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MBJ-Neutrosophic Structure Applied to BP-algebras: BP-subalgebras and α-ideals

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Abstract: In this article, we introduce the concepts of MBJ-neutrosophic BP-subalgebras and MBJ-neutrosophic α-ideals in BP-algebra by applying MBJ-neutrosophic logic to algebraic structure BP-algebra. We prove that the intersection of two MBJ-neutrosophic α-ideals and the inverse image of an MBJ-neutrosophic α-ideal are also MBJ-neutrosophic α-ideals. Furthermore, we prove an MBJ-neutrosophic set is an MBJ-neutrosophic BP-subalgebra if and only if its level sets are BP-subalgebras.

Keywords: BP-algebra; BP-subalgebra; α-ideal; MBJ-neutrosophic set; MBJ-neutrosophic BP-subalgebra; MBJ-neutrosophic α-ideal.

1. Introduction

Imai, Y. and Iséki, K. [1, 2] established two distinct categories of abstract algebraic structures: BCK-algebras and BCI-algebras. Since all BCK-algebras are BCI-algebras, but not conversely, BCK-algebras form a specific subclass of BCI-algebras. Hu, Q. P. and Li, X. [3, 4] introduced the vast and diverse class of abstract algebraic structures known as BCH-algebras. Their work established that all BCI-algebras are also BCH-algebras, but not conversely, BCI-algebras constitute a proper subclass of BCH-algebras. Neggers, J., Ahn, S. S., and Kim, H. S. [5] developed Q-algebras as a generalization of BCI/BCK-algebras and derived several important results. In 2002, Neggers and Kim [6, 7] developed the concept of a B-algebra and established several important results. In 2013, Ahn and Han [8] introduced a new type of algebra, called BP-algebra, which is related to various algebraic structures.

In 1965, Zadeh, L. A. [9] introduced the concept of the "degree of membership/truth" (t) and used it to define fuzzy sets. In 1986, Atanassov [10] introduced the notion of intuitionistic fuzzy sets by adding a "degree of nonmembership/falsehood" (f) to the existing idea of fuzzy sets. In 1995, Smarandache, F., [11-13] introduced the idea of "degree of indeterminacy/neutrality" (i) as an independent component and used it to define neutrosophic sets, which have three parts: truth, indeterminacy, and falsehood. For further information see [14].

Neutrosophic sets serve as a comprehensive platform by expanding upon classic sets, fuzzy sets, intuitionistic fuzzy sets, and interval-valued intuitionistic fuzzy sets. Takallo, M. M., Borzooei, R. A., and Jun, Y. B. [15] generalized the neutrosophic set to the MBJ-neutrosophic set. They utilized interval-valued fuzzy sets as the indeterminate membership function in the MBJ-neutrosophic set.

In 2015, Christopher Jefferson Y and Chandramouleeswara M. [16] introduced the concept of fuzzy BP-algebras and provided some results. They also defined fuzzy BP-Ideal [17], fuzzy T-Ideal [18], and intuitionistic L-fuzzy ideals in BP-Algebra [19]. In 2020, Osama Rashad El-Gendy [20] introduced the concept of fuzzy α-ideal in BP algebra.

The study of MBJ-neutrosophic structures in BCI/BCK-algebras has been significantly enriched by contributions from several researchers, who introduced various types of ideals, including positive implicative [21], implicative [22], commutative [23, 24], and hyper BCK-ideals [25]. MBJ-neutrosophic structures have been extensively applied to various algebraic systems, such as KU-algebras [26], B-algebras [27], BE-algebras [28], β-algebras [29], lattice implication algebras [30], and many more. In this paper, we apply MBJ-neutrosophic logic to BP-subalgebras and α-ideals in BP-algebras, introduce MBJ-neutrosophic BP-subalgebras and MBJ-neutrosophic α-ideals, along with their characterizations.

Throughout this article, we frequently utilize various symbols and their respective meanings. These symbols are outlined in Table 1.

Table 1. Symbols.

Symbol	Abbreviation
BP-A	BP-algebra
BP-SA	BP-subalgebra
$\boldsymbol{\alpha}$**I**	α-ideal
FS	Fuzzy set
MBJ-NSS	MBJ-neutrosophic set
MBJ-NSBPSA	MBJ-neutrosophic BP-subalgebra
MBJ-NS$\boldsymbol{\alpha}$**I**	MBJ-neutrosophic α-ideal

2. Preliminaries

Definition 2.1 [8] A BP-A is a non-empty set M with a constant 0 and a binary operation '$\Diamond$' satisfying the following conditions, for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$

(BP-A 1) $\mathrm{u} \Diamond \mathrm{u} = 0$

(BP-A 2) $\mathrm{u} \Diamond (\mathrm{u} \Diamond \mathrm{v}) = \mathrm{v}$

(BP-A 3) $(\mathrm{u} \Diamond \mathrm{w}) \Diamond (\mathrm{v} \Diamond \mathrm{w}) = \mathrm{u} \Diamond \mathrm{v}$.

In M, we can define a binary relation "$\mathrm{u} \leq \mathrm{v} \Leftrightarrow \mathrm{u} \Diamond \mathrm{v} = 0$".

Definition 2.2 [16] A subset $I(\neq \phi)$ of a BP-A M is said to be a BP-SA if $\mathrm{u} \Diamond \mathrm{v} \in I$, for all $\mathrm{u}, \mathrm{v} \in I$.

Definition 2.3 [20] A subset $I(\neq \phi)$ of a BP-A M is said to be an α-ideal if, for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$:

i. $0 \in \mathrm{I}$.

ii. $\mathrm{u} \Diamond \mathrm{w} \in \mathrm{I}$ and $\Diamond \mathrm{v} \in \mathrm{I} \Rightarrow \mathrm{v} \Diamond \mathrm{w} \in \mathrm{I}$.

Definition 2.4 [9] Let $\mathrm{M}(\neq \phi)$ be a set. A mapping $\alpha_{\mathrm{T}}: \mathrm{M} \to [0,1]$ is called a FS on M. The complement of a FS is denoted by $\alpha_{\mathrm{T}}{}^{C}(\mathrm{u})$, and is defined as $\alpha_{\mathrm{T}}{}^{C}(\mathrm{u}) = 1 - \alpha_{\mathrm{T}}(\mathrm{u})$.

Definition 2.5 [16] A FS α_{T} of a BP-A is called a FBP-SA of M if it satisfies

$$\alpha_{\mathrm{T}}(\mathrm{u} \Diamond \mathrm{v}) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u}), \alpha_{\mathrm{T}}(\mathrm{v})\}, \text{ for all } \mathrm{u}, \mathrm{v} \in \mathrm{M}.$$

Definition 2.6 [20] A FS α_{T} of a BP-A is called a Fα-I of M if it satisfies

$\alpha_{\mathrm{T}}(0) \geq \alpha_{\mathrm{T}}(\mathrm{u})$ *and* $\alpha_{\mathrm{T}}(\mathrm{v} \Diamond \mathrm{w}) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u} \Diamond \mathrm{w}), \alpha_{\mathrm{T}}(\mathrm{u} \Diamond \mathrm{v})\}$, for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$.

Definition 2.7 [20] Let $(\mathrm{M}, \Diamond, 0)$ and $(\mathrm{M}', \Diamond', 0')$ be BP-As. A mapping $f: \mathrm{M} \to \mathrm{M}'$ is called a homomorphism if $f(\mathrm{u} \Diamond \mathrm{v}) = f(\mathrm{u}) \Diamond' f(\mathrm{v})$, for all $\mathrm{u}, \mathrm{v} \in \mathrm{M}$.

Definition 2.8 By an interval number we mean a closed subinterval $\tilde{\mathcal{E}} = [\mathcal{E}^{L}, \mathcal{E}^{U}]$ of $[I]$, where $0 \leq \mathcal{E}^{L} \leq \mathcal{E}^{U} \leq 1$ Denote by $[I]$ The set of all interval numbers. Let us define 'refined minimum' (briefly, rmin), 'refined maximum' (briefly, rmax), '$\succcurlyeq$', '$\preccurlyeq$', and '$=$' of $\widetilde{\mathcal{E}_1} = [\mathcal{E}_1^{L}, \mathcal{E}_1^{U}]$ and $\widetilde{\mathcal{E}_2} = [\mathcal{E}_2^{L}, \mathcal{E}_2^{U}]$ in $[I]$.

i. $rmin\{\widetilde{\mathcal{E}_1}, \widetilde{\mathcal{E}_2}\} = [min\{\mathcal{E}_1^{L}, \mathcal{E}_2^{L}\}, min\{\mathcal{E}_1^{U}, \mathcal{E}_2^{U}\}]$.

ii. $rmax\{\widetilde{\mathcal{E}_1}, \widetilde{\mathcal{E}_2}\} = [max\{\mathcal{E}_1^{L}, \mathcal{E}_2^{L}\}, max\{\mathcal{E}_1^{U}, \mathcal{E}_2^{U}\}]$.

iii. $\widetilde{\mathcal{E}_1} \succcurlyeq (\preccurlyeq, =) \widetilde{\mathcal{E}_2} \Leftrightarrow \mathcal{E}_1^{L} \geq (\leq, =) \mathcal{E}_2^{L}, \mathcal{E}_1^{U} \geq (\leq, =) \mathcal{E}_2^{U}$.

Definition 2.9 [31] An Interval-valued fuzzy set in M is a function $\widetilde{\alpha_{\mathrm{T}}}: \mathrm{M} \to [I]$.

Definition 2.10 [15] Let $\mathrm{M}(\neq \phi)$ be a set. An MBJ-Neutrosophic Structure in M is in the form

$\mathcal{A} = \{\langle \mathrm{u}; \alpha_{\mathrm{T}}(\mathrm{u}), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}), \alpha_{\mathrm{F}}(\mathrm{u}) \rangle \mid \mathrm{u} \in \mathrm{M}\}$, where α_{T} and α_{F} are fuzzy sets in M, which are called a truth membership function and a false membership function, respectively, and $\widetilde{\alpha_{\mathrm{I}}}$ is an interval-valued fuzzy set in M which is called an indeterminate interval-valued membership function. For simplicity, we will use the symbol $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ to denote the MBJ-NSS.

3. MBJ-Neutrosophic BP-subalgebra

Definition 3.1 Let M be a BP-A. An MBJ-NSS $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ is called an MBJ-NSBPSA of M if, for all $\mathrm{u}, \mathrm{v} \in \mathrm{M}$, the following conditions are satisfied:

$$\alpha_{\mathrm{T}}(\mathrm{u} \Diamond \mathrm{v}) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u}), \alpha_{\mathrm{T}}(\mathrm{v})\},$$
$$\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u} \Diamond \mathrm{v}) \succcurlyeq rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{v})\},$$
$$\alpha_{\mathrm{F}}(\mathrm{u} \Diamond \mathrm{v}) \leq max\{\alpha_{\mathrm{F}}(\mathrm{u}), \alpha_{\mathrm{F}}(\mathrm{v})\}.$$

Example 3.2 Let $\mathrm{M} = \{0, \varsigma_1, \varsigma_2, \varsigma_3\}$ be a set with the binary operation "$\Diamond$", which is given in Table 2. Then $(\mathrm{M}, \Diamond, 0)$ is a BP-A. Let $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ be an MBJ-NSS in M, defined by Table 3:

Table 2. BP-algebra.

$\Diamond$	**0**	ς_1	ς_2	ς_3
0	0	ς_1	ς_2	ς_3
ς_1	ς_1	0	ς_3	ς_2
ς_2	ς_2	ς_3	0	ς_1
ς_3	ς_3	ς_2	ς_1	0

Table 3. MBJ-Neutrosophic BP-Subalgebra.

M	$\boldsymbol{\alpha}_{\mathrm{T}}(\mathrm{u})$	$\widetilde{\boldsymbol{\alpha}_{\mathrm{I}}}(\mathrm{u})$	$\boldsymbol{\alpha}_{\mathrm{F}}(\mathrm{u})$
0	0.73	[0.65,0.91]	0.23
ς_1	0.32	[0.11,0.34]	0.82
ς_2	0.59	[0.47,0.72]	0.45
ς_3	0.32	[0.11,0.34]	0.82

It is common to check that $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ is an MBJ-NSBPSA of M.

Theorem 3.3 The intersection of any two MBJ-NSBPSAs of M is again an MBJ-NSBPSA.

Proof: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two MBJ-NSBPSAs of M. $\mathcal{A}_1 \cap \mathcal{A}_2 = \left(\alpha_{\mathrm{T}_1} \cap \alpha_{\mathrm{T}_2}, \widetilde{\alpha}_{\mathrm{I}_1} \cap \widetilde{\alpha}_{\mathrm{I}_2}, \alpha_{\mathrm{F}_1} \cap \alpha_{\mathrm{F}_2}\right)$.

$$\begin{aligned}
\left(\alpha_{\mathrm{T}_1} \cap \alpha_{\mathrm{T}_2}\right)(\mathrm{u} \Diamond \mathrm{v}) &= min\{\alpha_{\mathrm{T}_1}(\mathrm{u} \Diamond \mathrm{v}), \alpha_{\mathrm{T}_2}(\mathrm{u} \Diamond \mathrm{v})\} \\
&\geq min\{min\{\alpha_{\mathrm{T}_1}(\mathrm{u}), \alpha_{\mathrm{T}_1}(\mathrm{v})\}, min\{\alpha_{\mathrm{T}_2}(\mathrm{u}), \alpha_{\mathrm{T}_2}(\mathrm{v})\}\} \\
&= min\{min\{\alpha_{\mathrm{T}_1}(\mathrm{u}), \alpha_{\mathrm{T}_2}(\mathrm{u})\}, min\{\alpha_{\mathrm{T}_1}(\mathrm{v}), \alpha_{\mathrm{T}_2}(\mathrm{v})\}\} \\
&= min\{\left(\alpha_{\mathrm{T}_1} \cap \alpha_{\mathrm{T}_2}\right)(\mathrm{u}), \left(\alpha_{\mathrm{T}_1} \cap \alpha_{\mathrm{T}_2}\right)(\mathrm{v})\},
\end{aligned}$$

$$\begin{aligned}
\left(\widetilde{\alpha}_{\mathrm{I}_1} \cap \widetilde{\alpha}_{\mathrm{I}_2}\right)(\mathrm{u} \Diamond \mathrm{v}) &= rmin\{\widetilde{\alpha}_{\mathrm{I}_1}(\mathrm{u} \Diamond \mathrm{v}), \widetilde{\alpha}_{\mathrm{I}_2}(\mathrm{u} \Diamond \mathrm{v})\} \\
&\succcurlyeq rmin\{rmin\{\widetilde{\alpha}_{\mathrm{I}_1}(\mathrm{u}), \widetilde{\alpha}_{\mathrm{I}_1}(\mathrm{v})\}, rmin\{\widetilde{\alpha}_{\mathrm{I}_2}(\mathrm{u}), \widetilde{\alpha}_{\mathrm{I}_2}(\mathrm{v})\}\} \\
&= min\{rmin\{\widetilde{\alpha}_{\mathrm{I}_1}(\mathrm{u}), \widetilde{\alpha}_{\mathrm{I}_2}(\mathrm{u})\}, rmin\{\widetilde{\alpha}_{\mathrm{I}_1}(\mathrm{v}), \widetilde{\alpha}_{\mathrm{I}_2}(\mathrm{v})\}\} \\
&= rmin\{\left(\widetilde{\alpha}_{\mathrm{I}_1} \cap \widetilde{\alpha}_{\mathrm{I}_2}\right)(\mathrm{u}), \left(\widetilde{\alpha}_{\mathrm{I}_1} \cap \widetilde{\alpha}_{\mathrm{I}_2}\right)(\mathrm{v})\},
\end{aligned}$$

$$\begin{aligned}
\left(\alpha_{\mathrm{F}_1} \cap \alpha_{\mathrm{F}_2}\right)(\mathrm{u} \Diamond \mathrm{v}) &= max\{\alpha_{\mathrm{F}_1}(\mathrm{u} \Diamond \mathrm{v}), \alpha_{\mathrm{F}_2}(\mathrm{u} \Diamond \mathrm{v})\} \\
&\leq max\{max\{\alpha_{\mathrm{F}_1}(\mathrm{u}), \alpha_{\mathrm{F}_1}(\mathrm{v})\}, max\{\alpha_{\mathrm{F}_2}(\mathrm{u}), \alpha_{\mathrm{F}_2}(\mathrm{v})\}\} \\
&= max\{max\{\alpha_{\mathrm{F}_1}(\mathrm{u}), \alpha_{\mathrm{F}_2}(\mathrm{u})\}, max\{\alpha_{\mathrm{F}_1}(\mathrm{v}), \alpha_{\mathrm{F}_2}(\mathrm{v})\}\} \\
&= max\{\left(\alpha_{\mathrm{F}_1} \cap \alpha_{\mathrm{F}_2}\right)(\mathrm{u}), \left(\alpha_{\mathrm{F}_1} \cap \alpha_{\mathrm{F}_2}\right)(\mathrm{v})\}.
\end{aligned}$$

Therefore, $\mathcal{A}_1 \cap \mathcal{A}_2$ is an MBJ-NSBPSA of M.

For a given MBJ-NSS $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ in a BP-A M, we consider the following sets [15] :

$$U_1(\alpha_{\mathrm{T}}, m) = \{\mathrm{u} \in \mathrm{M} \mid \alpha_{\mathrm{T}}(\mathrm{u}) \geq m\},$$
$$U_2(\widetilde{\alpha_{\mathrm{I}}}, [b_1, b_2]) = \{\mathrm{u} \in \mathrm{M} \mid \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}) \succcurlyeq [b_1, b_2]\},$$
$$L(\alpha_{\mathrm{F}}, j) = \{\mathrm{u} \in \mathrm{M} \mid \alpha_{\mathrm{F}}(\mathrm{u}) \leq j\},$$

where $m, j \in [0,1]$ and $[b_1, b_2] \in [I]$.

Lemma 3.4 Let $\mathcal{A} = (\alpha_T, \widetilde{\alpha_I}, \alpha_F)$ be an MBJ-NSBPSA of a BP-A M. Then

i. $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$ and $L(\alpha_F, j)$ are either empty or BP-SAs of M.

ii. $\alpha_T(0) \geq \alpha_T(u)$, $\widetilde{\alpha_I}(0) \succcurlyeq \widetilde{\alpha_I}(u)$, and $\alpha_F(0) \leq \alpha_F(u)$, for all $u \in \mathrm{M}$.

where $m, j \in [0,1]$ and $[b_1, b_2] \in [I]$.

Proof. Suppose that $\mathcal{A} = (\alpha_T, \widetilde{\alpha_I}, \alpha_F)$ is an MBJ-NSBPSA of M. Let $m, j \in [0,1]$ and $[b_1, b_2] \in [0,1]$ be such that $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $L(\alpha_F, j)$ are non-empty. For any $u_1, v_1, u_2, v_2, u_3, v_3 \in \mathrm{M}$, if $u_1, v_1 \in U_1(\alpha_T, m)$, $u_2, v_2 \in U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $u_3, v_3 \in L(\alpha_F, j)$, then
$\alpha_T(u_1 \lozenge v_1) \geq min\{\alpha_T(u_1), \alpha_T(v_1)\} \geq min\{m, m\} = m,$
$\widetilde{\alpha_I}(u_2 \lozenge v_2) \succcurlyeq rmin\{\widetilde{\alpha_I}(u_2), \widetilde{\alpha_I}(v_2)\} \succcurlyeq rmin\{[b_1, b_2], [b_1, b_2]\} = [b_1, b_2],$
$\alpha_F(u_3 \lozenge v_3) \leq max\{\alpha_F(u_3), \alpha_F(v_3)\} \leq max\{j, j\} = j$
and so $u_1 \lozenge v_1 \in U_1(\alpha_T, m)$, $u_2 \lozenge v_2 \in U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $u_3 \lozenge v_3 \in L(\alpha_F, j)$.
Therefore, $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $L(\alpha_F, j)$ are BP-SAs of M.

ii. $\alpha_T(0) = \alpha_T(u \lozenge u) \geq min\{\alpha_T(u), \alpha_T(u)\} = \alpha_T(u),$
$\widetilde{\alpha_I}(0) = \widetilde{\alpha_I}(u \lozenge u) \succcurlyeq rmin\{\widetilde{\alpha_I}(u), \widetilde{\alpha_I}(u)\} = \widetilde{\alpha_I}(u),$
$\alpha_F(0) = \alpha_F(u \lozenge u) \leq max\{\alpha_F(u), \alpha_F(u)\} = \alpha_F(u).$

Therefore, $\alpha_T(0) \geq \alpha_T(u)$, $\widetilde{\alpha_I}(0) \succcurlyeq \widetilde{\alpha_I}(u)$ and $\alpha_F(0) \leq \alpha_F(u)$, for all $u \in \mathrm{M}$.

Lemma 3.5 An MBJ-NSS $\mathcal{A} = (\alpha_T, \widetilde{\alpha_I}, \alpha_F)$ in a BP-A M is an MBJ-NSBPSA of M if and only if for all $m, j \in [0,1]$ and $[b_1, b_2] \in [0,1]$, the non-empty sets $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $L(\alpha_F, j)$ are BP-SAs of M.

Proof. The proof of the sufficient part follows from Lemma 3.4 (1).
Conversely, assume that $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $L(\alpha_F, j)$ are BP-SAs of M. If
$\alpha_T(u_1 \lozenge v_1) < min\{\alpha_T(u_1), \alpha_T(v_1)\},$
$\widetilde{\alpha_I}(u_2 \lozenge v_2) \prec rmin\{\widetilde{\alpha_I}(u_2), \widetilde{\alpha_I}(v_2)\},$
$\alpha_F(u_3 \lozenge v_3) > max\{\alpha_F(u_3), \alpha_F(v_3)\},$
for some $u_1, v_1, u_2, v_2, u_3, v_3 \in \mathrm{M}$. Then $u_1, v_1 \in U_1(\alpha_T, m_0)$, $u_2, v_2 \in U_2(\widetilde{\alpha_I}, [b_{01}, b_{02}])$, and $u_3, v_3 \in L(\alpha_F, j_0)$, but $u_1 \lozenge v_1 \notin U_1(\alpha_T, m_0)$, $u_2 \lozenge v_2 \notin U_2(\widetilde{\alpha_I}, [b_{01}, b_{02}])$, and $u_3 \lozenge v_3 \notin L(\alpha_F, j_0)$, for $m_0 = min\{\alpha_T(u_1), \alpha_T(v_1)\}$, $[b_{01}, b_{02}] = min\{\widetilde{\alpha_I}(u_2), \widetilde{\alpha_I}(v_2)\}$, and $j_0 = max\{\alpha_F(u_3), \alpha_F(v_3)\}$.
This is a contradiction with the fact that $U_1(\alpha_T, m)$, $U_2(\widetilde{\alpha_I}, [b_1, b_2])$, and $L(\alpha_F, j)$ are BP-SAs of M for all $m, j \in [0,1]$ and $[b_1, b_2] \in [I]$. Thus,
$\alpha_T(u \lozenge v) \geq min\{\alpha_T(u), \alpha_T(v)\},$
$\widetilde{\alpha_I}(u \lozenge v) \succcurlyeq rmin\{\widetilde{\alpha_I}(u), \widetilde{\alpha_I}(v)\},$
$\alpha_F(u \lozenge v) \leq max\{\alpha_F(u), \alpha_F(v)\}$, for all $u, v \in \mathrm{M}$. Consequently, $\mathcal{A} = (\alpha_T, \widetilde{\alpha_I}, \alpha_F)$ is an MBJ-NSBPSA of M.

Theorem 3.6 Any BP-SA of a BP-A M can be realized as a level subalgebra of some MBJ-NSBPSA of M.

Proof: Let I be a BP-SA of M, and $\mathcal{A} = (\alpha_T, \widetilde{\alpha_I}, \alpha_F)$ be an MBJ-NSS in M defined by

$$\alpha_T(u) = \begin{cases} m, \text{ if } u \in I, \\ 0, \text{ otherwise,} \end{cases} \quad \widetilde{\alpha_I}(u) = \begin{cases} [b_1, b_2], \text{ if } u \in I, \\ [0,0], \text{ otherwise,} \end{cases} \quad \text{and } \alpha_F(u) = \begin{cases} j, \text{ if } u \in I, \\ 1, \text{ otherwise,} \end{cases}$$

where $m, b_1, b_2 \in [0,1]$ with $b_1 \leq b_2$ and $j \in [0,1]$. Let $u, v \in \mathrm{M}$.

If $u, v \in I$, then $u \lozenge v \in I$
$\alpha_T(u \lozenge v) = m = min\{m, m\} = min\{\alpha_T(u), \alpha_T(v)\},$
$\widetilde{\alpha_I}(u \lozenge v) = [b_1, b_2] = rmin\{[b_1, b_2], [b_1, b_2]\} = rmin\{\widetilde{\alpha_I}(u), \widetilde{\alpha_I}(v)\},$
$\alpha_F(u \lozenge v) = j = max\{j, j\} = max\{\alpha_F(u), \alpha_F(v)\}.$

If both $u, v \notin I$, then
$\alpha_T(u \lozenge v) \geq 0 = min\{0,0\} = min\{\alpha_T(u), \alpha_T(v)\},$
$\widetilde{\alpha_I}(u \lozenge v) \succcurlyeq [0,0] = rmin\{[0,0], [0,0]\} = rmin\{\widetilde{\alpha_I}(u), \widetilde{\alpha_I}(v)\},$
$\alpha_F(u \lozenge v) \leq 1 = max\{1,1\} = max\{\alpha_F(u), \alpha_F(v)\}.$

If $u \in I$ and $v \notin I$, then
$\alpha_T(u \lozenge v) \geq 0 = min\{m, 0\} = min\{\alpha_T(u), \alpha_T(v)\},$
$\widetilde{\alpha_I}(u \lozenge v) \succcurlyeq [0,0] = rmin\{[b_1, b_2], [0,0]\} = rmin\{\widetilde{\alpha_I}(u), \widetilde{\alpha_I}(v)\},$

$\alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{v}) \leq 1 = max\{j, 1\} = max\{\alpha_{\mathrm{F}}(\mathrm{u}), \alpha_{\mathrm{F}}(\mathrm{v})\}$.

If $\mathrm{u} \notin I$ and $\mathrm{v} \in I$, then

$\alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v}) \geq 0 = min\{0, m\} = min\{\alpha_{\mathrm{T}}(\mathrm{u}), \alpha_{\mathrm{T}}(\mathrm{v})\}$,

$\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v}) \succcurlyeq [0,0] = rmin\{[0,0], [b_1, b_2]\} = rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{v})\}$,

$\alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{v}) \leq 1 = max\{1, j\} = max\{\alpha_{\mathrm{F}}(\mathrm{u}), \alpha_{\mathrm{F}}(\mathrm{v})\}$.

This shows that I is a level subalgebra of M corresponding to the MBJ-NSBPSA of M.

4. MBJ-neutrosophic α-Ideal in BP-algebra

Definition 4.1 Let M be a BP-A. An MBJ-NSS in M is called an MBJ-NSαI of M if it satisfies the following conditions

(MBJ-NSα-I 1) $\alpha_{\mathrm{T}}(0) \geq \alpha_{\mathrm{T}}(\mathrm{u})$, $\widetilde{\alpha_{\mathrm{I}}}(0) \succcurlyeq \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u})$, and $\alpha_{\mathrm{F}}(0) \geq \alpha_{\mathrm{F}}(\mathrm{u})$

(MBJ-NSα-I 2) $\alpha_{\mathrm{T}}(\mathrm{v}\Diamond\mathrm{w}) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})\}$

(MBJ-NSα-I 3) $\widetilde{\alpha_{\mathrm{I}}}(\mathrm{v}\Diamond\mathrm{w}) \succcurlyeq rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})\}$

(MBJ-NSα-I 4) $\alpha_{\mathrm{F}}(\mathrm{v}\Diamond\mathrm{w}) \leq max\{\alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{v})\}$, for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$.

Example 4.2 Consider a BP-A $\mathrm{M} = \{0, \varsigma_1, \varsigma_2, \varsigma_3\}$ in which the "$\Diamond$" operation is given in Table 4.

Table 4. BP-algebra.

$\Diamond$	**0**	ς_1	ς_2	ς_3
0	0	ς_1	ς_2	ς_3
ς_1	ς_1	ς_3	ς_2	ς_1
ς_2	ς_2	ς_3	0	ς_1
ς_3	ς_3	ς_1	0	ς_2

Let $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ be an MBJ-NSS in M defined by Table 5.

Table 5. MBJ-neutrosophic α-ideal.

M	$\alpha_{\mathrm{T}}(\mathrm{u})$	$\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u})$	$\alpha_{\mathrm{F}}(\mathrm{u})$
0	0.85	[0.75, 0.93]	0.22
ς_1	0.69	[0.57, 0.89]	0.51
ς_2	0.31	[0.27, 0.46]	0.97
ς_3	0.01	[0.27, 0.46]	0.97

Straightforward computations reveal that $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ is an MBJ-neutrosophic α-ideal of M.

Theorem 4.3 Let $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ be an MBJ-NSαI of a BP-A M. If $\mathrm{u} \leq \mathrm{v}$ holds in M, then $\alpha_{\mathrm{T}}(\mathrm{v}\Diamond\mathrm{w}) \geq \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{w})$, $\widetilde{\alpha_{\mathrm{I}}}(\mathrm{v}\Diamond\mathrm{w}) \succcurlyeq \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{w})$, and $\alpha_{\mathrm{F}}(\mathrm{v}\Diamond\mathrm{w}) \leq \alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{w})$, for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$.

Proof: Suppose that $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ is an MBJ-NSαI of M and $\mathrm{u} \leq \mathrm{v}$ holds. Then, $\mathrm{u}\Diamond\mathrm{v} = 0$. Now, utilizing (MBJ-NSαI 1) we obtain

$\alpha_{\mathrm{T}}(\mathrm{v}\Diamond\mathrm{w}) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})\} = min\{\alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{T}}(0)\} = \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{w})$,

$\widetilde{\alpha_{\mathrm{I}}}(\mathrm{v}\Diamond\mathrm{w}) \succcurlyeq rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})\} = rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\mathrm{I}}}(0)\} = \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{w})$,

$\alpha_{\mathrm{F}}(\mathrm{v}\Diamond\mathrm{w}) \leq max\{\alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{v})\} = max\{\alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\mathrm{F}}(0)\} = \alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{w})$.

Theorem 4.4 Let $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ be an MBJ-NSαI of a BP-A M. If $\mathrm{u} \leq \mathrm{v}\Diamond\mathrm{u}$ holds in M, then $\alpha_{\mathrm{T}}(\mathrm{u}) \geq \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})$, $\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}) \succcurlyeq \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})$ and $\alpha_{\mathrm{F}}(\mathrm{u}) \geq \alpha_{\mathrm{F}}(\mathrm{u}\Diamond\mathrm{v})$, for all $\mathrm{u}, \mathrm{v} \in \mathrm{M}$.

Proof. Suppose that $\mathcal{A} = (\alpha_{\mathrm{T}}, \widetilde{\alpha_{\mathrm{I}}}, \alpha_{\mathrm{F}})$ is an MBJ-NSαI of M and $\mathrm{u} \leq \mathrm{v}\Diamond\mathrm{u}$ holds in M. Then, $\mathrm{u}\Diamond(\mathrm{v}\Diamond\mathrm{u}) = 0$. Now, replacing w by $\mathrm{v}\Diamond\mathrm{u}$ in (MBJ-NSα-I 2,3,4) and utilizing (MBJ-NSα-I 1), (BP-A 2) we obtain

$\alpha_{\mathrm{T}}(\mathrm{v}\Diamond(\mathrm{v}\Diamond\mathrm{u})) \geq min\{\alpha_{\mathrm{T}}(\mathrm{u}\Diamond(\mathrm{v}\Diamond\mathrm{u})), \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})\} = min\{\alpha_{\mathrm{T}}(0), \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})\} = \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})$

$\Rightarrow \alpha_{\mathrm{T}}(\mathrm{u}) \geq \alpha_{\mathrm{T}}(\mathrm{u}\Diamond\mathrm{v})$,

$\widetilde{\alpha_{\mathrm{I}}}(\mathrm{v}\Diamond(\mathrm{v}\Diamond\mathrm{u})) \succcurlyeq rmin\{\widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond(\mathrm{v}\Diamond\mathrm{u})), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})\} = rmin\{\widetilde{\alpha_{\mathrm{I}}}(0), \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})\} = \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})$

$\Rightarrow \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}) \succcurlyeq \widetilde{\alpha_{\mathrm{I}}}(\mathrm{u}\Diamond\mathrm{v})$,

$\alpha_{\rm F}\big(\mathrm{v}\Diamond(\mathrm{v}\Diamond\mathrm{u})\big) \le max\{\alpha_{\rm F}\big(\mathrm{u}\Diamond(\mathrm{v}\Diamond\mathrm{u})\big), \alpha_{\rm F}(\mathrm{u}\Diamond\mathrm{v})\} = max\{\alpha_{\rm F}(0), \alpha_{\rm F}(\mathrm{u}\Diamond\mathrm{v})\} = \alpha_{\rm F}(\mathrm{u}\Diamond\mathrm{v})$

$\Rightarrow \alpha_{\rm F}(\mathrm{u}) \le \alpha_{\rm F}(\mathrm{u}\Diamond\mathrm{v})$.

Theorem 4.5 The intersection of any two MBJ-NSαIs in a BP-A M is also an MBJ-NSαI of M.

Proof: Let $\mathcal{A}_1 = \big(\alpha_{\rm T_1}, \widetilde{\alpha_{\rm I}}_1, \alpha_{\rm F_1}\big)$ and $\mathcal{A}_2 = \big(\alpha_{\rm T_2}, \widetilde{\alpha_{\rm I}}_2, \alpha_{\rm F_2}\big)$ be any two MBJ-NSαIs of M. Then,
$\mathcal{A}_1 \cap \mathcal{A}_2 = \big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}, \widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2, \alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)$.

$\big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}\big)(0) = min\{\alpha_{\rm T_1}(0), \alpha_{\rm T_2}(0)\} \ge min\{\alpha_{\rm T_1}(\mathrm{u}), \alpha_{\rm T_2}(\mathrm{u})\} \ge \big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}\big)(\mathrm{u})$,

$$\begin{aligned}\big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}\big)(\mathrm{v}\Diamond\mathrm{w}) &= min\{\alpha_{\rm T_1}(\mathrm{v}\Diamond\mathrm{w}), \alpha_{\rm T_2}(\mathrm{v}\Diamond\mathrm{w})\}\\ &\ge min\{min\{\alpha_{\rm T_1}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm T_1}(\mathrm{u}\Diamond\mathrm{v})\}, min\{\alpha_{\rm T_2}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm T_2}(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= min\{min\{\alpha_{\rm T_1}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm T_2}(\mathrm{u}\Diamond\mathrm{w})\}, min\{\alpha_{\rm T_1}(\mathrm{u}\Diamond\mathrm{v}), \alpha_{\rm T_2}(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= min\{\big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}\big)(\mathrm{v}\Diamond\mathrm{w}), \big(\alpha_{\rm T_1} \cap \alpha_{\rm T_2}\big)(\mathrm{v}\Diamond\mathrm{w})\},\end{aligned}$$

$\big(\widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2\big)(0) = rmin\{\widetilde{\alpha_{\rm I}}_1(0), \widetilde{\alpha_{\rm I}}_2(0)\} \succcurlyeq rmin\{\widetilde{\alpha_{\rm I}}_1(\mathrm{u}), \widetilde{\alpha_{\rm I}}_2(\mathrm{u})\} = \big(\widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2\big)(\mathrm{u})$,

$$\begin{aligned}\big(\widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2\big)(\mathrm{v}\Diamond\mathrm{w}) &= rmin\{\widetilde{\alpha_{\rm I}}_1(\mathrm{v}\Diamond\mathrm{w}), \widetilde{\alpha_{\rm I}}_2(\mathrm{v}\Diamond\mathrm{w})\}\\ &\succcurlyeq rmin\{rmin\{\widetilde{\alpha_{\rm I}}_1(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\rm I}}_1(\mathrm{u}\Diamond\mathrm{v})\}, rmin\{\widetilde{\alpha_{\rm I}}_2(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\rm I}}_2(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= rmin\{rmin\{\widetilde{\alpha_{\rm I}}_1(\mathrm{u}\Diamond\mathrm{w}), \widetilde{\alpha_{\rm I}}_2(\mathrm{u}\Diamond\mathrm{w})\}, rmin\{\widetilde{\alpha_{\rm I}}_1(\mathrm{u}\Diamond\mathrm{v}), \widetilde{\alpha_{\rm I}}_2(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= rmin\{\big(\widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2\big)(\mathrm{v}\Diamond\mathrm{w}), \big(\widetilde{\alpha_{\rm I}}_1 \cap \widetilde{\alpha_{\rm I}}_2\big)(\mathrm{v}\Diamond\mathrm{w})\},\end{aligned}$$

$\big(\alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)(0) = max\{\alpha_{\rm F_1}(0), \alpha_{\rm F_2}(0)\} \le max\{\alpha_{\rm F_1}(\mathrm{u}), \alpha_{\rm F_2}(\mathrm{u})\} = \big(\alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)(\mathrm{u})$,

$$\begin{aligned}\big(\alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)(\mathrm{v}\Diamond\mathrm{w}) &= max\{\alpha_{\rm F_1}(\mathrm{v}\Diamond\mathrm{w}), \alpha_{\rm F_2}(\mathrm{v}\Diamond\mathrm{w})\}\\ &\le max\{max\{\alpha_{\rm F_1}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm F_1}(\mathrm{u}\Diamond\mathrm{v})\}, max\{\alpha_{\rm F_2}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm F_2}(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= max\{max\{\alpha_{\rm F_1}(\mathrm{u}\Diamond\mathrm{w}), \alpha_{\rm F_2}(\mathrm{u}\Diamond\mathrm{w})\}, max\{\alpha_{\rm F_1}(\mathrm{u}\Diamond\mathrm{v}), \alpha_{\rm F_2}(\mathrm{u}\Diamond\mathrm{v})\}\}\\ &= max\{\big(\alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)(\mathrm{v}\Diamond\mathrm{w}), \big(\alpha_{\rm F_1} \cap \alpha_{\rm F_2}\big)(\mathrm{v}\Diamond\mathrm{w})\}.\end{aligned}$$

Therefore, $\mathcal{A}_1 \cap \mathcal{A}_2$ is an MBJ-NSαI of M.

Theorem 4.6 Let $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ be an MBJ-NSαI of a BP-A M. Then, for every $m, j \in [0,1]$ and $[b_1, b_2] \in [I]$, $U_1(\alpha_{\rm T}, m)$, $U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$, and $L(\alpha_{\rm F}, j)$ are α-ideals of BP-A M.

Proof: Assume that $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ is an MBJ-NSαI of M. For any $\mathrm{u}_1, \mathrm{v}_1, \mathrm{w}_1 \in$ M, if $\mathrm{u}_1 \in U_1(\alpha_{\rm T}, m)$, $\mathrm{v}_1 \in U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$, and $\mathrm{w}_1 \in L(\alpha_{\rm F}, j)$, then we obtain

$\alpha_{\rm T}(0) \ge \alpha_{\rm T}(\mathrm{u}_1) \ge m \Rightarrow 0 \in U_1(\alpha_{\rm T}, m)$

$\widetilde{\alpha_{\rm I}}(0) \succcurlyeq \widetilde{\alpha_{\rm I}}(\mathrm{v}_1) \succcurlyeq [b_1, b_2] \Rightarrow 0 \in U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$

$\alpha_{\rm F}(0) \le \alpha_{\rm F}(\mathrm{w}_1) \le j \Rightarrow 0 \in L(\alpha_{\rm F}, j)$.

For any $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3 \in$ M,
if $\mathrm{u}_1\Diamond\mathrm{u}_3, \mathrm{u}_1\Diamond\mathrm{u}_2 \in U_1(\alpha_{\rm T}, m)$; $\mathrm{v}_1\Diamond\mathrm{v}_3, \mathrm{v}_1\Diamond\mathrm{v}_2 \in U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$; and $\mathrm{w}_1\Diamond\mathrm{w}_3, \mathrm{w}_1\Diamond\mathrm{w}_2 \in L(\alpha_{\rm F}, j)$, then $\alpha_{\rm T}(\mathrm{u}_1\Diamond\mathrm{u}_3) \ge m$, $\alpha_{\rm T}(\mathrm{u}_1\Diamond\mathrm{u}_2) \ge m$, $\widetilde{\alpha_{\rm I}}(\mathrm{v}_1\Diamond\mathrm{v}_3) \succcurlyeq [b_1, b_2]$, $\widetilde{\alpha_{\rm I}}(\mathrm{v}_1\Diamond\mathrm{v}_2) \succcurlyeq [b_1, b_2]$, $\alpha_{\rm F}(\mathrm{w}_1\Diamond\mathrm{w}_3) \le j$ and $\alpha_{\rm F}(\mathrm{w}_1\Diamond\mathrm{w}_2) \le j$.

Now, by utilizing (MBJ-NSα-I 2), (MBJ-NSα-I 3), and (MBJ-NSα-I 4), we obtain

$\alpha_{\rm T}(\mathrm{u}_2\Diamond\mathrm{u}_3) \ge min\{\alpha_{\rm T}(\mathrm{u}_1\Diamond\mathrm{u}_3), \alpha_{\rm T}(\mathrm{u}_1\Diamond\mathrm{u}_2)\} \ge min\{m, m\} = m \Longrightarrow \mathrm{u}_2\Diamond\mathrm{u}_3 \in U_1(\alpha_{\rm T}, m)$,

$\widetilde{\alpha_{\rm I}}(\mathrm{v}_2\Diamond\mathrm{v}_3) \succcurlyeq rmin\{\widetilde{\alpha_{\rm I}}(\mathrm{v}_1\Diamond\mathrm{v}_3), \widetilde{\alpha_{\rm I}}(\mathrm{v}_1\Diamond\mathrm{v}_2)\} \succcurlyeq rmin\{[b_1, b_2], [b_1, b_2]\} = [b_1, b_2]$

$\Longrightarrow \mathrm{v}_2\Diamond\mathrm{v}_3 \in U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$,

$\alpha_{\rm F}(\mathrm{w}_2\Diamond\mathrm{w}_3) \le max\{\alpha_{\rm F}(\mathrm{w}_1\Diamond\mathrm{w}_3), \alpha_{\rm F}(\mathrm{w}_1\Diamond\mathrm{w}_2)\} \le max\{j, j\} = j \Longrightarrow \mathrm{w}_2\Diamond\mathrm{w}_3 \in L(\alpha_{\rm F}, j)$.

Therefore, $U_1(\alpha_{\rm T}, m)$, $U_2(\widetilde{\alpha_{\rm I}}, [b_1, b_2])$, and $L(\alpha_{\rm F}, j)$ are α-ideals of BP-A M, for any $m, j \in [0,1]$ and $[b_1, b_2] \in [I]$.

Theorem 4.7 Let $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ be an MBJ-NSS of a BP-algebra M. If $\alpha_{\rm T}, {\alpha_{\rm I}}^L, {\alpha_{\rm I}}^U$ and ${\alpha_{\rm F}}^C$ are fuzzy α-ideals of M, then $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ is an MBJ-NSαI of M.

Theorem 4.8 Let $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ be an MBJ-NSS of a BP-algebra M. If $\alpha_{\rm T}, {\alpha_{\rm I}}^L, {\alpha_{\rm I}}^U$ and ${\alpha_{\rm F}}^C$ are BP-SAs of M, then $\mathcal{A} = (\alpha_{\rm T}, \widetilde{\alpha_{\rm I}}, \alpha_{\rm F})$ is an MBJ-NSBPSA of M.

Definition 4.9 Let $(\mathrm{M}, \Diamond, 0)$ and $(\mathrm{M}', \Diamond', 0')$ be BP-As, and let f be a mapping from the set M to the set M′. If $\mathcal{A}_1 = \big(\alpha_{\rm T_1}, \widetilde{\alpha_{\rm I}}_1, \alpha_{\rm F_1}\big)$ and $\mathcal{A}_2 = \big(\alpha_{\rm T_2}, \widetilde{\alpha_{\rm I}}_2, \alpha_{\rm F_2}\big)$ are MBJ-NSSs of M and M′ respectively, then

$$f\big(\alpha_{\rm T_1}\big)(\mathrm{v}) = \alpha_{\rm T_2}(\mathrm{v}) = \begin{cases} \sup\limits_{\mathrm{u}\in f^{-1}(\mathrm{v})} \alpha_{\rm T_1}(\mathrm{u}), & if\, f^{-1}(\mathrm{v}) \neq \emptyset,\\ 0, & otherwise.\end{cases}$$

$$f(\widetilde{\alpha_{\mathrm{I}}}_1)(\mathrm{v}) = \widetilde{\alpha_{\mathrm{I}}}_2(\mathrm{v}) = \begin{cases} \underset{\mathrm{u}\in f^{-1}(\mathrm{v})}{rsup} \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}), & if\, f^{-1}(\mathrm{v}) \neq \emptyset, \\ [0,0] & otherwise. \end{cases}$$

$$f(\alpha_{\mathrm{F}_1})(\mathrm{v}) = \alpha_{\mathrm{F}_2}(\mathrm{v}) = \begin{cases} \underset{\mathrm{u}\in f^{-1}(\mathrm{v})}{inf} \alpha_{\mathrm{F}_1}(\mathrm{u}), & if\, f^{-1}(\mathrm{v}) \neq \emptyset, \\ 1, & otherwise. \end{cases}$$

is called the image of $\mathcal{A}_1 = (\alpha_{\mathrm{T}_1}, \widetilde{\alpha_{\mathrm{I}}}_1, \alpha_{\mathrm{F}_1})$ under f, for all $\mathrm{v} \in Y$.

Similarly, for an MBJ-NSS $\mathcal{A}_2 = (\alpha_{\mathrm{T}_2}, \widetilde{\alpha_{\mathrm{I}}}_2, \alpha_{\mathrm{F}_2})$ in M' an MBJ-NSS $\mathcal{A}_1 = \mathcal{A}_2 \circ f$ in M is defined as

$$\alpha_{\mathrm{T}_2}(f(\mathrm{u})) = \alpha_{\mathrm{T}_1}(\mathrm{u}), \quad \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u})) = \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}), \quad \text{and} \quad \alpha_{\mathrm{F}_2}(f(\mathrm{u})) = \alpha_{\mathrm{F}_1}(\mathrm{u}),$$

for all $\mathrm{u} \in \mathrm{M}$ and is called the preimage of $\mathcal{A}_2$ in M.

Theorem 4.10 A monomorphic pre-image of an MBJ-NSαI of BP-A is also an MBJ-NSαI.

Proof. Let $f: \mathrm{M} \to \mathrm{M}'$ be a monomorphism of BP-As. Assume that $\mathcal{A}_2 = (\alpha_{\mathrm{T}_2}, \widetilde{\alpha_{\mathrm{I}}}_2, \alpha_{\mathrm{F}_2})$ is an MBJ-NSαI in M' and $\mathcal{A}_1 = (\alpha_{\mathrm{T}_1}, \widetilde{\alpha_{\mathrm{I}}}_1, \alpha_{\mathrm{F}_1})$ is the preimage of $\mathcal{A}_2$ under f. Then

$\alpha_{\mathrm{T}_2}(f(\mathrm{u})) = \alpha_{\mathrm{T}_1}(\mathrm{u})$, $\widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u})) = \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u})$, $\alpha_{\mathrm{F}_2}(f(\mathrm{u})) = \alpha_{\mathrm{F}_1}(\mathrm{u})$, for all $\mathrm{u} \in \mathrm{M}$. Now,

$$\alpha_{\mathrm{T}_1}(0) = \alpha_{\mathrm{T}_2}(f(0)) \geq \alpha_{\mathrm{T}_2}(f(\mathrm{u})) = \alpha_{\mathrm{T}_1}(\mathrm{u}),$$
$$\widetilde{\alpha_{\mathrm{I}}}_1(0) = \widetilde{\alpha_{\mathrm{I}}}_2(f(0)) \succcurlyeq \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u})) = \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}),$$
$$\alpha_{\mathrm{F}_1}(0) = \alpha_{\mathrm{F}_2}(f(0)) \leq \alpha_{\mathrm{F}_2}(f(\mathrm{u})) = \alpha_{\mathrm{F}_1}(\mathrm{u})$$

Now let $\mathrm{u}, \mathrm{v}, z \in \mathrm{M}$. Then

$$\begin{aligned} \alpha_{\mathrm{T}_1}(\mathrm{v}\diamond\mathrm{w}) &= \alpha_{\mathrm{T}_2}(f(\mathrm{v}\diamond\mathrm{w})) = \alpha_{\mathrm{T}_2}\big(f(\mathrm{v})\diamond' f(\mathrm{w})\big) \\ &\geq min\left\{\alpha_{\mathrm{T}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{w})\big), \alpha_{\mathrm{T}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big)\right\} \\ &= min\{\alpha_{\mathrm{T}_2}(f(\mathrm{u}\diamond\mathrm{w})), \alpha_{\mathrm{T}_2}(f(\mathrm{u}\diamond\mathrm{v}))\} \\ &= min\{\alpha_{\mathrm{T}_1}(\mathrm{u}\diamond\mathrm{w}), \alpha_{\mathrm{T}_1}(\mathrm{u}\diamond\mathrm{v})\}, \\ \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{v}\diamond\mathrm{w}) &= \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{v}\diamond\mathrm{w})) = \widetilde{\alpha_{\mathrm{I}}}_2\big(f(\mathrm{v})\diamond' f(\mathrm{w})\big) \\ &\succcurlyeq rmin\left\{\widetilde{\alpha_{\mathrm{I}}}_2\big(f(\mathrm{u})\diamond' f(\mathrm{w})\big), \widetilde{\alpha_{\mathrm{I}}}_2\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big)\right\} \\ &= rmin\{\widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u}\diamond\mathrm{w})), \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u}\diamond\mathrm{v}))\} \\ &= rmin\{\widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}\diamond\mathrm{w}), \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}\diamond\mathrm{v})\}, \\ \alpha_{\mathrm{F}_1}(\mathrm{v}\diamond\mathrm{w}) &= \alpha_{\mathrm{F}_2}(f(\mathrm{v}\diamond\mathrm{w})) = \alpha_{\mathrm{F}_2}\big(f(\mathrm{v})\diamond' f(\mathrm{w})\big) \\ &\leq max\{\alpha_{\mathrm{F}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{w})\big), \alpha_{\mathrm{F}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big)\} \\ &= max\{\alpha_{\mathrm{F}_2}(f(\mathrm{u}\diamond\mathrm{w})), \alpha_{\mathrm{F}_2}(f(\mathrm{u}\diamond\mathrm{v}))\} \\ &= max\{\alpha_{\mathrm{F}_1}(\mathrm{u}\diamond\mathrm{w}), \alpha_{\mathrm{F}_1}(\mathrm{u}\diamond\mathrm{v})\}. \end{aligned}$$

Hence, the preimage of an MBJ-NSαI of a B-algebra is also an MBJ-NSαI.

Theorem 4.11 Let $f: \mathrm{M} \to \mathrm{M}'$ be a monomorphism of BP-algebras. If $\mathcal{A}_2 = (\alpha_{\mathrm{T}_2}, \widetilde{\alpha_{\mathrm{I}}}_2, \alpha_{\mathrm{F}_2})$ is an MBJ-NSBPSA of M', then its preimage $\mathcal{A}_1 = (\alpha_{\mathrm{T}_1}, \widetilde{\alpha_{\mathrm{I}}}_1, \alpha_{\mathrm{F}_1})$ is also an MBJ-NSBPSA of M.

Proof. Suppose that $\mathcal{A}_2 = (\alpha_{\mathrm{T}_2}, \widetilde{\alpha_{\mathrm{I}}}_2, \alpha_{\mathrm{F}_2})$ is an MBJ-NSBPSA of M'.

Now, let $\mathrm{u}, \mathrm{v} \in \mathrm{M}$, then

$$\begin{aligned} \alpha_{\mathrm{T}_1}(\mathrm{u}\diamond\mathrm{v}) = \alpha_{\mathrm{T}_2}(f(\mathrm{u}\diamond\mathrm{v})) = \alpha_{\mathrm{T}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big) &\geq min\{\alpha_{\mathrm{T}_2}(f(\mathrm{u})), \alpha_{\mathrm{T}_2}(f(\mathrm{v}))\} \\ &= min\{\alpha_{\mathrm{T}_1}(\mathrm{u}), \alpha_{\mathrm{T}_1}(\mathrm{v})\}, \\ \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}\diamond\mathrm{v}) = \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u}\diamond\mathrm{v})) = \widetilde{\alpha_{\mathrm{I}}}_2\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big) &\succcurlyeq rmin\{\widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{u})), \widetilde{\alpha_{\mathrm{I}}}_2(f(\mathrm{v}))\} \\ &= rmin\{\widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{u}), \widetilde{\alpha_{\mathrm{I}}}_1(\mathrm{v})\}, \\ \alpha_{\mathrm{F}_1}(\mathrm{u}\diamond\mathrm{v}) = \alpha_{\mathrm{F}_2}(f(\mathrm{u}\diamond\mathrm{v})) = \alpha_{\mathrm{F}_2}\big(f(\mathrm{u})\diamond' f(\mathrm{v})\big) &\leq max\{\alpha_{\mathrm{F}_2}(f(\mathrm{u})), \alpha_{\mathrm{F}_2}(f(\mathrm{v}))\} \\ &= max\{\alpha_{\mathrm{F}_1}(\mathrm{u}), \alpha_{\mathrm{F}_1}(\mathrm{v})\}. \end{aligned}$$

Hence, $\mathcal{A}_1 = (\alpha_{\mathrm{T}_1}, \widetilde{\alpha_{\mathrm{I}}}_1, \alpha_{\mathrm{F}_1})$ is an MBJ-NSBPSA of M.

5. Conclusion

In this study, we applied MBJ-neutrosophic structures to the algebraic structure BP-A and introduced the concepts of MBJ-NSBPSAs and MBJ-NSαIs with examples. We proved that the intersection of two MBJ-NSBPSAs is also an MBJ-NSBPSA, and similarly, the intersection of two MBJ-NSαIs is also an

MBJ-NSαI. Furthermore, we showed that under a homomorphism, the preimage of an MBJ-NSBPSA is an MBJ-NSBPSA, and the preimage of an MBJ-NSαI is an MBJ-NSαI.
These findings significantly advance our theoretical understanding of MBJ-neutrosophic structures in the field of BP-As. The methodology used in this article is also applicable to many other algebraic structures. To further expand on these results, future studies may focus on

- MBJ-neutrosophic T-ideals in BP-algebra.
- MBJ-neutrosophic BP-ideals in BP-algebra.
- MBJ-neutrosophic translations in BP-algebra.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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Author Contribution

All authors contributed equally to this research.

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