067]
μ_{x1N}	[11.5889, 35.6556]	C_{yN}	[0.5664, 0.1192]
μ_{x2N}	[40.5889, 93.7333]	C_{x1N}	[0.9258, 0.7679]
Md_{x1N}	[7.8, 51.6]	C_{x2N}	[0.7927, 0.8014]
Md_{x2N}	[34.4, 77]	ρ_{yx1N}	[0.5309, -0.4740]
σ^2_{yN}	[4.9351, 4.8289]	ρ_{yx2N}	[-0.0575, -0.6677]
σ^2_{x1N}	[115.1054, 749.6603]	ρ_{x1x2N}	[0.4282, 0.7363]

Table 2. Neutrosophic MSEs and PRE for various estimators.

Estimator	MSE	I_N	PRE
t_{0N}	[0.68543, 0.67068]	[0, 0.02199]	[100, 100]
t_{1N}	[1.32706, 32.59434]	[0, 0.95929]	[51.64993, 2.05765]
t_{2N}	[3.70626, 24.40345]	[0, 0.84813]	[18.49371, 2.74830]
t_{3N}	[0.54843, 9.67545]	[0, 0.94332]	[124.97873, 6.93176]
t_{4N}	[1.73803, 5.58001]	[0, 0.68853]	[39.43680, 12.01932]
t_{5N}	[1.22225, 33.40441]	[0, 0.96341]	[56.07901, 2.00776]
t_{6N}	[0.49223, 0.52000]	[0, 0.05340]	[139.24924, 128.97709]
t_{7N}	[0.49321, 3.54986]	[0, 0.86106]	[138.97349, 18.89310]
t_{8N}	[0.44479, 0.64988]	[0, 0.31557]	[154.09966, 103.20035]
t^*_N	**[0.42413, 0.37117]**	[0, 0.14269]	**[161.60580, 180.69208]**

Population 2: In this analysis, we use the data from [17], where rice yield is our neutrosophic study variable (Y_N). Temperature ripening (X_{1N}) and rain ripening (X_{2N}) are the two neutrosophic auxiliary variables. This data is used to evaluate the performance of various estimators for the neutrosophic study variable $Y_N \in [Y_L, Y_U]$.

Table 3. The descriptive statistics of the population being studied.

Parameter	Neutrosophic Value	Parameter	Neutrosophic Value
μ_{yN}	[3.9222, 18.4333]	σ^2_{x2N}	[1035.0988, 5643.2067]
μ_{x1N}	[24.4000, 36.6222]	C_{yN}	[0.5664, 0.1192]
μ_{x2N}	[40.5889, 93.7333]	C_{x1N}	[0.0448, 0.0511]
Md_{x1N}	[24.7, 36]	C_{x2N}	[0.7927, 0.8014]
Md_{x2N}	[34.4, 77]	ρ_{yx1N}	[-0.2086, 0.2412]
σ^2_{yN}	[4.9351, 4.8289]	ρ_{yx2N}	[-0.0575, -0.6677]
σ^2_{x1N}	[1.1956, 3.5062]	ρ_{x1x2N}	[-0.7674, -0.7445]

Table 4. Neutrosophic MSEs and PRE for various estimators.

Estimator	MSE	I_N	PRE
t_{0N}	[0.68543, 0.67068]	[0, 0.02199]	[100, 100]
t_{1N}	[0.71234, 0.65527]	[0, 0.08709]	[96.22171, 102.35132]
t_{2N}	[0.66709, 0.93283]	[0, 0.28487]	[102.74819, 71.89708]
t_{3N}	[0.69781, 0.63213]	[0, 0.10390]	[98.22524, 106.09793]
t_{4N}	[0.67519, 0.77091]	[0, 0.12417]	[101.51649, 86.99809]
t_{5N}	[0.71261, 0.65457]	[0, 0.08867]	[96.18533, 102.46097]
t_{6N}	[0.65560, 0.63165]	[0, 0.03792]	[104.54885, 106.17873]
t_{7N}	[1.00735, 5.91925]	[0, 0.82982]	[68.04210, 11.33048]
t_{8N}	[0.60374, 0.60106]	[0, 0.00446]	[113.53039, 111.58335]
t^*	[0.57667, 0.27305]	[0, 1.11195]	[118.85879, 245.62368]

7. Discussion

Table 2 shows that the variance of the usual neutrosophic mean estimator (t_{0N}) is [0.68543, 0.67068], whereas the MSE for the neutrosophic exponential product estimator (t_{4N}) is [1.73803, 5.58001], and MSEs of the neutrosophic estimator (t_{5N}) range from [1.22225, 33.40441]. Thus, the estimator (t_{0N}) shows an improvement over some of the reviewed neutrosophic estimators. The cause of the higher MSEs for neutrosophic estimators (t_{4N}) and (t_{5N}), in contrast to estimators (t_{0N}), is the low correlation between the study variable and the auxiliary variables. The MSE of the proposed neutrosophic estimator (t^*_N) is lower than that of other population mean estimators in the competition, ranging from **[0.42413, 0.37117],** and demonstrates higher PRE.

Table 4 similarly outlines the MSEs and PREs of the proposed and existing estimators, highlighting that the proposed estimators have lower MSEs and higher PREs compared to the estimators already in use. In Table 4, the neutrosophic PRE of the proposed estimator, derived from empirical findings, is given as **[118.85879, 245.62368]**, with the value highlighted in bold text.

Accordingly, Tables 2 and 4 validate these findings, demonstrating that the proposed estimators outperform the existing ones.

8. Conclusion

This paper presents a new modified ratio-cum-product log-type estimator for the estimation of the population mean using known medians of two auxiliary variables when working with Ambiguous and indeterminate data. We focused on first-order approximations and computed the bias and MSE to evaluate the precision of the proposed estimator. We compared our proposed estimator with existing ones by using agricultural data to demonstrate its practical application. Conducting an empirical study provided strong evidence that our suggested estimator is superior to the existing ones in the neutrosophic sampling context. It is crucial to note that neutrosophic estimators are the most appropriate for better population mean estimates when the study variable's observations are uncertain or nondeterministic. Consequently, neutrosophic estimators are a reliable option for handling indeterminate data. While our research involved only two auxiliary variables on agricultural secondary data, there is an opportunity to further refine the estimator by including more auxiliary variables.

In future work, extending the ratio-product log-type estimator to handle multiple auxiliary variables, nonlinear relationships, and datasets with missing or skewed data will improve its flexibility and practical relevance. It is advisable to use the proposed estimators to estimate the neutrosophic population means in real-world contexts, including economics, social sciences, agriculture, mathematics, and biology. Researchers can further conduct empirical studies in diverse fields focusing on population mean estimation. Future statistical research beyond sampling theory opens up exploration in control charts, inference, reliability analysis, non-parametric estimation, hypothesis testing, and other science fields.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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