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A Reconsideration of Advanced Concepts in Neutrosophic Graphs: Smart, Zero Divisor, Layered, Weak, Semi, and Chemical Graphs

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Abstract: One of the most powerful tools in graph theory is the classification of graphs into distinct classes based on shared properties or structural features. Over time, many graph classes have been introduced, each aimed at capturing specific behaviors or characteristics of a graph. Neutrosophic Set Theory, a method for handling uncertainty, extends fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity. Building on this framework, Neutrosophic Graphs [84, 9, 135] have emerged as significant generalizations of fuzzy graphs. In this paper, we extend several classes of fuzzy graphs to Neutrosophic graphs and analyze their properties.

Keywords: Neutrosophic graph, Fuzzy Graph, Graph class, Neutrosophic set

1 Introduction

1.1 Graph Theory and Graph Classes

Graph theory, a fundamental branch of mathematics, investigates the relationships between nodes (vertices) and edges (connections) that form networks. It emphasizes the study of their structures, paths, and properties [50]. This field has been extensively explored due to its wide-ranging applications across diverse domains, including computer science, biology, and network analysis (ex.[107, 141, 36]).

One of the core aspects of graph theory is the classification of graphs into distinct classes based on shared properties or structural characteristics. Such classifications enable the development of efficient algorithms, facilitate problem-solving, and provide deeper insights into computational complexity. Moreover, these graph classes serve as essential frameworks for studying specific graph behaviors and their applications in various disciplines (cf.[40, 109, 23, 37]).

Notable examples of graph classes include Tree Graphs [156], Path Graphs [163], Complete Graphs [46], Circle Graphs [39], Unit Disk Graphs [47], Edge-Transitive Graphs [100, 101], Ultrahomogeneous Graphs [90], Visibility Graphs [89], Outerplanar Graphs [76], Petersen Graphs [75], and Total Graphs [157]. Studying these classes allows researchers to identify common properties, create specialized and efficient algorithms, and apply these insights to practical and theoretical problems.

1.2 Fuzzy Graph and Neutrosophic graph

Uncertainty refers to the lack of complete knowledge or predictability, influencing decision-making across disciplines like economics, science, and risk management. Zadeh [166] introduced fuzzy set theory in 1965 to address uncertainty, and Rosenfeld [131, 117] extended this concept to fuzzy graph theory in 1975. A fuzzy set is widely used to model uncertainty across various real-life domains [96, 86, 69, 172, 167, 169, 171, 168, 170]. It is defined by a membership function, denoted as l, which maps values to the range $[0, 1]$.

Fuzzy graphs assign membership values to both vertices and edges, allowing for the analysis of relationships with imprecision, and have been applied in fields such as logic, information theory, robotics, and nanotechnology [107, 141]. Within fuzzy graph theory, various graph classes have been proposed to generalize fuzzy graphs or adapt them for real-world applications. These include Intuitionistic Fuzzy Graphs [118], Bipolar Fuzzy Graphs [4], Fuzzy Planar Graphs [138], Hyperfuzzy graph (Hyperfuzzy set)[56, 82, 154, 64], Superhyperfuzzy graph (Superhyperfuzzy set) [56], Irregular Bipolar Fuzzy Graphs [137], and Complex Hesitant Fuzzy Graphs [1], among others[57]. Studying these classes helps researchers uncover common properties, develop specialized algorithms, and apply findings to practical problems.

In addition to fuzzy graphs, other frameworks have been developed to handle uncertainty and real-life parameters, such as weighted graphs [74], rough graph[139, 49], vague graph[129, 38], and Plithogenic Graphs [85, 60, 146, 152, 140].

Neutrosophic Set Theory, an alternative approach to handling uncertainty, was proposed to extend fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity[142, 143, 56, 144, 161, 42]. Similar to fuzzy set theory, Neutrosophic Sets have been widely researched for their applications across various fields [51, 165, 149]. Intuitively speaking, when Neutrosophic Set Theory is applied to graphs, it leads to the concept of Neutrosophic Graphs. Neutrosophic Graphs [84, 53, 58, 61, 52, 9, 135] and Neutrosophic Hypergraphs [95, 11] have emerged as significant generalizations of fuzzy graphs. These frameworks have attracted attention due to their applications in areas closely related to fuzzy graph theory [53, 117, 55].

Numerous classes of Neutrosophic Graphs have been studied, including Bipolar Neutrosophic Graphs [11], Neutrosophic Incidence Graphs [153], HyperNeutrosophic graph (HyperNeutrosophic set) [56], SuperhyperNeutrosophic graph (SuperhyperNeutrosophic set) [56], and Complex Neutrosophic Hypergraphs [95]. Investigating these graph classes allows researchers to identify shared properties, refine algorithms, and explore new applications in various fields.

1.3 Our Contribution

Based on the above, the study of graph classes holds great significance. In this paper, we extend several classes of fuzzy graphs to Neutrosophic graphs and analyze their properties. Specifically, we explore graph classes related to Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semigraphs, Double/Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

2 Preliminaries and definitions

In this section, we will briefly explain the definitions and notations used in this paper. We begin by introducing fundamental concepts related to graphs and rings, followed by an explanation of fuzzy graphs and Neutrosophic Graphs. Subsequently, we will present the definitions and examples of various graph classes, including Smart Fuzzy Graphs, Fuzzy Zero Divisor Graphs, Weak Fuzzy Graphs, Fuzzy Semigraphs, Mild Balanced Intuitionistic Fuzzy Graphs, Double/Triple Layered Fuzzy Graphs, and Connected Fuzzy Chemical Graphs.

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. In addition to graph concepts, this paper also utilizes fundamental concepts from set theory. Readers may refer to lecture notes or surveys on set theory as needed [91, 73, 80].

Definition 1 (Graph)**.** [50] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Graphs can be used to model a wide range of real-world concepts. Although just one example, the following definition illustrates this capability.

Example 2 (Social Network Graph)**.** In social networks, individuals can be represented as *vertices*, and their connections (such as friendships, interactions, or communications) can be represented as *edges*. This type of graph models how people are connected to each other and allows for analysis of social dynamics, such as identifying influencers, clusters of closely connected individuals, or finding the shortest path between people.

- **Vertices**: Each person in the network is represented by a vertex (node).
- **Edges**: A connection or relationship between two people (e.g., a "friendship" in Facebook) is represented by an edge between two vertices.

Definition 3 (Subgraph)**.** [50] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 4. Consider the graph $G = (V, E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$$

A subgraph $H = (V_H, E_H)$ is created by selecting a subset of the vertices and the edges from G.

Let the subgraph H be defined by the following vertex and edge sets:

$$V_H = \{v_2, v_3, v_4\}, \quad E_H = \{(v_2, v_3), (v_3, v_4)\}$$

The subgraph H includes the vertices v_2, v_3, v_4 and the edges that connect these vertices in the original graph G, specifically the edges (v_2, v_3) and (v_3, v_4). It excludes the vertices v_1 and v_5 and their incident edges.

Definition 5 (Degree)**.** [50] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 6 (Connectedness)**.** A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a path $P \subseteq G$ that connects u and v. Formally, G is connected if:

$$\forall u, v \in V, \exists P \subseteq G \text{ such that } P \text{ is a path from } u \text{ to } v.$$

Example 7 (Connected and Unconnected Graphs)**.** Consider the following two graphs:

- **Connected Graph:** Let $G_1 = (V_1, E_1)$ be a graph with the vertex set $V_1 = \{v_1, v_2, v_3, v_4\}$ and edge set $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$.

  In this graph, there exists a path between any pair of vertices. For example:

  $$P_{v_1 \to v_4} = (v_1, v_2, v_3, v_4)$$

  is a path connecting v_1 to v_4. Thus, G_1 is a connected graph.

- **Unconnected Graph:** Let $G_2 = (V_2, E_2)$ be a graph with the vertex set $V_2 = \{u_1, u_2, u_3, u_4\}$ and edge set $E_2 = \{(u_1, u_2), (u_3, u_4)\}$.

  In this graph, there is no path between the vertices u_1 and u_3 (or between u_2 and u_4, for example). Hence, G_2 is an unconnected graph because it contains two distinct subgraphs, one with u_1, u_2 and the other with u_3, u_4, and there is no path connecting these subgraphs.

Definition 8 (Path)**.** (cf.[41]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Example 9 (Path)**.** Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the set of vertices and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$ is the set of edges.

A path from v_1 to v_5 can be written as:

$$P = (v_1, v_2, v_3, v_4, v_5),$$

where each pair of consecutive vertices (v_i, v_{i+1}) is connected by an edge in E.

Thus, P is a valid path in the graph G, connecting v_1 and v_5 through distinct vertices.

Definition 10 (Tree)**.** A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 11 (Tree)**.** Consider the following graph $T = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the set of vertices and $E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$ is the set of edges. Thus, T is a connected, acyclic graph, satisfying the definition of a tree.

Definition 12 (Complete)**.** (cf.[28]) A graph $G = (V, E)$ is said to be *complete* if for every pair of distinct vertices $u, v \in V$, there exists an edge $(u, v) \in E$ connecting them. In other words, every pair of vertices in G is adjacent.

The number of edges in a complete graph with n vertices is given by:

$$|E| = \frac{n(n-1)}{2},$$

where $n = |V|$ is the number of vertices in the graph.

Example 13. Consider a complete graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$. Since G is complete, every pair of distinct vertices must be connected by an edge. The vertex set and edge set of G are as follows:

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$$

In this case, the number of vertices is $n = 4$, and the number of edges is:

$$|E| = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = \frac{12}{2} = 6.$$

Thus, the complete graph G with four vertices has six edges, and each pair of vertices is connected by an edge.

Graphically, the complete graph G can be represented as a set of vertices where every vertex is connected to every other vertex.

Definition 14 (union)**.** The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = (V, E)$ where:

- The vertex set V is the union of the vertex sets of G_1 and G_2:
$$V = V_1 \cup V_2.$$
- The edge set E is the union of the edge sets of G_1 and G_2:
$$E = E_1 \cup E_2.$$

Thus, the union of G_1 and G_2 combines the vertices and edges of both graphs, without duplicating any elements.

Definition 15 (Partition of a Graph)**.** A *partition* of a graph $G = (V, E)$ is a division of the vertex set V into disjoint, non-empty subsets $V_1, V_2, \ldots, V_k$ such that:

$$V = V_1 \cup V_2 \cup \cdots \cup V_k \quad \text{and} \quad V_i \cap V_j = \emptyset \quad \text{for all} \quad i \neq j.$$

The subsets $V_1, V_2, \ldots, V_k$ are called the *parts* or *blocks* of the partition.

Definition 16 (Bipartite Graph)**.** (cf.[20, 103]) A graph $G = (V, E)$ is called a *bipartite graph* if the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge $e \in E$ connects a vertex in V_1 to a vertex in V_2. In other words, there are no edges between vertices within the same set V_1 or V_2. Formally, G is bipartite if $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for all $e = (u, v) \in E$, $u \in V_1$ and $v \in V_2$.

Example 17. Consider the graph $G = (V, E)$ where the vertex set V is given by

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and the edge set E is given by

$$E = \{(v_1, v_4), (v_1, v_5), (v_2, v_4), (v_3, v_5), (v_3, v_6)\}.$$

We can partition the vertex set V into two disjoint sets:

$$V_1 = \{v_1, v_2, v_3\}, \quad V_2 = \{v_4, v_5, v_6\}.$$

Every edge in E connects a vertex from V_1 to a vertex from V_2, and there are no edges between vertices within the same set. For example, (v_1, v_4) connects $v_1 \in V_1$ to $v_4 \in V_2$, and similarly for all other edges.

Thus, G is a bipartite graph.

Definition 18. In graph theory, a *triangle graph*, also known as the *3-cycle graph* or the *complete graph* K_3, is a simple, undirected graph that consists of three vertices connected by three edges. Each vertex in the triangle graph is connected to every other vertex, forming a cycle of length three.

The triangle graph can be denoted as:

$$T = (V, E)$$

where:

- $V = \{v_1, v_2, v_3\}$ is the set of three vertices, and
- $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ is the set of three edges, representing the connections between all pairs of vertices.

For more foundational graph concepts and notations, please refer to [50, 111, 68, 67, 162].

2.2 Basic Ring Concepts

A ring is an algebraic structure equipped with two operations, addition and multiplication, that satisfy the properties of associativity, distributivity, and the existence of an additive identity (cf.[79]). Since this paper focuses on the Zero-Divisor Graph, we begin by introducing the fundamental concepts of rings. Several definitions are outlined below.

Definition 19 (Commutative Ring)**.** (cf.[99, 18, 97]) A *commutative ring* is a set R equipped with two binary operations, addition $+$ and multiplication $\cdot$, such that:

- $(R, +)$ is an abelian group.
- $(R, \cdot)$ is a monoid with an identity element $1 \in R$ (i.e., multiplication is associative, and there exists a multiplicative identity 1).
- Multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in R$.
- Multiplication is distributive over addition, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

Example 20. Let $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, the set of integers modulo 6. This set forms a commutative ring under addition and multiplication modulo 6. We will verify that this is a commutative ring by performing some calculations for addition and multiplication.

The addition table modulo 6 is as follows:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

This table confirms that: - $(R, +)$ forms an abelian group (commutative, with 0 as the identity and each element having an inverse, for example, $1 + 5 = 0$).

The multiplication table modulo 6 is as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

This table confirms the following: - $(R, \cdot)$ is a monoid with 1 as the multiplicative identity. - Multiplication is commutative, for example, $2 \cdot 3 = 0$ and $3 \cdot 2 = 0$. - Multiplication is distributive over addition, for example, $2 \cdot (3 + 4) = 2 \cdot 1 = 2$ and $2 \cdot 3 + 2 \cdot 4 = 0 + 2 = 2$.

Thus, $\mathbb{Z}_6$ satisfies all the conditions of a commutative ring.

Definition 21 (Zero-Divisor)**.** (cf.[3, 112]) In a commutative ring R, an element $a \in R$ is called a *zero-divisor* if there exists a non-zero element $b \in R$ such that $a \cdot b = 0$.

Example 22 (Zero-Divisor)**.** Consider the commutative ring $\mathbb{Z}_6$ (the integers modulo 6). The elements of this ring are $\{0, 1, 2, 3, 4, 5\}$, and addition and multiplication are performed modulo 6.

In this ring, the element 2 is a zero-divisor because:

$$2 \times 3 = 6 \equiv 0 \pmod{6}.$$

Here, $2 \neq 0$ and $3 \neq 0$, yet their product is 0. Therefore, 2 is a zero-divisor in $\mathbb{Z}_6$. Similarly, 3 is also a zero-divisor, as:

$$3 \times 2 = 6 \equiv 0 \pmod{6}.$$

Definition 23 (Zero-divisor Graph)**.** (cf.[17, 94, 19, 2]) Let R be a commutative ring with unity, and let $Z(R)$ denote the set of zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is an undirected graph defined as follows:

- The vertex set of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, i.e., the set of nonzero zero-divisors of R.
- For distinct $x, y \in Z(R)^*$, there is an edge between x and y if and only if $xy = 0$ in R.

Thus, the graph $\Gamma(R)$ captures the relationships between the nonzero zero-divisors of R. If R is an integral domain, $\Gamma(R)$ is the empty graph.

Example 24 (Zero-divisor Graph)**.** Consider the commutative ring $\mathbb{Z}_6$ (the integers modulo 6). The elements of this ring are $\{0, 1, 2, 3, 4, 5\}$, and the set of zero-divisors is $Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$.

To construct the zero-divisor graph $\Gamma(\mathbb{Z}_6)$, we first remove the element 0, so the vertex set of $\Gamma(\mathbb{Z}_6)$ is $\{2, 3, 4\}$.

Next, we determine the edges:

- $2 \times 3 = 6 \equiv 0 \pmod 6$, so there is an edge between 2 and 3.
- $2 \times 4 = 8 \equiv 0 \pmod 6$, so there is an edge between 2 and 4.
- $3 \times 4 = 12 \equiv 0 \pmod 6$, so there is an edge between 3 and 4.

Therefore, the zero-divisor graph $\Gamma(\mathbb{Z}_6)$ has vertices $\{2, 3, 4\}$, and it forms a complete graph K_3, where every pair of distinct vertices is connected by an edge.

2.3 Fuzzy graph and Intuitionistic fuzzy graph

A Fuzzy Graph represents relationships under uncertainty by assigning membership degrees to both vertices and edges, enabling more flexible and detailed analysis. Due to its importance, Fuzzy Graphs have been widely studied in various fields [13, 8, 15, 14, 12, 160, 113, 104, 7, 83, 30, 93, 155, 108, 32]. The formal definition of a Fuzzy Graph is given in [105, 131]. This concept extends the set-theoretic ideas of Fuzzy Sets [166] and Intuitionistic Fuzzy Sets [27] into graph theory.

Definition 25 (Crisp Graph)**.** A *crisp graph* is a mathematical structure $G = (V, E)$, where:

(1) V is a non-empty finite or infinite set, referred to as the set of vertices or nodes.

(2) $E \subseteq V \times V$ is a set of edges, representing relationships between pairs of vertices.

(3) For any edge $e = (u, v) \in E$, $u, v \in V$, and there is no uncertainty in the membership of u, v in V or e in E.

Depending on the nature of the edges:

- In an *undirected graph*, if $(u, v) \in E$, then $(v, u) \in E$.
- In a *directed graph*, edges have a specific direction; if $(u, v) \in E$, it does not necessarily imply $(v, u) \in E$.

Definition 26. [131] A fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ, such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$
- $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$

A fuzzy subgraph $H = (\sigma', \mu')$ of G is defined as follows:

- There exists $X \subseteq V$ such that $\sigma' : X \to [0, 1]$ is a fuzzy subset.
- $\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ', satisfying $\mu'(x, y) \leq \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 27. (cf.[45]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$.

The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this case, the fuzzy graph G has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees.
- The fuzzy relations ensure that $\mu(x, y)$ for any edge (x, y) does not exceed the minimum membership of the corresponding vertices.

Definition 28. [62] A fuzzy graph $G = (\sigma, \mu)$ is called *complete* if for all $u, v \in V$, the following condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Proposition 29. *A complete fuzzy graph is a special case of a fuzzy graph.*

Proof: This is evident. □

The intuitionistic fuzzy graph, which generalizes the fuzzy graph, is also defined in a similar manner[5, 119, 48, 164, 118]. The definition is provided below.

Definition 30. [5] An *intuitionistic fuzzy graph* $G = (A, B)$ on an underlying set V is defined as follows:

(i) The functions $\mu_A : V \to [0, 1]$ and $\nu_A : V \to [0, 1]$ represent the degree of membership and non-membership of each vertex $x \in V$, respectively. These functions satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in V.$$

(ii) The functions $\mu_B : E \subseteq V \times V \to [0, 1]$ and $\nu_B : E \subseteq V \times V \to [0, 1]$ represent the degree of membership and non-membership of each edge $\{x, y\} \in E$, respectively. These functions satisfy the following conditions for all $\{x, y\} \in E$:

$$\mu_B(\{x, y\}) \leq \min(\mu_A(x), \mu_A(y)),$$

$$\nu_B(\{x, y\}) \geq \max(\nu_A(x), \nu_A(y)),$$

$$0 \leq \mu_B(\{x, y\}) + \nu_B(\{x, y\}) \leq 1.$$

Here:

- A is the *intuitionistic fuzzy vertex set* of G, and
- B is the *intuitionistic fuzzy edge set* of G, which represents a symmetric intuitionistic fuzzy relation on A.

The intuitionistic fuzzy graph $G = (A, B)$ corresponds to the crisp graph $G^* = (V, E)$ if the following conditions hold for all $\{x, y\} \in E$:

$$\mu_B(\{x, y\}) \leq \min(\mu_A(x), \mu_A(y)), \quad \nu_B(\{x, y\}) \geq \max(\nu_A(x), \nu_A(y)).$$

2.4 Neutrosophic Graph and Intuitionistic Neutrosophic Graph

We introduce the concept of a neutrosophic graph [9, 135, 77, 70, 44, 150], which extends the framework of fuzzy graphs [83].

A neutrosophic graph can also be viewed as a graphical representation of a neutrosophic set. Therefore, we begin by providing the definition of a neutrosophic set below.

Definition 31 (Neutrosophic Set)**.** (cf.[145, 136, 110, 142, 151, 145, 43]) Let X be a space of points and let $x \in X$. A *neutrosophic set* S in X is characterized by three membership functions: a truth membership function T_S, an indeterminacy membership function I_S, and a falsity membership function F_S. For each point $x \in X$, $T_S(x), I_S(x)$, and $F_S(x)$ are real standard or non-standard subsets of the interval $]0^-, 1^+[$, where:

$$T_S, I_S, F_S : X \to [0^-, 1^+].$$

The neutrosophic set S can be represented as:

$$S = \{(x, T_S(x), I_S(x), F_S(x)) \mid x \in X\}.$$

There are no restrictions on the sum of $T_S(x), I_S(x)$, and $F_S(x)$, so:

$$0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3^+.$$

Definition 32 (Single Valued Neutrosophic Set)**.** Let X be a universal set, and let $x \in X$ be an element of X. A *Single Valued Neutrosophic Set (SVNS)* A on X is characterized by three functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where:

- $T_A(x)$ represents the truth-membership degree of x in A,
- $I_A(x)$ represents the indeterminacy-membership degree of x in A,
- $F_A(x)$ represents the falsity-membership degree of x in A.

These functions satisfy:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \quad \text{for all } x \in X.$$

When X is continuous, A can be expressed as:

$$A = \{(T_A(x), I_A(x), F_A(x)) \mid x \in X\}.$$

When X is discrete, A can be expressed as:

$$A = \left\{(T_A(x_i), I_A(x_i), F_A(x_i))/x_i \mid x_i \in X\right\}.$$

Below are three real-world scenarios where neutrosophic sets can be applied.

Example 33. Let X be the set of possible conditions a patient might have, and $x \in X$ represent the condition "diabetes." A neutrosophic set S is used to characterize the diagnosis:

- $T_S(x) = 0.8$: The truth degree, derived from positive glucose tolerance test results and family history.
- $I_S(x) = 0.15$: The indeterminacy degree, reflecting uncertainties due to borderline HbA1c levels and inconsistent symptoms.
- $F_S(x) = 0.05$: The falsity degree, representing evidence against diabetes, such as normal fasting glucose levels.

Thus, the neutrosophic representation of the diagnosis is:

$$S = \{(x, 0.8, 0.15, 0.05) \mid x \in X\}.$$

Example 34. Consider an environmental monitoring system where X represents regions, and $x \in X$ is "Region A." The neutrosophic set S evaluates the pollution status of x:

- $T_S(x) = 0.6$: The truth degree that the area is polluted, based on sensor data showing moderate PM2.5 levels.
- $I_S(x) = 0.3$: The indeterminacy degree due to missing data from certain sensors and conflicting readings.
- $F_S(x) = 0.1$: The falsity degree, based on visual inspections showing clear skies and absence of visible pollution.

The neutrosophic representation of the pollution status is:

$$S = \{(x, 0.6, 0.3, 0.1) \mid x \in X\}.$$

Example 35. Let X represent products on an e-commerce platform, and $x \in X$ denote a specific product, "Smartphone A." The neutrosophic set S evaluates customer satisfaction:

- $T_S(x) = 0.7$: The truth degree that customers are satisfied with the product, based on 70% positive reviews.
- $I_S(x) = 0.2$: The indeterminacy degree due to mixed reviews where customers were neutral about features such as battery life.
- $F_S(x) = 0.1$: The falsity degree, reflecting dissatisfaction based on complaints about delivery delays and defective units.

The neutrosophic representation of customer satisfaction is:

$$S = \{(x, 0.7, 0.2, 0.1) \mid x \in X\}.$$

Next, the definition of a neutrosophic graph is provided below.

Definition 36. [150] A *neutrosophic graph* $NTG = (V, E, \sigma, \mu)$ is defined as follows:

- V is the set of vertices.
- E is the set of edges, where $E \subseteq V \times V$.
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a tuple of vertex membership functions:
$$\sigma_i : V \to [0, 1], \quad i = 1, 2, 3,$$
where:
  - $\sigma_1(v)$: Truth degree of vertex v,
  - $\sigma_2(v)$: Indeterminacy degree of vertex v,
  - $\sigma_3(v)$: Falsity degree of vertex v.
- $\mu = (\mu_1, \mu_2, \mu_3)$ is a tuple of edge membership functions:
$$\mu_i : E \to [0, 1], \quad i = 1, 2, 3,$$
where:
  - $\mu_1(e)$: Truth degree of edge e,
  - $\mu_2(e)$: Indeterminacy degree of edge e,
  - $\mu_3(e)$: Falsity degree of edge e.

The following condition must hold for each edge $e = \{v_i, v_j\} \in E$:

$$\mu(e) \leq \sigma(v_i) \wedge \sigma(v_j),$$

where $\wedge$ denotes the minimum operation.

Additionally, the following terminology is used:

(1) σ is referred to as the *neutrosophic vertex set*.

(2) μ is referred to as the *neutrosophic edge set*.

(3) $|V|$ is called the *order* of NTG, denoted by $O(NTG)$.

(4) The sum of all vertex membership values, $\sum_{v \in V} \sigma(v)$, is called the *neutrosophic order* of NTG, denoted by $On(NTG)$.

(5) $|E|$ is called the *size* of NTG, denoted by $S(NTG)$.

(6) The sum of all edge membership values, $\sum_{e\in E}\mu(e)$, is called the *neutrosophic size* of NTG, denoted by $Sn(NTG)$.

The Examples of neutrosophic graph is following.

Example 37. (cf.[45]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_iv_j \in E$, $\mu(v_iv_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

Similarly to Fuzzy Graphs, an Intuitionistic Neutrosophic Graph has been defined as a generalization of Neutrosophic Graphs [6, 10, 78]. The definition is provided as follows.

Definition 38 (Intuitionistic Neutrosophic Graph)**.** (cf.[6, 10, 78]) An *intuitionistic neutrosophic graph* $G = (\eta, \rho)$ is defined as follows:

- Let V be the set of vertices.
- For each vertex $u \in V$, the intuitionistic neutrosophic membership functions $T(u), I(u), F(u)$ represent the truth, indeterminacy, and falsity memberships, respectively, where $0 \leq T(u) + I(u) + F(u) \leq 2$ and the following conditions hold:
$$\min\{T(u), I(u)\} \leq 0.5, \quad \min\{F(u), I(u)\} \leq 0.5, \quad \min\{T(u), F(u)\} \leq 0.5.$$
- Let $E \subseteq V \times V$ be the set of edges.
- For each edge $(u, v) \in E$, the intuitionistic neutrosophic membership functions $T(u, v), I(u, v), F(u, v)$ represent the truth, indeterminacy, and falsity memberships of the edge, subject to the following conditions:
$$T(u, v) \leq T(u) \wedge T(v), \quad I(u, v) \leq I(u) \wedge I(v), \quad F(u, v) \leq F(u) \vee F(v),$$
where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.
- Additionally, the following conditions must be satisfied for all edges $(u, v) \in E$:
$$T(u, v) \wedge I(u, v) \leq 0.5, \quad T(u, v) \wedge F(u, v) \leq 0.5, \quad I(u, v) \wedge F(u, v) \leq 0.5,$$
and
$$0 \leq T(u, v) + I(u, v) + F(u, v) \leq 2.$$

This definition describes a graph structure where both the vertices and edges are characterized by their intuitionistic neutrosophic truth, indeterminacy, and falsity memberships, ensuring balanced contributions of these components to the overall uncertainty in the graph.

Example 39. (cf.[6, 10, 78]) Consider a graph $G = (V, E, \eta, \rho)$ where the vertex set $V = \{V_1, V_2, V_3\}$ and edge set $E = \{(V_1, V_2), (V_2, V_3)\}$.

- For each vertex $V_i \in V$, the intuitionistic neutrosophic membership functions for truth $T(u)$, indeterminacy $I(u)$, and falsity $F(u)$ are as follows:

$$T(V_1) = 0.2, \quad I(V_1) = 0.2, \quad F(V_1) = 0.3,$$
$$T(V_2) = 0.3, \quad I(V_2) = 0.3, \quad F(V_2) = 0.4,$$
$$T(V_3) = 0.5, \quad I(V_3) = 0.4, \quad F(V_3) = 0.5.$$

- For the edges $(V_i, V_j) \in E$, the intuitionistic neutrosophic membership functions $T(u, v), I(u, v), F(u, v)$ represent the truth, indeterminacy, and falsity memberships of the edge:

$$T(V_1, V_2) = 0.2, \quad I(V_1, V_2) = 0.2, \quad F(V_1, V_2) = 0.4,$$
$$T(V_2, V_3) = 0.3, \quad I(V_2, V_3) = 0.3, \quad F(V_2, V_3) = 0.5.$$

The conditions outlined in the definition of an intuitionistic neutrosophic graph are satisfied for all vertices and edges. Specifically:

- For each vertex V_i, we ensure that $T(V_i) + I(V_i) + F(V_i) \leq 2$.
- For each edge $(V_i, V_j) \in E$, the relationships between the intuitionistic neutrosophic membership functions of vertices and edges hold, as described by the following conditions:

$$T(V_i, V_j) \leq T(V_i) \wedge T(V_j), \quad I(V_i, V_j) \leq I(V_i) \wedge I(V_j), \quad F(V_i, V_j) \leq F(V_i) \vee F(V_j),$$

  where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

This graph structure provides an example of how the intuitionistic neutrosophic framework balances truth, indeterminacy, and falsity at both the vertex and edge levels, following the rules set out in the definition.

Theorem 40. *An intuitionistic neutrosophic graph generalizes a neutrosophic graph.*

Proof: To prove that an intuitionistic neutrosophic graph generalizes a neutrosophic graph, we establish the following relationship between their definitions.

A neutrosophic graph $NTG = (V, E, \sigma, \mu)$ is defined such that:

- For each vertex $u \in V$, the membership functions $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ represent the truth (σ_1), indeterminacy (σ_2), and falsity (σ_3) memberships, satisfying:

$$0 \leq \sigma_1(u) + \sigma_2(u) + \sigma_3(u) \leq 1.$$

- For each edge $(u, v) \in E$, the edge memberships $\mu = (\mu_1, \mu_2, \mu_3)$ satisfy:

$$\mu_i(u, v) \leq \sigma_i(u) \wedge \sigma_i(v), \quad \forall i = 1, 2, 3.$$

An intuitionistic neutrosophic graph $ING = (\eta, \rho)$ is defined such that:

- For each vertex $u \in V$, the membership functions $\eta = (T, I, F)$ represent truth (T), indeterminacy (I), and falsity (F), satisfying:

$$0 \leq T(u) + I(u) + F(u) \leq 2.$$

- Additional conditions ensure pairwise contributions to uncertainty:

$$\min\{T(u), I(u)\} \leq 0.5, \quad \min\{I(u), F(u)\} \leq 0.5, \quad \min\{T(u), F(u)\} \leq 0.5.$$

- For each edge $(u, v) \in E$, the edge memberships $\rho = (T, I, F)$ satisfy:

$$T(u, v) \leq T(u) \wedge T(v), \quad I(u, v) \leq I(u) \wedge I(v), \quad F(u, v) \leq F(u) \vee F(v),$$

  and:

$$0 \leq T(u, v) + I(u, v) + F(u, v) \leq 2.$$

If we constrain the intuitionistic neutrosophic graph ING by setting:

$$T(u) + I(u) + F(u) \leq 1, \quad \text{and} \quad T(u,v) + I(u,v) + F(u,v) \leq 1,$$

then ING reduces to a neutrosophic graph NTG. This is because:

- The sum of the memberships in NTG is bounded by 1, which is a stricter condition than the bound of 2 in ING.
- The relationships $\mu_i(u,v) \leq \sigma_i(u) \wedge \sigma_i(v)$ in NTG are equivalent to the edge conditions in ING when the sum of memberships is constrained to 1.

Thus, every neutrosophic graph is a special case of an intuitionistic neutrosophic graph, where the total memberships are further restricted. Therefore, an intuitionistic neutrosophic graph generalizes a neutrosophic graph. □

Theorem 41. *An intuitionistic neutrosophic graph generalizes an intuitionistic fuzzy graph.*

Proof: To obtain an intuitionistic fuzzy graph from an intuitionistic neutrosophic graph $G = (\eta, \rho)$, set:

$$T(u) = \mu_A(u), \quad F(u) = \nu_A(u), \quad I(u) = 0 \quad \forall u \in V,$$

and:

$$T(u,v) = \mu_B(u,v), \quad F(u,v) = \nu_B(u,v), \quad I(u,v) = 0 \quad \forall (u,v) \in E.$$

Under this reduction:

- For vertices:
$$0 \leq T(u) + I(u) + F(u) = \mu_A(u) + \nu_A(u) \leq 1.$$
- For edges:
$$T(u,v) = \mu_B(u,v) \leq \min(T(u), T(v)) = \min(\mu_A(u), \mu_A(v)),$$
$$F(u,v) = \nu_B(u,v) \geq \max(F(u), F(v)) = \max(\nu_A(u), \nu_A(v)).$$
- Indeterminacy ($I(u)$ and $I(u,v)$) is set to zero, as it is not considered in the intuitionistic fuzzy graph.

An intuitionistic fuzzy graph can be obtained as a special case of an intuitionistic neutrosophic graph by setting the indeterminacy memberships to zero ($I(u) = I(u,v) = 0$). Therefore, the intuitionistic neutrosophic graph is a generalization of the intuitionistic fuzzy graph. □

An example of operations in a Neutrosophic Graph is provided below. Since it is difficult to list all operations, please refer to the relevant references as needed.

Definition 42 (Complement of a Neutrosophic Graph)**.** Let $G = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a Neutrosophic graph, where σ represents the neutrosophic membership functions of the vertices, and μ represents the neutrosophic membership functions of the edges. The *complement* of G, denoted as $\overline{G} = (V, E', \sigma, \mu')$, is a neutrosophic graph defined as follows:

- The vertex set remains the same: $V(\overline{G}) = V(G)$.
- The edge set E' is the complement of the original edge set E, meaning $E' = \{(u,v) \mid (u,v) \notin E\}$.
- The neutrosophic membership functions for edges in the complement graph μ' are defined as:
$$\mu_1'(u,v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u,v),$$
$$\mu_2'(u,v) = \left|\sigma_2(u) \vee \sigma_2(v) - \mu_2(u,v)\right|,$$
$$\mu_3'(u,v) = \left|\sigma_3(u) \vee \sigma_3(v) - \mu_3(u,v)\right|,$$
for all $u, v \in V$, where $\wedge$ denotes the minimum operation, $\vee$ denotes the maximum operation, and the absolute value ensures non-negative membership values. The functions $\mu_1(u,v), \mu_2(u,v), \mu_3(u,v)$ represent the truth, indeterminacy, and falsity memberships of the edge in the original graph.

Thus, the complement graph $\overline{G}$ adjusts the truth, indeterminacy, and falsity memberships based on the complement of the original edge relations.

Definition 43 (μ-Complement of a Neutrosophic Graph)**.** Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic graph. The *μ-complement* of G, denoted by $G^\mu = (V, E, \sigma, \mu^\mu)$, is a neutrosophic graph where the neutrosophic edge membership functions μ^μ are defined as follows:

- If $\mu_1(u, v) = 0$, then:
$$\mu_1^\mu(u, v) = 0,$$
meaning there is no truth membership for non-edges.

- If $\mu_1(u, v) > 0$, then:
$$\mu_1^\mu(u, v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u, v),$$
where $\mu_1^\mu(u, v)$ is the complement of the truth membership for edges that exist in the original graph.

- The indeterminacy and falsity memberships for the μ-complement are defined as:
$$\mu_2^\mu(u, v) = |\sigma_2(u) \vee \sigma_2(v) - \mu_2(u, v)|, \quad \mu_3^\mu(u, v) = |\sigma_3(u) \vee \sigma_3(v) - \mu_3(u, v)|,$$
meaning the complement operation adjusts the indeterminacy and falsity memberships of the edges based on the maximum operation, with absolute values ensuring non-negative results.

This definition adjusts the truth, indeterminacy, and falsity memberships based on the complement of the original edge weights.

Theorem 44. *The complement of a Neutrosophic Graph G is itself a Neutrosophic Graph.*

Proof: Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Graph, and let $\overline{G} = (V, E', \sigma, \mu')$ be its complement as defined above. To show that $\overline{G}$ is a Neutrosophic Graph, we verify the properties of neutrosophic membership functions:

1. **Vertex membership functions:** The vertex membership functions $\sigma_1, \sigma_2, \sigma_3$ remain unchanged, satisfying:
$$0 \leq \sigma_1(u) + \sigma_2(u) + \sigma_3(u) \leq 3 \quad \text{for all } u \in V.$$

2. **Edge membership functions:** For μ_1', μ_2', μ_3', we verify that:
$$0 \leq \mu_1'(u, v) + \mu_2'(u, v) + \mu_3'(u, v) \leq 3,$$

since:
$$\mu_1'(u, v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u, v),$$
$$\mu_2'(u, v) = |\sigma_2(u) \vee \sigma_2(v) - \mu_2(u, v)|,$$
$$\mu_3'(u, v) = |\sigma_3(u) \vee \sigma_3(v) - \mu_3(u, v)|.$$

Each component $\mu_i'(u, v)$ satisfies the required bounds because $\mu_1, \mu_2, \mu_3 \in [0, 1]$, and the operations $\wedge, \vee$, subtraction, and absolute value do not violate the constraints.

Hence, $\overline{G}$ is a valid Neutrosophic Graph. □

2.5 Smart Fuzzy Graph

A Smart Fuzzy Graph models real-world systems with uncertain relationships, utilizing fuzzy sets, and is widely applied in IoT and connectivity problems [25, 26] [1]. Related graph classes include the Regular Smart Fuzzy Graph and the Totally Regular Smart Fuzzy Graph [25, 26]. The definitions and examples are presented as follows.

Definition 45. [25, 26] A *Smart Fuzzy Graph* $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on σ, such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$, where $\wedge$ denotes the minimum operation.

[1] The Internet of Things (IoT) connects devices, sensors, and systems, enabling data exchange and automation for enhanced efficiency and decision-making(cf.[102]).

The Smart Fuzzy Graph must satisfy the following conditions:

- If $u \neq v$, then:
$$\sum_{u,v \in V} \mu(u,v) \leq \sum_{u,v \in V} \sigma(u) \wedge \sigma(v) \leq 1.$$
- If $u = v$, then:
$$\sum_{u \in V} \mu(u,u) = \sum_{u \in V} \sigma(u) \wedge \sigma(u) = 0.$$

Example 46. [25, 26] Consider a Smart Fuzzy Graph with 5 vertices $V = \{V_1, V_2, V_3, V_4, V_5\}$, where the membership degrees of the vertices are as follows:

$$\sigma(V_1) = 0.8, \quad \sigma(V_2) = 0.6, \quad \sigma(V_3) = 0.5, \quad \sigma(V_4) = 0.9, \quad \sigma(V_5) = 0.5$$

The edges are defined with the following membership values:

	$V_1(0.8)$	$V_2(0.6)$	$V_3(0.5)$	$V_4(0.9)$	$V_5(0.5)$
$V_1(0.8)$	0	0.09	0.11	0.07	0.13
$V_2(0.6)$	0.09	0	0.2	0	0.1
$V_3(0.5)$	0.11	0.2	0	0.1	0
$V_4(0.9)$	0.07	0	0.1	0	0.1
$V_5(0.5)$	0.13	0.1	0	0.1	0

For this Smart Fuzzy Graph, the condition $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ must hold for all pairs of vertices. For example:

- For $V_1(0.8)$ and $V_5(0.5)$, we have $\mu(V_1, V_5) = 0.13$ and $\sigma(V_1) \wedge \sigma(V_5) = 0.5$, so the condition $0.13 \leq 0.5$ is satisfied.
- For $V_2(0.6)$ and $V_3(0.5)$, we have $\mu(V_2, V_3) = 0.2$ and $\sigma(V_2) \wedge \sigma(V_3) = 0.5$, so the condition $0.2 \leq 0.5$ is satisfied.

Thus, the graph satisfies the conditions of a Smart Fuzzy Graph, where the strength of the fuzzy relations between vertices is constrained by their membership degrees.

2.6 Fuzzy zero divisor graph

A fuzzy zero divisor graph is a fuzzy graph where vertices represent nonzero zero-divisors of a ring, and edges exist if their product equals zero[88, 87]. The definitions and examples are presented as follows.

Notation 47. (cf.[87]) Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is defined as follows:

- The vertex set of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero zero-divisors of R.
- Two distinct vertices $x, y \in Z(R)^*$ are connected by an edge if and only if their product in R is zero, i.e., $xy = 0$.

In this way, $\Gamma(R)$ encodes the relationships between the nonzero zero-divisors of R. If R is an integral domain (i.e., R has no nonzero zero-divisors), the graph $\Gamma(R)$ is empty, as $Z(R)^* = \emptyset$.

Definition 48. [87] A *fuzzy zero divisor graph* $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$ is defined as follows:

- V is a non-empty set of vertices, representing the elements of a commutative ring R with 1.
- $\sigma : V \to (0, 1]$ is a fuzzy membership function that assigns a membership degree $\sigma(v)$ to each vertex $v \in V$, where $\sigma(v)$ reflects the relevance or strength of the zero divisor element v.

- $\mu : V \times V \to (0, 1]$ is a fuzzy relation on σ, defined as:

$$\mu(v_i, v_j) = \frac{\sigma(v_i) \cdot \sigma(v_j)}{\sigma(v_i) + \sigma(v_j)}, \quad \forall v_i, v_j \in V.$$

  The relation $\mu(v_i, v_j)$ represents the degree of adjacency between two vertices v_i and v_j, which is influenced by their fuzzy membership degrees.

The fuzzy zero divisor graph represents the relationships between the zero-divisor elements of a commutative ring R. Each vertex corresponds to a nonzero zero divisor of R, and two distinct vertices v_i and v_j are adjacent if their product is zero, i.e., $v_i v_j = 0$.

Theorem 49. *A Fuzzy Zero Divisor Graph generalizes both a Zero-Divisor Graph and a Fuzzy Graph.*

Proof: Let $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$ be a fuzzy zero divisor graph of a commutative ring R.

If we ignore the fuzzy membership function σ and the fuzzy relation μ, the vertex set V corresponds to $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R. An edge exists between two vertices $v_i, v_j \in V$ if and only if $v_i v_j = 0$, matching the definition of a zero-divisor graph $\Gamma(R)$. Thus, Γ_{fuzzy} reduces to $\Gamma(R)$, the zero-divisor graph, by removing fuzzy characteristics.

A fuzzy graph $G = (V, \sigma, \mu)$ has:

- A vertex set V with fuzzy membership $\sigma : V \to (0, 1]$,
- A fuzzy relation $\mu : V \times V \to (0, 1]$.

For $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$, the membership function σ assigns a fuzzy degree to each vertex $v \in V$, and the adjacency relation $\mu(v_i, v_j)$ is defined as:

$$\mu(v_i, v_j) = \frac{\sigma(v_i) \cdot \sigma(v_j)}{\sigma(v_i) + \sigma(v_j)}.$$

This satisfies the requirements of a fuzzy graph structure. Thus, Γ_{fuzzy} reduces to a fuzzy graph by considering the fuzzy memberships and adjacency relations without enforcing the ring-theoretic zero-divisor constraints.

A Fuzzy Zero Divisor Graph Γ_{fuzzy} generalizes the Zero-Divisor Graph by incorporating fuzzy memberships for vertices and edges, and it generalizes a Fuzzy Graph by embedding the algebraic properties of zero-divisors within the fuzzy structure. □

2.7 Fuzzy semigraph

A fuzzy semigraph is a fuzzy graph that generalizes semigraphs, combining fuzzy vertices and fuzzy edges, often applied in network systems like roads or telecommunications[24, 127, 106, 115]. The definitions are presented as follows.

Definition 50. [126] A *fuzzy semigraph* $G = (V, X, \sigma, \mu, \eta)$ is defined as follows:

- V is a non-empty set of vertices.
- X is a set of edges, where each edge is an n-tuple of distinct vertices from V, i.e., $e = (v_1, v_2, \ldots, v_n)$, with $n \geq 2$.
- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(v)$ represents the membership degree of vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on the vertices, where $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.
- $\eta : X \to [0, 1]$ is a fuzzy subset of edges, where $\eta(e)$ is the membership degree of the edge $e = (v_1, v_2, \ldots, v_n)$ and satisfies:

$$\eta(e) \leq \mu(v_1, v_2) \wedge \mu(v_2, v_3) \wedge \cdots \wedge \mu(v_{n-1}, v_n) \wedge \sigma(v_1) \wedge \sigma(v_n).$$

In this fuzzy semigraph, the vertices v_1 and v_n are called the *end vertices*, while the vertices $v_2, v_3, \dots, v_{n-1}$ are called the *middle vertices*. If a middle vertex is also an end vertex of another edge, it is called a *middle-end vertex*.

Definition 51. [126] A *fuzzy subsemigraph* $H = (\gamma, \rho, \delta)$ of a fuzzy semigraph $G = (\sigma, \mu, \eta)$ is defined as follows:

- All edges of H are subedges of G.
- For every vertex $u \in V$, the membership degree of u in H is less than or equal to its membership degree in G, i.e., $\gamma(u) \leq \sigma(u)$.
- For every pair of vertices $(u, v) \in V \times V$, the fuzzy relation $\rho(u, v)$ in H is less than or equal to the fuzzy relation $\mu(u, v)$ in G, i.e., $\rho(u, v) \leq \mu(u, v)$.
- For every edge $e \in X$, the fuzzy membership degree $\delta(e)$ in H is less than or equal to the fuzzy membership degree $\eta(e)$ in G, i.e., $\delta(e) \leq \eta(e)$.

Definition 52. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph on the vertex set V and edge set X. The *End Vertex Fuzzy Graph* (e-Fuzzy Graph), denoted as $G_e = (\sigma_e, \eta_e)$, is defined as follows:

- The vertex set is V, where $\sigma_e(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_e if and only if they are end vertices of an edge in G, with $\eta_e(uv) = \eta(uv)$ for every pair of end vertices u and v.

Definition 53. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph. The *Adjacency Fuzzy Graph* (a-Fuzzy Graph), denoted as $G_a = (\sigma_a, \eta_a)$, is defined as follows:

- The vertex set is V, where $\sigma_a(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_a if they are adjacent in G. The adjacency membership function is given by:
$$\eta_a(uv) = \mu(uv_1) \wedge \mu(v_1v_2) \wedge \dots \wedge \mu(v_{k-1}v_k),$$
for every pair of adjacent vertices u, v where $(u, v_1, \dots, v_k)$ is an edge or partial edge of G.

Definition 54. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph. The *Consecutive Adjacency Fuzzy Graph* (ca-Fuzzy Graph), denoted as $G_{ca} = (\sigma_{ca}, \mu_{ca})$, is defined as follows:

- The vertex set is V, where $\sigma_{ca}(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_{ca} if and only if they are consecutively adjacent in G. The adjacency membership function is given by $\mu_{ca}(uv) = \mu(uv)$ for every pair of consecutively adjacent vertices u and v.

2.8 Double Layered Fuzzy Graph and Triple Layered Fuzzy Graph

A Layered Fuzzy Graph is an extension of fuzzy graphs with multiple layers, where each layer represents distinct fuzzy relations or membership degrees between vertices and edges. The Double Layered Fuzzy Graph [123, 133, 124] and Triple Layered Fuzzy Graph [98, 63, 134] are defined accordingly. Related graph classes include the Intuitionistic Double Layered Fuzzy Graph [132], Balanced Double Layered Bipolar Fuzzy Graph [122, 128], and Complete Double Layered Fuzzy Graph [16]. Additionally, generalized forms such as the K-Partitioned Fuzzy Graph [121, 120] and the Quadruple Layered Fuzzy Graph [81] have also been introduced.

Definition 55. [123, 133, 124] A *Double Layered Fuzzy Graph (DLFG)* is a fuzzy graph $G = (\sigma, \mu)$ with an underlying crisp graph $G^* = (\sigma^*, \mu^*)$. The *Double Layered Fuzzy Graph* is denoted as $DL(G) = (\sigma_{DL}, \mu_{DL})$, and it is defined as follows:

The **node set** of $DL(G)$ is the union $\sigma^* \cup \mu^*$, where σ^* is the set of vertices and μ^* is the set of edges from the original fuzzy graph.

The **fuzzy subset** σ_{DL} is defined as:

$$\sigma_{DL}(u) = \begin{cases} \sigma(u) & \text{if } u \in \sigma^*, \\ \mu(uv) & \text{if } uv \in \mu^*. \end{cases}$$

This definition assigns the membership degree for both vertices and edges within the fuzzy subset.

The **fuzzy relation** μ_{DL} on $\sigma^* \cup \mu^*$ is defined as:

$$\mu_{DL}(u,v) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u, v \in \sigma^*, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in \mu^*, \text{ and they share a common vertex}, \\ \sigma(u) \wedge \mu(e) & \text{if } u \in \sigma^*, e \in \mu^*, \text{ and } u \text{ is incident to } e, \\ 0 & \text{otherwise}. \end{cases}$$

Here, μ_{DL} is a fuzzy relation that defines the interaction between nodes and edges based on their membership degrees. For any $u, v \in \sigma^* \cup \mu^*$, the relation satisfies:

$$\mu_{DL}(u,v) \leq \sigma_{DL}(u) \wedge \sigma_{DL}(v).$$

Thus, the pair $DL(G) = (\sigma_{DL}, \mu_{DL})$ is referred to as the *Double Layered Fuzzy Graph*.

Theorem 56. *A Double Layered Fuzzy Graph (DLFG) generalizes a Fuzzy Graph.*

Proof: If we constrain $\sigma_{DL}(u)$ to only represent vertex memberships and ignore edge memberships, the definitions of σ_{DL} and μ_{DL} reduce to the fuzzy subset σ and fuzzy relation μ of a Fuzzy Graph. Under this condition:

$$\sigma(u) = \sigma_{DL}(u), \quad \mu(u,v) = \mu_{DL}(u,v) \quad \forall u, v \in \sigma^*.$$

Thus, DLFG generalizes a Fuzzy Graph by incorporating edge memberships as part of the fuzzy subset. □

Definition 57. [98, 63, 134] A *Triple Layered Fuzzy Graph (TLFG)* is a fuzzy graph $G = (\sigma, \mu)$ defined with the following properties. Let $G^* = (\sigma^*, \mu^*)$ represent the underlying crisp graph of G.

The **node set** of the Triple Layered Fuzzy Graph $TL(G)$ is the union of the sets $\sigma^* \cup \mu^*$, where σ^* denotes the fuzzy subset of nodes and μ^* denotes the fuzzy subset of edges.

The fuzzy subset σ_{TL} is defined as:

$$\sigma_{TL}(u) = \begin{cases} \sigma(u) & \text{if } u \in \sigma^*, \\ 2\mu(uv) & \text{if } uv \in \mu^*. \end{cases}$$

This represents the membership of vertices and edges in the fuzzy subset.

The fuzzy relation μ_{TL} on $\sigma^* \cup \mu^*$ is defined as follows:

$$\mu_{TL}(u,v) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u, v \in \sigma^*, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in \mu^*, \text{and they share a common node}, \\ \sigma(u) \wedge \mu(e) & \text{if } u \in \sigma^*, e \in \mu^*, \text{and } u \text{ is incident to } e, \\ 0 & \text{otherwise}. \end{cases}$$

Here, σ_{TL} is a fuzzy subset of nodes and edges, and μ_{TL} is a fuzzy relation that satisfies:

$$\mu_{TL}(u,v) \leq \sigma_{TL}(u) \wedge \sigma_{TL}(v) \quad \text{for all } u, v \in \sigma^* \cup \mu^*.$$

Thus, the pair $TL(G) = (\sigma_{TL}, \mu_{TL})$ is defined as the *Triple Layered Fuzzy Graph*.

Theorem 58. *A Triple Layered Fuzzy Graph (TLFG) generalizes a Double Layered Fuzzy Graph (DLFG).*

Proof: If we constrain σ_{TL} such that $\sigma_{TL}(uv) = \mu(uv)$ instead of $2\mu(uv)$, the definitions of σ_{TL} and μ_{TL} align with σ_{DL} and μ_{DL}, respectively. Under this constraint:

$$\sigma_{DL}(u) = \sigma_{TL}(u), \quad \mu_{DL}(u,v) = \mu_{TL}(u,v) \quad \forall u, v \in \sigma^* \cup \mu^*.$$

Thus, TLFG generalizes DLFG by allowing flexibility in the membership degree assignments for edges. □

2.9 Weak fuzzy graph

A Weak Fuzzy Graph is defined as a fuzzy graph where the membership degree of each edge is strictly less than the minimum of the membership degrees of its connected vertices [72, 114, 125]. A related fuzzy graph class is the General Fuzzy Graph, which is also well-known [113]. The definition is provided as follows[125].

Definition 59. [125] A *weak fuzzy graph* $F = (\sigma, \mu)$ is defined as follows:

- Let V be the set of vertices.
- $\sigma : V \to [0,1]$ represents the membership degree of each vertex $v \in V$.
- $\mu : V \times V \to [0,1]$ is the fuzzy relation on σ, which represents the strength of the relationship between vertices.

The fuzzy graph F is called a *weak fuzzy graph* if for all pairs of vertices $(a,b) \in V \times V$, the fuzzy relation satisfies the condition:

$$\mu(a,b) < \sigma(a) \wedge \sigma(b),$$

where $\wedge$ denotes the minimum operation.

In a weak fuzzy graph, the strength of the connection (or ”flow”) between any two vertices is always strictly less than the minimum membership degree of those two vertices. This ensures that the edges in the graph have weaker relationships than the vertices they connect.

Example 60. Let $F = (\sigma, \mu)$ be a weak fuzzy graph with the following properties (We call Weak fuzzy Triangle graph):

- **Vertices**: $V = \{v_1, v_2, v_3\}$
- **Vertex Membership Degrees**:

$$\sigma(v_1) = 0.7,$$
$$\sigma(v_2) = 0.8,$$
$$\sigma(v_3) = 0.9$$

- **Edge Membership Degrees** (Fuzzy relations between vertices):

$$\mu(v_1, v_2) = 0.6,$$
$$\mu(v_1, v_3) = 0.3,$$
$$\mu(v_2, v_3) = 0.7$$

For each edge in the graph, we check whether the fuzzy relation $\mu(v_i, v_j)$ is less than the minimum of the membership degrees of the two vertices:

- For edge (v_1, v_2):

$$\sigma(v_1) = 0.7, \quad \sigma(v_2) = 0.8$$

$$\sigma(v_1) \wedge \sigma(v_2) = \min(0.7, 0.8) = 0.7$$

$$\mu(v_1, v_2) = 0.6 < 0.7 \quad \text{(Condition satisfied)}$$

- For edge (v_1, v_3):

$$\sigma(v_1) = 0.7, \quad \sigma(v_3) = 0.9$$
$$\sigma(v_1) \wedge \sigma(v_3) = \min(0.7, 0.9) = 0.7$$
$$\mu(v_1, v_3) = 0.3 < 0.7 \quad \text{(Condition satisfied)}$$

- For edge (v_2, v_3):

$$\sigma(v_2) = 0.8, \quad \sigma(v_3) = 0.9$$
$$\sigma(v_2) \wedge \sigma(v_3) = \min(0.8, 0.9) = 0.8$$
$$\mu(v_2, v_3) = 0.7 < 0.8 \quad \text{(Condition satisfied)}$$

In this example, all edges in the weak fuzzy graph satisfy the condition $\mu(v_i, v_j) < \sigma(v_i) \wedge \sigma(v_j)$, making this a valid weak fuzzy graph.

2.10 Mild balanced intuitionistic fuzzy graph

A Mild Balanced Intuitionistic Fuzzy Graph (IFG) is defined as an Intuitionistic Fuzzy Graph in which all connected subgraphs are intense. This implies that the membership and non-membership degrees of any connected subgraph are less than or equal to those of the original graph [116]. The formal definition is provided below.

Definition 61 (Intense Subgraph)**.** [116] Let $G = (V, E, \mu, \nu)$ be an intuitionistic fuzzy graph, where μ and ν denote the membership and non-membership functions, respectively. A connected subgraph $H = (V(H), E(H), \mu_H, \nu_H)$ of G is called an *intense subgraph* if the following conditions hold:

(1) $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$,

(2) The degree of membership and non-membership in H satisfy:

$$D_\mu(H) \geq D_\mu(G) \quad \text{and} \quad D_\nu(H) \leq D_\nu(G),$$

where $D_\mu(G)$ and $D_\nu(G)$ are the degree functions of G defined as:

$$D_\mu(G) = \sum_{v \in V(G)} \mu(v) + \sum_{e \in E(G)} \mu(e),$$
$$D_\nu(G) = \sum_{v \in V(G)} \nu(v) + \sum_{e \in E(G)} \nu(e),$$

and similarly for H.

Definition 62 (Feeble Subgraph)**.** [116] Let $G = (V, E, \mu, \nu)$ be an intuitionistic fuzzy graph, where μ and ν denote the membership and non-membership functions, respectively. A connected subgraph $H = (V(H), E(H), \mu_H, \nu_H)$ of G is called a *feeble subgraph* if the following conditions hold:

(1) $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$,

(2) The degree of membership and non-membership in H satisfy:

$$D_\mu(H) < D_\mu(G) \quad \text{and} \quad D_\nu(H) > D_\nu(G),$$

where $D_\mu(G)$ and $D_\nu(G)$ are the degree functions of G, defined as:

$$D_\mu(G) = \sum_{v \in V(G)} \mu(v) + \sum_{e \in E(G)} \mu(e),$$
$$D_\nu(G) = \sum_{v \in V(G)} \nu(v) + \sum_{e \in E(G)} \nu(e),$$

and similarly for H.

Definition 63 (Mild Balanced Intuitionistic Fuzzy Graph)**.** [116] An intuitionistic fuzzy graph $G = (V, E)$ is called a *mild balanced intuitionistic fuzzy graph* if all connected subgraphs of G are intense subgraphs.

Question 64. Is it possible to define a Mild Balanced Neutrosophic Graph?

2.11 Connected Fuzzy Chemical Graph

The definition of a Chemical Graph is provided below. This is a graph widely used in the field of chemistry [158, 159, 35].

Definition 65. (cf.[158, 159, 35]) A *chemical graph* $G_C = (A, B)$ is a simple graph representing the molecular structure of a chemical compound, where:

- A is the set of vertices representing atoms in the molecule,
- B is the set of edges representing chemical bonds between the atoms in the molecule.

Each edge $(a, b) \in B$ connects two distinct atoms $a, b \in A$, indicating the existence of a bond between these atoms. In this representation, the degree of a vertex corresponds to the valency of the atom, i.e., the number of bonds that an atom forms with other atoms.

The definitions of a Connected Fuzzy Chemical Graph and a Neighborly Irregular Fuzzy Chemical Graph are provided below. These definitions extend the fundamental concepts of graph theory used to represent molecular structures to "fuzzy graphs" and "neighborly irregular chemical graphs," mathematically capturing the uncertainty present in molecular structures [31, 21, 22].

Definition 66. [22] A *fuzzy chemical graph* is a fuzzy graph $G = (V, \sigma, \mu)$, where:

- V is the set of vertices representing atoms in a molecule,
- $\sigma : V \to [0, 1]$ is a membership function representing the degree of membership of each atom in the graph,
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the degree of membership of bonds (edges) between atoms, and $\mu(u, v) \leq \min(\sigma(u), \sigma(v))$ for all $u, v \in V$.

A fuzzy chemical graph G is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a sequence of vertices $u = v_0, v_1, \ldots, v_k = v$ such that $\mu(v_i, v_{i+1}) > 0$ for all $0 \leq i < k$. This ensures that all atoms in the molecular structure are connected by chemical bonds.

Example 67 (Methane Molecule)**.** Consider the chemical graph for methane (CH_4):

- The vertex set $A = \{C, H_1, H_2, H_3, H_4\}$, where C represents the carbon atom and H_1, H_2, H_3, H_4 represent the four hydrogen atoms.
- The edge set $B = \{(C, H_1), (C, H_2), (C, H_3), (C, H_4)\}$, representing the chemical bonds between the carbon atom and each hydrogen atom.

The corresponding fuzzy chemical graph can be defined with:

- $\sigma(C) = 1$ and $\sigma(H_i) = 1$ for $i = 1, 2, 3, 4$,
- $\mu(C, H_i) = 1$ for $i = 1, 2, 3, 4$, and $\mu(u, v) = 0$ otherwise.

This representation can be extended by assigning fuzzy membership values σ and μ to account for uncertainty in the molecular structure.

Theorem 68. *Fuzzy chemical graphs generalize chemical graphs.*

Proof: Let $G_C = (A, B)$ be a chemical graph. We construct a fuzzy chemical graph $G_F = (V, \sigma, \mu)$ corresponding to G_C as follows:

- Let $V = A$, the set of atoms in the chemical graph.
- Define $\sigma(v) = 1$ for all $v \in V$, indicating that all vertices fully belong to the fuzzy graph.
- Define $\mu(u, v) = 1$ if $(u, v) \in B$, and $\mu(u, v) = 0$ otherwise, indicating crisp edges between atoms.

Clearly, G_F satisfies the conditions for a fuzzy chemical graph:

(1) For all $u, v \in V$,

$$\mu(u, v) \leq \min(\sigma(u), \sigma(v)) = 1.$$

(2) If G_C is connected, any pair of vertices $u, v \in V$ is connected by a sequence of edges in G_C. Since $\mu(u, v) = 1$ for edges in B, this connectivity is preserved in G_F.

Thus, G_F is a valid fuzzy chemical graph. Furthermore, by assigning membership values $\sigma(v)$ and $\mu(u, v)$ in the interval $[0, 1]$, fuzzy chemical graphs allow for the representation of uncertainty or partial relationships between atoms and bonds, generalizing the crisp structure of G_C. □

Definition 69. [22] A *neighborly irregular chemical graph* $G_{NIC} = (A, B)$ is a graph where:

- A is the set of vertices representing atoms in the molecular structure,
- B is the set of edges representing chemical bonds between atoms,
- For every edge $(a, b) \in B$, the degrees of the adjacent atoms a and b are distinct. That is, $\deg(a) \neq \deg(b)$ for all $(a, b) \in B$.

In the context of molecular structures, a neighborly irregular chemical graph typically models molecules in which atoms have varying valency in their adjacent atoms.

Definition 70. [22] A *neighborly irregular fuzzy chemical graph* $G_{NIFC} = (V, \sigma, \mu)$ is a fuzzy chemical graph where:

- V is the set of vertices representing atoms in the molecular structure,
- $\sigma : V \to [0, 1]$ is a membership function representing the degree of membership of each atom,
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the degree of membership of bonds between atoms,
- Any two adjacent vertices $u, v \in V$ have distinct degrees, i.e., $\deg(u) \neq \deg(v)$, with their corresponding membership values.

Theorem 71. *A Neighborly irregular fuzzy chemical graphs is a Fuzzy chemical graphs.*

Proof: This is evident. □

Theorem 72. *A Neighborly Irregular Fuzzy Chemical Graph generalizes a Neighborly Irregular Chemical Graph.*

Proof: To prove this theorem, we demonstrate that the definition of a Neighborly Irregular Chemical Graph is a special case of the definition of a Neighborly Irregular Fuzzy Chemical Graph when the membership functions are binary.

(1) **Definition Comparison:**

- In a Neighborly Irregular Chemical Graph $G_{NIC} = (A, B)$:
  - A is the set of vertices (atoms),
  - B is the set of edges (chemical bonds),
  - For every edge $(a, b) \in B$, $\deg(a) \neq \deg(b)$, where $\deg(a)$ and $\deg(b)$ are the degrees of vertices a and b in the graph.
- In a Neighborly Irregular Fuzzy Chemical Graph $G_{NIFC} = (V, \sigma, \mu)$:
  - V is the set of vertices (atoms),
  - $\sigma : V \to [0, 1]$ is the membership function of the vertices,
  - $\mu : V \times V \to [0, 1]$ is the fuzzy relation of the edges,

– For every edge (u,v) with $\mu(u,v) > 0$, $\deg(u) \neq \deg(v)$, where $\deg(u)$ and $\deg(v)$ are the fuzzy degrees of vertices u and v.

(2) **Special Case:**

- In a Neighborly Irregular Chemical Graph, all membership functions are binary:

$$\sigma(a) = 1 \quad \text{for all } a \in A,$$

$$\mu(a,b) = 1 \quad \text{for all } (a,b) \in B.$$

- The condition $\deg(a) \neq \deg(b)$ remains the same in both definitions since the degrees are defined in terms of the adjacent vertices.

(3) **Generalization:**

- A Neighborly Irregular Fuzzy Chemical Graph allows the membership values $\sigma(u)$ and $\mu(u,v)$ to take any value in the interval $[0,1]$, introducing a measure of uncertainty or partial membership.
- The Neighborly Irregular Chemical Graph is a special case of the Neighborly Irregular Fuzzy Chemical Graph when all membership values are crisp (binary).

Thus, the Neighborly Irregular Fuzzy Chemical Graph G_{NIFC} generalizes the Neighborly Irregular Chemical Graph G_{NIC} by incorporating fuzzy memberships for vertices and edges while preserving the irregularity condition for adjacent vertices. □

3 Result in this paper

We will outline the results presented in this paper. The fuzzy graph is extended to a Neutrosophic Graph, and its properties are analyzed and explored as necessary. Specifically, as mentioned in the introduction, we extend several classes of fuzzy graphs to Neutrosophic graphs and examine their characteristics. In this section, we focus on graph classes related to Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semigraphs, Double/Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

First, we will provide a proof of the relationship between fuzzy graphs and neutrosophic graphs.

Theorem 73. *A Neutrosophic Graph can be transformed into a Fuzzy Graph.*

Proof: To prove that a Neutrosophic Graph can be transformed into a Fuzzy Graph, we will show that the vertices and edges of the Neutrosophic Graph NTG can be represented using the truth membership component of the neutrosophic functions.

Each vertex $v \in V$ in the Neutrosophic Graph is assigned a neutrosophic membership function:

$$\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v)),$$

where $\sigma_1(v)$, $\sigma_2(v)$, and $\sigma_3(v)$ represent the truth, indeterminacy, and falsity memberships, respectively.

In a Fuzzy Graph, each vertex is assigned a single membership value $\sigma_f(v) \in [0,1]$, representing the degree of belonging of the vertex. We define this fuzzy membership as:

$$\sigma_f(v) = \sigma_1(v),$$

where $\sigma_1(v)$ is the truth component of the neutrosophic membership function. This step preserves the most certain (truth) aspect of the neutrosophic representation while discarding indeterminacy and falsity for the fuzzy graph.

Each edge $e = (u,v) \in E$ in the Neutrosophic Graph is assigned a neutrosophic membership function:

$$\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e)),$$

where $\mu_1(e)$, $\mu_2(e)$, and $\mu_3(e)$ represent the truth, indeterminacy, and falsity memberships of the edge.

In a Fuzzy Graph, each edge is assigned a single membership value $\mu_f(e) \in [0,1]$, representing the degree of connection between vertices. We define this fuzzy membership as:

$$\mu_f(e) = \mu_1(e),$$

where $\mu_1(e)$ is the truth component of the neutrosophic membership function.

After applying the transformation to both the vertices and edges, the Neutrosophic Graph $NTG = (V, E, \sigma, \mu)$ is transformed into a Fuzzy Graph $FG = (V, \sigma_f, \mu_f)$, where:

$$\sigma_f(v) = \sigma_1(v) \quad \text{for all } v \in V,$$

$$\mu_f(e) = \mu_1(e) \quad \text{for all } e \in E.$$

The transformation retains the structure of the graph while simplifying the neutrosophic membership functions to a single fuzzy membership. Hence, we have shown that any Neutrosophic Graph can be transformed into a Fuzzy Graph by using the truth membership values of the vertices and edges. Thus, the theorem is proved. □

3.1 Smart Neutrosophic Graph

The definition of a Smart Neutrosophic Graph is provided below.

Definition 74. A *Smart Neutrosophic Graph* is a generalization of the neutrosophic graph, incorporating smart structures and connectivity properties used in Internet of Things (IoT) applications. Formally, a Smart Neutrosophic Graph $SNG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as follows:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges connecting the vertices.
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a neutrosophic vertex set, where:

  $$\sigma_i : V \to [0,1], \quad \text{for } i = 1, 2, 3,$$

  where $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma_3(x)$ represent the degrees of truth, indeterminacy, and falsity of vertex $x \in V$, respectively.
- $\mu = (\mu_1, \mu_2, \mu_3)$ is a neutrosophic edge set, where:

  $$\mu_i : E \to [0,1], \quad \text{for } i = 1, 2, 3,$$

  and $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ represent the degrees of truth, indeterminacy, and falsity of the edge $(x,y) \in E$.
- For every $(x,y) \in E$, the following conditions hold:

  $$\mu_1(x,y) \leq \min(\sigma_1(x), \sigma_1(y)),$$

  $$\mu_2(x,y) \geq \max(\sigma_2(x), \sigma_2(y)),$$

  $$\mu_3(x,y) \geq \max(\sigma_3(x), \sigma_3(y)).$$

Additional conditions for smart connectivity is following.

- The sum of the neutrosophic membership values of all edges must satisfy:

  $$\sum_{(x,y)\in E} \mu_1(x,y) + \mu_2(x,y) + \mu_3(x,y) \leq 3.$$

- The vertices must satisfy smart connectivity rules, ensuring efficient communication in the graph structure, particularly in IoT applications.

Theorem 75. *A Smart Neutrosophic Graph (SNG) generalizes both Smart Fuzzy Graphs and Neutrosophic Graphs.*

Proof: We will show that a Smart Neutrosophic Graph (SNG) can reduce to both a Smart Fuzzy Graph and a Neutrosophic Graph under specific conditions.

Consider a Smart Neutrosophic Graph $G = (V, E, \sigma, \mu)$ with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. Define the following transformations:

$$\sigma_{\text{fuzzy}}(x) = \sigma_1(x), \quad \mu_{\text{fuzzy}}(x, y) = \mu_1(x, y).$$

This transformation ignores the indeterminacy (σ_2, μ_2) and falsity (σ_3, μ_3) components, leaving only the truth degrees. Under these mappings:

$$\mu_{\text{fuzzy}}(x, y) \leq \min(\sigma_{\text{fuzzy}}(x), \sigma_{\text{fuzzy}}(y)),$$

which satisfies the conditions for a Smart Fuzzy Graph. Additionally, the connectivity and membership sum conditions are inherited from G.

Consider the same Smart Neutrosophic Graph $G = (V, E, \sigma, \mu)$. By dropping the smart connectivity constraints and IoT-specific rules, we retain only the neutrosophic structure:

$$\sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x)), \quad \mu(x, y) = (\mu_1(x, y), \mu_2(x, y), \mu_3(x, y)).$$

This structure satisfies the conditions of a Neutrosophic Graph:

$$\mu_1(x, y) \leq \min(\sigma_1(x), \sigma_1(y)), \quad \mu_2(x, y) \geq \max(\sigma_2(x), \sigma_2(y)), \quad \mu_3(x, y) \geq \max(\sigma_3(x), \sigma_3(y)).$$

Thus, G reduces to a Neutrosophic Graph when IoT-specific constraints are removed.

By the above transformations, a Smart Neutrosophic Graph generalizes both Smart Fuzzy Graphs and Neutrosophic Graphs. □

3.2 Neutrosophic Zero Divisor Graph

The definition of the Neutrosophic Zero Divisor Graph is provided as follows.

Definition 76. Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R. A *Neutrosophic Zero Divisor Graph* $\Gamma_N = (V, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as follows:

- $V = Z(R)^* = Z(R) \setminus \{0\}$ represents the set of nonzero zero-divisors of R. Each element in V corresponds to a zero-divisor in the ring.
- $\sigma : V \to [0, 1]^3$ is a neutrosophic membership function that assigns three values to each vertex $v \in V$, i.e., $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$, where:
  - $\sigma_1(v)$ represents the truth degree of the membership of v,
  - $\sigma_2(v)$ represents the indeterminacy degree of the membership of v,
  - $\sigma_3(v)$ represents the falsity degree of the membership of v.
- $\mu : V \times V \to [0, 1]^3$ is a neutrosophic relation function that defines the adjacency between two vertices $v_i, v_j \in V$. The adjacency relation $\mu(v_i, v_j) = (\mu_1(v_i v_j), \mu_2(v_i v_j), \mu_3(v_i v_j))$ is described by the following conditions:
  - The truth degree of adjacency $\mu_1(v_i v_j)$ satisfies:

    $$\mu_1(v_i v_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$

    meaning that the truth degree of the adjacency between v_i and v_j is bounded by the minimum truth membership of the two vertices.
  - The indeterminacy degree of adjacency $\mu_2(v_i v_j)$ satisfies:

    $$\mu_2(v_i v_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$

    meaning that the indeterminacy degree of the adjacency between v_i and v_j is at least the maximum indeterminacy membership of the two vertices.

- The falsity degree of adjacency $\mu_3(v_iv_j)$ satisfies:
$$\mu_3(v_iv_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)),$$
meaning that the falsity degree of the adjacency between v_i and v_j is at least the maximum falsity membership of the two vertices.

Additionally, for every pair $v_i, v_j \in V$, the sum of the truth, indeterminacy, and falsity degrees of adjacency satisfies:
$$\mu_1(v_iv_j) + \mu_2(v_iv_j) + \mu_3(v_iv_j) \leq 3.$$

In this graph, two vertices v_i and v_j are adjacent (i.e., there is an edge between them) if and only if their product in the ring R is zero, that is, $v_i \cdot v_j = 0$. This graph structure represents the relationships between zero-divisors in a neutrosophic context, capturing degrees of truth, indeterminacy, and falsity.

Theorem 77. *A Non-zero Divisor Neutrosophic Graph can be transformed into a Non-zero Divisor Fuzzy Graph.*

Proof: Obviously holds. □

Theorem 78. *A Neutrosophic Zero Divisor Graph $\Gamma_N = (V, \sigma, \mu)$ reduces to a Neutrosophic Graph when the Zero Divisor condition $v_i \cdot v_j = 0$ is ignored.*

Proof: By ignoring the zero-divisor condition $v_i \cdot v_j = 0$:

(1) The vertex set V remains the same, and the neutrosophic membership function $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$ continues to satisfy:
$$\sigma_1(v) + \sigma_2(v) + \sigma_3(v) \leq 3, \quad \forall v \in V.$$

(2) The adjacency relation between v_i and v_j is now determined solely by the neutrosophic relation $\mu(v_i, v_j) = (\mu_1(v_iv_j), \mu_2(v_iv_j), \mu_3(v_iv_j))$, satisfying:
$$\mu_1(v_iv_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$
$$\mu_2(v_iv_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$
$$\mu_3(v_iv_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)).$$

(3) The sum of the neutrosophic degrees of adjacency satisfies:
$$\mu_1(v_iv_j) + \mu_2(v_iv_j) + \mu_3(v_iv_j) \leq 3.$$

This structure corresponds exactly to the definition of a Neutrosophic Graph, as it retains all the neutrosophic membership and adjacency properties while disregarding the algebraic constraints of zero-divisors.

Hence, a Neutrosophic Zero Divisor Graph reduces to a Neutrosophic Graph when the zero-divisor condition $v_i \cdot v_j = 0$ is ignored. □

Theorem 79. *If $n = p^2$, where p is a prime number and $p > 2$, then the non-zero Neutrosophic zero divisor graph is a 2-partite graph.*

Proof: Let $R = \mathbb{Z}_{p^2}$, where p is a prime number greater than 2. The elements of $\mathbb{Z}_{p^2}$ are $\{0, 1, 2, \dots, p^2 - 1\}$. The set of zero divisors $Z(R)$ in $\mathbb{Z}_{p^2}$ consists of the multiples of p, because for any $a \in \mathbb{Z}_{p^2}$, $a \cdot p \equiv 0 \pmod{p^2}$. Therefore, the set of non-zero zero divisors $Z(R)^*$ is:
$$Z(R)^* = \{p, 2p, 3p, \dots, (p-1)p\}.$$

Each element in this set is a multiple of p, and the product of any two such elements is zero in $\mathbb{Z}_{p^2}$. Thus, these are the vertices of the non-zero Neutrosophic zero divisor graph $\Gamma_N(R)$.

We assign to each vertex $v \in V = Z(R)^*$ a neutrosophic membership function $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$, where:

- $\sigma_1(v)$ represents the truth membership degree,
- $\sigma_2(v)$ represents the indeterminacy membership degree,
- $\sigma_3(v)$ represents the falsity membership degree.

For any two vertices $v_i, v_j \in V$, the neutrosophic adjacency relation $\mu(v_i, v_j) = (\mu_1(v_iv_j), \mu_2(v_iv_j), \mu_3(v_iv_j))$ is defined as:

$$\mu_1(v_iv_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$
$$\mu_2(v_iv_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$
$$\mu_3(v_iv_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)),$$

with the condition that $\mu_1(v_iv_j) + \mu_2(v_iv_j) + \mu_3(v_iv_j) \leq 3$.

We now partition the vertex set $V = \{p, 2p, 3p, \dots, (p-1)p\}$ into two disjoint subsets based on whether the multiple of p is odd or even. Specifically, define:

$$V_1 = \{p, 3p, 5p, \dots, (p-2)p\} \quad \text{(odd multiples of } p\text{)},$$
$$V_2 = \{2p, 4p, 6p, \dots, (p-1)p\} \quad \text{(even multiples of } p\text{)}.$$

Note that:

- V_1 contains all odd multiples of p,
- V_2 contains all even multiples of p.

To show that the non-zero Neutrosophic zero divisor graph is 2-partite, we need to verify that:

(1) No two vertices in V_1 are adjacent,

(2) No two vertices in V_2 are adjacent,

(3) Any vertex in V_1 is adjacent to any vertex in V_2.

Consider any two vertices $v_i, v_j \in V_1$. Since both v_i and v_j are odd multiples of p, their product is not zero modulo p^2. Thus, no edges exist between vertices in V_1.

Similarly, for any two vertices $v_i, v_j \in V_2$, their product is not zero modulo p^2, as they are both even multiples of p. Hence, no edges exist between vertices in V_2.

For any $v_i \in V_1$ and $v_j \in V_2$, their product $v_i \cdot v_j = 0 \pmod{p^2}$, because one is an odd multiple and the other is an even multiple of p. Therefore, an edge exists between any vertex in V_1 and any vertex in V_2.

Since the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that:

- No edges exist within V_1,
- No edges exist within V_2,
- Edges exist between vertices in V_1 and V_2,

we conclude that the non-zero Neutrosophic zero divisor graph $\Gamma_N(R)$ is a 2-partite graph.

This completes the proof. □

3.3 Weak Neutrosophic Graph

The definition of the Weak Neutrosophic Graph is provided as follows.

Definition 80 (Weak Neutrosophic Graph)**.** A *Weak Neutrosophic Graph* $G = (V, E, \eta, \rho)$ is a graph characterized by the following components:

- V is the set of vertices.

- $\eta : V \to [0, 1] \times [0, 1] \times [0, 1]$ is the neutrosophic vertex membership function. For each vertex $v \in V$,
$$\eta(v) = (\eta_1(v), \eta_2(v), \eta_3(v)),$$
where:
  - $\eta_1(v)$: Truth membership of vertex v,
  - $\eta_2(v)$: Indeterminacy membership of vertex v,
  - $\eta_3(v)$: Falsity membership of vertex v.
- $\rho : E \to [0, 1] \times [0, 1] \times [0, 1]$ is the neutrosophic edge membership function. For each edge $(u, v) \in E$,
$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v)),$$
where:
  - $\rho_1(u, v)$: Truth membership of edge (u, v),
  - $\rho_2(u, v)$: Indeterminacy membership of edge (u, v),
  - $\rho_3(u, v)$: Falsity membership of edge (u, v).

The graph G is called a *Weak Neutrosophic Graph* if the following conditions hold for all edges $(u, v) \in E$:

(1) **Truth Membership Condition:**
$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)),$$
where $\min(\cdot, \cdot)$ denotes the minimum operation between the truth memberships of the connected vertices.

(2) **Indeterminacy Membership Condition:**
$$\rho_2(u, v) > \max(\eta_2(u), \eta_2(v)),$$
where $\max(\cdot, \cdot)$ denotes the maximum operation between the indeterminacy memberships of the connected vertices.

(3) **Falsity Membership Condition:**
$$\rho_3(u, v) > \max(\eta_3(u), \eta_3(v)),$$
where $\max(\cdot, \cdot)$ denotes the maximum operation between the falsity memberships of the connected vertices.

Theorem 81. *A Weak Neutrosophic Graph generalizes a Weak Fuzzy Graph.*

Proof: Let $G_N = (V, E, \eta, \rho)$ be a Weak Neutrosophic Graph, where:
$$\eta(v) = (\eta_1(v), \eta_2(v), \eta_3(v)) \quad \text{for all } v \in V,$$
$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v)) \quad \text{for all } (u, v) \in E.$$

To show that G_N generalizes a Weak Fuzzy Graph, consider the special case where:
$$\eta_2(v) = 0, \quad \eta_3(v) = 0, \quad \rho_2(u, v) = 0, \quad \rho_3(u, v) = 0 \quad \text{for all } v \in V \text{ and } (u, v) \in E.$$

In this case:
$$\eta(v) = (\eta_1(v), 0, 0), \quad \rho(u, v) = (\rho_1(u, v), 0, 0),$$
and the conditions of a Weak Neutrosophic Graph reduce to:
$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)).$$

This matches the definition of a Weak Fuzzy Graph $G_F = (V, \sigma, \mu)$, where:
$$\sigma(v) = \eta_1(v), \quad \mu(u, v) = \rho_1(u, v).$$

Thus, a Weak Neutrosophic Graph includes Weak Fuzzy Graphs as a special case and therefore generalizes them. □

Theorem 82. *A Weak Neutrosophic Graph reduces to a Neutrosophic Graph under specific conditions.*

Proof: Let $G_N = (V, E, \eta, \rho)$ be a Weak Neutrosophic Graph. To show that G_N can reduce to a Neutrosophic Graph $G' = (V, E, \sigma, \mu)$, assume the following conditions hold:

- $\rho_1(u, v) = \min(\eta_1(u), \eta_1(v))$,
- $\rho_2(u, v) = \max(\eta_2(u), \eta_2(v))$,
- $\rho_3(u, v) = \max(\eta_3(u), \eta_3(v))$,

for all $(u, v) \in E$.

Under these conditions, the edge membership functions of G_N match the edge membership functions of a Neutrosophic Graph, and the conditions for a Weak Neutrosophic Graph:

$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)),$$
$$\rho_2(u, v) > \max(\eta_2(u), \eta_2(v)),$$
$$\rho_3(u, v) > \max(\eta_3(u), \eta_3(v)),$$

become equalities, matching the adjacency relations in a Neutrosophic Graph.

Therefore, the Weak Neutrosophic Graph reduces to a Neutrosophic Graph when these specific conditions are satisfied. □

Theorem 83. *The union of two weak Neutrosophic graphs is a weak Neutrosophic graph.*

Proof: Let $G_1 = (V_1, E_1, \eta_1, \rho_1)$ and $G_2 = (V_2, E_2, \eta_2, \rho_2)$ be two weak Neutrosophic graphs, where $\eta_1(v) = (\eta_{1,1}(v), \eta_{1,2}(v), \eta_{1,3}(v))$ and $\eta_2(v) = (\eta_{2,1}(v), \eta_{2,2}(v), \eta_{2,3}(v))$ are the Neutrosophic vertex membership functions, and $\rho_1(u, v) = (\rho_{1,1}(u, v), \rho_{1,2}(u, v), \rho_{1,3}(u, v))$ and $\rho_2(u, v) = (\rho_{2,1}(u, v), \rho_{2,2}(u, v), \rho_{2,3}(u, v))$ are the Neutrosophic edge membership functions.

The union of G_1 and G_2, denoted by $G = G_1 \cup G_2 = (V, E, \eta, \rho)$, is defined as follows:

- $V = V_1 \cup V_2$,
- $E = E_1 \cup E_2$,
- $\eta(v) = \max(\eta_1(v), \eta_2(v))$ for $v \in V_1 \cap V_2$,
- $\eta(v) = \eta_1(v)$ if $v \in V_1 \setminus V_2$,
- $\eta(v) = \eta_2(v)$ if $v \in V_2 \setminus V_1$,
- $\rho(u, v) = \max(\rho_1(u, v), \rho_2(u, v))$ for $(u, v) \in E_1 \cap E_2$,
- $\rho(u, v) = \rho_1(u, v)$ if $(u, v) \in E_1 \setminus E_2$,
- $\rho(u, v) = \rho_2(u, v)$ if $(u, v) \in E_2 \setminus E_1$.

We must show that the union graph G satisfies the conditions of a weak Neutrosophic graph. That is, for all $(u, v) \in E$, the following inequalities hold:

(1) $\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v)$,

(2) $\rho_2(u, v) > \eta_2(u) \vee \eta_2(v)$,

(3) $\rho_3(u, v) > \eta_3(u) \vee \eta_3(v)$.

When $(u, v) \in E_1 \setminus E_2$, $\rho(u, v) = \rho_1(u, v)$. Since G_1 is a weak Neutrosophic graph, we have:

$$\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v),$$
$$\rho_2(u, v) > \eta_2(u) \vee \eta_2(v),$$
$$\rho_3(u, v) > \eta_3(u) \vee \eta_3(v),$$

which satisfies the weak Neutrosophic graph conditions for the union graph.

When $(u, v) \in E_2 \setminus E_1$, $\rho(u, v) = \rho_2(u, v)$. Since G_2 is a weak Neutrosophic graph, we have:

$$\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v),$$
$$\rho_2(u, v) > \eta_2(u) \vee \eta_2(v),$$
$$\rho_3(u, v) > \eta_3(u) \vee \eta_3(v),$$

which also satisfies the weak Neutrosophic graph conditions.

When $(u, v) \in E_1 \cap E_2$, $\rho(u, v) = \max(\rho_1(u, v), \rho_2(u, v))$ and $\eta(v) = \max(\eta_1(v), \eta_2(v))$. We now show that the weak Neutrosophic graph conditions hold:

- For the truth membership:

$$\rho_1(u, v) = \max(\rho_{1,1}(u, v), \rho_{2,1}(u, v)) < \max(\eta_1(u), \eta_2(u)) \wedge \max(\eta_1(v), \eta_2(v)).$$

  Since both G_1 and G_2 satisfy the weak Neutrosophic conditions, this inequality holds.

- For the indeterminacy membership:

$$\rho_2(u, v) = \max(\rho_{1,2}(u, v), \rho_{2,2}(u, v)) > \eta_2(u) \vee \eta_2(v).$$

- For the falsity membership:

$$\rho_3(u, v) = \max(\rho_{1,3}(u, v), \rho_{2,3}(u, v)) > \eta_3(u) \vee \eta_3(v).$$

Thus, in all cases, the union graph $G = G_1 \cup G_2$ satisfies the weak Neutrosophic graph conditions, completing the proof. □

3.4 Neutrosophic Semigraph

The definition of the Neutrosophic Semigraph is provided as follows.

Definition 84 (Neutrosophic Semigraph)**.** A *neutrosophic semigraph* $G = (V, X, \eta, \rho)$ is defined as follows:

- V is a non-empty set of vertices.
- X is a set of edges, where each edge is an n-tuple of distinct vertices from V, i.e., $e = (v_1, v_2, \dots, v_n)$, with $n \geq 2$.
- $\eta : V \to [0, 1] \times [0, 1] \times [0, 1]$ is a neutrosophic membership function that assigns each vertex $v \in V$ a triple $(\eta_1(v), \eta_2(v), \eta_3(v))$, representing the truth, indeterminacy, and falsity memberships, respectively.
- $\rho : V \times V \to [0, 1] \times [0, 1] \times [0, 1]$ is a neutrosophic relation on vertices, where

$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v))$$

  represents the neutrosophic truth, indeterminacy, and falsity memberships of the relationship between vertices u and v.
- For each edge $e = (v_1, v_2, \dots, v_n)$, the neutrosophic edge membership function $\rho(e) = (\rho_1(e), \rho_2(e), \rho_3(e))$ satisfies:

$$\rho_1(e) \leq \rho_1(v_1, v_2) \wedge \rho_1(v_2, v_3) \wedge \cdots \wedge \rho_1(v_{n-1}, v_n) \wedge \eta_1(v_1) \wedge \eta_1(v_n),$$
$$\rho_2(e) \geq \rho_2(v_1, v_2) \vee \rho_2(v_2, v_3) \vee \cdots \vee \rho_2(v_{n-1}, v_n) \vee \eta_2(v_1) \vee \eta_2(v_n),$$
$$\rho_3(e) \geq \rho_3(v_1, v_2) \vee \rho_3(v_2, v_3) \vee \cdots \vee \rho_3(v_{n-1}, v_n) \vee \eta_3(v_1) \vee \eta_3(v_n),$$

  where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

In this neutrosophic semigraph, the vertices v_1 and v_n are referred to as the *end vertices*, and the vertices $v_2, v_3, \dots, v_{n-1}$ are called the *middle vertices*. A middle vertex that is also an end vertex of another edge is termed a *middle-end vertex*.

Theorem 85. *A Neutrosophic Semigraph can be transformed into both a Fuzzy Semigraph and a Neutrosophic Graph.*

Proof: Let $G = (V, X, \eta, \rho)$ be a Neutrosophic Semigraph where:

- V is the set of vertices.
- X is the set of edges, where each edge is an n-tuple of distinct vertices.
- $\eta : V \to [0,1]^3$ assigns a neutrosophic membership $(\eta_1(v), \eta_2(v), \eta_3(v))$ to each vertex.
- $\rho : V \times V \to [0,1]^3$ assigns a neutrosophic membership $(\rho_1(u,v), \rho_2(u,v), \rho_3(u,v))$ to the relationship between two vertices u and v.
- For each edge $e \in X$, the neutrosophic membership $\rho(e) = (\rho_1(e), \rho_2(e), \rho_3(e))$ satisfies conditions defined in the Neutrosophic Semigraph.

To transform G into a Fuzzy Semigraph $G' = (V, X, \sigma, \mu, \eta')$, define:

$$\sigma(v) = \eta_1(v), \quad \mu(u,v) = \rho_1(u,v), \quad \eta'(e) = \rho_1(e),$$

for all $v \in V$, $u, v \in V$, and $e \in X$, where $\eta_1(v)$ and $\rho_1(u,v)$ are the truth memberships from the neutrosophic graph.

The conditions for a Fuzzy Semigraph hold because the neutrosophic truth memberships satisfy the membership constraints of a Fuzzy Semigraph:

$$\eta'(e) \leq \mu(v_1, v_2) \wedge \mu(v_2, v_3) \wedge \cdots \wedge \mu(v_{n-1}, v_n) \wedge \sigma(v_1) \wedge \sigma(v_n).$$

Thus, G reduces to a Fuzzy Semigraph under this transformation.

To transform G into a Neutrosophic Graph $G' = (V, E, \eta, \rho')$, where E is the edge set of unordered vertex pairs, redefine:

$$\rho'(u,v) = \rho(u,v),$$

for all $u, v \in V$. The conditions of the Neutrosophic Graph are naturally satisfied because the neutrosophic memberships for vertices and edges are preserved.

Edges in the semigraph X, which are tuples, become unordered pairs in E. Therefore, G is transformed into a Neutrosophic Graph. □

3.5 Double/Triple Layered Neutrosophic Graph

Double/Triple Layered Neutrosophic Graph are provided as follows.

Definition 86 (Double Layered Neutrosophic Graph (DLNG))**.** A *Double Layered Neutrosophic Graph* is an extension of the standard neutrosophic graph where both vertices and edges are characterized by neutrosophic memberships. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph with vertex set V, edge set E, neutrosophic vertex membership functions $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and neutrosophic edge membership functions $\mu = (\mu_1, \mu_2, \mu_3)$.

Define the *Double Layered Neutrosophic Graph* $DLNG(G) = (V^*, E^*, \sigma_{DL}, \mu_{DL})$ as follows:

- The **node set** V^* is the union of vertices and edges from the original graph: $V^* = V \cup E$.
- The **neutrosophic vertex membership function** σ_{DL} is defined by:

  $$\sigma_{DL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E. \end{cases}$$

  where $\sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x))$ for vertices and $\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e))$ for edges.
- The **neutrosophic edge membership function** μ_{DL} on $V^* \times V^*$ is defined as:

  $$\mu_{DL}(x,y) = \begin{cases} \sigma(x) \wedge \sigma(y) & \text{if } x, y \in V, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in E, \text{and they share a common vertex,} \\ \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E, \text{and } x \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the pair $DLNG(G) = (\sigma_{DL}, \mu_{DL})$ represents the *Double Layered Neutrosophic Graph*.

Theorem 87. *Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph. The order of the Double Layered Neutrosophic Graph (DLNG) is given by:*

$$O(DLNG) = O(G) + S(G),$$

where $O(G)$ is the order of G, and $S(G)$ is the size of G.

Proof: Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph with the following components:

- V is the set of vertices,
- E is the set of edges,
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the neutrosophic membership function for vertices,
- $\mu = (\mu_1, \mu_2, \mu_3)$ is the neutrosophic membership function for edges.

The order of the neutrosophic graph G, denoted $O(G)$, is the number of vertices in G, and the size of the graph, denoted $S(G)$, is the number of edges in G.

The Double Layered Neutrosophic Graph (DLNG) extends the graph G by including the edges as additional vertices. Hence, the node set V^* of the DLNG is defined as $V^* = V \cup E$.

The order of the DLNG, denoted $O(DLNG)$, is the total number of elements in V^*, i.e., the sum of the number of vertices and edges in G. Therefore,

$$O(DLNG) = |V^*| = |V| + |E|.$$

Thus, we can express the order of the DLNG as:

$$O(DLNG) = O(G) + S(G).$$

Additionally, the neutrosophic membership function for the DLNG, denoted σ_{DL}, is defined as follows:

- For $v \in V$, $\sigma_{DL}(v) = \sigma(v)$,
- For $e \in E$, $\sigma_{DL}(e) = \mu(e)$.

Thus, the neutrosophic order of DLNG, denoted $On(DLNG)$, is given by the sum of the neutrosophic memberships of all vertices and edges in V^*, i.e.,

$$On(DLNG) = \sum_{v \in V} \sigma(v) + \sum_{e \in E} \sum_{i=1}^{3} \mu_i(e).$$

This can be rewritten as:

$$On(DLNG) = On(G) + Sn(G),$$

where $On(G)$ is the neutrosophic order of G, and $Sn(G)$ is the neutrosophic size of G.

Thus, we have shown that the order of the Double Layered Neutrosophic Graph is the sum of the order and size of the original neutrosophic graph. □

Theorem 88. *A Double Layered Neutrosophic Graph (DLNG) can be transformed into a Double Layered Fuzzy Graph (DLFG).*

Proof: Let $DLNG(G) = (V^*, E^*, \sigma_{DL}, \mu_{DL})$. By ignoring the neutrosophic components $\sigma_2(x), \sigma_3(x), \mu_2(e), \mu_3(e)$, the vertex and edge membership functions reduce to:

$$\sigma_{DLFG}(x) = \sigma_1(x), \quad \mu_{DLFG}(x, y) = \mu_1(x, y),$$

where σ_1 and μ_1 are the truth membership functions in σ_{DL} and μ_{DL}, respectively. The resulting graph satisfies the definitions of a Double Layered Fuzzy Graph:

$$DLFG(G) = (\sigma_{DLFG}, \mu_{DLFG}),$$

proving that $DLNG(G)$ generalizes $DLFG(G)$. □

Definition 89 (Triple Layered Neutrosophic Graph (TLNG))**.** A *Triple Layered Neutrosophic Graph* is a further extension of the Double Layered Neutrosophic Graph, incorporating an additional layer. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph. The *Triple Layered Neutrosophic Graph* $TLNG(G) = (V^*, E^*, \sigma_{TL}, \mu_{TL})$ is defined as follows:

- The **node set** V^* is the union of vertices and edges from the original graph: $V^* = V \cup E$.
- The **neutrosophic vertex membership function** σ_{TL} is defined by:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ 2 \cdot \mu(e) & \text{if } e \in E. \end{cases}$$

  where the factor of 2 represents the additional layer's increased influence on the neutrosophic memberships.

- The **neutrosophic edge membership function** μ_{TL} on $V^* \times V^*$ is defined as:

$$\mu_{TL}(x, y) = \begin{cases} \sigma(x) \wedge \sigma(y) & \text{if } x, y \in V, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in E, \text{and they share a common vertex}, \\ \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E, \text{and } x \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the pair $TLNG(G) = (\sigma_{TL}, \mu_{TL})$ represents the *Triple Layered Neutrosophic Graph*.

Theorem 90. *The order of a Triple Layered Neutrosophic Graph (TLNG) is given by:*

$$O(TLNG) = O(G) + 2 \cdot S(G),$$

where $O(G)$ is the order of the neutrosophic graph G, and $S(G)$ is the size of G.

Proof: Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

- V is the set of vertices,
- E is the set of edges,
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the neutrosophic vertex membership function,
- $\mu = (\mu_1, \mu_2, \mu_3)$ is the neutrosophic edge membership function.

The order of the neutrosophic graph G, denoted $O(G)$, is the number of vertices in G, and the size of the graph, denoted $S(G)$, is the number of edges in G.

In a Triple Layered Neutrosophic Graph (TLNG), both the vertices and edges of the original graph G contribute to the node set. However, in TLNG, each edge is counted twice because of the additional layer. Specifically:

$$V^* = V \cup E,$$

where E appears with double the influence.

Thus, the neutrosophic vertex membership function for the TLNG, denoted σ_{TL}, is defined as:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ 2 \cdot \mu(e) & \text{if } e \in E. \end{cases}$$

This means that the membership for the vertices in E is doubled, reflecting the "triple layer" nature.

The order of the Triple Layered Neutrosophic Graph, $O(TLNG)$, is the sum of the vertices and twice the number of edges:

$$O(TLNG) = |V^*| = |V| + 2 \cdot |E|.$$

Thus, we have:

$$O(TLNG) = O(G) + 2 \cdot S(G).$$

Additionally, the neutrosophic order of TLNG, denoted $On(TLNG)$, is the sum of the neutrosophic membership values for all vertices and edges:

$$On(TLNG) = \sum_{v \in V} \sigma(v) + 2 \cdot \sum_{e \in E} \mu(e),$$

which can be rewritten as:

$$On(TLNG) = On(G) + 2 \cdot Sn(G),$$

where $On(G)$ is the neutrosophic order of the graph G and $Sn(G)$ is the neutrosophic size of the graph.

Thus, the order of the Triple Layered Neutrosophic Graph is indeed $O(G) + 2 \cdot S(G)$, as required. □

Theorem 91. *A Triple Layered Neutrosophic Graph (TLNG) can be transformed into a Triple Layered Fuzzy Graph (TLFG) or a Double Layered Neutrosophic Graph (DLNG).*

Proof: Let $TLNG(G) = (V^*, E^*, \sigma_{TL}, \mu_{TL})$ represent a Triple Layered Neutrosophic Graph. To show the transformations:

By constraining the neutrosophic vertex and edge membership functions σ_{TL} and μ_{TL} such that:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E, \end{cases}$$

and replacing σ_{TL} with the fuzzy membership function σ_{TLFG}, we obtain a Triple Layered Fuzzy Graph. The neutrosophic parameters $\sigma_2(x), \sigma_3(x), \mu_2(e), \mu_3(e)$ are ignored, reducing the representation to:

$$TLFG(G) = (\sigma_{TLFG}, \mu_{TLFG}).$$

By removing the additional layer factor (e.g., $2\mu(e)$) in σ_{TL} and ensuring that $V^* = V \cup E$, the Triple Layered Neutrosophic Graph reduces to a Double Layered Neutrosophic Graph:

$$\sigma_{DL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E. \end{cases}$$

The edge membership μ_{TL} reduces to μ_{DL}, completing the transformation.

Thus, $TLNG(G)$ generalizes both $TLFG(G)$ and $DLNG(G)$. □

3.6 Connected Neutrosophic Chemical Graph

We define a Connected Neutrosophic Chemical Graph as follows. This graph concept combines the principles of a Connected Fuzzy Chemical Graph and a Neutrosophic Graph.

Definition 92 (Connected Neutrosophic Chemical Graph)**.** A *Connected Neutrosophic Chemical Graph* $G = (V, E, \sigma, \mu)$ is a neutrosophic graph where:

- V is the set of vertices representing atoms in a molecule.
- E is the set of edges representing chemical bonds between atoms.
- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ is the neutrosophic membership function for each atom, where:
  - $\sigma_1(v)$ represents the truth membership degree of atom v,
  - $\sigma_2(v)$ represents the indeterminacy membership degree of atom v,
  - $\sigma_3(v)$ represents the falsity membership degree of atom v.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ is the neutrosophic relation representing the degree of chemical bond membership, where:
  - $\mu_1(e)$ represents the truth membership of bond e,
  - $\mu_2(e)$ represents the indeterminacy membership of bond e,

  - $\mu_3(e)$ represents the falsity membership of bond e.

- The membership degrees for the edges must satisfy the condition $\mu_i(u, v) \leq \sigma_i(u) \wedge \sigma_i(v)$, where $\wedge$ denotes the minimum operation, for all $u, v \in V$ and $i \in \{1, 2, 3\}$.

The graph is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a path of vertices $u = v_0, v_1, \ldots, v_k = v$ such that $\mu(v_i, v_{i+1}) > 0$ for all $0 \leq i < k$, ensuring that all atoms in the molecule are connected by chemical bonds.

Theorem 93. *A Connected Neutrosophic Chemical Graph (CNCG) can be transformed into both a Connected Fuzzy Chemical Graph (CFCG) and a Neutrosophic Graph.*

Proof: Let $G_C = (V, E, \sigma, \mu)$ be a Connected Neutrosophic Chemical Graph where:

- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ represents the neutrosophic vertex memberships.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ represents the neutrosophic edge memberships.

To transform G_C into a Connected Fuzzy Chemical Graph $G_F = (V, E, \sigma', \mu')$, we define:

$$\sigma'(v) = \sigma_1(v), \quad \mu'(u, v) = \mu_1(u, v), \quad \forall v \in V, \forall (u, v) \in E,$$

where $\sigma_1(v)$ and $\mu_1(u, v)$ are the truth memberships from the neutrosophic graph.

The connectivity condition is preserved because the truth memberships govern the connectedness of G_C. Thus, G_C reduces to a CFCG.

To transform G_C into a Neutrosophic Graph $G_N = (V, E, \sigma, \mu)$, no changes are needed as G_C already satisfies the definition of a Neutrosophic Graph. Therefore, G_C is inherently a Neutrosophic Graph. □

Definition 94 (Neighborly Irregular Neutrosophic Chemical Graph)**.** A *Neighborly Irregular Neutrosophic Chemical Graph* $G_{NIC} = (V, E, \sigma, \mu)$ is a neutrosophic chemical graph where:

- V is the set of vertices representing atoms in the molecular structure,
- E is the set of edges representing chemical bonds between atoms,
- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ is the neutrosophic membership function representing the truth, indeterminacy, and falsity memberships of each atom,
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ is the neutrosophic relation representing the degrees of membership of the bonds, with the same conditions on membership as above.
- For any two adjacent vertices $u, v \in V$, their neutrosophic degrees are distinct. Specifically, $\deg(u) \neq \deg(v)$ holds with respect to the neutrosophic membership values of the atoms, ensuring that adjacent atoms have different connection strengths or roles in the molecule. The degree $\deg(v)$ of a vertex is calculated based on its neutrosophic memberships in the edges connected to it.

Theorem 95. *A Neighborly Irregular Neutrosophic Chemical Graph (NICG) can be transformed into both a Neighborly Irregular Fuzzy Chemical Graph (NIFCG) and a Neutrosophic Graph.*

Proof: Let $G_{NIC} = (V, E, \sigma, \mu)$ be a Neighborly Irregular Neutrosophic Chemical Graph where:

- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ represents the neutrosophic vertex memberships.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ represents the neutrosophic edge memberships.

To transform G_{NIC} into a Neighborly Irregular Fuzzy Chemical Graph $G_{NIFC} = (V, E, \sigma', \mu')$, we define:

$$\sigma'(v) = \sigma_1(v), \quad \mu'(u, v) = \mu_1(u, v), \quad \forall v \in V, \forall (u, v) \in E,$$

where $\sigma_1(v)$ and $\mu_1(u, v)$ are the truth memberships from the neutrosophic graph.

The degree distinctness property in G_{NIC}, $\deg(u) \neq \deg(v)$, is preserved under this transformation because the truth memberships uniquely determine the vertex and edge roles in G_{NIFC}. Thus, G_{NIC} reduces to a NIFCG.

To transform G_{NIC} into a Neutrosophic Graph $G_N = (V, E, \sigma, \mu)$, no structural changes are necessary as G_{NIC} is already a neutrosophic graph by definition. The graph retains its neutrosophic membership functions for vertices and edges. Therefore, G_{NIC} is inherently a Neutrosophic Graph. □

4 Conclusion and Future Work

This paper has explored various graph classes associated with Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semigraphs, Double and Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

In terms of future research directions, our primary objective is to investigate the potential for defining more refined or generalized classes of graphs. This will involve both theoretical analysis and computational experiments based on the graph classes discussed in this work. By applying these definitions to real-world scenarios, we aim to evaluate their practicality and identify opportunities for the introduction of novel graph definitions.

Additionally, we plan to extend this study to hypergraphs [66, 29, 65, 92] and superhypergraphs [148, 71, 147, 54, 59], as well as investigate their applicability to directed graphs.

Furthermore, we intend to explore width parameters for these graph classes [33, 130, 34], which will allow us to examine graph-related problems and develop algorithms tailored to these advanced structures. This research aims to deepen our understanding of Neutrosophic Graphs and their applications across different domains.

Declarations

Ethics Approval and Consent to Participate

The results, data, and figures presented in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and no permissions are required.

Consent for Publication

This article does not contain any studies involving human participants or animals performed by any of the authors.

Availability of Data and Materials

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

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References

1. Eman A AbuHijleh. Complex hesitant fuzzy graph. *Fuzzy Information and Engineering*, 15(2):149–161, 2023.
2. S Akbari, HR Maimani, and S Yassemi. When a zero-divisor graph is planar or a complete r-partite graph. *Journal of Algebra*, 270(1):169–180, 2003.
3. S Akbari and A Mohammadian. On the zero-divisor graph of a commutative ring. *Journal of algebra*, 274(2):847–855, 2004.
4. Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.
5. Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.
6. Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.
7. Muhammad Akram and Anam Luqman. *Fuzzy hypergraphs and related extensions*. Springer, 2020.
8. Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.
9. Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.
10. Muhammad Akram and Sundas Shahzadi. *Representation of graphs using intuitionistic neutrosophic soft sets*. Infinite Study, 2016.
11. Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.
12. T Al-Hawary, S Al-Shalaldeh, and MUHAMMAD Akram. Certain matrices and energies of fuzzy graphs. *TWMS Journal of Applied and Engineering Mathematics*, 2:2146–1147, 2021.
13. Talal Al-Hawary and Laith AlMomani. -balanced fuzzy graphs. *arXiv preprint arXiv:1804.08677*, 2018.
14. Talal Ali Al-Hawary. Certain classes of fuzzy graphs. *European Journal of Pure and Applied Mathematics*, 10(3):552–560, 2017.
15. TALAL ALI AL-HAWARY and MAREF YM ALZOUBI. β-product of product fuzzy graphs. *Journal of applied mathematics & informatics*, 42(2):283–290, 2024.
16. M Anandhi and B Keerthiga. Vertex coloring of double layered complete fuzzy graph using α-cut.
17. David F Anderson and Ayman Badawi. On the zero-divisor graph of a ring. *Communications in Algebra*, 36(8):3073–3092, 2008.
18. David F Anderson and Ayman Badawi. The total graph of a commutative ring. *Journal of algebra*, 320(7):2706–2719, 2008.
19. David F Anderson and John D LaGrange. Some remarks on the compressed zero-divisor graph. *Journal of Algebra*, 447:297–321, 2016.
20. Patrizio Angelini, Michael A Bekos, Michael Kaufmann, Maximilian Pfister, and Torsten Ueckerdt. Beyond-planarity: density results for bipartite graphs. *arXiv preprint arXiv:1712.09855*, 2017.
21. S Anjalmose and J Arockia Aruldoss. Neighborly irregular fuzzy chemical graphs. *Stochastic modeling & Applications*, 26:161–173, 2022.
22. S Anjalmose and J Arockia Aruldoss. Chromatic index and line graphs of neighbourly irregular fuzzy chemical graphs (gnifc) among s-block and p-block elements. *Punjab University Journal of Mathematics*, 54(9), 2023.
23. Lakshmanan S Aparna and A Vijayakumar. *Studies on Some Graph Classes*. PhD thesis, Cochin University of Science and Technology, 2008.
24. S Archana and PREETHI KUTTIPULACKAL. Line regular fuzzy semigraphs. *Baghdad Science Journal*, 20(1 (SI)):0288–0288, 2023.
25. J Jon Arockiaraj and PM Chaarumathi. Smart fuzzy graph. In *Journal of Physics: Conference Series*, volume 1850, page 012058. IOP Publishing, 2021.
26. J Jon Arockiaraj and Chaarumathi PM. Max connectivity and inherent value of a smart fuzzy graph. *South East Asian Journal of Mathematics & Mathematical Sciences*, 17, 2021.
27. Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.
28. Mehdi Behzad, Gary Chartrand, and John K Cooper Jr. The colour numbers of complete graphs. *Journal of the London Mathematical Society*, 1(1):226–228, 1967.
29. Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.
30. Leonid S Bershtein and Alexander V Bozhenyuk. Fuzzy graphs and fuzzy hypergraphs. In *Encyclopedia of Artificial Intelligence*, pages 704–709. IGI Global, 2009.
31. T Bharathi, S Antony Vinoth, and S Leo. Fuzzy radio reciprocal l-labeling on certain chemical graphs. *Indian Journal of Science and Technology*, 14(27):2284–2292, 2021.
32. Prabir Bhattacharya. Some remarks on fuzzy graphs. *Pattern recognition letters*, 6(5):297–302, 1987.
33. Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1–21, 1993.
34. Hans L Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical computer science*, 209(1-2):1–45, 1998.
35. Danail Bonchev. *Chemical graph theory: introduction and fundamentals*. Routledge, 2018.
36. John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*, volume 290. Macmillan London, 1976.
37. Flavia Bonomo-Braberman, Guillermo Durán, Martín D Safe, and Annegret K Wagler. On some graph classes related to perfect graphs: A survey. *Discrete Applied Mathematics*, 281:42–60, 2020.
38. Rajab Ali Borzooei and Hossein Rashmanlou. New concepts of vague graphs. *International Journal of Machine Learning and Cybernetics*, 8:1081–1092, 2017.
39. André Bouchet. Circle graph obstructions. *Journal of Combinatorial Theory, Series B*, 60(1):107–144, 1994.
40. Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.
41. Haitze J Broersma and Cornelis Hoede. Path graphs. *Journal of graph theory*, 13(4):427–444, 1989.

42. Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.
43. Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.
44. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.
45. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
46. Paul Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the second annual symposium on Computational geometry*, pages 169–177, 1986.
47. Brent N Clark, Charles J Colbourn, and David S Johnson. Unit disk graphs. *Discrete mathematics*, 86(1-3):165–177, 1990.
48. Bijan Davvaz, Naeem Jan, Tahir Mahmood, and Kifayat Ullah. Intuitionistic fuzzy graphs of n th type with applications. *Journal of Intelligent & Fuzzy Systems*, 36(4):3923–3932, 2019.
49. R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.
50. Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.
51. Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.
52. Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.
53. Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.
54. Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.
55. Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *ResearchGate*, July 2024.
56. Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.
57. Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.
58. Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.
59. Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.
60. Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.
61. Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.
62. A Nagoor Gani and K Radha. On regular fuzzy graphs. 2008.
63. B Gayathri and D Gnanavel. Structural coregraph of triple layered fuzzy graph.
64. Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.
65. Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.
66. Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.
67. Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.
68. David Guichard. An introduction to combinatorics and graph theory. *Whitman College-Creative Commons*, 2017.
69. Alfred L Guiffrida and Rakesh Nagi. Fuzzy set theory applications in production management research: a literature survey. *Journal of intelligent manufacturing*, 9:39–56, 1998.
70. Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.
71. Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.
72. B Mohamed Harif and TM Nishad. Interval valued weak fuzzy graphs and fermat's weak fuzzy graphs. *Advances and Applications in Mathematical Sciences*, 22(2):487–501, 2022.
73. Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.
74. Tong He, Yong Chen, and Kaiquan Shi. Weighted rough graph and its application. In *Sixth International Conference on Intelligent Systems Design and Applications*, volume 1, pages 486–491. IEEE, 2006.
75. Derek Allan Holton and John Sheehan. *The Petersen Graph*, volume 7. Cambridge University Press, 1993.
76. Tamás Horváth, Jan Ramon, and Stefan Wrobel. Frequent subgraph mining in outerplanar graphs. In *Proceedings of the 12th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 197–206, 2006.
77. Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.
78. S Satham Hussain, Isnaini Rosyida, Hossein Rashmanlou, and Farshid Mofidnakhaei. Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis. *Computational and Applied Mathematics*, 40(4):121, 2021.
79. Nathan Jacobson. *Structure of rings*, volume 37. American Mathematical Soc., 1956.
80. Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

81. L Jethruth Emelda Mary and R Kiruthikaswari. A study on fuzzy coloring in quadruple layered fuzzy graph. In *Fuzzy Mathematical Analysis and Advances in Computational Mathematics*, pages 91–97. Springer, 2022.
82. Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.
83. Cengiz Kahraman. *Fuzzy multi-criteria decision making: theory and applications with recent developments*, volume 16. Springer Science & Business Media, 2008.
84. Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.
85. WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.
86. Vahid Khatibi and Gholam Ali Montazer. Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition. *Artificial intelligence in medicine*, 47(1):43–52, 2009.
87. A Kuppan and J Ravi Sankar. Fuzzy zero divisor graph in a commutative ring. *TWMS Journal of Applied and Engineering Mathematics*, 11(Jaem Vol 11-Special Issue, 2021):42, 2021.
88. A Kuppan, J RAVI SANKAR, et al. Complete and cycle graph covers in a zero divisor graph. *Advances in Mathematics: Scientific Journal*, 9(8):5965–5970, 2020.
89. Lucas Lacasa, Bartolo Luque, Fernando Ballesteros, Jordi Luque, and Juan Carlos Nuno. From time series to complex networks: The visibility graph. *Proceedings of the National Academy of Sciences*, 105(13):4972–4975, 2008.
90. Alistair H Lachlan and Robert E Woodrow. Countable ultrahomogeneous undirected graphs. *Transactions of the American Mathematical Society*, pages 51–94, 1980.
91. Azriel Levy. *Basic set theory*. Courier Corporation, 2012.
92. Xiaowei Liao, Yong Xu, and Haibin Ling. Hypergraph neural networks for hypergraph matching. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 1266–1275, 2021.
93. Puyin Liu and Hong-Xing Li. *Fuzzy neural network theory and application*, volume 59. World Scientific, 2004.
94. Thomas G Lucas. The diameter of a zero divisor graph. *Journal of Algebra*, 301(1):174–193, 2006.
95. Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: new social network models. *Algorithms*, 12(11):234, 2019.
96. J Maiers and Yosef S Sherif. Applications of fuzzy set theory. *IEEE Transactions on Systems, Man, and Cybernetics*, (1):175–189, 1985.
97. Merle E Manis. Valuations on a commutative ring. *Proceedings of the American Mathematical Society*, 20(1):193–198, 1969.
98. L Jethruth Emelda Mary and U Gogulalakshmi. Domination and its related parameters in a triple layered fuzzy graph. In *Journal of Physics: Conference Series*, volume 1850, page 012060. IOP Publishing, 2021.
99. Hideyuki Matsumura. *Commutative ring theory*. Number 8. Cambridge university press, 1989.
100. Jixiang Meng. Connectivity of vertex and edge transitive graphs. *Discrete applied mathematics*, 127(3):601–613, 2003.
101. Jixiang Meng. Optimally super-edge-connected transitive graphs. *Discrete Mathematics*, 260(1-3):239–248, 2003.
102. Mesfer Ahmed Mesfer. Blockchain with single-valued neutrosophic hypersoft sets assisted threat detection for secure iot assisted consumer electronics. *International Journal of Neutrosophic Science*.
103. Sarbari Mitra and Soumya Bhoumik. Graceful labeling of triangular extension of complete bipartite graph. *Electron. J. Graph Theory Appl.*, 7:11–30, 2019.
104. John N Mordeson and Peng Chang-Shyh. Operations on fuzzy graphs. *Information sciences*, 79(3-4):159–170, 1994.
105. John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.
106. John N Mordeson, Sunil Mathew, and G Gayathri. *Fuzzy Graph Theory: Applications to Global Problems*, volume 424. Springer Nature, 2023.
107. John N Mordeson and Premchand S Nair. Applications of fuzzy graphs. In *Fuzzy graphs and fuzzy hypergraphs*, pages 83–133. Springer, 2000.
108. Susana Munoz, M Teresa Ortuno, Javier Ramírez, and Javier Yanez. Coloring fuzzy graphs. *Omega*, 33(3):211–221, 2005.
109. Jaroslav Nešetřil, Patrice Ossona de Mendez, and David R Wood. Characterisations and examples of graph classes with bounded expansion. *European Journal of Combinatorics*, 33(3):350–373, 2012.
110. Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.
111. Bogdan Nica. A brief introduction to spectral graph theory. *arXiv preprint arXiv:1609.08072*, 2016.
112. Mohamad Javad Nikmehr, Abdolreza Azadi, and Reza Nikandish. The weakly zero-divisor graph of a commutative ring. *Revista de la Unión Matemática Argentina*, 62(1):105–116, 2021.
113. TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.
114. TM Nishad, B Mohamed Harif, and A Prasanna. Group decision making in conditions of uncertainty using fermat's weak fuzzy graphs and beal's weak fuzzy graphs. *Ratio Mathematica*, 43:208–216, 2022.
115. S Nithishraj, AN Gani, and P Muruganantham. Domination and independence in fuzzy semigraphs. *Indian Journal of Science and Technology*, 17(34):3580–3588, 2024.
116. V Nivethana and A Parvathi. Mild balanced intuitionistic fuzzy graphs. *Int. Journal of Engineering Research and Application*, 7:13–20, 2017.
117. Madhumangal Pal, Sovan Samanta, and Ganesh Ghorai. *Modern trends in fuzzy graph theory*. Springer, 2020.
118. R Parvathi and MG Karunambigai. Intuitionistic fuzzy graphs. In *Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer, 2006.
119. R Parvathi, MG Karunambigai, and Krassimir T Atanassov. Operations on intuitionistic fuzzy graphs. In *2009 IEEE international conference on fuzzy systems*, pages 1396–1401. IEEE, 2009.

120. T PATHINATHAN and A KIRUPA. Intuitionistic k-partitioned fuzzy graph.
121. T Pathinathan and A Kirupa. k-partitioned fuzzy graph. *South East Asian Journal of Mathematics and Mathematical Sciences*, 16(01):223–240, 2020.
122. T Pathinathan and M Peter. Balanced double layered fuzzy graph. *International Journal of Multidisciplinary Research and Modern Education (IJMRME)*, 3(1):208–217, 2017.
123. T Pathinathan and J Jesintha Rosline. Double layered fuzzy graph. *Annals of pure and applied mathematics*, 8(1):135–143, 2014.
124. T Pathinathan and J Jesintha Rosline. Matrix representation of double layered fuzzy graph and its properties. *Annals of Pure and Applied Mathematics*, 8(2):51–58, 2014.
125. A Prasanna and TM Nishad. Some properties of weak fuzzy graphs. *J. Math. Comput. Sci.*, 11(3):3594–3601, 2021.
126. K Radha and P Renganathan. Effective fuzzy semigraphs. *Adv Appl Math Sci*, 20(5):895–904, 2021.
127. K RADHA and P RENGANATHAN. Isomorphic properties of fuzzy semigraphs. 2022.
128. M Rajeshwari, R Murugesan, and KA Venkatesh. Balanced double layered bipolar fuzzy graph. *Technology*, 10(02):743–751, 2019.
129. Hossein Rashmanlou and Rajab Ali Borzooei. Vague graphs with application. *Journal of Intelligent & Fuzzy Systems*, 30(6):3291–3299, 2016.
130. Neil Robertson and Paul D. Seymour. Graph minors. i. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.
131. Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.
132. J Jesintha Rosline and T Pathinathan. Intuitionistic double layered fuzzy graph. *ARPN Journel of Engineering and Applied Science, 10 (12)(2015) 5413*, 5417, 2006.
133. J Jesintha Rosline and T Pathinathan. Structural core graph of double layered fuzzy graph. *Intern. J. Fuzzy Mathematical Archive*, 8(2):59–67, 2015.
134. J Jesintha Rosline and T Pathinathan. Triple layered fuzzy graph. *Int. J. Fuzzy Mathematical Arhive*, 8(1):36–42, 2015.
135. Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.
136. AA Salama and SA Alblowi. Neutrosophic set and neutrosophic topological spaces. 2012.
137. Sovan Samanta and Madhumangal Pal. Irregular bipolar fuzzy graphs. *arXiv preprint arXiv:1209.1682*, 2012.
138. Sovan Samanta and Madhumangal Pal. Fuzzy planar graphs. *IEEE Transactions on Fuzzy Systems*, 23(6):1936–1942, 2015.
139. Jiao Shi, Xin Zhao, Nan Zhang, Yu Lei, and Lingtong Min. Rough-fuzzy graph learning domain adaptation for fake news detection. *IEEE Transactions on Computational Social Systems*, 2023.
140. Prem Kumar Singh. *Intuitionistic Plithogenic Graph*. Infinite Study, 2022.
141. Muzzamal Sitara, Muhammad Akram, and Muhammad Yousaf Bhatti. Fuzzy graph structures with application. *Mathematics*, 7(1):63, 2019.
142. Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
143. Florentin Smarandache. *Definitions derived from neutrosophics*. Infinite Study, 2003.
144. Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.
145. Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.
146. Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.
147. Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.
148. Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
149. Florentin Smarandache. *Indeterminacy in neutrosophic theories and their applications*. Infinite Study, 2021.
150. Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.
151. Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.
152. Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.
153. Florentin Smarandache, Siti Nurul Fitriah Mohamad, and Roslan Hasni. Novel concepts on domination in neutrosophic incidence graphs with some applications. 2023.
154. Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.
155. P Srivastava and RL Gupta. Fuzzy proximity structures and fuzzy ultrafilters. *Journal of Mathematical Analysis and Applications*, 94(2):297–311, 1983.
156. Ben C Stöver and Kai F Müller. Treegraph 2: combining and visualizing evidence from different phylogenetic analyses. *BMC bioinformatics*, 11:1–9, 2010.
157. Abolfazl Tehranian and Hamid Reza Maimani. A study of the total graph. 2011.
158. Nenad Trinajstic. *Chemical graph theory*. CRC press, 2018.
159. Stephan Wagner and Hua Wang. *Introduction to chemical graph theory*. Chapman and Hall/CRC, 2018.
160. Danyang Wang and Ping Zhu. Fuzzy rough digraph based on strength of connectedness with application. *Neural Computing and Applications*, 35(16):11847–11866, 2023.

161. Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.
162. Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.
163. Yiding Yang, Xinchao Wang, Mingli Song, Junsong Yuan, and Dacheng Tao. Spagan: Shortest path graph attention network. *arXiv preprint arXiv:2101.03464*, 2021.
164. Naveed Yaqoob, Muhammad Gulistan, Seifedine Kadry, and Hafiz Abdul Wahab. Complex intuitionistic fuzzy graphs with application in cellular network provider companies. *Mathematics*, 7(1):35, 2019.
165. Jun Ye. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. *Neural computing and Applications*, 26:1157–1166, 2015.
166. Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
167. Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.
168. Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.
169. Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.
170. Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.
171. Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.
172. Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

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