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*"***Neutrosophic Systems with Applications***"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

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This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e., notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjointed two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of *]-0, 1+[*.

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Contents

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BER Analysis of BPSK Modulation Scheme for Multiple Combining Schemes over Flat Fading Channel

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Abstract: Focus of the study was to provide error-free communication in mobile communication with higher data rates, spectral efficiency, and energy efficient. Basically, work was done to investigate the performance of the Binary Phase Shift Keying (BPSK) modulation technique for multiple combining schemes and the behavior of signal in wireless communication where multipath propagation and uncertainty in the system. We use the Multiple-input Multiple-output (MIMO) and antenna diversity to get many copies of the same signal; some of them were faded but some had sufficient information. Then the next step was to combine or select the best signal to achieve the optimum results from them. Moreover, the Bit Error Rate (BER) of multiple combining schemes for 2*1 and 2*2 were found and the results of each combining scheme were compared. Later on, the comparison of these combining schemes was collectively represented; it is very important because getting the required data from the received copies of the same signal is not easy.

Keywords: BER Analysis; BPSK; Modulation Scheme; MIMO; Antenna Diversity.

1. Introduction

As new technologies are invented the users grow faster; because of the limitations of physical connections in wireless networks it generates a challenging task for researchers [1]. The main challenge is meeting the demand for high-quality wireless services while taking into account the frequency spectrum that is limited and expensive, it is required that it should be operating as efficiently as possible [2]. Due to the multipath propagation in wireless communication, the transmitted data experience fading, co-channel interference, and uncertainty because of outsiders accessing your network and the most credible situation is that the reception is affected by large distances, obstacles, and interference [3]. Now a day's, everyone demands fast data rates, which has boosted demand for technologies that supply bigger capacities and reliable links which should be targeted by given current systems [4]. Multipath signal propagation in wireless communication is a basic source of fading; the abrupt and random change in the received signal is known as fading [5]. The signal experiences diffraction, reflection, and refraction as it travels across a radio channel and the communication environment changes rapidly and adds more complexity to the channel response, especially in suburban and metropolitan areas where cell phones are mostly used [6]. Antenna diversity is a good scheme to reduce the effects of fading [7], hence increasing the system routine performance, reliability, and expanding the capacity of the channel [8]. While several antennas are employed at the receiver in receiver diversity, many antennas are a main component of the transmitter portion in transmitter diversity; communication benefits from the technique of modulation, which transmits data through varying low-powered signals [9]. The main goal of utilizing the modulation technique is to increase data rate with better quality while using the least amount of bandwidth and signal power when there are channel imperfections possible [10]. The majority of first-generation systems, which use an analog transmission technique, have been introduced in the middle of the 1980s; the disadvantages of this technology are the relatively low data rate and very poor noise, these are the drawbacks of this scheme[11]. In the early 1990s, the second-generation system that uses digital communication was introduced and it has a lot of advantages over analog technology, which enhances communication system performance [12]. The adoption of the digital modulation method depends on factors including power efficiency, spectrum efficiency, and bit error rate performance [13]; while designing a modulation scheme, power, and spectral efficiency are always trade-offs. Furthermore, more bandwidth and strong signal strength can also be assigned to achieve improved Bit Error Rate (BER) performance [14]. Mobile communication has been evolving since 1990; the goal of the next generation is to achieve a wider bandwidth, a high data rate, and a seamless handoff [15].

The main work for researchers is on offering flawless services across a large wireless network and the next generation of mobile communication systems will be able to offer a comprehensive solution, delivering voice, data, and streamed multimedia to consumers when needed at high data rates [15]. It will mark a significant development toward ubiquitous communications networks and seamless, high-quality communication services [16]. Additionally, Multiple-input Multiple-output (MIMO) communication systems can accomplish the majority of our goals [17]. These concepts of a wireless communication link create a new avenue for reliable communication and significantly increase system performance and dependability [18]. The idea behind MIMO is to get multiple copies of the same signal by designing the transmit antennas at one end and the receive antennas at the other[19]; after combining these signals the BER and SER for each user are enhanced [20]. Such technologies gained a lot of interest in mobile communication as a result of this tremendous capacity growth and it makes use of the particular diversity obtained in a dense multipath scattering environment by specially spaced antennas [21]. Theoretical investigations have shown that the number of transmit antennas utilized causes a sudden shift in the capabilities of MIMO systems if we add more antennas, BER and SER will also be modified [22].

This research work presents the performance analysis of a system for multiple combining schemes and the present communication system receives multiple copies of the same signal at the receiving end. In this scenario, for Binary Phase Shift Keying (BPSK) modulation scheme in a flat fading channel, using a space-time block coding scheme and multiple diversities in the presence of channel-estimation error is studied for different combining schemes. BER analysis of wireless communication over fading channels is an important performance metric to measure the quality and full end-to-end system performance including transmitter, receiver, and transmission medium between them.

2. System Model and Performance Analysis

In this article, we discussed the performance analysis and mathematical modeling of multiple diversity techniques with space time coding scheme in Rayleigh channel. We presented transmit diversity and also discussed linear combination schemes which are less complex with the assumptions that the channel state information at receiver, two relays and a destination node over Rayleigh fading channel. Due to multipath propagation and fading changes with time in wireless communication introduce uncertainty and the communication will be non-comparable with the fiber cable, coaxial cable and in satellite communication [23]. There are multiple ways to reduce the fading effect but here we used antenna diversity. In MRRC scheme for 1*2, it mean one antenna at transmitter and two at receiver are used but we get same results by using 2*1, two antennas as transmitter and one receiver. After this same result it can be made general for two antenna diversity at transmission and for any M number of antennas at receiver.

2.1 BER and its Probability (P_B) for BPSK

It is an important measure of performance used for comparing digital modulation schemes [24]; the total number of bit errors per unit at a time time is known as bit error rate. It is determined by dividing the bit errors by the total bits transmitted during the course of the time period under consideration and the expected value of the bit error ratio is the probability of bit error [25].On the other hand, bit error rate can be considered as approximate estimation of probability of bit error rate; this estimation is accurate for long time interval and high bit errors. Suppose that the transmitted signals are W_1 and W_2. Then the probability of BER of transmitted signals will be

$$p_B = -\int_{(a_1-a_2)/2\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{u^2}{2}\right) du = Q\left(\frac{(a_1-a_2))}{2\sigma_0}\right) \quad (1)$$

2.2 Selective Combining Diversity

This approach provides the best criteria to trade off of receiver complexity and performance of any system as the receiver will pick the signal with highest SNR value [26]. To explain this technique, consider the single transmit antenna and multiple receiving antennas. There are N multiple copies of transmitted symbols. By using the SC we combine these copies of data and extract the optimal solution [27] and we already know that the BER with BPSK modulation scheme in AWGN channel is

$$P_b = \frac{1}{2} erfc(\sqrt{\frac{Eb}{N_o}}) \quad (2)$$

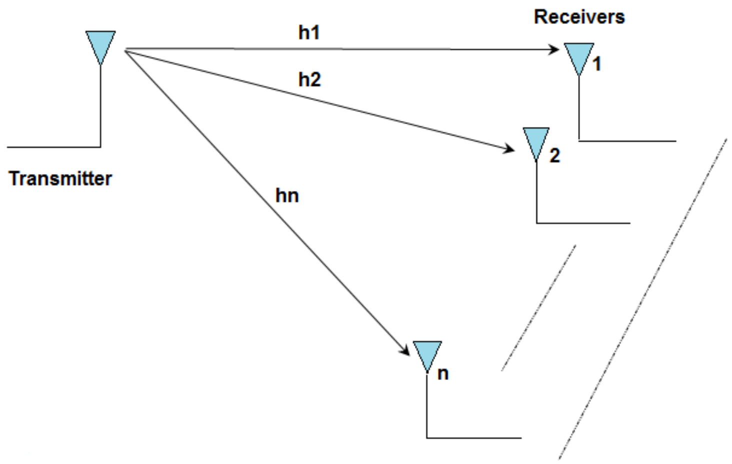

Figure 1. Receiving diversity in a wireless link.

The Figure 1 represents receiving diversity in a wireless link, suppose that r is the effective bit energy to noise ratio by this scheme, then the total BER will be integration of all possible BER values of r.

$$P_b = \int_0^{\infty} \frac{1}{2} erfc(\sqrt{r}) P(r) dr \quad (3)$$

$$= \int_0^{\infty} \frac{1}{2} erfc(\sqrt{r}) \frac{N}{Eb/N_o} \; e^{-\frac{r}{Eb/N_o}} \left[1 - e^{-\frac{r}{Eb/N_o}}\right]^{N-1} dr \quad (4)$$

After solving this equation will become

$$P_e = \frac{1}{2} \Sigma_{k=0}^{N} (-1)^k \binom{N}{K} \left(1 + \frac{k}{Eb/N_o}\right)^{-1/2} \quad (5)$$

2.3 Maximal ratio combining diversity scheme (MRC)

In this technique, each copy of received signal is multiplied by the branch factor which directly proportional to the amplitude of signal [28]. It further boosts up the strong signal and attenuated the weak signal. In this approach, all branch signals are taking and the gain of all branches will be proportional to the RMS value of signal, moreover it is inversely proportional to mean square of given channel. The signals from all branches will be weighted as compared to their SNR's and added them and all signals will be considered for phase alignment before adding them. At receiver the signal will be,

$$r_1=h_1s_0+n_1 \tag{6}$$

$$r_2=h_2s_0+n_2 \tag{7}$$

After combining signals,

$$s_0=h_1^*r_1+h_2^*n_2 \tag{8}$$

The BER for BPSK is

$$P_b=\frac{1}{2}erfc\left(\sqrt{\frac{Eb}{N_o}}\right) \tag{9}$$

The effective BER to noise ratio with MRC is r, the total BER will integral over all possible values of

r,

$$P_e=\int_0^\infty \frac{1}{2}erfc(\sqrt{r})P(r)dr \tag{10}$$

$$=\int_0^\infty \frac{1}{2}erfc(\sqrt{r})\frac{1}{(N-1)!\left(\frac{Eb}{N_o}\right)^N}\ r^{N-1}\left[e^{\frac{-r}{Eb/N_o}}\right]\ dr \tag{11}$$

This will be reduces into

$$P_e=P^N\ \Sigma_{k=0}^{N-1}\left(\binom{N-1+k}{K}(1-p)^k\right) \tag{12}$$

Where

$$P=\frac{1}{2}-\frac{1}{2}\left(1+\frac{1}{Eb/N_o}\right)^{-1/2} \tag{13}$$

2.4 Equal Gain Combining Diversity

According to MRC technique, there should be exact estimate of channel amplitude gain, which increases the complexity of receiver[29] but in this approach, it considers all the signals equally after coherent detection. This is very simple diversity method, because it simply added all coherent detected signals and given to the decision device. The exact estimation of fading will not be required for any receiver, so that why its complexity automatically decrease. The BER by using this diversity with two receiving antennas are,

$$P_e=\frac{1}{2}\left[1-\frac{\sqrt{Eb/N_o\left(\frac{Eb}{N_o}+2\right)}}{Eb/N_{o+1}}\right] \tag{14}$$

3. Performance Analysis of Various Combining Schemes over Rayleigh Fading Channel

Radio spectrum is becoming more significant for service providers as a result of the examination of new mobile communication technologies [30]. Therefore, there is a constant search for digital modulation techniques that will be efficient with bandwidth and have very low BER [31]. Due to the benefits of analog modulation systems, researchers are now focusing their study on digital modulation techniques. On the other hand, different modulation techniques are chosen depending on the application, cost, power efficiency, bandwidth efficiency, error rate, and other performance characteristics of digital modulation techniques are in conflict with one another. We simultaneously cannot optimize everything.

The best usage strategy is one that offers these requirements the best trade-off. Amplitude Shift Keying (ASK), Frequency Shift Keying (FSK), Phase Shift Keying (PSK), and Quadrature Amplitude Modulation (QAM) will be the major modulation methods. The carrier in digital modulation is analog in nature, while the signal is digital in nature. There are multiple parameters, including amplitude, frequency, and phase of analog carrier signal are changing based on the base band digital signal [32]. As a result of its poor quality, ASK are utilized in slow-moving communication processes like telemetry circuits. FSK has very low error performance when there is channel noise. Phase Shift Keying (PSK), one of several digital modulation schemes, performs better in terms of errors and band width efficiency. While the carrier's peak amplitude and fundamental frequency will not change, the carrier's phase will change in response to the baseband digital signal in PSK to represent multiple signals[33]. Modulation schemes are studied because of their numerous performances of valuable characteristics. As the number of M rises, the error performance of 8-PSK and 16-QAM over AWGN channel improves [34]. It has been determined through performance analysis of various M-ary PSK modulation schemes, including BPSK, QPSK, and 8-PSK, that error rate lowers as the value of M rises.

3.1 BPSK modulation scheme with Maximum likelihood (ML) combining scheme

The results as seen in the graph with 2*2 ML equalization helped us to get the good performance which is comparable with other schemes. These results are very closely related with the 1*2 MRC scheme. If we want to compute the results for higher constellation, then ML combining scheme will become very complex. For example, in case of 64 QAM we have to find the minimum from = 4096 combinations. So that’s why it might be difficult to compute for higher order constellation. Figure 2 represents BER for BPSK modulation with 2x1, 2x2 MIMO and ML equalizer by using Rayleigh fading channel.

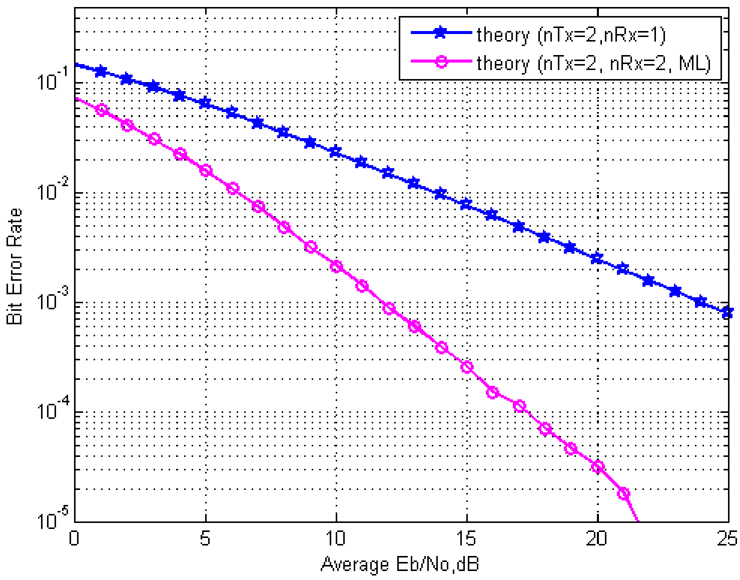

Figure 2. BER for BPSK modulation with 2x1, 2x2 MIMO and ML equalizer by using Rayleigh fading channel.

3.2 BPSK modulation scheme with Equal Gain Combining (EGC) scheme

The results as seen in the Figure 3 are much correlated with theoretical results. It is also same with the MRC scheme but the gain of all the branches are set to equal and cannot changed. If we use multiple antennas the branch signals are multiplied with the same branch gain. We see that for calculation the bit error rate with respect to E_b/N_o, have lessor value for 2*2 EGC as compared with 2*1 combining scheme. So by the analysis of this it can be conclude that the performance of the EGC receiver is better than selection combining scheme and marginally close to the MRC. This scheme is practically better to use because of its complexity is less as compared with the optimum MRC. In this technique there is no knowledge about the amplitude of each branch signal.

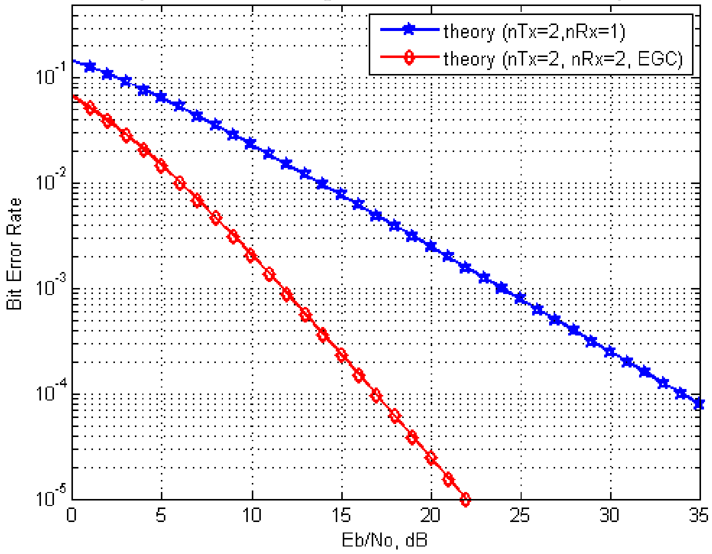

Figure 3. BER for BPSK modulation with 2*1, 2*2 MIMO using Equal Gain Combining in Rayleigh channel.

3.3 BPSK modulation scheme with Maximum Ratio Combining (MRC) scheme

In this technique each branch signals are considered and weighted according to their instantaneous energy to noise ratios. The branch signals are co-phased before summation to insure that the signals which have to be considered will be in- phase. The resultant signal consider as received signal and forward to the demodulator. As we seen that the performance of 2*2 MRC is better as compared with the 2*1 scheme. It is very complicated and correct estimates of signal level and average noise power will be required to get better performance. By this scheme the improvement can be made if both branches are completely correlated.

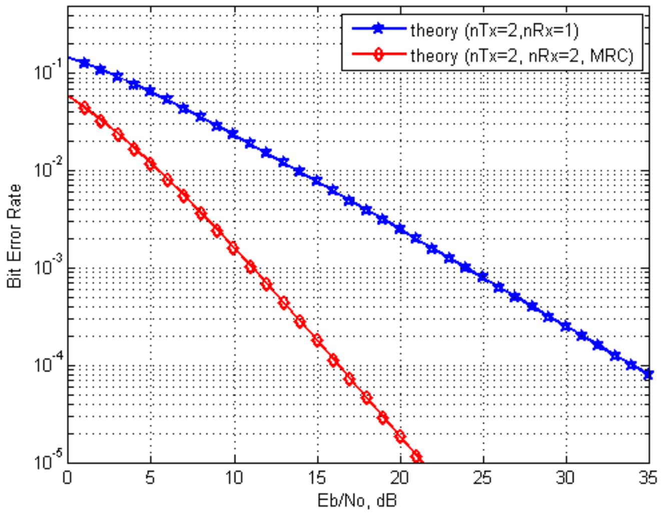

Figure 4. BER for BPSK modulation with 2*1, 2*2 MIMO using Maximum Ratio Combining in Rayleigh channel.

3.4 BPSK modulation scheme with Selection Combining (SC) scheme

The branch signal with higher signal to noise ratio is selected from the receiving signals by the SC scheme. The selected signal then forward to the demodulator. We see that there is large difference of BER between 2*1 and 2*2 selection combining scheme. Generally for larger the number of receiving branches, there is more probability that having greater SNR at the output. This technique is comparable with other combining schemes and easy to implement.

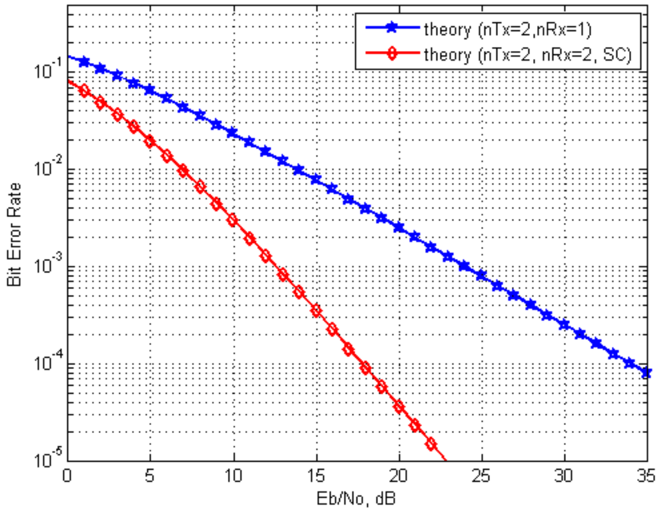

Figure 5. BER for BPSK modulation with 2*1, 2*2 MIMO using Selection Combining in Rayleigh channel.

3.5 BPSK modulation scheme with Zero Forcing Combining (ZF) scheme

In the scheme, it tries to nullify the received signals which are interfered. So the diagonal terms which are not zero tries to be zero due to ZF equalizers. Moreover to solve the $w_1$the interference due to w_2 is tries to be cancelled out and vice versa. Due to this technique there can be noise amplification. On the other hand, however it is easy and simple to implement. Moreover, it is clear that the signal from spatial dimension is a like 1*1 diversity. So that's why the BER analysis for 2*2 and 2*1 ZF combining scheme with Rayleigh fading channel is same.

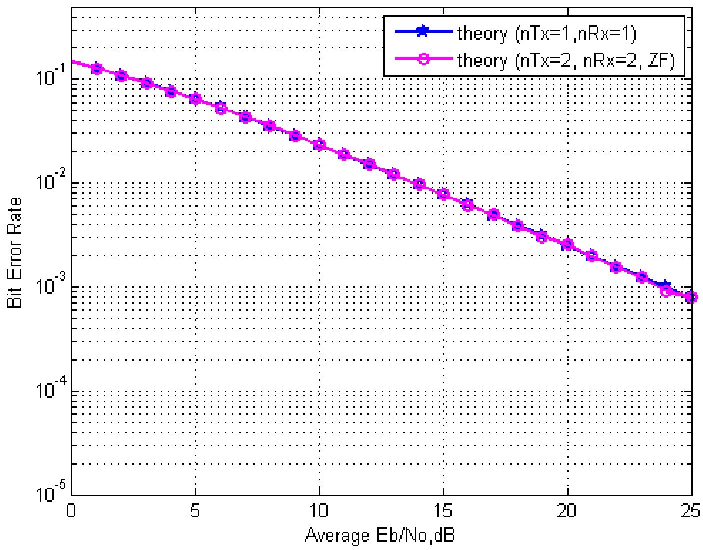

Figure 6. BER for BPSK modulation with 2*1, 2*2 MIMO using Selection Combining in Rayleigh channel.

4. BER Comparison of Multiple Combining Techniques

In previous section each combining scheme was compare with different receiving antenna diversity but now we will compare the performance of the multiple techniques in terms of efficiency in BER and complexity. As expected the best improvement is for MRC, while the worst is for ZF combining scheme. Here we determine the code by which multiple combining schemes can be compared. In terms of the required processing, the selection combining scheme should be easiest, because it required only the value of SNR of each signal, not the phase and amplitude, this combiner also needs not be coherent. On the other hand, MRC and EG combiner required the phase information. The MRC also require the accurate information of the gain too. So that's why this is difficult to implement, because the range of Rayleigh fading signal is very large.

4.1 BPSK modulation scheme with Maximum Mean Square Error (MMSE) scheme and zero forcing

In the MMSE, it tries to minimize the interference between symbols and recover the signal having good SNR. This equalizer also proves the BER properties of the recovered signal. Moreover it should be clear that it is an equalizer which minimizes the mean square error. As seen in the graph, it is very comparable with the zero forcing equalil2zer, actually if noise term is zero the MMSE equalizer reduces to ZFE. However this scheme is simple but not practically good as other combining schemes. Moreover, it is observed that if the number of transmitter kept constant and increases of receivers the BER will be decreases.

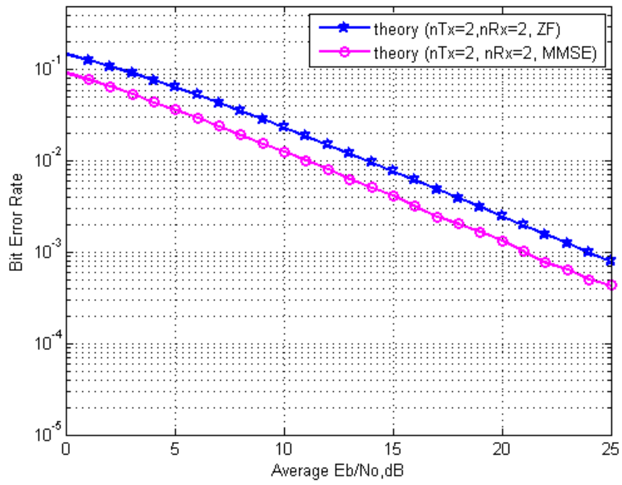

Figure 7. BER for BPSK modulation with MIMO using ZF and MMSE Combining scheme in Rayleigh channel.

4.2 EGC, MRC and ML combining scheme

We will compare here three combining schemes on the bases of BER and complexity. As seen in the diagram that BER improvement is a function of number of elements. Moreover the best improvement is only for MRC and worst for ML for 2*2 MIMO. There is much improvement in EGC scheme which is comparable with MRC and other combining schemes. But there is a problem with these two combining schemes is that because of complexity and therefore not good for abruptly changing environment. That's why these two combining schemes are not uses for ultra-high frequency and mobile communication because the channel is not properly co-phased and tracked. On the other, hand ML combining is very simple that the combiner compares the received signals with the pre-defined reference signal value in terms of BER. It selects the signals with which is very close to the reference signal. This technique is very easy to use as compared to above mentioned schemes.

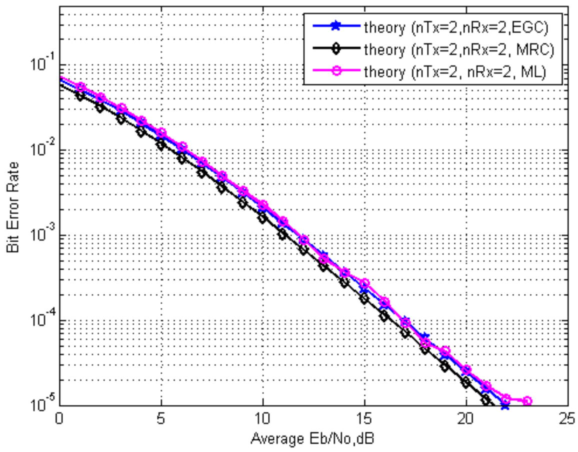

Figure 8. BER comparisons for BPSK modulation with MIMO using EGC, MRC and MMSE combining scheme in Rayleigh channel.

The similarity between EGC and MRC is that all the branches should be co-phased, but because we know that in MRC all branches are consider and weighted on the basis of their amplitude's. On the other hand, in EGC it not likes that of MRC and only channel vector will be required.

4.3 *ZF, MRC, MMSE and EGC scheme*

As seen in Figure, it is clear that which technique is better to use and which one is not. There is too much difference of BER between these four schemes. It is clear for designers of the network that which technique is better and which one not. For 2*2 MIMO the ZE and MMSE combining scheme having BER is very high as compared with the MRC and EGC schemes. But the BER of MMSE and ZF are close to each. If the noise is minimized by MMSE then it will show the results very close to the ZF. We observed that the BER for BPSK is same in MRC and EGC scheme for 2*2 antenna's but is not same with higher of receiving antenna's. Then result will be different. This situation is same with other combining schemes.

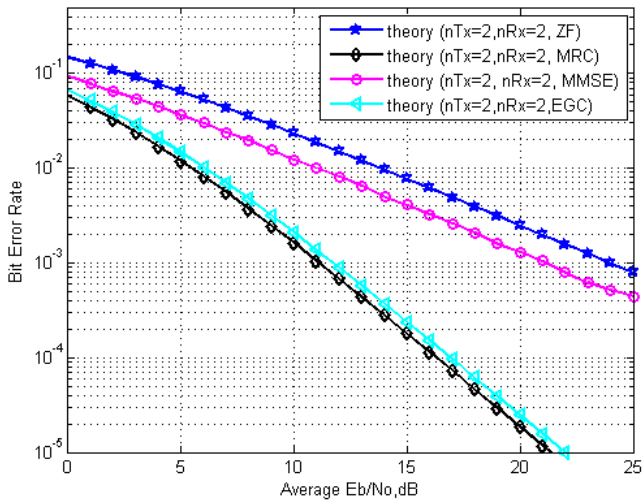

Figure 9. BER comparisons for BPSK modulation with MIMO using EGC, MRC MMSE and ZF Combining scheme in Rayleigh channel.

4.4 *EGC, MRC, ML and SC technique*

There is an interesting result which we observe in the above Figure is that the results of SC and EGC scheme are exactly same but complexity is different. In the selection combining scheme it selects the signal with higher BER and does not require the amplitude and phase of the received signals. But on the other hand, in MRC and EGC it requires the phase and amplitude. It is clear that MRC show good BER performance but it also requires accurate measurement of gain. So it difficult to implement because there is wide range of Rayleigh fading signal. For this additional cost in MRC it improves only 0.6db over the EGC scheme at a BER of 1%.

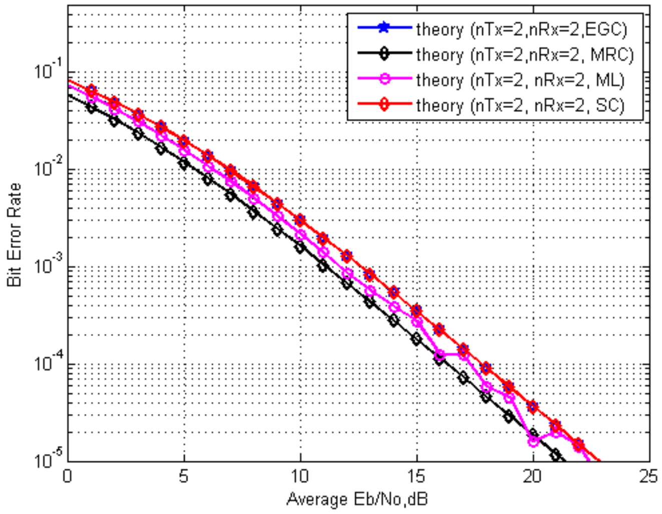

Figure 10. BER comparisons for BPSK modulation with MIMO using EGC, MRC, SC and ML Combining scheme in Rayleigh channel.

5. Conclusions

The focus of the study was on the mathematical analysis of different combining techniques in wireless fading environments. More important and specifically discussed in this study was the BER analysis of the BPSK modulation technique with MIMO by using multiple diversity combining schemes in the environment where the signal strength is constantly changing due to wireless communication. One of the important contributions of this work was the development of a generalized mathematical framework to compute the BER of multiple combining schemes by using MIMO. To complete this study, a space-time block coding scheme is applied for 2*1 and 2*2 MIMO, and the results of these schemes are presented mathematically and graphically. On the other hand, the result shows that the SC and EGC schemes are also good for use because the results are very close to the optimum MRC and are easy to use. Moreover, from the obtained results it is shown that the diversity combiner using EGC for multipath diversity gives satisfactory throughput results with much lesser implementation complexity than the MRC approach.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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An Introduction to Bipolar Pythagorean Refined Sets

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Abstract: The aim of this paper is to introduce the new concept of a Bipolar Pythagorean refined set by combining the two notions called a Bipolar Pythagorean set and a Pythagorean refined set. Also, basic operations and algebraic properties of the Bipolar Pythagorean refined set are discussed with suitable examples.

Keywords: Pythagorean Set; Bipolar Pythagorean Set; Pythagorean Refined Set; Bipolar Pythagorean Refined Set.

1. Introduction

Fuzzy sets were first initiated by Zadeh [24] and he examined the membership function. After introducing some more concepts with fuzzy set theory, Atanassov [1, 2, 3, 4] generalized and introduced the new concept called intuitionistic fuzzy set (IFS) which is a generalized form of FS. Atanassov [5, 6] extended the set to Intuitionistic fuzzy Multi-dimensional sets. Also, Intuitionistic fuzzy topological spaces were introduced by Coker [11].

Yager [22] familiarized the model of Pythagorean fuzzy sets. Peng and Yang [20] presented the basic operators for PFNs. In [21, 23] similarity measures, distance measures, and multiple decision-making problems of Pythagorean fuzzy sets were discussed.

Bosc and Pivert [9] stated "Bipolarity refers to the tendency of the human mind to make decisions on the basis of positive and negative effects. Positive information states what is desired, satisfactory, possible, or considered as being acceptable. At the same time, negative statements express what is rejected, impossible, or forbidden. Negative preferences correspond to constraints while positive preferences correspond to wishes, Later Lee [15] introduced the concept of bipolar fuzzy sets which is a generalization of the fuzzy sets. Recently, bipolar fuzzy models have been studied by many authors on algebraic structures. Chen et. al. [10] studied of m-polar fuzzy set. Then, they examined many results which are related to these concepts and can be generalized to the case of m-polar fuzzy sets. They also proposed numerical examples to show how to apply m-polar fuzzy sets in real-world problems. In [19] Naeem discussed Pythagorean m polar fuzzy sets. In [12, 14] Florentin Smarandache introduced the concept of neutrosophic refined and bipolar neutrosophic sets, as an extension of this [13] Smarandache came with the topic Bipolar Neutrosophic refined sets. R. Jhansi [16] introduced the concept of bipolar Pythagorean fuzzy sets which is an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets. The score function, accuracy function, and some basic operators are also discussed in this paper with real-life applications.

Contrary to ordinary sets, multisets permit us to have multiple occurrences of the members. Blizard [7, 8] introduced multiset theory as a generalization of crisp set theory. As an extension of multiset, Yager introduced the notion of fuzzy multiset (FMS). Muhammad Riaz, Khalid Naeem, Xindong Peng, Deeba Afzal [17] introduced Pythagorean fuzzy multisets that have real-life applications by applying the concept of multiple-valued logic. Pythagorean fuzzy multisets provide a strong mathematical model to deal with multi-attribute group decision-making (MAGDM). While tackling real-world problems, intuitionistic fuzzy multiset cannot deal with the situation if the sum of the membership degree and non-membership degree of the parameter gets larger than 1. It makes decision-making demarcated and affects the optimum decision. PFM sets assist us in handling such situations. [18] Muhammad Saeed explained the properties, Set-Theoretic Operations, and Axiomatic results of Refined Pythagorean fuzzy sets.

In this paper, we introduce the concept of a bipolar Pythagorean refined set which is the combination of the bipolar Pythagorean fuzzy sets and Pythagorean refined sets. Also, we give some basic operators and algebraic properties of bipolar Pythagorean refined set operations with desirable examples.

2. Preliminaries

In this section, we recall the basic definitions and related results for developing the desired set.

Definition 2.1. (Fuzzy set) [23] Let M be a fixed set, then a fuzzy sets Q in M can be define as: Q = {(m, μ_Q (m)) / m ϵ M } Where μ_Q : M → [0,1] is called the membership degree of m ϵ M.

Definition 2.2: (Pythagorean Fuzzy set) [21] Let X be a non-empty set and I the unit interval [0, 1]. A PF set S is an object having the form $P=\{\langle x,\mu_p(x),v_p(x)\rangle : x\in X\}$ where the functions $\mu_p(x): X\to[0,1]$ and $v_p(x): X\to[0,1]$ denote respectively the degree of membership and degree of non-membership of each element x ϵ X to the set P, and $0\le(\mu_p(x))^2+(v_p(x))^2\le 1$ for each x ϵ X.

Definition 2.3: (Bipolar Pythagorean Fuzzy set) [16] Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A=\{\langle x,(\mu_A^P,\eta_A^P),(\mu_A^N,\eta_A^N)\rangle : x\in X\}$ where $\mu_A^P: X\to[0,1]$, $\eta_A^P: X\to[0,1]$ $\mu_A^N: X\to[-1,0]$, $\eta_A^N: X\to[-1,0]$ are the mappings such that

$0\le(\mu_p(x))^2+(v_p(x))^2\le 1$,

$-1\le-((\mu_p(x))^2+(v_p(x))^2)\le 0$ and

μ_A^P denote the positive membership degree, η_A^P denote the positive non-membership degree, μ_A^N denote the negative membership degree and η_A^N denote the negative non membership degree.

Definition 2.4. [16] Let $A=\{\langle x,(\mu_A^P,\eta_A^P),(\mu_A^N,\eta_A^N)\rangle : x\in X\}$ and $B=\{\langle x,(\mu_B^P,\eta_B^P),(\mu_B^N,\eta_B^N)\rangle : x\in X\}$ be two BPFSs, then their operations are defined as follows:

(i) $A\cup B=\{x,\max(\mu_A^P,\mu_B^P),\min(\eta_A^P,\eta_B^P),\min(\mu_A^N,\mu_B^N),\max(\eta_A^N,\eta_B^N): x\in X\}$

(ii) $A\cap B=\{x,\min(\mu_A^P,\mu_B^P),\max(\eta_A^P,\eta_B^P),\max(\mu_A^N,\mu_B^N),\min(\eta_A^N,\eta_B^N): x\in X\}$

(iii) $A^C=\{x,(\eta_A^P,\mu_A^P),(\eta_A^N,\mu_A^N): x\in X\}$

Definition 2.5. (Refined Pythagorean Fuzzy Set) [18] A Refined Pythagorean fuzzy set (rpfs) A_{RP} in U is given by $A_{RP}=\{\langle x,{\mu_A}^{\alpha}(x),{v_A}^{\beta}(x)\rangle : \omega\in N_1^{\gamma},\lambda\in N_1^{\delta},\alpha+\beta\ge 3,u\in U\}$ where $\alpha,\beta\in N$ such that

${\mu_A}^{\alpha}(x),{v_A}^{\beta}(x): U\to I$ with the condition that, $0\le\sum_{\alpha=1}^{\mu}(\mu_A^{\alpha})^2+\sum_{\beta=1}^{\delta}(v_A^{\beta})^2\le 1$.

3. Bipolar Pythagorean Refined Sets

Despite the fact that electric cars have the potential to greatly decrease GHG emissions and enhance air quality, there are still obstacles that must be overcome before their widespread adoption can be achieved.

Definition 3.1. (Bipolar Pythagorean refined set) Let X be the non - empty set in U. A Bipolar Pythagorean refined set (in short BPRs) A_{BPR} on X can be defined by the form

$$A_{BPR} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{A_{BPR}}(x), \vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{A_{BPR}}(x), \vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

Where ,

$$\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),\ldots.,\varsigma^{P+}_{A_{BPR}}(x), \vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),\ldots,\vartheta^{P+}_{A_{BPR}}(x) : X \to [0,1]$$

$$\varsigma^{1-}_{A_{BPR}}(x),\ \varsigma^{2-}_{A_{BPR}}(x),\ldots,\varsigma^{P-}_{A_{BPR}}(x)\ ,\ \vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),\ldots,\vartheta^{P-}_{A_{BPR}}(x) :\quad X \to [-1,0]$$

Such that
$$0 \le (\varsigma^{i+}_{A_{BPR}}(x))^2 + (\vartheta^{i+}_{A_{BPR}}(x))^2 \le 1$$

$$-1 \le -(\varsigma^{i-}_{A_{BPR}}(x))^2 + (\vartheta^{i-}_{A_{BPR}}(x))^2 \le 0$$ for i=1,2,….,p for any element x ϵ X

$\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{A_{BPR}}(x)$ denote the positive membership degree.

$\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{A_{BPR}}(x)$ denote the positive non membership degree.

$\varsigma^{1-}_{A_{BPR}}(x),\ \varsigma^{2-}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{A_{BPR}}(x)$ denote the negative membership degree.

$\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{A_{BPR}}(x)$ denote the negative non membership degree.

***Definition** 3.2.* **(Subset)** Let A_{BPR}, B_{BPR} ϵ BPRS(X) ,where

$$A_{BPR} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$B_{BPR} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x),\ \varsigma^{2+}_{B_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x),\ \vartheta^{2+}_{B_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x),\ \vartheta^{2-}_{B_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{B_{BPR}}(x)) : x \in X \}$$

Then A_{BPR} is said to be BPR Subset of B_{BPR} and is denoted by $A_{BPR} \subseteq B_{BPR}$ if

$$\varsigma^{i+}_{A_{BPR}}(x) \le \varsigma^{i+}_{B_{BPR}}(x),\ \vartheta^{i+}_{A_{BPR}}(x) \ge \vartheta^{i+}_{B_{BPR}}(x)$$

$$\varsigma^{i-}_{A_{BPR}}(x) \geq \varsigma^{i-}_{B_{BPR}}(x), \quad \vartheta^{i-}_{A_{BPR}}(x) \leq \vartheta^{i-}_{B_{BPR}}(x)$$

for every x ϵ X and i=1,2,…..,p

Example 3.3:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x, ([0.3,.0.4,0.6],[0.5,0.2,0.8])([-0.6,-0.5,-0.4],[-0.2,-0.4,-0.3])\rangle : x \in X\}$$

We can say that $A_{\text{BPR}} \subseteq B_{\text{BPR}}$

Definition 3.4: (Equality) Let A_{BPR} , B_{BPR} ϵ BPRS(X) , where

$$A_{\text{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

$$B_{\text{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X\}$$

Then A_{BPR} is said to be BPR set equal of B_{BPR} and is denoted by $A_{\text{BPR}} = B_{\text{BPR}}$ if

$$\varsigma^{i+}_{A_{BPR}}(x) = \varsigma^{i+}_{B_{BPR}}(x), \quad \vartheta^{i+}_{A_{BPR}}(x) = \vartheta^{i+}_{B_{BPR}}(x)$$

$$\varsigma^{i-}_{A_{BPR}}(x) = \varsigma^{i-}_{B_{BPR}}(x), \quad \vartheta^{i-}_{A_{BPR}}(x) = \vartheta^{i-}_{B_{BPR}}(x)$$

for every x ϵ X and i=1,2,…..,p

Example 3.5.

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\text{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

$$B_{\text{BPR}} = \{\langle x, ([0.2,.0.3,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

We can say that $A_{\text{BPR}} = B_{\text{BPR}}$

Definition 3.6. (Complement)

Let A_{BPR} ϵ BPRS(X) . where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x),……, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x),……, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),……, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x),……, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X$$

The complement of A_{BPR} denoted by $A^{\mathrm{C}}_{\mathrm{BPR}}$ and is defined by

$$A_{BPR}{}^{C} = \{(x, (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x),……, \vartheta^{P+}_{A_{BPR}}(x), \varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x),……, \varsigma^{P+}_{A_{BPR}}(x)), (\vartheta^{1-}_{A_{BPR}}(x),$$

$$\vartheta^{2-}_{A_{BPR}}(x),……, \vartheta^{P-}_{A_{BPR}}(x), \varsigma^{1-}_{A_{BPR}}(x), \varsigma^{2-}_{A_{BPR}}(x),……, \varsigma^{P-}_{A_{BPR}}(x)) : x \in X \}$$

for every x ϵ X and i=1,2,…..,p

Example 3.7:

Let X be a non empty set in U. If A_{BPR} is bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.2,.03,0.5],[0.7,0.6,0.9])([-0.5,-0.4,-0.3],[-0.4,-0.6,-0.7])\rangle : x \in X\}$$

Then the complement of A_{BPR}

$$A^{C}_{BPR} = \{\langle x, ([0.7,0.6,0.9],[0.2,.0.3,0.5])([-0.4,-0.6,-0.7],[-0.5,-0.4,-0.3])\rangle : x \in X\}$$

Definition 3.8. (Union) Let A_{BPR} , B_{BPR} ϵ BPRS(X) . , where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x),……, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x),……, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x),……, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x),……, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x),……, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x),……, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x),……, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x),……, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X \}$$

The union of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \cup B_{\mathrm{BPR}} = C_{\mathrm{BPR}}$ and is defined by

$$C_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{C_{\mathrm{BPR}}}(x), \varsigma^{2+}_{C_{\mathrm{BPR}}}(x),……, \varsigma^{P+}_{C_{\mathrm{BPR}}}(x), \vartheta^{1+}_{C_{\mathrm{BPR}}}(x), \vartheta^{2+}_{C_{\mathrm{BPR}}}(x),……, \vartheta^{P+}_{C_{\mathrm{BPR}}}(x)), (\varsigma^{1-}_{C_{\mathrm{BPR}}}(x),$$

$$\varsigma^{2-}_{C_{\mathrm{BPR}}}(x),……, \varsigma^{P-}_{C_{\mathrm{BPR}}}(x), (\vartheta^{1-}_{C_{\mathrm{BPR}}}(x), \vartheta^{2-}_{C_{\mathrm{BPR}}}(x),……, \vartheta^{P-}_{C_{\mathrm{BPR}}}(x)) : x \in X \}$$

Where,

$$\varsigma^{i+}_{C_{\mathrm{BPR}}}(x) = \max\{\varsigma^{i+}_{A_{\mathrm{BPR}}}(x), \varsigma^{i+}_{B_{\mathrm{BPR}}}(x)\}$$
$$\vartheta^{i+}_{C_{\mathrm{BPR}}}(x) = \min\{\vartheta^{i+}_{A_{\mathrm{BPR}}}(x), \vartheta^{i+}_{B_{\mathrm{BPR}}}(x)\}$$

$$\varsigma^{i-}_{C_{\mathrm{BPR}}}(x) = \min\{\varsigma^{i-}_{A_{\mathrm{BPR}}}(x), \varsigma^{i-}_{B_{\mathrm{BPR}}}(x)\}$$
$$\vartheta^{i-}_{C_{\mathrm{BPR}}}(x) = \max\{\vartheta^{i-}_{A_{\mathrm{BPR}}}(x), \vartheta^{i-}_{B_{\mathrm{BPR}}}(x)\}$$ for every x ϵ X and i=1,2,…..,p

Example 3.9:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

$$B_{\mathrm{BPR}} = \{\langle x, ([0.2,.0.3,0.6],[0.4,0.8,0.3])([-0.3,-0.2,-0.7],[-0.7,-0.8,-0.5])\rangle : x \in X\}$$

then the union of two sets is

$$C_{\mathrm{BPR}} = \{\langle x, ([0.3,.0.5,0.7],[0.4,0.8,0.3])([-0.3,-0.5,-0.7],[-0.5,-0.6,-0.5])\rangle : x \in X\}$$

Definition 3.10. (Intersection)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$

$$\varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X\}$$

The intersection of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \cap B_{\mathrm{BPR}} = D_{\mathrm{BPR}}$

$$D_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{D_{\mathrm{BPR}}}(x), \varsigma^{2+}_{D_{\mathrm{BPR}}}(x), \ldots\ldots, \varsigma^{P+}_{D_{\mathrm{BPR}}}(x), \vartheta^{1+}_{D_{\mathrm{BPR}}}(x), \vartheta^{2+}_{D_{\mathrm{BPR}}}(x), \ldots\ldots, \vartheta^{P+}_{D_{\mathrm{BPR}}}(x)),$$

$$(\varsigma^{1-}_{D_{\mathrm{BPR}}}(x)$$

$$, \varsigma^{2-}_{D_{\mathrm{BPR}}}(x), \ldots\ldots, \varsigma^{P-}_{D_{\mathrm{BPR}}}(x), (\vartheta^{1-}_{D_{\mathrm{BPR}}}(x), \vartheta^{2-}_{D_{\mathrm{BPR}}}(x), \ldots\ldots, \vartheta^{P-}_{D_{\mathrm{BPR}}}(x)) : x \in X\}$$

Where,

$$\varsigma^{i+}_{D_{\mathrm{BPR}}}(x) = \min\{\varsigma^{i+}_{A_{\mathrm{BPR}}}(x), \varsigma^{i+}_{B_{\mathrm{BPR}}}(x)\}$$
$$\vartheta^{i+}_{D_{\mathrm{BPR}}}(x) = \max\{\vartheta^{i+}_{A_{\mathrm{BPR}}}(x), \vartheta^{i+}_{B_{\mathrm{BPR}}}(x)\}$$

$$\varsigma^{i-}_{D_{\mathrm{BPR}}}(x) = \max\{\varsigma^{i-}_{A_{\mathrm{BPR}}}(x), \varsigma^{i-}_{B_{\mathrm{BPR}}}(x)\}$$

$$\vartheta^{i-}_{D_{\mathrm{BPR}}}(x) = \min\{\vartheta^{i-}_{A_{\mathrm{BPR}}}(x), \vartheta^{i-}_{B_{\mathrm{BPR}}}(x)\}$$ for every x ϵ X and i=1,2,…..,p

Example 3.11:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.3, .0.5, 0.7], [0.6, 0.8, 0.9])([-0.2, -0.5, -0.6], [-0.5, -0.6, -0.9])\rangle : x \in X\}$$

$$B_{\mathrm{BPR}} = \{\langle x, ([0.2, .0.3, 0.6], [0.4, 0.8, 0.3])([-0.3, -0.4, -0.7], [-0.7, -0.8, -0.5])\rangle : x \in X\}$$

then the intersection of two sets is

$$D_{\mathrm{BPR}} = \{\langle x, ([0.2, .0.3, 0.6, [0.6, 0.8, 0.9])([-0.2, -0.4, -0.6], [-0.7, -0.8, -0.9])\rangle : x \in X\}$$

Definition 3.12: (Addition)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$
$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x), \varsigma^{2+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x), \vartheta^{2+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),$$
$$\varsigma^{2-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x), \vartheta^{2-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{B_{BPR}}(x)) : x \in X\}$$

Then the addition of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \oplus B_{\mathrm{BPR}}$

$$A_{\mathrm{BPR}} \oplus B_{\mathrm{BPR}} = \begin{pmatrix} \left(\varsigma^{i+}_{A_{BPR}}(x) + \varsigma^{i+}_{B_{BPR}}(x) - \varsigma^{i+}_{A_{BPR}}(x)\varsigma^{i+}_{B_{BPR}}(x), \vartheta^{i+}_{A_{BPR}}(x)\vartheta^{i+}_{B_{BPR}}(x)\right), \\ \left(-\varsigma^{i-}_{A_{BPR}}(x)\varsigma^{i-}_{B_{BPR}}(x), -(\vartheta^{i-}_{A_{BPR}}(x) + \vartheta^{i-}_{B_{BPR}}(x) - \vartheta^{i-}_{A_{BPR}}(x)\vartheta^{i-}_{B_{BPR}}(x))\right) \end{pmatrix}$$

For x ϵ X , i=1,2,…,p

Definition 3.13: (Multiplication)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) . where

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$
$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

B_{BPR} = {(x, ($\varsigma^{1+}_{B_{BPR}}(x)$, $\varsigma^{2+}_{B_{BPR}}(x)$,……, $\varsigma^{P+}_{B_{BPR}}(x)$,($\vartheta^{1+}_{B_{BPR}}(x)$, $\vartheta^{2+}_{B_{BPR}}(x)$,……, $\vartheta^{P+}_{B_{BPR}}(x)$) , ($\varsigma^{1-}_{B_{BPR}}(x)$,

$\varsigma^{2-}_{B_{BPR}}(x)$,……, $\varsigma^{P-}_{B_{BPR}}(x)$,($\vartheta^{1-}_{B_{BPR}}(x)$, $\vartheta^{2-}_{B_{BPR}}(x)$,……, $\vartheta^{P-}_{B_{BPR}}(x)$) : x ϵ X }

Then the multiplication of A_{BPR} and B_{BPR} is denoted by $A_{\mathrm{BPR}} \otimes B_{\mathrm{BPR}}$

$$A_{\mathrm{BPR}} \otimes B_{\mathrm{BPR}} = \begin{pmatrix} \left(\varsigma^{i+}_{A_{BPR}}(x)\varsigma^{i+}_{B_{BPR}}(x), \vartheta^{i+}_{A_{BPR}}(x) + \vartheta^{i+}_{B_{BPR}}(x) - \vartheta^{i+}_{A_{BPR}}(x)\vartheta^{i+}_{B_{BPR}}(x)\right), \\ \left(-(\varsigma^{i-}_{A_{BPR}}(x) + \varsigma^{i-}_{B_{BPR}}(x), -\varsigma^{i-}_{A_{BPR}}(x)\varsigma^{i-}_{B_{BPR}}(x)), -\vartheta^{i-}_{A_{BPR}}(x)\vartheta^{i-}_{B_{BPR}}(x))\right) \end{pmatrix}$$

For x ϵ X, i=1, 2, …, p.

4. Algebraic Properties of Bipolar Pythagorean Refined Set Operations

Proposition 4.1: (Commutative Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .Then

(a) $A_{BPR} \cup B_{BPR} = B_{BPR} \cup A_{BPR}$

(b) $A_{BPR} \cap B_{BPR} = B_{BPR} \cap A_{BPR}$

Proof: The proofs can be easily made.

Proposition 4.2 : (Associative Law)

Let A_{BPR} , B_{BPR} ϵ BPRS(X) .Then

(a) $A_{BPR} \cup (B_{BPR} \cup C_{BPR}) = (A_{BPR} \cup B_{BPR}) \cup C_{BPR}$

(b) $A_{BPR} \cap (B_{BPR} \cap C_{BPR}) = (A_{BPR} \cap B_{BPR}) \cap C_{BPR}$

Proof **:** Let A_{BPR}, B_{BPR} and C_{BPR} be three bipolar Pythagorean refined sets defined as follows.

A_{BPR} = {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$) , ($\varsigma^{1-}_{A_{BPR}}(x)$,

$\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }

B_{BPR} = {(x, ($\varsigma^{1+}_{B_{BPR}}(x)$, $\varsigma^{2+}_{B_{BPR}}(x)$,……, $\varsigma^{P+}_{B_{BPR}}(x)$,($\vartheta^{1+}_{B_{BPR}}(x)$, $\vartheta^{2+}_{B_{BPR}}(x)$,……, $\vartheta^{P+}_{B_{BPR}}(x)$) , ($\varsigma^{1-}_{B_{BPR}}(x)$,

$\varsigma^{2-}_{B_{BPR}}(x)$,……, $\varsigma^{P-}_{B_{BPR}}(x)$,($\vartheta^{1-}_{B_{BPR}}(x)$, $\vartheta^{2-}_{B_{BPR}}(x)$,……, $\vartheta^{P-}_{B_{BPR}}(x)$) : x ϵ X }

C_{BPR} = {(x, ($\varsigma^{1+}_{C_{BPR}}(x)$, $\varsigma^{2+}_{C_{BPR}}(x)$,……, $\varsigma^{P+}_{C_{BPR}}(x)$,($\vartheta^{1+}_{C_{BPR}}(x)$, $\vartheta^{2+}_{C_{BPR}}(x)$,……, $\vartheta^{P+}_{C_{BPR}}(x)$) , ($\varsigma^{1-}_{C_{BPR}}(x)$,

$\varsigma^{2-}_{C_{BPR}}(x)$,……, $\varsigma^{P-}_{C_{BPR}}(x)$,($\vartheta^{1-}_{C_{BPR}}(x)$, $\vartheta^{2-}_{C_{BPR}}(x)$,……, $\vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

Then $(A_{BPR} \cup B_{BPR}) \cup C_{BPR} =$

$\{(x, (\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x)) \ , (\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x)) \ , (\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{B_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x)), (\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{B_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x)) : x \in X \} \cup \{(x, (\varsigma^{1+}_{C_{BPR}}(x), \ldots\ldots\ldots, \varsigma^{P+}_{C_{BPR}}(x), (\vartheta^{1+}_{C_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{C_{BPR}}(x)), (\varsigma^{1-}_{C_{BPR}}(x), \ldots\ldots\ldots\ldots, \varsigma^{P-}_{C_{BPR}}(x), (\vartheta^{1-}_{C_{BPR}}(x), \ldots\ldots\ldots\ldots, \vartheta^{P-}_{C_{BPR}}(x)) : x \in X \}$

$= \{(x, (\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x)), \ldots\ldots, (\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x) \vee \varsigma^{P+}_{C_{BPR}}(x)) \ , (\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{B_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x)), \ldots\ldots, (\vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x) \vee \vartheta^{P+}_{C_{BPR}}(x)) \ , (\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{B_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x)), \ldots\ldots\ldots\ldots, (\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x)), (\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{B_{BPR}}(x) \vee, \vartheta^{1-}_{C_{BPR}}(x)) , \ldots\ldots\ldots.., (\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x) \vee, \vartheta^{P-}_{C_{BPR}}(x)) \ : x \in X \}.$

$= \{(x, \ \varsigma^{1+}_{A_{BPR}}(x) \vee (\varsigma^{1+}_{B_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x)), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x) \vee (\varsigma^{P+}_{B_{BPR}}(x) \vee \varsigma^{P+}_{C_{BPR}}(x)) \ , \vartheta^{1+}_{A_{BPR}}(x) \vee (\vartheta^{1+}_{B_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x)), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x) \vee (\vartheta^{P+}_{B_{BPR}}(x) \vee F^{P+}_{C_{BPR}}(x)) \ , \ \varsigma^{1-}_{A_{BPR}}(x) \vee (\varsigma^{1-}_{B_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x)), \ldots\ldots\ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x) \vee (\varsigma^{P-}_{B_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x)), \ \vartheta^{1-}_{A_{BPR}}(x) \vee (\vartheta^{1-}_{B_{BPR}}(x) \vee, \vartheta^{1-}_{C_{BPR}}(x)) , \ldots\ldots\ldots.., \vartheta^{P-}_{A_{BPR}}(x) \vee (\vartheta^{P-}_{B_{BPR}}(x) \vee, \vartheta^{P-}_{C_{BPR}}(x)) \ : x \in X \}.$

$= A_{BPR} \cup (B_{BPR} \cup C_{BPR})$

(b) The proof is obvious.

Proposition 4.3: (Idempotent Law)

Let A_{BPR} , B_{BPR} $\in$ BPRS(X) .Then

(a) $A_{BPR} \cup A_{BPR} = A_{BPR}$

(b) $A_{BPR} \cap A_{BPR} = A_{BPR}$

Proof: The proofs can be easily made.

Example 4.4:

Let X be a non empty set in U. If A_{BPR} and B_{BPR} are bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

Then ,

$$A_{BPR} \cup A_{BPR} = \{\langle x, ([0.3,.0.5,0.7],[0.6,0.8,0.9])([-0.2,-0.5,-0.6],[-0.5,-0.6,-0.9])\rangle : x \in X\}$$

Hence , $A_{BPR} \cup A_{BPR} = A_{BPR}$

(b) The proof is obvious.

Proposition 4.5: (Demorgan's Law)

Let A_{BPR} , $B_{\mathrm{BPR}} \in$ BPRS(X) .

(a) $(A_{BPR} \cup B_{BPR})^C = B_{BPR}{}^C \cap A_{BPR}{}^C$

(b) $(A_{BPR} \cap B_{BPR})^C = B_{BPR}{}^C \cup A_{BPR}{}^C$

Proof: The proofs can be easily made.

Proposition 4.6: (Distributive Law)

Let A_{BPR} , $B_{\mathrm{BPR}} \in$ BPRS(X) .

(a) $A_{BPR} \cup (B_{BPR} \cap C_{BPR}) = (A_{BPR} \cup B_{BPR}) \cap (A_{BPR} \cup C_{BPR})$

(b) $A_{BPR} \cap (B_{BPR} \cup C_{BPR}) = (A_{BPR} \cap B_{BPR}) \cup (A_{BPR} \cap C_{BPR})$

Proof : Let A_{BPR}, B_{BPR} and C_{BPR} be three bipolar Pythagorean refined sets defined as follows.

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x),\ \varsigma^{2+}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x),\ \vartheta^{2+}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),\ \varsigma^{2-}_{A_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x),\ \vartheta^{2-}_{A_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{A_{BPR}}(x)) : x \in X\}$$

$$B_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{B_{BPR}}(x),\ \varsigma^{2+}_{B_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{B_{BPR}}(x), (\vartheta^{1+}_{B_{BPR}}(x),\ \vartheta^{2+}_{B_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{B_{BPR}}(x)), (\varsigma^{1-}_{B_{BPR}}(x),\ \varsigma^{2-}_{B_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{B_{BPR}}(x), (\vartheta^{1-}_{B_{BPR}}(x),\ \vartheta^{2-}_{B_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{B_{BPR}}(x)) : x \in X\}$$

$$C_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{C_{BPR}}(x),\ \varsigma^{2+}_{C_{BPR}}(x),\ldots\ldots,\varsigma^{P+}_{C_{BPR}}(x), (\vartheta^{1+}_{C_{BPR}}(x),\ \vartheta^{2+}_{C_{BPR}}(x),\ldots\ldots,\vartheta^{P+}_{C_{BPR}}(x)), (\varsigma^{1-}_{C_{BPR}}(x),\ \varsigma^{2-}_{C_{BPR}}(x),\ldots\ldots,\varsigma^{P-}_{C_{BPR}}(x), (\vartheta^{1-}_{C_{BPR}}(x),\ \vartheta^{2-}_{C_{BPR}}(x),\ldots\ldots,\vartheta^{P-}_{C_{BPR}}(x)) : x \in X\}$$

Then $A_{BPR} \cup (B_{BPR} \cap C_{BPR})$

= {(x, ($\varsigma^{1+}_{A_{BPR}}(x)$, $\varsigma^{2+}_{A_{BPR}}(x)$,……, $\varsigma^{P+}_{A_{BPR}}(x)$,($\vartheta^{1+}_{A_{BPR}}(x)$, $\vartheta^{2+}_{A_{BPR}}(x)$,……, $\vartheta^{P+}_{A_{BPR}}(x)$), ($\varsigma^{1-}_{A_{BPR}}(x)$, $\varsigma^{2-}_{A_{BPR}}(x)$,……, $\varsigma^{P-}_{A_{BPR}}(x)$,($\vartheta^{1-}_{A_{BPR}}(x)$, $\vartheta^{2-}_{A_{BPR}}(x)$,……, $\vartheta^{P-}_{A_{BPR}}(x)$) : x ϵ X }U {(x, ($\varsigma^{1+}_{B_{BPR}}(x) \wedge \varsigma^{1+}_{C_{BPR}}(x)$) ,……, ($\varsigma^{P+}_{B_{BPR}}(x) \wedge \varsigma^{P+}_{C_{BPR}}(x)$) ,($\vartheta^{1+}_{B_{BPR}}(x) \wedge \vartheta^{1+}_{C_{BPR}}(x)$) ,……, ($\vartheta^{P+}_{B_{BPR}}(x) \wedge \vartheta^{P+}_{C_{BPR}}(x)$), ($\varsigma^{1-}_{B_{BPR}}(x) \wedge \varsigma^{1-}_{C_{BPR}}(x)$) ,……, ($\varsigma^{P-}_{B_{BPR}}(x) \wedge \varsigma^{P-}_{C_{BPR}}(x)$),($\vartheta^{1-}_{B_{BPR}}(x) \wedge \vartheta^{1-}_{C_{BPR}}(x)$) ,……, ($\vartheta^{P-}_{B_{BPR}}(x) \wedge \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

= {(x, $\varsigma^{1+}_{A_{BPR}}(x) \vee$ ($\varsigma^{1+}_{B_{BPR}}(x) \wedge \varsigma^{1+}_{C_{BPR}}(x)$) ,……, $\varsigma^{P+}_{A_{BPR}}(x) \vee (\varsigma^{P+}_{B_{BPR}}(x) \wedge \varsigma^{P+}_{C_{BPR}}(x))$, $\vartheta^{1+}_{A_{BPR}}(x) \vee$ ($\vartheta^{1+}_{B_{BPR}}(x) \wedge \vartheta^{1+}_{C_{BPR}}(x)$) ,……, $\vartheta^{P+}_{A_{BPR}}(x)) \vee (\vartheta^{P+}_{B_{BPR}}(x) \wedge \vartheta^{P+}_{C_{BPR}}(x))$, $\varsigma^{1-}_{A_{BPR}}(x) \vee$ ($\varsigma^{1-}_{B_{BPR}}(x) \wedge \varsigma^{1-}_{C_{BPR}}(x)$) ,……, $\varsigma^{P-}_{A_{BPR}}(x) \vee (\varsigma^{P-}_{B_{BPR}}(x) \wedge \varsigma^{P-}_{C_{BPR}}(x))$, $\vartheta^{1-}_{A_{BPR}}(x) \vee$ ($\vartheta^{1-}_{B_{BPR}}(x) \wedge \vartheta^{1-}_{C_{BPR}}(x)$) ,……, $\vartheta^{P-}_{A_{BPR}}(x) \vee (\vartheta^{P-}_{B_{BPR}}(x) \wedge \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

= {(x, $(\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{B_{BPR}}(x)) \wedge (\varsigma^{1+}_{A_{BPR}}(x) \vee \varsigma^{1+}_{C_{BPR}}(x))$,……, $(\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{B_{BPR}}(x)) \wedge (\varsigma^{P+}_{A_{BPR}}(x) \vee \varsigma^{P+}_{C_{BPR}}(x))$, $(\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{B_{BPR}}(x)) \wedge (\vartheta^{1+}_{A_{BPR}}(x) \vee \vartheta^{1+}_{C_{BPR}}(x))$,……, $(\vartheta^{P+}_{A_{BPR}}(x) \vee \vartheta^{P+}_{B_{BPR}}(x)) \wedge (\vartheta^{P+}_{A_{BPR}}(x)) \vee \vartheta^{P+}_{C_{BPR}}(x))$, $(\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{B_{BPR}}(x)) \wedge (\varsigma^{1-}_{A_{BPR}}(x) \vee \varsigma^{1-}_{C_{BPR}}(x))$,……, $(\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{B_{BPR}}(x)) \wedge (\varsigma^{P-}_{A_{BPR}}(x) \vee \varsigma^{P-}_{C_{BPR}}(x))$, $(\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{B_{BPR}}(x)) \wedge (\vartheta^{1-}_{A_{BPR}}(x) \vee \vartheta^{1-}_{C_{BPR}}(x))$,……, $(\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{B_{BPR}}(x)) \wedge (\vartheta^{P-}_{A_{BPR}}(x) \vee \vartheta^{P-}_{C_{BPR}}(x)$) : x ϵ X }

$= (A_{BPR} \cup B_{BPR}) \cap (A_{BPR} \cup C_{BPR})$

(b) The proof is obvious.

Proposition 4.7: (Double complement Law)

Let A_{BPR} ϵ BPRS(X) , then

$(A^{c}_{BPR})^{c} = A_{\mathrm{BPR}}$

Proof: Let A_{BPR} be the bipolar Pythagorean refined set defined as follows.

$$A_{\mathrm{BPR}} = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), (\varsigma^{1-}_{A_{BPR}}(x),$$

$$\varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$(A^c_{BPR}) = \{(x, (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x), \varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x)), (\vartheta^{1-}_{A_{BPR}}(x),$$

$$\vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x), \varsigma^{1-}_{A_{BPR}}(x), \varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x)) : x \in X \}$$

$$(A^c_{BPR})^c = \{(x, (\varsigma^{1+}_{A_{BPR}}(x), \varsigma^{2+}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P+}_{A_{BPR}}(x), (\vartheta^{1+}_{A_{BPR}}(x), \vartheta^{2+}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P+}_{A_{BPR}}(x)), ($$

$$\varsigma^{1-}_{A_{BPR}}(x), \varsigma^{2-}_{A_{BPR}}(x), \ldots\ldots, \varsigma^{P-}_{A_{BPR}}(x), (\vartheta^{1-}_{A_{BPR}}(x), \vartheta^{2-}_{A_{BPR}}(x), \ldots\ldots, \vartheta^{P-}_{A_{BPR}}(x)) : x \in X \}$$

Hence $(A^c_{BPR})^c = A_{\mathrm{BPR}}$

Proposition 4.8: (Absorption Law)

Let A_{BPR} , $B_{\mathrm{BPR}} \in$ BPRS(X) .

(a) $A_{BPR} \cup (A_{BPR} \cap B_{BPR}) = A_{BPR}$

(b) $A_{BPR} \cap (A_{BPR} \cup B_{BPR}) = A_{BPR}$

Proof: The proofs can be easily made.

5. Conclusion

This paper ensures the work of introducing the new set namely the Bipolar Pythagorean refined set by using the theory of the Bipolar Pythagorean set and Pythagorean refined(multi) set. Several operations and laws have been discussed along with some examples. In the future, Bipolar Pythagorean refined topological spaces can be introduced. And also, decision-making problems on bipolar Pythagorean refined sets can be introduced.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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New Statistical Methodology for Capacitor Data Analysis via LCR Meter

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Abstract: This research work introduces a novel methodology to establish the relationship between capacitance and resistance when dealing with imprecise data obtained from LCR meters. The proposed relationship is based on the principles of neutrosophic statistics, enabling the utilization of interval data of resistance or capacitance without losing the indeterminacy of the intervals. By employing this relationship, we can accurately determine capacitance values from interval data of resistance, thereby generating more flexible and informative graphs. Additionally, we have applied the neutrosophic analysis method to the interval data of resistance to further enhance our findings. The comparative analysis demonstrates the superiority of the proposed approach and neutrosophic analysis over classical or pre-existing methods, highlighting their enhanced flexibility and information content.

Keywords: LCR meter; Resistance; Capacitance; Informative.

1. Introduction

Energy storage devices have been gotten importance at the industrial level in recent years. This thing has increased the fabrication of energy storage devices such as batteries and supercapacitors etc. Generally, highly efficient and flexible fabrication of such energy storage devices is always required. For example, if we talk about supercapacitors, a number of research have used different materials in the fabrication of highly efficient supercapacitors. To understand the behavior of the supercapacitor different models have been like transmission line model [1], thermal, frequency, and voltage model [2], two branch model [3], resistance and capacitance comparison models [4, 5]. These all have different efficient ways to study the supercapacitors behavior. Similarly, there are three methods used for the measurement and observation of the capacitance of the supercapacitors like impedance spectroscopy [6], galvanostatic charging [7], digital meter (like LCR meter) [8] and cyclic voltammetry [9]. Generally, it is seen that through impedance spectroscopy one can gets a differential capacitance but through galvanostatic charging and cyclic voltammetry one can gets integrate experimental value of capacitance [10]. Similarly, through the digital meters especially LCR meter we get the imprecise value of capacitance i.e. in interval form [11-15]. If we talk about the data analysis of supercapacitors in term of material statistics, there are two methods for data analysis. One is classical method (based on classical graphs, tables and formulas of analyzing) which is used when data is single value. For example the use of classical table to represent the data of ultra-capacitor's capacitance can be seen in following reference [8] Table 1. The classical method of analysis is good when some one is dealing with the fix-point/deterministic data. But if there is interval/indeterministic data, the classical method is not used directly untill the indeterministic data is not convered into deterministic. For example, generally, research takes the average of a interval and used in their work.

In this way, the variance of interval turn in to a fix value as the result a person will not able to analyze the whole variance of data (may be capacitance or resistance) for that interval. That's we turn to the indeterministic model. This model is based on the neutrosophic statistics. Neutrosophic statistics is a tool or method of statistics which is used to analyze the interval value without losing the indeterminacy of each interval. It was proposed by F. Smarandache, which is more flexible and informative then all other methods of the statistics. Recently, it is observed that the use of the neutrosophic statistics has been increased in different field like in medicine to analyze diagnosis data [16], in applied sciences [17], in astrophysics to analyze the wind data [18] and in material science for sensor data analysis [12], conductor resistance analysis [11] and graphene foam analysis [19].

1.1 Aim of Study

This research endeavors to develop an innovative indeterministic relationship between capacitance and resistance in LCR data measurement, with a specific focus on supercapacitors and sensors. Leveraging principles from materials statistics, we employ the neutrosophic statistics framework to formulate the proposed relationship. By using this indeterministic approach, we aim to directly calculate capacitance from interval values of resistance and vice versa, enabling more accurate and comprehensive characterizations of these electronic devices. Through the incorporation of materials statistics and the application of the neutrosophic framework, we anticipate significant advancements in materials science and electronic engineering analyses.

2. Methodology

The present study introduces an innovative Indeterministic model for capacitance and resistance, firmly rooted in the principles of neutrosophic statistics. This model will be instrumental in accurately calculating capacitance and resistance from interval data. Before delving into the details of our proposed approach, let us explore some of the prior definitions and concepts of neutrosophic statistics:

2.1 Definition of Neutrosophic Statistic

Let X_N is a neutrosophic variable with interval of variance $X_N \epsilon [X_L, X_U]$ having size $n_N \epsilon [n_{NL}, n_{NU}]$ with indeterminacy $I_N \epsilon [I_{NL}, I_{NU}]$, so the neutrosophic formula can be written as written as follows [20]:

$$X_{iN} = X_{iL} + X_{iU} I_N \ (i = 1, 2, 3, \dots, n_N) \tag{1}$$

$X_{iN} \epsilon [X_{iL}, X_{iU}]$has two parts: X_{iL} expressing the lower value which is under classical statistics and $X_{iU} I_N$ is an upper part with indeterminacy$I_N \epsilon [I_L, I_U]$. Moreover for intervals, the lower value of the indeterminacy interval is always taken as zero under classical statistics extension i.e. $I_L = 0$. But the upper value of indeterminacy interval can be found by $I_U = [(X_{iU} - X_{iL})/X_{iU}]$. In this the indeterminacy interval with respect to each is as $I_N \epsilon [0, (X_{iU} - X_{iL})/X_{iU}]$. Similarly, the neutrosophic mean interval $\bar{X}_N \epsilon [\bar{X}_L, \bar{X}_U]$ is defined as follows:

$$\bar{X}_N = \bar{X}_L - \bar{X}_U I_N;\ I_N \epsilon [I_L, I_U] \tag{2}$$

Where $\bar{X}_L = \sum_{i=1}^{nU} \left(\frac{X_{iL}}{n_L}\right)$ is the lower value of the neutrosophic mean and $\bar{X}_U = \sum_{i=1}^{nU} \left(\frac{X_{iU}}{n_U}\right)$ is the highest value of the neutrosophic means with the indeterminacy interval $I_N \in [I_L, I_U]$. By using the above neutrosophic statistics definition, the variation for resistance (R) interval $[R_L, R_U]$ and for capacitance (C) interval$[C_L, C_U]$:

$$R_N = R_L + R_U I_N;\ I_N \epsilon [I_L, I_U] \tag{3}$$
$$C_N = C_L + C_U I_N;\ I_N \epsilon [I_L, I_U] \tag{4}$$

2.2 Development of Indeterministic Relationship between Capacitance and Resistance:

In the process of establishing the indeterministic relationship between capacitance and resistance, our initial step involves the classical relationship existing between capacitance and resistance within an LCR meter framework. Given the prevalent nature of AC circuits in LCR meter applications, it becomes possible to express the impedance pertaining to a supercapacitor concerning its capacitance as follows:

$$Z = \frac{1}{2j\pi fC} \tag{5}$$

The frequency 'f' can be written as $f = \frac{\omega}{2\pi}$, so the equation (5) becomes:

$$Z = \frac{1}{j\omega C} \tag{6}$$

Here 'j' is a complex no having value '$\sqrt{-1}$' so we ignore it. Also if we consider that resistance and impedance are equal for LCR meter the above equation can be written as:

$$R = \frac{1}{\omega C} \tag{7}$$

This is a deterministic/classical relationship between resistance and capacitance. It is seen that both have inverse relation, i.e. as the capacitance of a supercapacitor increases the resistance starts to decrease. So, the indeterministic relationship by using the definition of neutrosophic statistics can be written as:

$$R = \frac{1}{\omega * (C_L + C_U I_N; I_N \epsilon\, [I_L, I_U])} \tag{8}$$

And

$$C = \frac{1}{\omega * (R_L + R_U I_N; I_N \epsilon\, [I_L, I_U])} \tag{9}$$

The Eq. (8) will be used if someone wants to calculate the resistance when he has measured capacitance in interval from LCR for a supercapacitor. Similarly, the Eq. (9) will be used if someone wants to calculate the capacitance when he has measured resistance in interval.

3. Data Collection

In this scenario, we employ a capacitor for which the resistance is subjected to measurement through employment of an LCR meter. The measurement is conducted concerning alterations in the current at a frequency of 1 kHz, while maintaining a constant input of 1.0 V. The process involves determining the resistance in intervals, wherein at specific current levels, both the maximum and minimum variations in resistance are meticulously recorded. This is visually represented in Figure 1, showcasing the observed range of resistance as [minimum resistance, maximum resistance].

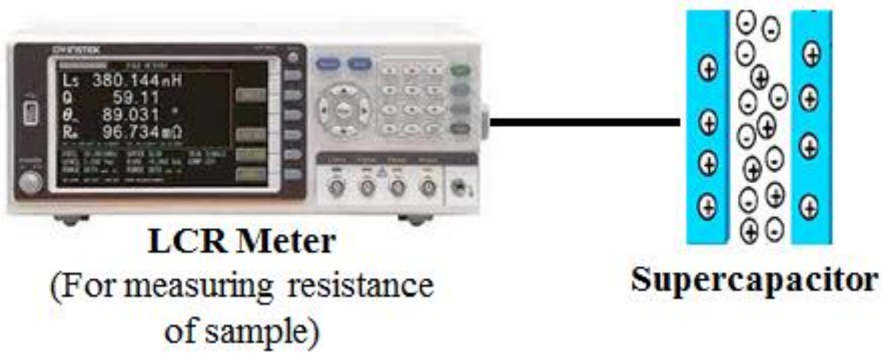

Figure 1. Characterization setup.

In our investigation, it is pertinent to clarify that no experimental procedures were conducted by our team. Instead, we acquired the relevant resistance data from a published paper, specifically by extracting the data points from its graphical representation [21]. Subsequently, we employed the aforementioned indeterministic formula to derive the corresponding capacitance values in conjunction with the collected resistance data, carefully analyzed within specific intervals. In our analytical pursuit, we applied both classical and neutrosophic methodologies to comprehensively analyze the dataset encompassing resistance and capacitance values. These methodologies allowed us to gain valuable insights and draw meaningful conclusions from the gathered information.

4. Results and Discussion

The collected resistance data (by reading the graph of Figure 2 (b) from [10]) of the supercapacitor with respect to change in current as shown in Table 1.

Table 1. Data of resistance of supercapacitor with respect to current.

Current	Resistance
(A)	*(Ω)*
0.1	[30.18, 40.92]
0.2	[18.54, 26.67]
0.3	[12.86, 18.34]
0.4	[10.01, 13.19]
0.5	[7.10, 9.21]
0.6	[7.01, 7.64]
0.7	[5.90, 6.17]
0.8	[5.25, 5.29]

Now, let we use the above indeterministic relation to calculate the capacitance from the above resistance data of supercapacitor. For example we calculate the capacitance for interval of resistance [30.18, 40.92] with respect to 0.1 A current at 1 kHz = 1000 Hz. From the Eq. (9), $'R_L' = 30.18$, $'R_U' = 40.92$, $'I_L' = 0$(according to the definition of neutrosophic statistics) and $'I_U' = 0.26$ (according to the definition of neutrosophic statistics) so the capacitance can be written as:

$$C = \frac{1}{(1000)*(30.18 + 40.92I_N; I_N \epsilon\ [0,0.26])} \tag{10}$$

If we choose the '0.02' interval difference for indeterminacy interval, we get the calculated values of the capacitance showing in the Table 2 for single interval i.e. [30.18, 40.92] from the above equation:

Table 2. Calculated value of capacitance and resistance values by indeterminacy relationship.

Indeterminacy Variation [0, 0.26]	Resistance [30.18, 40.92] (Ω)	Capacitance (F)
0	30.18	0.033135
0.02	30.9984	0.03226
0.04	31.8168	0.03143
0.06	32.6352	0.030642
0.08	33.4536	0.029892
0.10	34.272	0.029178
0.12	35.0904	0.028498
0.14	35.9088	0.027848
0.16	36.7272	0.027228

0.18	37.5456	0.026634
0.20	38.364	0.026066
0.22	39.1824	0.025522
0.24	40.0008	0.025
0.26	40.9232	0.024498

The Table 2 is expressing the calculated values of capacitance for a single interval of the resistance. The capacitance values are calculated from the equation (10) and resistance values are calculated from the equation (3). Now, we plot the graph between the capacitance vs resistance as shown in Figure 2.

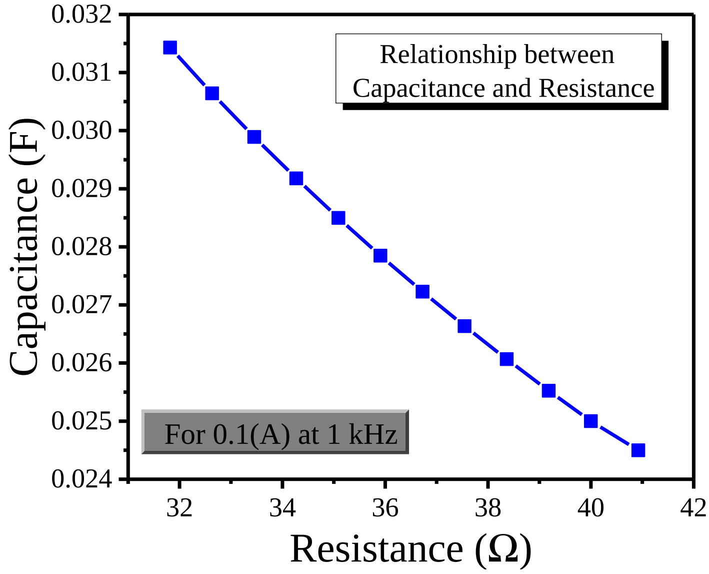

Figure 2. Relationship of capacitance and resistance for 0.1 (A) current at 1 kHz.

The Figure 2 is expressing the relationship between capacitance and resistance for the first interval of resistance i.e. [30.18, 40.92]. It is seen that the resistance and capacitance values calculated from the interval satisfy the relationship for LCR meter as expressed in equation (7) i.e. with increased in resistance capacitance decreased. In the same way, we can calculate the value of capacitance for each interval of table 1. And can also draw the graph for each interval.

4.1 Computational Approach for Indeterministic Relationship

Now we use the computational approach for calculating the values from the above indeterministic relationship. So the algorithm to run the above relation/formula on computer as following:

Step 1: Start program

Step 2: Run a loop from $I_N = I_L \; to \; I_N = I_U$ (For given interval)

Step 3: Execute formulas:

$$C = \frac{1}{(1000)*(30.18 + 40.92 I_N)}$$ (For calculating the variance value of capacitance)

$$R = R_L + R_U I_N$$ (For analyzing the variance value of resistance)

Step 4: Collect data in table and draw graph

Step 5: Increment of 0.02 and go to step 3

Step 6: End loop

Step 7: End program

The above algorithm will be used to run the indeterministic relationship on any computer program.

4.2 Advantages of Indeterministic Relationship

- Utilizing the aforementioned indeterministic relation, it becomes feasible to directly input the values as interval values without necessitating any additional adjustments within the interval. Consequently, this enables a straightforward computation of both resistance and capacitance.Through the utilization of a singular interval, it is possible to construct a comprehensive graph that facilitates the analysis of capacitance and resistance for various devices, such as sensors and capacitors, among others.
- These formulas exhibit high computational efficiency and can be effortlessly applied using any standard computer software, thereby streamlining the analysis process.
- Employing this approach allows for a more precise examination of data variance, leading to the acquisition of more insightful and valuable information from the dataset.

4.3 Limitation of Indeterministic Relationship

- The aforementioned indeterministic relationship is specifically designed for the purpose of analyzing resistance or capacitance data obtained from LCR meter measurements.
- To apply this relationship effectively, it is imperative that the dataset of capacitance and resistance comprises interval values. The validity and applicability of the relationship rely on the availability of such interval data in the dataset.

4.4 Analysis of data

We now proceed to perform a comprehensive analysis of the entire dataset, employing the neutrosophic method. Furthermore, we aim to make a comparative assessment between the outcomes obtained from the neutrosophic analysis and the previously utilized classical method of graph representation.

To initiate this comparative study, we have utilized the resistance data presented in Table 1. Employing the neutrosophic analysis technique, we examine the data and draw relevant conclusions. Subsequently, we contrast the results with those derived through the classical method of graph plotting. The findings from both the classical and neutrosophic analyses are tabulated in Table 3. It is important to emphasize that, for this phase of analysis; we have exclusively utilized the resistance data available in the dataset.

Table 3. Classical and neutrosophic analysis of resistance interval data.

Current (A)	Classical Method	Neutrosophic Method
0.1	35.74 ± 5.18	30.18+ $40.92I_N; I_N \epsilon$ [0, 0.26]
0.2	21.12 ± 5.55	18.54+ $26.6I_N; I_N \epsilon$ [0, 0.31]
0.3	15.92 ± 2.42	12.86+ $18.34I_N; I_N \epsilon$ [0, 0.30]
0.4	11.59 ± 1.6	10.01+ $13.19I_N; I_N \epsilon$ [0, 0.24]
0.5	8.01 ± 1.2	7.10+ $9.21I_N; I_N \epsilon$ [0, 0.23]
0.6	7.32 ± 0.32	7.01+ $7.64I_N; I_N \epsilon$ [0, 0.08]
0.7	6.06 ± 0.11	5.90+ $6.17I_N; I_N \epsilon$ [0, 0.04]
0.8	5.27 ± 0.02	5.25+ $5.29I_N; I_N \epsilon$ [0, 0.01]

Based on the findings presented in Table 3, it becomes evident that the classical analysis of resistance data was executed using the classical mean/average formula. However, it is notable that the classical method only yields a single value, accompanied by an associated error, at a specific data point. For instance, at a current of 0.1A, the resistance value derived through the classical approach is 35.74 ± 5.18. It is evident that the concept of intervals, which represents the indeterminacy, is lost within the classical analysis. Consequently, we do not recommend the classical method for analyzing interval data of resistance due to its limited reliability in decision-making and its lack of flexibility in problem resolution.

This limitation necessitates a shift toward the neutrosophic method for interval data analysis. The neutrosophic method provides an equation for each data interval, along with its corresponding indeterminate interval. As observed, the neutrosophic analysis proves to be more reliable since it effectively addresses indeterminacy and furnishes comprehensive information concerning the variance of resistance changes at specific current levels. For instance, at 0.1A current, the neutrosophic analysis provides an equation of the form R = 30.18 + 40.92 I_N, where $I_N \epsilon\ [0, 0.26]$, with an indeterminate interval of [0, 0.26]. According to the neutrosophic analysis, the resistance value lies within the range of 30.18 and 40.92 by incorporating different indeterminacy values ranging from 0 to 0.26.

To better illustrate these distinct analytical approaches, we have depicted the classical and neutrosophic graphs in Figure 3, providing a visual representation of their respective outcomes.

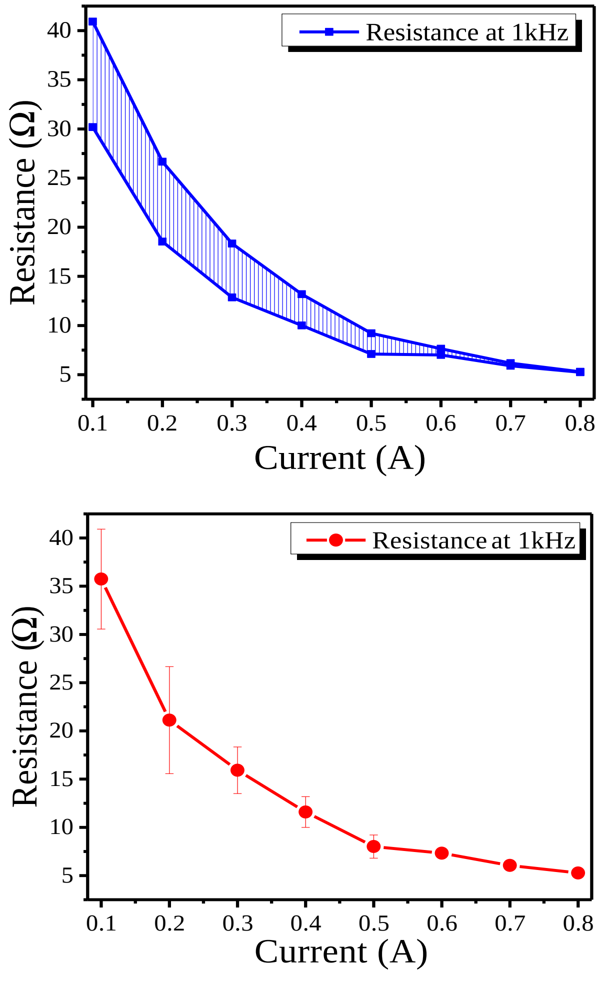

Figure 3. Left side is neutrosophic graph and right side is classical graph.

The plots displayed in Figure 3 facilitate a comprehensive comparison between the classical and neutrosophic graphs. The classical graph, while commonly employed, demonstrates limited flexibility and information conveyance when elucidating the resistance characteristics of a supercapacitor. It primarily adheres to fixed-point values or represents data points solely at specific fixed values. On the contrary, the neutrosophic graph exhibits a remarkable degree of flexibility, rendering it capable of providing valuable and decision-relevant information. The ability to work with interval data values allows for more informed conclusions to be drawn.

In our visual representation of the classical analysis, we have utilized error bars, as commonly seen in the works of other researchers. However, it is essential to acknowledge that error bars, while widely used, are not statistically effective in showcasing data variance. Instead, they depict the error found in data, which may result from personal error, sample handling, or mechanical discrepancies. This distinction is critical, as error bars do not effectively represent the variation present within the data.

Conversely, the neutrosophic method emerges as a highly valuable and effective approach for interval data analysis, as it requires no manipulation or modification of intervals. By directly utilizing the interval data, this method offers a comprehensive understanding of the variance in the supercapacitor's resistance measured through the LCR meter in response to current fluctuations. In light of its superior performance in dealing with interval data, we have opted for the neutrosophic approach to conduct our analysis, enabling us to gain meaningful insights into the behavior of the supercapacitor under study.

5. Conclusions

The present study represents an application of modern material statistics, specifically leveraging the principles of neutrosophic statistics. In this work, we introduce an innovative indeterministic relationship for interval values of capacitance and resistance, both of which are measured using an LCR meter for a supercapacitor. The primary objective of this relationship is to facilitate the calculation of capacitance or resistance values directly from their corresponding interval values, without necessitating any alteration or adjustment to the intervals and their associated indeterminacy. To demonstrate the efficacy of the proposed indeterministic relationship, we have included a practical example, showcasing how capacitance values can be calculated from the interval data of resistance for a supercapacitor. Through this example, we illustrate the ease and accuracy with which this relationship can be applied to real-world scenarios. Furthermore, to comprehensively understand the analysis methodologies of both neutrosophic and classical statistics, we have incorporated an additional example and compared the output tables and graphs derived from each approach. The findings from this comparative analysis allowed us to draw conclusive insights. As a result of our investigations, we have arrived at the noteworthy conclusion that the proposed neutrosophic statistical method proves to be more effective and informative in analyzing interval data collected from an LCR meter. The ability to work directly with interval values without losing indeterminacy enhances the reliability and utility of the neutrosophic approach, underscoring its significance in modern material statistics and its potential for practical applications in various research domains.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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New Types of Soft Sets" HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set": An Improved Version

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Abstract: This is an improved paper of [10], where we recall the definitions together with practical applications of the Soft Set and its extensions to HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set.

Keywords: Soft Set; HyperSoft Set; IndetermSoft Set; IndetermHyperSoft Set; MultiSoft Set; TreeSoft Set.

1. Introduction

The Soft Set was introduced by Molodtsov [1] in 1999.
Further on, the HyperSoft Set (2018), IndetermSoft Set (2022), IndetermHyperSoft Set (2022), and TreeSoft Set (2022) were introduced by Smarandache [2-7]. The definitions of IndetermSoft Set and IndetermHyperSoft Set [9] are updated now.
The MultiSoft Set (2010) was introduced by Alkhazaleh et al. [8].
The soft set and its extensions have many applications in our real world.

Many hybrid version of the soft set have been proposed and used, combined with fuzzy or fuzzy extension sets, such as: fuzzy soft set, intuitionistic fuzzy soft set, neutrosophic soft set, picture fuzzy soft set, spherical fuzzy soft set, plithogenic soft set, and similarly for fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoftsoft set, picture fuzzy hypersoft set, spherical fuzzy hypersoft soft set, plithogenic sift set.

Future research may also investigate and apply the newly forms of soft sets, in combinations with fuzzy and fuzzy extension sets, that will result in fuzzy / intuitionistic fuzzy / neutrosophic / picture fuzzy / spherical fuzzy / Pythagorean fuzzy etc. / plithogenic IndetermSoft / IndetermHyperSoft / TreeSoft Set, respectively.

Let us recall their definitions together with real examples.

2. Definition of Soft Set

Let U be a universe of discourse, $P(U)$ the power set of U, and A a set of attributes. Then, the pair (F, U), where $F: A \rightarrow P(U)$ is called a Soft Set over U.

3. Real Example of Soft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.
Let the attribute be: a = size, and its attribute' values respectively:

$$\text{Size} = A_1 = \{\text{small, medium, tall}\}.$$

Let the function be: $F: A_1 \rightarrow P(U)$.

Then, for example:

$$F(\text{tall}) = \{\text{Helen, Mary}\},$$

which means that both Helen and Mary are tall.

4. Definition of IndetermSoft Set

Let *U* be a universe of discourse, *H* a non-empty subset of *U*, and *P(H)* be the powerset of *H*. Let *a* be an attribute, and *A* be a set of this attribute-values. Then $F: A \to P(H)$ is called an IndetermSoft Set if at least one of the bellow occurs:

i) The set *A* has some indeterminacy;
ii) At least one of the sets *H* or *P(H)* has some indeterminacy;
iii) The function *F* has some indeterminacy, i.e. there exist at least one relationship $F(a) = M$ where *a* or *M* have some indeterminacy (not unique, unclear, incomplete, unknown).

IndetermSoft Set, as an extension of the classical (determinate) Soft Set, deals with indeterminate data, because there are sources unable to provide exact or complete information on the sets *A*, *H*, or *P(H)*, nor on the function *F*. We did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set, as such we still need to deal with such situations.

Herein, 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, incomplete, not unique outcome).

Similarly, distinctions between determinate and indeterminate operators are taken into consideration. Afterwards, an IndetermSoft Algebra is built, using a determinate soft operator (joinAND), and three indeterminate soft operators (disjoinOR, exclussiveOR, NOT), whose properties are further on studied.

Smarandache has generalized the Soft Set to the HyperSoft set by transforming the function F into a multi-attribute function, and then he introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic HyperSoft Set.

The classical Soft Set is based on a determinate function (whose values are certain, and unique), but in our world there are many sources that, because of lack of information or ignorance, provide indeterminate (uncertain, and not unique – but hesitant or alternative) information. They can be modeled by operators having some degree of indeterminacy due to the imprecision of our world.

5. Real Example of IndetermSoft Set

The present study represents an application of modern material statistics, specifically leveraging the principles of neutrosophic statistics.

Assume a town has many houses.

1) Indeterminacy with respect to the set *A* of attributes.

You ask the source:

- What are all colors of the houses?

The source:

- I know for sure that there are houses of colors red, yellow, and blue, but I do not know if there are houses of other colors (?)

This is the IndetermSoft Set.

2) Indeterminacy with respect to the set *H* of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the function $F: A \rightarrow P(H)$.

3a) You ask a source:
- What houses have the red color in the town?
The source:
- I am not sure, I think the houses h_1 or h_2.
Therefore, F(red) = h_1 or h_2 (indeterminate / uncertain answer).
3b) You ask again:
- But, what houses are yellow?
The source:
- I do not know, the only thing I know is that the house h_5 is not yellow because I have visited it.
Therefore, F(yellow) = not h_5 (again indeterminate / uncertain answer).
3c) Another question you ask:
- Then what houses are blue?
The source:
- For sure, either h_8 or h_9.
Therefore, F(blue) = either h_8 or h_9 (again indeterminate / uncertain answer).

6. Definition of HyperSoft Set

The soft set was extended to the hypersoft set by transforming the function F into a multi-attribute function. Afterwards, the hybrids of HyperSoft Set with the Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic Set were introduced.

Let U be a universe of discourse, $P(U)$ the power set of U. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$, for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(U)$, is called a HyperSoft Set over U.

7. Real Example of HyperSoft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attributes be: a_1 = size, a_2 = color, a_3 = gender, a_4 = nationality, and their attributes' values respectively:

Size = A_1 = {small, medium, tall},

Color = A_2 = {white, yellow, red, black},

Gender = A_3 = {male, female},

Nationality = A_4 = {American, French, Spanish, Italian, Chinese}.

Let the function be: $F: A_1 \times A_2 \times A_3 \times A_4 \rightarrow P(U)$.
Then, for example:
F({tall, white, female, Italian}) = {Helen, Mary}, which means that both Helen and Mary are tall, and white, and female, and Italian.
Notice that this is an extension of the previous Real Example of Soft Set.

8. Definition of IndetermHyperSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$ for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(H)$, is called an IndetermHyperSoft Set over U if at least one of the bellow occurs:

i) at least one of the sets $A_1, A_2, \dots, A_n$ has some indeterminacy;
ii) at least one of the sets H or $P(H)$ has some indeterminacy;
iii) the function F has some indeterminacy, i.e. there exist at least one relationship

$F(e_1, e_2, \ldots, e_n) = M$, where some of $e_1, e_2, \ldots, e_n$, or *M have indeterminacy* (not unique, unclear, incomplete, unknown).

Similarly, IndetermHyperSoft Set ia an extension of the HyperSoft Set, when there is indeterminate data, or indeterminate functions, or indeterminate sets.

9. Real Example of IndetermHyperSoft Set

Assume a town has many houses.

1) Indeterminacy with respect to the product set $A_1 \times A_2 \times \ldots \times A_n$ of attributes.

You ask the source:

- What are all colors and sizes of the houses?

The source:

- I know for sure that there are houses of the following colors: red, yellow, and blue, but I do not know if there are houses of other colors (?)

About the size, I saw many houses that are small, but I do not remember to have seeing big houses.

2) Indeterminacy with respect to the set H of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the function $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(H)$.

3a) You ask a source:

- What houses are of red color and big size in the town?

The source:

- I am not sure, I think the houses h_1 or h_2.

Therefore, F(red, big) = h_1 or h_2 (indeterminate / not unique / uncertain answer).

3b) You ask again:

- But, what houses are yellow and small?

The source:

- I do not know, the only thing I know is that the house h_5 is neither yellow nor small because I have visited it.

Therefore, F(yellow, small) = not h_5 (again indeterminate / uncertain answer).

3c) Another question you ask:

- Then what houses are blue and big?

The source:

- For sure, either h_8 or h_9.

Therefore, F(blue, big) = either h_8 or h_9 (again indeterminate / uncertain answer).

This is the IndetermHyperSoft Set.

10. Definition of TreeSoft Set

Let U be a universe of discourse, and H a non-empty subset of U, with $P(H)$ the powerset of H.

Let A be a set of attributes (parameters, factors, etc.),

$A= \{A_1, A_2, \ldots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \ldots, A_n$ are considered <u>attributes of first level</u> (since they have one-digit indexes).

Each attribute A_i, $1 \leq i \leq n$, is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, \ldots\}$$

$$A_2 = \{A_{2,1}, A_{2,2}, \ldots\}$$

....

$$A_n = \{A_{n,1}, A_{n,2}, \ldots\}$$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).
Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$$A_{i,j,k}$$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes):

$$A_{i1,i2,\ldots,im}$$

Therefore, a graph-tree is formed, that we denote as Tree(A), whose root is A (considered of level zero), then nodes of level 1, level 2, up to level m.

We call *leaves* of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is:

$$F: P(Tree(A)) \rightarrow P(H)$$

Tree(A) is the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and P(Tree(A)) is the powerset of the Tree(A).
All node sets of the *TreeSoft Set of level m* are:

$$\text{Tree}(A) = \{A_{i1} \mid i1 = 1, 2, \ldots\}$$

The first set is formed by the nodes of level 1, second set by the nodes of level 2, third set by the nodes of level 3, and so on, the last set is formed by the nodes of level m. If the graph-tree has only two levels (m = 2), then the TreeSoft Set is reduced to a MultiSoft Set [8].

11. Practical Example of TreeSoft Set of Level 3

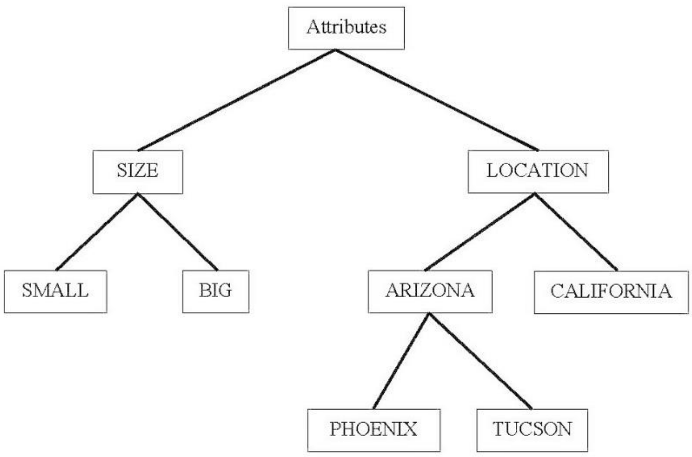

Figure 1. A TreeSoft Set of Level 3.

This is a classical tree, whose:
Level 0 (the root) is the node Attributes;
Level 1 is formed by the nodes: Size, Location;
Level 2 is formed by the nodes Small, Big, Arizona, California;
Level 3 is formed by the nodes Phoenix, Tucson.
Let's consider $H = \{h_1, h_2, \ldots, h_{10}\}$ be a set of houses, and $P(H)$ the powerset of H.
And the set of Attributes: $A = \{A_1, A_2\}$, where A_1 = Size, A_2 = Location.
Then $A_1 = \{A_{11}, A_{12}\}$ = {Small, Big}, $A_2 = \{A_{21}, A_{22}\}$ = {Arizona, California} as American states.
Further on, $A_{22} = \{A_{211}, A_{212}\}$ = {Phoenix, Tucson} as Arizonian cities.

Let's assume that the function F gets the following values:

$$F(\text{Big, Arizona, Phoenix}\} = \{ h_9, h_{10} \}$$

$$F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4 \}$$

F(Big, Arizona) = all big houses from both cities, Phoenix and Tucson

$$= F(\text{Big, Arizona, Phoenix}) \cup F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4, h_9, h_{10} \}.$$

12. Conclusion

The present study represents an application of modern material statistics, specifically leveraging the principles of neutrosophic statistics.

The HyperSoft Set (2018) is a generalization of Soft Set (1999) and MultiSoft Set (2010), from a uni-variate function to a multi-variate function F;

IndetermSoft Set (2022) is an extension of the Soft Set, from the determinate data to indeterminate data;

IndetermHyperSoft Set (2022) is an extension of the HyperSoft Set, from the determinate data to indeterminate data;

and TreeSoft Set (2022) that is a generalization of the MultiSoft Set.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Neutrosophic gsα^*-Open and Closed Maps in Neutrosophic Topological Spaces

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Abstract : The main aim of this paper is to introduce a new concept of $N_{eu}-$ mapping namely $N_{eu}gs\alpha^*-$ open maps and $N_{eu}gs\alpha^*-$ closed maps in $N_{eu}-$ topological spaces . Additionally we relate the properties and characterizations of these mappings with the other mappings in $N_{eu}-$ topological spaces .

Keywords: $N_{eu}gs\alpha^*-$ open set , $N_{eu}gs\alpha^*-$ closed set , $N_{eu}gs\alpha^*-$open map , $N_{eu}gs\alpha^*-$closed map.

1. Introduction

Then the idea of $N_{eu}-$set theory was introduced by F.Smarandache[7] . It includes three components, truth , indeterminancy and false membership function . R.Dhavaseelan and S.Jafari[5] has introduced the concept of $N_{eu}g-$closed sets . A.A.Salama[11] has very first discussed about $N_{eu}-$continuous function and he also discussed about $N_{eu}-$ open and closed mapping . The real life application of $N_{eu}-$topology is applied in Information Systems , Applied Mathematics etc .

In this paper, we introduce some new concepts in $N_{eu}-$topological spaces such as $N_{eu}gs\alpha^*-$closed map and $N_{eu}gs\alpha^*-$open map .

2. Preliminaries

Definition 2.1:[12] Let $\mathbb{P}$ be a non-empty fixed set . A $N_{eu}-$set Ꞥ on the universe $\mathbb{P}$ is defined as Ꞥ= $\{\langle \wp,(t_{Ꞥ}(\wp),i_{Ꞥ}(\wp),f_{Ꞥ}(\wp))\rangle : \wp \in \mathbb{P}\}$ where $t_{Ꞥ}(\wp),i_{Ꞥ}(\wp),f_{Ꞥ}(\wp)$ represent the degree of membership $t_{Ꞥ}(\wp)$, indeterminacy $i_{Ꞥ}(\wp)$ and non-membership function $f_{Ꞥ}(\wp)$ respectively for each element $\wp \in \mathbb{P}$ to the set Ꞥ . Also , $t_{Ꞥ}$, $i_{Ꞥ}$, $f_{Ꞥ}$: $\mathbb{P} \rightarrow]^-0,1^+[$ and $^-0 \leq t_{Ꞥ}(\wp)+i_{Ꞥ}(\wp)+f_{Ꞥ}(\wp) \leq 3^+$. Set of all Neutrosophic set over $\mathbb{P}$ is denoted by $N_{eu}(\mathbb{P})$.

Definition 2.2:[12] A neutrosophic topology (N_{eu}T) on a non-empty set $\mathbb{P}$ is a family $\tau_{N_{eu}}$ of $N_{eu}-$sets in $\mathbb{P}$ satisfying the following axioms ,

(i) $0_{N_{eu}}$, $1_{N_{eu}} \in \tau_{N_{eu}}$.

(ii) $\mathbb{A}_1 \cap \mathbb{A}_2 \in \tau_{N_{eu}}$ for any $\mathbb{A}_1$, $\mathbb{A}_2 \in \tau_{N_{eu}}$.

(iii) $\cup \mathbb{A}_i \in \tau_{N_{eu}}$ for every family $\{ \mathbb{A}_i \,/\, i \in \Omega\iota \} \subseteq \tau_{N_{eu}}$.

In this case , the ordered pair $(\mathbb{P},\tau_{N_{eu}})$ or simply $\mathbb{P}$ is called a $N_{eu}-$topological space (N_{eu}TS) . The elements of $\tau_{N_{eu}}$ is neutrosophic open set ($N_{eu}-OS$) and $\tau_{N_{eu}}{}^c$ is neutrosophic closed set ($N_{eu}-CS$) .

Definition 2.3:[1] A $N_{eu}-$set $\mathbb{A}$ in a N_{eu}TS $(\mathbb{P},\tau_{N_{eu}})$ is called a neutrosophic generalized semi alpha star closed set $(N_{eu}gs\alpha^*-CS)$ if $N_{eu}\alpha-int(N_{eu}\alpha-cl(\mathbb{A})) \subseteq N_{eu}-int(\mathcal{G})$, whenever $\mathbb{A} \subseteq \mathcal{G}$ and $\mathcal{G}$ is $N_{eu}\alpha^*-OS$.

Definition 2.4:[2] A $N_{eu}TS$ $(\mathbb{P},\tau_{N_{eu}})$ is called a $N_{eu}gs\alpha^* - T_{1/2}$ space if every $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}})$ is a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$.

Definition 2.5:[5] Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be any $N_{eu} -$ function and $\mathbb{A} = \{\langle p, (t_{\mathbb{A}}(p), i_{\mathbb{A}}(p), f_{\mathbb{A}}(p)) \rangle : p \in \mathbb{P}\}$ be any $N_{eu} -$ set in $(\mathbb{P},\tau_{N_{eu}})$, then the image of $\mathbb{A}$ under f is denoted by $f_N(\mathbb{A})$, is a $N_{eu} -$ set in $(\mathbb{Q}, \sigma_{N_{eu}})$ and is defined by $f_N(\mathbb{A}) = \{\langle q, (f(t_{\mathbb{A}}(q)), f(i_{\mathbb{A}}(q)), f(f_{\mathbb{A}}(q))) \rangle : q \in \mathbb{Q}\}$,

$$\text{where} \quad f_N\left(t_{\mathbb{A}}(q)\right) = \begin{cases} Sup_{p \epsilon f_N^{-1}(q)}\ t_{\mathbb{A}}(p), & if\ f_N^{-1}(q) \neq \emptyset \\ 0, & otherwise \end{cases}$$

$$f_N\left(i_{\mathbb{A}}(q)\right) = \begin{cases} Sup_{p \epsilon f_N^{-1}(q)}\ i_{\mathbb{A}}(p), & if\ f_N^{-1}(q) \neq \emptyset \\ 0, & otherwise \end{cases}$$

$$\left(1 - f_N(1 - f_{\mathbb{A}})\right)(q) = \begin{cases} inf_{p \epsilon f_N^{-1}(q)}\ f_{\mathbb{A}}(p), & if\ f_N^{-1}(q) \neq \emptyset \\ 1, & otherwise \end{cases} .$$

3. Neutrosophic $gs\alpha^*$ – Open and Closed Maps

Definition 3.1: A $N_{eu} -$ function $f_N: (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* -$ closed map if the image of every $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. (ie) $f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$, for every $N_{eu} - CS$ $\mathbb{A}$ in $(\mathbb{P},\tau_{N_{eu}})$. The complement of $N_{eu}gs\alpha^* -$ closed map is $N_{eu}gs\alpha^* -$ open map .

Theorem 3.2: Every $N_{eu} -$ closed map[10] is $N_{eu}gs\alpha^* -$ closed map , but not conversely.

Example 3.3: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.4,0.6,0.8)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.2,0.4,0.6)\rangle\}$. Define a map $f_N: (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.8,0.4,0.4)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.8, 0.4, 0.4)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map . But f_N is not $N_{eu} -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Theorem 3.4: Every $N_{eu}\alpha -$ closed map[10] is $N_{eu}gs\alpha^* -$ closed map , but not conversely.

Example 3.5: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.3,0.8,0.6)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.3,0.2,0.8)\rangle\}$. Define a map $f_N: (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.6,0.2,0.3)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.6, 0.2, 0.3)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map . But f_N is not $N_{eu}\alpha -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}\alpha - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Theorem 3.6: Every $N_{eu}S -$ closed map[6] is $N_{eu}gs\alpha^* -$ closed map , but not conversely.

Example 3.7: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.2,0.7,0.8)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.4,0.3,0.6)\rangle\}$. Define a map $f_N: (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.8,0.3,0.2)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.8, 0.3, 0.2)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map . But f_N is not $N_{eu}S -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}S - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Theorem 3.8: Every $N_{eu}gs\alpha^* -$ closed map is $N_{eu}\beta -$ closed map[10] , but not conversely.

Example 3.9: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.5,0.6,0.4)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.6,0.8,0.4)\rangle\}$. Define a map $f_N: (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.4,0.4,0.5)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then

$f_N(\mathbb{A}^c) = \{\langle q, (0.4,\ 0.4,\ 0.5)\rangle\}$ is $N_{eu}\beta - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}\beta -$ closed map . But f_N is not $N_{eu}gs\alpha^* -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Theorem 3.10: Every $N_{eu}gs\alpha^* -$ closed map is $N_{eu}\pi g\beta -$ closed map[9] , but not conversely.

Example 3.11: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.2,0.5,0.3)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.4,0.6,0.2)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.3,0.5,0.2)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.3, 0.5, 0.2)\rangle\}$ is $N_{eu}\pi g\beta - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}\pi g\beta -$ closed map . But f_N is not $N_{eu}gs\alpha^* -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Remark 3.12: The concept of $N_{eu}G^* -$ closed map[4] and $N_{eu}gs\alpha^* -$ closed map are independent .

Example 3.13: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.8,0.9,0.7)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.5,0.3,0.8)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.7,0.1,0.8)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.7, 0.1, 0.8)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map . But f_N is not $N_{eu}G^* -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}G^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Example 3.14: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.4,0.3,0.9)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.7,0.4,0.6)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.9,0.7,0.4)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.9, 0.7, 0.4)\rangle\}$ is $N_{eu}G^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}G^* -$closed map . But f_N is not $N_{eu}gs\alpha^* -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Remark 3.15: The concept of $N_{eu}g -$ closed map[10] and $N_{eu}gs\alpha^* -$ closed map are independent .

Example 3.16: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.7,0.7,0.2)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.4,0.3,0.6)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.2,0.3,0.7)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.2, 0.3, 0.7)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map . But f_N is not $N_{eu}g -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}g - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Example 3.17: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.9,0.8,0.8)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.6,0.8,0.4)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.8,0.2,0.9)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.8, 0.2, 0.9)\rangle\}$ is $N_{eu}g - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}g -$closed map . But f_N is not $N_{eu}gs\alpha^* -$closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Remark 3.18: The concept of $N_{eu}P -$ closed map[10] and $N_{eu}gs\alpha^* -$ closed map are independent .

Example 3.19: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.4,0.4,0.5)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.3,0.2,0.8)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.5,0.6,0.4)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.5, 0.6, 0.4)\rangle\}$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$closed map . But f_N is not $N_{eu}P -$ closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}P - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Example 3.20: Let $\mathbb{P} = \{p\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle p, (0.8,0.4,0.3)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.7,0.6,0.5)\rangle\}$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(p) = q$. Let $\mathbb{A}^c = \{\langle p, (0.3,0.6,0.8)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.3, 0.6, 0.8)\rangle\}$ is $N_{eu}P - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}P -$closed map. But f_N is not $N_{eu}gs\alpha^* -$closed map , because $f_N(\mathbb{A}^c)$ is not $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Remark 3.21: The concept of $N_{eu}bg-$ closed map[8] and $N_{eu}gs\alpha^*-$ closed map are independent .

Example 3.22: Let $\mathbb{P}=\{p\}$ and $\mathbb{Q}=\{q\}$. $\tau_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},A\}$ and $\sigma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},B\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q},\sigma_{N_{eu}})$, A= $\{\langle p,(0.6,0.1,0.7)\rangle\}$ and B= $\{\langle q,(0.5,0.3,0.8)\rangle\}$. Define a map $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ by $f_N(p)=q$. Let $A^c=\{\langle p,(0.7,0.9,0.6)\rangle\}$ be a $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(A^c)=\{\langle q,(0.7,\ 0.9,\ 0.6)\rangle\}$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$closed map . But f_N is not $N_{eu}bg-$ closed map , because $f_N(A^c)$ is not $N_{eu}bg-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$.

Example 3.23: Let $\mathbb{P}=\{p\}$ and $\mathbb{Q}=\{q\}$. $\tau_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},A\}$ and $\sigma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},B\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q},\sigma_{N_{eu}})$, A= $\{\langle p,(0.8,0.6,0.6)\rangle\}$ and B= $\{\langle q,(0.7,0.8,0.3)\rangle\}$. Define a map $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ by $f_N(p)=q$. Let $A^c=\{\langle p,(0.6,0.4,0.8)\rangle\}$ be a $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(A^c)=\{\langle q,\ (0.6,\ 0.4,\ 0.8)\rangle\}$ is $N_{eu}bg-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}bg-$ closed map . But f_N is not $N_{eu}gs\alpha^*-$closed map , because $f_N(A^c)$ is not $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$.

Remark 3.24: Let $f:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ and $g:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map , then $gof:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ need not be $N_{eu}gs\alpha^*-$ closed map .

Example 3.25: Let $\mathbb{P}=\{p\}$ and $\mathbb{Q}=\{q\}$. $\tau_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},A\}$ and $\sigma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},B\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q},\sigma_{N_{eu}})$, A = $\{\langle p,(0.6,0.3,0.9)\rangle\}$ and B = $\{\langle q,(0.4,0.5,0.7)\rangle\}$. Define a map $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ by $f_N(p)=q$. Let $A^c=\{\langle p,(0.9,0.7,0.6)\rangle\}$ be a $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(A^c)=\{\langle q,(0.9,0.7,0.6)\rangle\}$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$ closed map . Let $\mathbb{R}=\{r\}$. Also, C= $\{\langle r,(0.2,0.7,0.8)\rangle\}$ is $N_{eu}(\mathbb{R})$ and $\gamma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},C\}$ is N_{eu}TS on $(\mathbb{R},\gamma_{N_{eu}})$. Define a map $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ by $g_N(q-0.2)=r$. Let $B^c=\{\langle q,\ (0.7,0.5,0.4)\rangle\}$ be a $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Then $g_N(B^c)=\{\langle r,(0.5,0.3,0.2)\rangle\}$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_N$ is $N_{eu}gs\alpha^*-$ closed map . Define a map $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ by $g_Nof_N(p-0.2)=r$. But g_Nof_N is not $N_{eu}gs\alpha^*-$ closed map , because $g_Nof_N(A^c)$ is not $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})$.

Theorem 3.26: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ and $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map . Also , $(\mathbb{Q},\sigma_{N_{eu}})$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map .

Proof: Let A be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ closed map , then $f_N(A)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given $(\mathbb{Q},\sigma_{N_{eu}})$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $f_N(A)$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^*-$ closed map , then $g_N(f_N(A))=g_Nof_N(A)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_Nof_N$ is $N_{eu}gs\alpha^*-$ closed map .

4. Properties of Neutrosophic $gs\alpha^*$ −Open and Closed Maps

Theorem 4.1: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}\alpha-$ closed map and $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map . Also , $(\mathbb{Q},\sigma_{N_{eu}})$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map .

Proof: Let A be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}\alpha-$ closed map , then $f_N(A)$ is $N_{eu}\alpha-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N(A)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given $(\mathbb{Q},\sigma_{N_{eu}})$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $f_N(A)$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^*-$ closed map , then $g_N(f_N(A))=g_Nof_N(A)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_Nof_N$ is $N_{eu}gs\alpha^*-$ closed map .

Remark 4.1(a): The above theorem is true if we replace $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ as $N_{eu}S-$ closed map, $N_{eu}\alpha^*-$ closed map , $N_{eu}R-$ closed map , $N_{eu}S\alpha-$ closed map and $N_{eu}g\alpha-$ closed map .

Theorem 4.2: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}-$ closed map and $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map , then $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map .

Proof: Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}-$ closed map , then $f_N(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^*-$ closed map , then $g_N(f_N(\mathbb{A}))=g_Nof_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_Nof_N$ is $N_{eu}gs\alpha^*-$closed map .

Theorem 4.3: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}\alpha-$ continuous , surjective and $(\mathbb{P},\tau_{N_{eu}})$ be $N_{eu}gs\alpha^*-T_{1/2}$ space. Also, $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map , then $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$closed map .

Proof: Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given f_N is $N_{eu}\alpha-$continuous , then ${f_N}^{-1}(\mathbb{A})$ is $N_{eu}\alpha-CS$ in $(\mathbb{P},\tau_{N_{eu}})\Rightarrow {f_N}^{-1}(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $(\mathbb{P},\tau_{N_{eu}})$ be $N_{eu}gs\alpha^*-T_{1/2}$ space , then ${f_N}^{-1}(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map , then $g_Nof_N\left({f_N}^{-1}(\mathbb{A})\right)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})$. Given f_N is surjective , then $g_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_N$ is $N_{eu}gs\alpha^*-$closed map .

Remark 4.3(a): The above theorem is true if we replace f_N as $N_{eu}S-$continuous , $N_{eu}\alpha^*-$ continuous , $N_{eu}R-$ continuous , $N_{eu}S\alpha-$ continuous and $N_{eu}g\alpha-$ continuous .

Theorem 4.4: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}-$ continuous and surjective . Also, $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map , then $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map .

Proof: Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given f_N is $N_{eu}-$continuous , then ${f_N}^{-1}(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$closed map , then $g_Nof_N(f^{-1}(\mathbb{A}))$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})\Rightarrow g_N$ is $N_{eu}gs\alpha^*-$closed map .

Theorem 4.5: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ and $g_N:(\mathbb{Q},\sigma_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be two $N_{eu}-$ mappings, such that their composition $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map . Then the following statements are true .

(1) If g_N is $N_{eu}gs\alpha^*-$ irresolute[2] and injective , then f_N is $N_{eu}gs\alpha^*-$ closed map .

(2) If g_N is strongly $N_{eu}gs\alpha^*-$ continuous[3] and injective, then f_N is $N_{eu}gs\alpha^*-$ closed map .

(3) If g_N is perfectly $N_{eu}gs\alpha^*-$ continuous[3] and injective, then f_N is $N_{eu}gs\alpha^*-$ closed map .

Proof: (1) , Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map , then $g_Nof_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^*-$ irresolute , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given g_N is injective , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))=f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$closed map .

(2) , Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map, then $g_Nof_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})$. Given g_N is strongly $N_{eu}gs\alpha^*-$ continuous , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given g is injective , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))=f_N(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$closed map .

(3) , Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $g_Nof_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{R},\gamma_{N_{eu}})$ is $N_{eu}gs\alpha^*-$ closed map, then $g_Nof_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{R},\gamma_{N_{eu}})$. Given g_N is perfectly $N_{eu}gs\alpha^*-$ continuous , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))$ is both $N_{eu}-OS$ and $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$ .Given g_N is injective , then ${g_N}^{-1}(g_Nof_N(\mathbb{A}))=f_N(\mathbb{A})$ is both $N_{eu}-OS$ and $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$closed map .

Theorem 4.6: Let $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* -$ closed map and $g_N : (\mathbb{Q}, \sigma_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ be $N_{eu} -$ closed map , then $g_N o f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ is $N_{eu}gs\alpha^* -$ closed map , if $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space.

Proof: Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ closed map , then $f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then $f_N(\mathbb{A})$ is $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given g_N is $N_{eu} -$ closed map , then $g_N(f_N(\mathbb{A}))$ is $N_{eu} - CS$ in $(\mathbb{R}, \gamma_{N_{eu}}) \Rightarrow g_N o f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{R}, \gamma_{N_{eu}}) \Rightarrow g_N o f_N$ is $N_{eu}gs\alpha^* -$ closed map .

Theorem 4.7: Let $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ and $g_N : (\mathbb{Q}, \sigma_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ be two $N_{eu} -$ mappings. Then the following statements are true .

(1) If $g_N o f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ is $N_{eu}gs\alpha^* -$ open map and f_N is $N_{eu} -$ continuous , then g_N is $N_{eu}gs\alpha^* -$ open map .

(2) If $g_N o f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ is $N_{eu} -$ closed map and g_N is $N_{eu}gs\alpha^* -$ continuous , then f_N is $N_{eu}gs\alpha^* -$ closed map .

Proof: (1) , Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu} -$ continuous , then ${f_N}^{-1}(\mathbb{A})$ is $N_{eu} - OS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $g_N o f_N$ is $N_{eu}gs\alpha^* -$ open map , then $g_N o f_N\left({f_N}^{-1}(\mathbb{A})\right) = g_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - OS$ in $(\mathbb{R}, \gamma_{N_{eu}}) \Rightarrow g_N$ is $N_{eu}gs\alpha^* -$ open map .

(2) , Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $g_N o f_N$ is $N_{eu} -$ closed map , then $g_N o f_N(\mathbb{A})$ is $N_{eu} - CS$ in $(\mathbb{R}, \gamma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^* -$continuous , then ${g_N}^{-1}(g_N o f_N(\mathbb{A})) = f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map .

Theorem 4.8: Let $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be a bijective $N_{eu} -$ mapping . Then the following are equivalent.

(1) f_N is $N_{eu}gs\alpha^* -$ open map , (2) f_N is $N_{eu}gs\alpha^* -$ closed map , (3) ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$continuous[2] .

Proof: (1) ⇒ (2) , Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow \mathbb{A}^c$ is $N_{eu} - OS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ open map , then $f_N(\mathbb{A}^c) = (f_N(\mathbb{A}))^c$ is $N_{eu}gs\alpha^* - OS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow$ $f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ closed map .

(2) ⇒ (3) , Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ closed map , then $f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is bijective , then $\left({f_N}^{-1}\right)^{-1}(\mathbb{A}) = f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^* -$continuous .

(3) ⇒ (1) , Let $\mathbb{A}$ be any $N_{eu} - OS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$continuous , then $\left({f_N}^{-1}\right)^{-1}(\mathbb{A}) = f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - OS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ open map .

Theorem 4.9: Let $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* -$ closed map and $(\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - T_{1/2}$ space , then f_N is $N_{eu} -$ closed map .

Proof: Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ closed map , then $f_N(\mathbb{A})$ is $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then $f_N(\mathbb{A})$ is $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu} -$ closed map .

Theorem 4.10: Let $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* -$ closed map and $(\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - T_{1/2}$ space , then f_N is $N_{eu}\alpha -$ closed map .

Proof: Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ closed map, then $f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given $(\mathbb{Q},\sigma_{N_{eu}})$ is $N_{eu}gs\alpha^*-T_{1/2}$ space, then $f_N(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{Q},\sigma_{N_{eu}}) \Rightarrow f_N(\mathbb{A})$ is $N_{eu}\alpha-CS$ in $(\mathbb{Q},\sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}\alpha-$ closed map.

Remark 4.10(a): The above theorem is true if we replace f as $N_{eu}S-$closed map, $N_{eu}\alpha^*-$ closed map, $N_{eu}R-$ closed map, $N_{eu}S\alpha-$ closed map and $N_{eu}g\alpha-$ closed map.

Theorem 4.11: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be a bijective $N_{eu}-$mapping and f_N^{-1} is $N_{eu}gs\alpha^*-$ irresolute, then f_N is $N_{eu}gs\alpha^*-$ closed map.

Proof: Let $\mathbb{A}$ be any $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}}) \Rightarrow \mathbb{A}$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given $f_N{}^{-}{}_1$ is $N_{eu}gs\alpha^*-$ irresolute, then $(f_N{}^{-1})^{-1}(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Given f_N is bijective, then $(f_N{}^{-1})^{-1}(\mathbb{A})=f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^*-$closed map.

Theorem 4.12: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ open map iff $f_N(N_{eu}-int(\mathbb{A}))\subseteq N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))$, for each $N_{eu}-$set $\mathbb{A}$ in $(\mathbb{P},\tau_{N_{eu}})$.

Proof: Let $\mathbb{A}$ be any $N_{eu}-$ set in $(\mathbb{P},\tau_{N_{eu}}) \Rightarrow N_{eu}-int(\mathbb{A})$ is $N_{eu}-OS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ open map, then $f_N(N_{eu}-int(\mathbb{A}))$ is $N_{eu}gs\alpha^*-OS$ in $(\mathbb{Q},\sigma_{N_{eu}}) \Rightarrow N_{eu}gs\alpha^*-int\left(f_N(N_{eu}-int(\mathbb{A}))\right)=f_N(N_{eu}-int(\mathbb{A}))$. Given $f_N(N_{eu}-int(\mathbb{A}))\subseteq f_N(\mathbb{A})$, then $f_N(N_{eu}-int(\mathbb{A}))=N_{eu}gs\alpha^*-int\left(f_N(N_{eu}-int(\mathbb{A}))\right)\subseteq N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))$. Conversely, Suppose $\mathbb{A}$ is $N_{eu}-OS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then by hypothesis, $f_N(\mathbb{A})=f_N(N_{eu}-int(\mathbb{A}))\subseteq N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))\to ①$. Given $N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))$ is the largest $N_{eu}gs\alpha^*-OS$ which is contained in $f_N(\mathbb{A})$, then $N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))\subseteq f_N(\mathbb{A})\to ②$. From ① and ②, $f_N(\mathbb{A})=N_{eu}gs\alpha^*-int(f_N(\mathbb{A}))\Rightarrow f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-OS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$ open map.

Theorem 4.13: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map iff $f_N(N_{eu}-cl(\mathbb{A}))\supseteq N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))$, for each $N_{eu}-$set $\mathbb{A}$ in $(\mathbb{P},\tau_{N_{eu}})$.

Proof: Let $\mathbb{A}$ be any $N_{eu}-$ set in $(\mathbb{P},\tau_{N_{eu}}) \Rightarrow N_{eu}-cl(\mathbb{A})$ is $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ closed map, then $f_N(N_{eu}-cl(\mathbb{A}))$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow N_{eu}gs\alpha^*-cl\left(f_N(N_{eu}-cl(\mathbb{A}))\right)=f_N(N_{eu}-cl(\mathbb{A}))$. Given $f_N(\mathbb{A})\subseteq f_N(N_{eu}-cl(\mathbb{A}))$, then $f_N(N_{eu}-cl(\mathbb{A}))=N_{eu}gs\alpha^*-cl\left(f_N(N_{eu}-cl(\mathbb{A}))\right)\supseteq N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))$. Conversely, Suppose $\mathbb{A}$ is $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then by hypothesis, $f_N(\mathbb{A})=f_N(N_{eu}-cl(\mathbb{A}))\supseteq N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))\to ①$. Given $N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))$ is the smallest $N_{eu}gs\alpha^*-CS$ containing $f_N(\mathbb{A})$, then $f_N(\mathbb{A})\subseteq N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))\to ②$. From ① and ②, $f_N(\mathbb{A})=N_{eu}gs\alpha^*-cl(f_N(\mathbb{A}))\Rightarrow f_N(\mathbb{A})$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow f_N$ is $N_{eu}gs\alpha^*-$ closed map.

Theorem 4.14: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ closed map, then $N_{eu}-cl\left(f_N{}^{-1}(\mathbb{A})\right)\supseteq f_N{}^{-1}(N_{eu}gs\alpha^*-cl(\mathbb{A}))$ for every $N_{eu}-$set $\mathbb{A}$ of $(\mathbb{Q},\sigma_{N_{eu}})$.

Proof: Let $\mathbb{A}$ be a $N_{eu}-$ set in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow N_{eu}-cl\left(f_N{}^{-1}(\mathbb{A})\right)$ is $N_{eu}-CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ closed map, then $f_N\left(N_{eu}-cl\left(f_N{}^{-1}(\mathbb{A})\right)\right)$ is $N_{eu}gs\alpha^*-CS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. By theorem 4.13, $f_N\left(N_{eu}-cl\left(f_N{}^{-1}(\mathbb{A})\right)\right)\supseteq N_{eu}gs\alpha^*-cl\left(f_N\left(f_N{}^{-1}(\mathbb{A})\right)\right)\supseteq N_{eu}gs\alpha^*-cl(\mathbb{A}) \quad \Rightarrow N_{eu}-cl\left(f_N{}^{-1}(\mathbb{A})\right)\supseteq f_N{}^{-1}(N_{eu}gs\alpha^*-cl(\mathbb{A}))$.

Theorem 4.15: Let $f_N:(\mathbb{P},\tau_{N_{eu}})\to(\mathbb{Q},\sigma_{N_{eu}})$ be $N_{eu}gs\alpha^*-$ open map, then $N_{eu}-int\left(f_N{}^{-1}(\mathbb{A})\right)\subseteq f_N{}^{-1}(N_{eu}gs\alpha^*-int(\mathbb{A}))$ for every $N_{eu}-$ set $\mathbb{A}$ of $(\mathbb{Q},\sigma_{N_{eu}})$.

Proof: Let $\mathbb{A}$ be a $N_{eu}-$ set in $(\mathbb{Q},\sigma_{N_{eu}})\Rightarrow N_{eu}-int\left(f_N{}^{-1}(\mathbb{A})\right)$ is $N_{eu}-OS$ in $(\mathbb{P},\tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^*-$ open map, then $f_N\left(N_{eu}-int\left(f_N{}^{-1}(\mathbb{A})\right)\right)$ is $N_{eu}gs\alpha^*-OS$ in $(\mathbb{Q},\sigma_{N_{eu}})$. Now, $f_N\left(N_{eu}-int\left(f_N{}^{-1}(\mathbb{A})\right)\right)\subseteq N_{eu}gs\alpha^*-int\left(f_N\left(f_N{}^{-1}(\mathbb{A})\right)\right)\subseteq N_{eu}gs\alpha^*-int(\mathbb{A})$ (by theorem 4.12) $\Rightarrow N_{eu}-int\left(f_N{}^{-1}(\mathbb{A})\right)\subseteq f_N{}^{-1}(N_{eu}gs\alpha^*-int(\mathbb{A}))$.

5.Conclusions : In this paper we have discussed about the $N_{eu}gs\alpha^* -$ open and closed map . We had an idea to extend this paper to the next level about $N_{eu}gs\alpha^* -$ homeomorphism and also the application of this paper . In future work , we will discussed and find out the results of this paper application.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Generalized Double Statistical Convergence Sequences on Ideals in Neutrosophic Normed Spaces

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Abstract: In this present research, having view in the Neutrosophic norm $(\dot{\mu}, \ddot{v}, \dddot{\omega})$, which we presented $\mathcal{J}_2$-lacunary statistical convergence and $\mathcal{J}_2$-lacunary convergence strongly, looked into interactions between them, and made a few findings regarding the respective categories. At least went further to look at how both of such case approaches relate to $\mathcal{J}_2$-statistical convergence within the relevant Neutrosophic normed space.

Keywords: Banach Space; Ideal; Neutrosophic Normed Space; $\mathcal{J}_2$-lacunary Convergence of Statistical; Strongly Convergence.

1. Introduction

Fast invented statistical convergence in a sequence of real number. Research conducted by Das and Ghosal, et al. provide additional information along with applications using ideals. When Zadeh's [18], studied fuzzy set theory began to gain significance to be an academic subject. Atanassov [1] studied intuitionistic fuzzy sets; Atanassov et al. used this idea to analyze administrative decision-making challenges. The concept of an intuitionistic fuzzy metric space was proposed by Park.

Smarandache [15] introduced Neutrosophic Sets (NS) as a development of the IFS. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the neutrosophic set operators, various results can be acquired by applying the intuitionistic fuzzy operators, whereas the neutrosophic operators are taken into the cognizance of the indeterminacy at a degree akin to truth-membership and falsehood-non membership, the operators disregard the indeterminacy. Jeyaraman et al. [9] developed approximate fixed point theorems for weak contractions on neutrosophic normed spaces in 2022. In the present paper, our aim is to discuss Neutrosophic norm $(\dot{\mu}, \ddot{v}, \dddot{\omega})$, which we presented $\mathcal{J}_2$-lacunary statistical convergence and $\mathcal{J}_2$-lacunary convergence strongly, looked into interactions between them, and made a few findings regarding the respective categories.

2. Preliminaries

The formula $\breve{\delta}(\mathfrak{K}) = \lim_{\hat{n}\to\infty} \frac{1}{\hat{n}} |\{\widetilde{m} \leq \hat{n} : \widetilde{m} \in \mathfrak{K}\}|$, describes the natural density that exists for an integer set $\mathfrak{K}$ that includes positive numbers, whenever $|\widetilde{m} \leq \hat{n} : \widetilde{m} \in \mathfrak{K}|$ represents the maximum value less than or equal to $\widetilde{m}$ with many elements in $\mathfrak{K}$.

Since each value $\breve{\varsigma} > 0$, the numerical sequence $\mathfrak{x} = (\mathfrak{x}_{\widetilde{m}})$ can be considered being statistically convergent in terms of $\mathfrak{L}$.

$$\lim_{\hat{n}\to\infty} \frac{1}{\hat{n}} |\{\widetilde{m} \leq \hat{n} : |\mathfrak{x}_{\widetilde{m}} - \mathfrak{L}| \geq \breve{\varsigma}\}| = 0,$$

i.e., $|\mathfrak{x}_{\widetilde{m}} - \mathfrak{L}| < \breve{\varsigma}$ $(a.a.\ \widetilde{m})$ (1.1)

Here, which we state that $st - lim\, \mathfrak{x}_{\widetilde{m}} = \mathfrak{L}$. As an example, specify $\mathfrak{x}_{\widetilde{m}} = 1$ When $\widetilde{m}$ acts as square, else $\mathfrak{x}_{\widetilde{m}} = 0$. When $|\{\widetilde{m} \leq \hat{n} : \mathfrak{x}_{\widetilde{m}} \neq 0\}| \leq \sqrt{\hat{n}}, thus\ st - lim\, \mathfrak{x}_{\widetilde{m}} = 0$. Remember it $st - lim\, \mathfrak{x}_{\widetilde{m}} = 0$ holds true even if the developers had given $\mathfrak{x}_{\widetilde{m}}$ anything measures upon all if m has become square. Yet $\mathfrak{x}$ never converges nor bounded. This happens obvious which $lim\, \mathfrak{x}_{\widetilde{m}} = \mathfrak{L}$ while inequality of (1.1) exists over every a limited amount of $\widetilde{m}$. Usual convergence naturally generalizes to Statistical Convergence (SC). SC can become thought of just being regular summability convergent in addition to require never remain convergent nor bounded because $lim\, \mathfrak{x}_{\widetilde{m}} = \mathfrak{L}$ yields $st - lim\, \mathfrak{x}_{\widetilde{m}} = \mathfrak{L}$.

A ideal of non-trivial, if $\mathcal{J}$ consists only singletons, and then $\mathcal{J}$ has (i) an admissible ideal over S. The sequence $(\mathfrak{x}_{\widetilde{m}})$ is shown to make themselves ideal convergent towards $\mathfrak{L}$ and every $\breve{\varsigma} > 0$, i.e.,

$$\mathfrak{A}\,(\breve{\varsigma}) = \{\widetilde{m} \in \mathbb{N} : |\mathfrak{x}_{\widetilde{m}} - \mathfrak{L}| \geq \breve{\varsigma}\} \in \mathcal{J}.$$

Considering $\mathcal{J} = \mathcal{J}_{\breve{\delta}} = \{\mathfrak{A} \subseteq \mathbb{N} : \breve{\delta}(\mathfrak{A}) = 0\}$, while $\breve{\delta}(\mathfrak{A})$ denotes the convergent value for set $\mathfrak{A}$. Convergence of ideal occurs around identical interval as statistical convergence when there is a non-trivial admissible ideal $\mathcal{J}_{\breve{\delta}}$.

Ideal $\mathcal{J}_2$ is a nontrivial on $\mathbb{N} \times \mathbb{N}$ appears to be strongly admissible when $\{\imath\} \times \mathbb{N}$ and $\mathbb{N} \times \{\imath\}$ originate from $\mathcal{J}_2$ with every case $\imath \in \mathbb{N}$.

Therefore immediately apparent such strongly admissible ideal remains permissible.

Additionally, the article we take strongly admissible ideal $\mathcal{J}_2$ in $\mathbb{N} \times \mathbb{N}$, and ℓ^2_∞ denotes the space that contains all bounded double sequences.

A sequence of double lacunary represented by a double sequence $\bar{\theta} = \theta_{u\mathcal{s}} = \{(\grave{c}_u, \acute{d}_{\mathcal{s}})\}$ exist two increasing integer sequences $(\grave{c}_u)$ and $(\acute{d}_{\mathcal{s}})$ which means

$$\grave{c}_0 = 0, \hbar_u = \grave{c}_u - \grave{c}_{u-1} \to \infty\ and\ \acute{d}_0 = 0, \bar{\hbar}_{\mathcal{s}} = \acute{d}_{\mathcal{s}} - \acute{d}_{\mathcal{s}-1} \to \infty,\ u, \mathcal{s} \to \infty.$$

We shall utilize the term shown below, $\grave{c}_{u\mathcal{s}} := \grave{c}_u \acute{d}_{\mathcal{s}}, \hbar_{u\mathcal{s}} := \hbar_u \hbar_{\mathcal{s}}\ and\ \theta_{u\mathcal{s}}$ is determined as

$$\mathfrak{J}_{u\mathcal{s}} := \{(\grave{c}, \acute{d}) : \grave{c}_u - 1 < \grave{c} \leq \grave{c}_u \text{ and } \acute{d}_{\mathcal{s}} - 1 < \acute{d} \leq \acute{d}_{\mathcal{s}}\},$$

$$\breve{q}_u := \frac{\grave{c}_u}{\grave{c}_u - 1}, \bar{\breve{q}}_{\mathcal{s}} := \frac{\acute{d}_u}{\acute{d}_u - 1}\ \breve{q}_{u\mathcal{s}} := \breve{q}_u \bar{\breve{q}}_{\mathcal{s}}.$$

All through the research, with $\theta_2 = \theta_{u\mathcal{s}} = \{(\grave{c}_u, \acute{d}_{\mathcal{s}})\}$ that we shall designate a sequence of double lacunary nonnegative real numbers, respectively, wherever a different condition exists.

The double sequence with the integers $\mathfrak{x} = \{\mathfrak{x}_{\widetilde{m}\hat{n}}\}$ can be considered that it is $\mathcal{J}_2$-lacunary statistical convergent as well as $(\mathcal{J}_2)$-convergent towards $\mathfrak{L}$, when for every $\breve{\varsigma} > 0$ and $\breve{\delta} > 0$,

$$\left\{(u, \mathcal{s}) \in \mathbb{N} \times \mathbb{N} : \frac{1}{y_r \overline{y_u}} |\{(\widetilde{m}, \hat{n}) \in \mathfrak{J}_{ru} : |\mathfrak{x}_{\widetilde{m}} - \mathfrak{L}| \geq \breve{\varsigma}\}| \geq \breve{\delta}\right\} \in \mathcal{J}_2.$$

In the above illustration, we have to put

$$\mathfrak{x}_{\widetilde{m}\hat{n}} \to \mathfrak{L}\left(I_{\theta_2}(\mathcal{J}_2)\right) or\ I_{\theta_2}(\mathcal{J}_2) - \lim_{\widetilde{m},\hat{n}\to\infty} \mathfrak{x}_{\widetilde{m}\hat{n}} = \mathfrak{L}.$$

Definition 2.1. The 7-tuple $(\Xi, \dot{\mu}, \ddot{v}, \ddot{\omega}, \circledast, \Theta, \Delta)$ is said to be a Neutrosophic Normed Space [*NNS*], if $\circledast$ acts as t-norm which is continuous, Θ and Δ act as t-co norms which are continuous along with Ξ which acts as a real vector space, $\dot{\mu}, \ddot{v}, \ddot{\omega}$ which denote fuzzy sets through $\Xi \times \mathfrak{F}$.

(cn1) $\dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) + \ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) + \ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) \leq 3$,

(cn2) $0 \leq \dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) \leq 1$; $0 \leq \ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) \leq 1$ *and* $0 \leq \ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) \leq 1$,

(cn3) $\dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$ for all non-positive real number $\dot{\varrho}$,

(cn4) $\dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$ for all $\dot{\varrho} \in \mathbb{R}^{+} \Leftrightarrow \mathfrak{h} = 0$,

(cn5) $\dot{\mu}\left(\gamma\mathfrak{h}, \dot{\varrho}\right) = \dot{\mu}\left(\mathfrak{h}, \frac{\dot{\varrho}}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$,

(cn6) $\dot{\mu}\left(\mathfrak{h} + \mathfrak{z}, \dot{\varrho} + \widetilde{\alpha}\right) \geq min\{\dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right), \dot{\mu}\left(\mathfrak{z}, \widetilde{\alpha}\right)\}$,

(cn7) $\lim_{\dot{\varrho}\to\infty} \dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$ and $\lim_{\varrho\to\infty} \dot{\mu}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$,

(cn8) $\ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$ for all non-positive real number $\dot{\varrho}$,

(cn9) $\ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$ for all $\dot{\varrho} \in \mathbb{R}^{+} \Leftrightarrow \mathfrak{h} = 0$,

(cn10) $\ddot{v}\left(\gamma\mathfrak{h}, \dot{\varrho}\right) = \ddot{v}\left(\mathfrak{h}, \frac{\dot{\varrho}}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$,

(cn11) $\ddot{v}\left(\mathfrak{h} + \mathfrak{z}, \dot{\varrho} + \widetilde{\alpha}\right) \leq max\{\ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right), \ddot{v}\left(\mathfrak{z}, \widetilde{\alpha}\right)\}$,

(cn12) $\lim_{\dot{\varrho}\to\infty} \ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$ and $\lim_{\dot{\varrho}\to\infty} \ddot{v}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$,

(cn13) $\ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$ for all non-positive real number $\dot{\varrho}$,

(cn14) $\ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$ for all $\dot{\varrho} \in \mathbb{R}^{+} \Leftrightarrow \mathfrak{h} = 0$,

(cn15) $\ddot{\omega}\left(\gamma\mathfrak{h}, \dot{\varrho}\right) = \ddot{\omega}\left(\mathfrak{h}, \frac{\dot{\varrho}}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$,

(cn16) $\ddot{\omega}\left(\mathfrak{h} + \mathfrak{z}, \dot{\varrho} + \widetilde{\alpha}\right) \leq max\{\ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right), \ddot{\omega}\left(\mathfrak{z}, \widetilde{\alpha}\right)\}$,

(cn17) $\lim_{\dot{\varrho}\to\infty} \ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) = 0$ and $\lim_{\dot{\varrho}\to\infty} \ddot{\omega}\left(\mathfrak{h}, \dot{\varrho}\right) = 1$.

In the above case, $(\dot{\mu}, \ddot{v}, \ddot{\omega})$ is identified as a *NN* on Ξ. In addition, $(\Xi, \dot{\mu}, \ddot{v}, \ddot{\omega})$ is referred to be a *NNS*.

In *NNS,* we look at generalized sequence a statistical convergence through ideals. In the present article, we pay attention and in addition we have to investigate the interaction among two new ideas, as well as the author's introduction of $\mathcal{J}_2$-Lacunary Statistical Convergence (LSC) and strongly $\mathcal{J}_2$-Lacunary Convergence (LC) in a *NNS*.

3. Main Results

Definition 3.1. Let $(\Xi, \dot{\mu}, \ddot{v}, \ddot{\omega}, \circledast, \Theta, \Delta)$ be a *NNS*, $\mathcal{J}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ which is a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $\mathfrak{x} = (\mathfrak{x}_{az})$ is said to be $\mathcal{J}_2$ −Statistically Convergent (SC) to $\check{\xi} \in \Xi$ relate to the *NN* $(\dot{\mu}, \ddot{v}, \ddot{\omega})$, which is represented by $I(\mathcal{J}_2)^{(\dot{\mu},\ddot{v},\ddot{\omega})} - lim\ \mathfrak{x} = \check{\xi}$ or $\mathfrak{x}_{a,z} \xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})} \check{\xi}(I(\mathcal{J}_2))$, if for every $\check{\varsigma} > 0$, every $\check{\delta} > 0$, and $\dot{\varrho} > 0$,

$$\left\{(\widetilde{m},\widehat{n})\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{\widetilde{m}\widehat{n}}\left|\left\{\begin{matrix}a\le\widetilde{m},z\le\widehat{n}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\zeta}\ or\\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\ and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\end{matrix}\right\}\right|\ge\check{\delta}\right\}\in\mathcal{J}_2.$$

Definition 3.2. A sequence $\mathfrak{x}=(\mathfrak{x}_{az})$ is said to be $\mathcal{J}_2-$ LSC to $\grave{\xi}\in\Xi$ relate with the *NN* $(\dot{\mu},\ddot{v},\ddot{\omega})$ which is denoted by $I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{v},\ddot{\omega})}-lim\ \mathfrak{x}=\grave{\xi}$ or $\mathfrak{x}_{a,z}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(I_\theta(\mathcal{J}_2)\right)$, if for every $\check{\zeta}>0$, every $\check{\delta}>0$, and $\dot{\varrho}>0$,

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{matrix}(a,z)\in J_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\zeta}\ or\\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\ and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\end{matrix}\right\}\right|\ge\check{\delta}\right\}\in\mathcal{J}_2.$$

Definition 3.3. A sequence $\mathfrak{x}=(\mathfrak{x}_{az})$ is said as a strongly $\mathcal{J}_2-$ LC to $\grave{\xi}$ or $J_\theta(\mathcal{J}_2)-$convergent to $\grave{\xi}\in\Xi$ relate to the *NN* $(\dot{\mu},\ddot{v},\ddot{\omega})$ it can be denoted by $\mathfrak{x}_{az}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}$ $\grave{\xi}\left(J_\theta(\mathcal{J}_2)\right)$, if for every $\check{\delta}>0$ and $\dot{\varrho}>0$,

$$\left\{\begin{matrix}(r,u)\in\mathbb{N}\times\mathbb{N}\ :\dfrac{1}{y_r\overline{y_u}}\displaystyle\sum_{(a,z)\,\in\,\mathfrak{J}_{ru}}\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\delta}\ or\\ \dfrac{1}{y_r\overline{y_u}}\displaystyle\sum_{(a,z)\,\in\,\mathfrak{J}_{ru}}\ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\delta}\\ and\dfrac{1}{y_r\overline{y_u}}\displaystyle\sum_{(a,z)\,\in\,\mathfrak{J}_{ru}}\ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\delta}\end{matrix}\right\}\in\mathcal{J}_2.$$

Theorem 3.4. *Let* $(\Xi,\dot{\mu},\ddot{v},\ddot{\omega},\circledast,\Theta,\Delta)$ *be a NNS, Double Lacunary (DL) sequence* θ, *strongly admissible ideal* $\mathcal{J}_2$ *in* $\mathbb{N}$, *and* $\mathfrak{x}=(\mathfrak{x}_{az})\in\Xi$, *then*

(i) (a) If $\mathfrak{x}_{az}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(J_\theta(\mathcal{J}_2)\right)$, *then* $\mathfrak{x}_{az}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(I_\theta(\mathcal{J}_2)\right)$.

(b) If $\mathfrak{x}\in\ell_\infty^2(\Xi)$, *be a* Ξ *of all bounded sequence space with* $\mathfrak{x}_{az}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(I_\theta(\mathcal{J}_2)\right)$ *then* $\mathfrak{x}_{az}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(J_\theta(\mathcal{J}_2)\right)$.*n*

(ii) $I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{v},\ddot{\omega})}\cap\ell_\infty^2(\Xi)=J_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{v},\ddot{\omega})}\cap\ell_\infty^2(\Xi)$.

Proof. *(i) – (a).* Given hypothesis, for all $\check{\zeta}>0,\check{\delta}>0$, and $\dot{\varrho}>0$, let $\mathfrak{x}_{a,z}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(J_\theta(\mathcal{J}_2)\right)$. Then we can write

$$\sum_{(a,z)\,\in\,\mathfrak{J}_{ru}}\left(\dot{\mu}\ \left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)and\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)\right)$$

$$\ge\sum_{\substack{(a,z)\,\in\,\mathfrak{J}_{ru}\\ \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\zeta}\ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\ and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}}}\left(\dot{\mu}\ \left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)and\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\right)$$

$$\ge\check{\zeta}.\left|\left\{(a,z)\in\mathfrak{J}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\zeta}\ \ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\ \ and\ \ddot{\omega}\ \left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\right\}\right|.$$

Then observe that

$$\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{matrix}(a,z)\in\mathfrak{J}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le1-\check{\zeta}\ \ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\zeta}\end{matrix}\right\}\right|\ge\check{\delta}$$

and

$$\frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}} \dot{\mu}\,\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le (1-\check{\varsigma}\,)\breve{\delta}\ or\ \frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}} \ddot{v}\,\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\varsigma}\breve{\delta}$$

$$\text{and}\ \frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}} \ddot{\omega}\,\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\varsigma}\breve{\delta},$$

which implies $\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{matrix}(a,z)\in\Im_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le 1-\check{\varsigma}\\ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\varsigma}\\ and\\ \ddot{\omega}\ \left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\varsigma}\end{matrix}\right\}\right|\ge\breve{\delta}\right\}$

$$\subset\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{y_r\overline{y_u}}\ge\check{\varsigma}\breve{\delta}\right\}$$

Since $\mathfrak{x}_{a,z}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(J_\theta(\mathcal{I}_2)\right)$, we immediately see that $\mathfrak{x}_{a,z}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(I_\theta(\mathcal{I}_2)\right)$.

$(i)-(b)$. We assume that $\mathfrak{x}_{a,z}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\grave{\xi}\left(I_\theta(\mathcal{I}_2)\right)$ and $\mathfrak{x}\in\ell^2_\infty(\Xi)$. The inequalities $\mathcal{T}$ or hold for all a,z. Let $\check{\varsigma}\ >0$ be given. Then we have

$$\frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}}(\dot{\mu}\ \left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right))$$

$$=\frac{1}{y_r\overline{y_u}}\sum_{\substack{(a,z)\,\in\,\Im_{ru}\\ \dot{\mu}(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})\le 1-\check{\varsigma}\ or\ \ddot{v}\ (\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})\ge\check{\varsigma}\ and\ \ddot{\omega}\ (\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})\ge\check{\varsigma}}}\left(\begin{matrix}\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\end{matrix}\right)$$

$$+\frac{1}{y_r\overline{y_u}}\sum_{\substack{(a,z)\,\in\,\Im_{ru}\\ \dot{\mu}\ (\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})>1-\check{\varsigma}\ or\ \ddot{v}\ (\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})<\check{\varsigma}\ and\ \ddot{\omega}\ (\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho})<\check{\varsigma}}}\left(\begin{matrix}\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\end{matrix}\right)$$

$$\le\frac{\mathfrak{M}}{y_r\overline{y_u}}\left|\left\{\begin{matrix}(a,z)\in\Im_{ru}\ :\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)\le 1-\check{\varsigma}\ \ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)\ge\check{\varsigma}\ \ and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\ge\check{\varsigma}\end{matrix}\right\}\right|+\check{\varsigma}\,.$$

Note that

$$\mathfrak{A}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\varsigma}\,,\dot{\varrho}\ )=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{y_r\overline{y_u}}\left|\left\{\begin{matrix}(a,z)\in\Im_{ru}\ :\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\le 1-\check{\varsigma}\\ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)\ge\check{\varsigma}\ \ and\\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\ \right)\ge\check{\varsigma}\end{matrix}\right\}\right|\ge\frac{\check{\varsigma}}{\mathfrak{M}}\right\}$$

belongs to $\mathcal{I}_2$. If $r\in\left(\mathfrak{A}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\varsigma}\,,\dot{\varrho})\right)^c$ then we have

$$\frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}}\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)>(1-2\check{\varsigma}\ )$$

$$or\ \frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}}\ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)<2\check{\varsigma}$$

$$and\ \frac{1}{y_r\overline{y_u}}\sum_{(a,z)\,\in\,\Im_{ru}}\ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)<2\check{\varsigma}\,,$$

Now

$$\mathfrak{T}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\zeta},\dot{\varrho}) = \left\{\begin{array}{c}(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{y_r\overline{y_u}}\sum_{(a,z)\in\mathfrak{J}_{ru}} \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le (1-2\check{\zeta}) \\ or\,\frac{1}{y_r\overline{y_u}}\sum_{(a,z)\in\mathfrak{J}_{ru}} \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge 2\check{\zeta} \\ and\,\frac{1}{y_r\overline{y_u}}\sum_{(a,z)\in\mathfrak{J}_{ru}} \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) < 2\check{\zeta},\end{array}\right\}.$$

Hence $\mathfrak{T}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\zeta},\dot{\varrho}) \subseteq \mathfrak{A}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\zeta},\dot{\varrho})$, then along with on an ideal, $\mathfrak{T}_{\dot{\mu},\ddot{v},\ddot{\omega}}(\check{\zeta},\dot{\varrho}) \in \mathcal{I}_2$.

Therefore, we conclude that $\mathfrak{x}_{a,z} \xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})} \grave{\xi}\big(J_\theta(\mathcal{I}_2)\big)$,

(*ii*) It immediately following (*i*) – (*a*) and (*i*) – (*b*).

Theorem 3.5. *Let* $(\Xi,\dot{\mu},\ddot{v},\ddot{\omega},\circledast,\Theta,\Delta)$ *be a NNS. When a sequence* θ *of DL with* $\lim\inf_r q_r > 1$, $\lim\inf_u q_u > 1$ *thereafter*

$$\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})} \grave{\xi}\left(I(\mathcal{I}_2)\right) \Rightarrow \mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})} \grave{\xi}\left(I_\theta(\mathcal{I}_2)\right),$$

Proof. Assume initially that $\liminf_r q_r > 1$, $\lim\inf_u q_u > 1$ then there exists a $\tilde{\varphi},\hat{\psi} > 0$ so that $q_r \ge 1+\alpha, q_u > 1+\beta$ for sufficiently large r,u, which implies that

$$\frac{y_r\overline{y_u}}{z_{ru}} \ge \frac{\tilde{\varphi}\hat{\psi}}{(1+\tilde{\varphi})(1+\hat{\psi})}.$$

If $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})} \grave{\xi}\left(I(\mathcal{I}_2)\right)$, then for every $\check{\zeta} > 0$ and for sufficiently large r,u, we have

$$\frac{1}{z_{ru}}\left|\left\{a \le a_r, z \le z_u : \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le 1-\check{\zeta}\; or\; \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\; and\; \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\right\}\right|$$

$$\ge \frac{1}{z_{ru}}\left|\left\{(a,z) \in \mathfrak{J}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le 1-\check{\zeta}\; or\; \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\; and\; \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\right\}\right|$$

$$\ge \frac{\tilde{\varphi}\hat{\psi}}{(1+\tilde{\varphi})(1+\hat{\psi})}\left(\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{array}{c}(a,z) \in \mathfrak{J}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le 1-\check{\zeta}\; or \\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\; and\; \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\end{array}\right\}\right|\right)$$

Then for any $\breve{\delta} > 0$, we get

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{y_r\overline{y_u}}\left|\left\{\begin{array}{c}(a,z) \in \mathfrak{J}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le 1-\check{\zeta}\; or\; \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta} \\ and\; \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\end{array}\right\}\right| \ge \breve{\delta}\right\}$$

$$\subseteq \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{z_{ru}}\left|\left\{\begin{array}{c}a \le a_r, z \le z_u : \dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \le 1-\check{\zeta} \\ or\; \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta} \\ and\; \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right) \ge \check{\zeta}\end{array}\right\}\right| \ge \frac{\breve{\delta}\tilde{\varphi}\hat{\psi}}{(1+\tilde{\varphi})(1+\hat{\psi})}\right\}.$$

If $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}(I(\mathcal{J}_2))$ consequently, the set on the right-hand belongs to $\mathcal{J}_2$ and the set on the left-hand belongs to $\mathcal{J}_2$. It demonstrates that $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}(I_\theta(\mathcal{J}_2))$.

The following result depends on the hypothesis of the lacunary sequence θ provides satisfaction to the requirement to satisfy each set $\mathcal{C} \in \mathfrak{F}(\mathcal{J}_2)$, $\cup\{\hat{n} : z_{r-1} < \hat{n} \leq z_r,\ r \in \mathcal{C}\} \in \mathfrak{F}(\mathcal{J}_2)$.

Theorem 3.6. *Let $(\Xi, \dot{\mu}, \ddot{\nu}, \ddot{\omega}, \circledast, \Theta, \Delta)$ be a NNS. When a sequence θ of DL with* $\limsup_{r} q_r > \infty$, $\lim \sup_{u} q_u > \infty$ *thereafter*

$$\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \xi(I_\theta(\mathcal{J}_2)) \Rightarrow \mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \xi(I(\mathcal{J}_2)),$$

Proof. If $\lim \sup_{r} q_r > \infty$, $\lim \sup_{u} q_u > \infty$ then without limited uniformity we can assume that there exists a $\mathfrak{M}, \mathfrak{N} > 0$ such that $q_r < \mathfrak{M}$ and $q_u < \mathfrak{N}$ for every r, u. Assume $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}(I_\theta(\mathcal{J}_2))$and let

$$\mathcal{C}_{ru} := \left|\left\{(a, z) \in \mathfrak{I}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma}\ or\ \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma}\ and\ \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma}\right\}\right|.$$

Since $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}(I_\theta(\mathcal{J}_2))$, it becomes that for all $\check{\varsigma} > 0$, $\check{\delta} > 0$, and $\dot{\varrho} > 0$,

$$\left\{(r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{y_r \overline{y_u}} \left|\left\{\begin{matrix} a, z \in \mathfrak{I}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma}\ or \\ \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \\ and\ \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \end{matrix}\right\}\right| \geq \check{\delta}\right\} \in \mathcal{J}_2.$$

Hence, we can select a positive integers $r_0, u_0 \in \mathbb{N}$ so that $\frac{\mathcal{C}_{ru}}{y_r \overline{y_u}} < \check{\delta}$, for every $r > r_0, u > u_0$.

Now let $\mathfrak{K} := \max\{\mathcal{C}_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0\}$ and let ϱ and v be any integers satisfying $a_{r-1} < \dot{\varrho} \leq a_r$ and $z_{u-1} < v \leq z_u$. Then, we have

$$\frac{1}{\dot{\varrho} v}\left|\left\{a \leq \dot{\varrho}, z \leq v : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma}\ or\ \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma}\ and\ \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma}\right\}\right|$$

$$\leq \frac{1}{a_{r-1} z_{u-1}} \left|\left\{\begin{matrix} a \leq a_r, z \leq z_u : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma}\ or\ \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \\ and\ \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \end{matrix}\right\}\right|$$

$$\leq \frac{1}{a_{r-1} z_{u-1}} \left(\mathcal{C}_{11} + \mathcal{C}_{12} + \mathcal{C}_{21} + \mathcal{C}_{22} + \cdots + \mathcal{C}_{r_0 u_0} + \cdots \mathcal{C}_{ru}\right)$$

$$\leq \frac{1}{a_{r-1} z_{u-1}} . r_0 u_0 + \frac{1}{a_{r-1} z_{u-1}} \left(y_{r_0} \bar{y}_{u_{r_0}+1} \frac{\mathcal{C}_{r_0, u_0+1}}{y_{r_0} \bar{y}_{u_{r_0}+1}} + y_{r_{0+1}} \bar{y}_{u_0} \frac{\mathcal{C}_{r_{0+1}, u_0}}{y_{r_{0+1}} \bar{y}_{u_0}} + \cdots + y_r \bar{y}_u \frac{\mathcal{C}_{ru}}{y_r \bar{y}_u}\right)$$

$$\leq \frac{r_0 u_0 . l}{a_{r-1} z_{u-1}} + \frac{1}{a_{r-1} z_{u-1}} \left(\sup_{r > r_0, u > u_0} \frac{\mathcal{C}_{ru}}{y_r y_u}\right) \left(y_{r_0} \bar{y}_{u_0+1} + y_{r_{0+1}} \bar{y}_{u_0} + \cdots + y_r y_u\right)$$

$$\leq \frac{r_0 u_0 . a}{a_{r-1} z_{u-1}} + \check{\varsigma} . \frac{(a_r - a_{r_0})(z_u - z_{-1})}{a_{r-1} z_{u-1}}$$

$$\leq \frac{r_0 u_0 . z}{a_{r-1} z_{u-1}} + \check{\varsigma} . q_r . q_u \leq \frac{r_0 u_0 . l}{a_{r-1} z_{u-1}} + \check{\varsigma} . \mathfrak{M} . \mathfrak{N}$$

Since $a_{r-1} z_u \to \infty$ as $\dot{\varrho}, \upsilon \to \infty$, it follows that

$$\frac{1}{\dot{\varrho}\upsilon} \left|\left\{a \leq \dot{\varrho}, z \leq \upsilon : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma} \text{ or } \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \text{ and } \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma}\right\}\right| \to 0$$

and for all $\check{\delta}_1 > 0$, the set

$$\left\{(\dot{\varrho}, \upsilon) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\dot{\varrho}\upsilon} \left|\left\{\begin{matrix} a \leq \dot{\varrho}, z \leq \upsilon : \dot{\mu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \leq 1 - \check{\varsigma} \text{ or } \ddot{\nu}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \\ \text{and } \ddot{\omega}\left(\mathfrak{x}_{az} - \dot{\xi}, \dot{\varrho}\right) \geq \check{\varsigma} \end{matrix}\right\}\right|\right\} \in \mathcal{J}_2.$$

This shows that $\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}\left(I(\mathcal{J}_2)\right)$.

Joining Theorem 3.5 and Theorem 3.6 we get

Theorem 3.7. *Let sequence θ obtain strongly lacunary. NNS. If* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, *and* $1 < \lim$

$\inf_u q_u \leq \lim \sup_u q_u < \infty$ *then*

$$\mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}\left(I_\theta(\mathcal{J}_2)\right) \Leftrightarrow \mathfrak{x}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \dot{\xi}\left(I(\mathcal{J}_2)\right),$$

Proof. *It follows immediately from Theorem 2.2 and Theorem 2.3.*

Theorem 3.8. *Let* $(\Xi, \dot{\mu}, \ddot{\nu}, \ddot{\omega}, \circledast, \Theta, \Delta)$ *be a NNS so that*

$$\tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}} \Theta \tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}} < \tfrac{1}{2}\check{\varsigma}_{\widetilde{m}\hat{n}}, \ \left(1 - \tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}}\right) \circledast \left(1 - \tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}}\right) > 1 - \tfrac{1}{2}\check{\varsigma}_{\widetilde{m}\hat{n}} \text{ and } \tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}} \Delta \tfrac{1}{4}\check{\varsigma}_{\widetilde{m}\hat{n}} < \tfrac{1}{2}\check{\varsigma}_{\widetilde{m}\hat{n}}$$

If a Banach space Ξ *then closed subset* $I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \cap \ell^2_\infty(\Xi)$ *of* $\ell^2_\infty(\Xi)$.

Proof. We initially assume that $\left(\mathfrak{x}^{\widetilde{m}\hat{n}}\right) = \left(\mathfrak{x}^{\widetilde{m}\hat{n}}_{az}\right)$ be a convergent sequence in $I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \cap \ell^2_\infty(\Xi)$.
Suppose $\mathfrak{x}^{(\widetilde{m}\hat{n})}$ convergent to $\mathfrak{x}$. It is clear $\mathfrak{x} \in \ell^2_\infty(\Xi)$.
We must demonstrate this $\mathfrak{x} \in I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \cap \ell^2_\infty(\Xi)$. Since $\mathfrak{x}^{\widetilde{m}\hat{n}} \in I_\theta(\mathcal{J}_2)^{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \cap l^2_\infty(\Xi)$ there are some real numbers $\mathfrak{L}_{\widetilde{m}\hat{n}}$ in such a way that $\mathfrak{x}^{\widetilde{m}\hat{n}}_{az} \xrightarrow{(\dot{\mu},\ddot{\nu},\ddot{\omega})} \mathfrak{L}_{\widetilde{m}\hat{n}}\left(I_\theta(\mathcal{J}_2)\right)$ for $\widetilde{m}, \hat{n} = 1, 2, 3, \ldots$

Consider a strictly decreasing positive value double sequence $\{\check{\varsigma}_{\widetilde{m}\hat{n}}\}$ converging to zero. For all $\widetilde{m}, \hat{n} = 1, 2, 3 \ldots$ that is positive $\mathfrak{N}_{\widetilde{m}\hat{n}}$ in such way that if $\widetilde{m}, \hat{n} \geq \mathfrak{N}_{\widetilde{m}\hat{n}}$ then

$$\sup_{\widetilde{m},\hat{n}}, \ddot{\nu}\left(\mathfrak{x} - \mathfrak{x}^{\widetilde{m}\hat{n}}, \dot{\varrho}\right) \leq \frac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{\mathbf{4}}.$$

Without limited normality assume that $\mathfrak{N}_{\widetilde{m}\hat{n}} = \widetilde{m}\hat{n}$ and select a $\check{\delta} > 0$ in such a way that $\check{\delta} < \frac{1}{5}$.

Now set

$$\mathfrak{A}_{\mu,\nu,\omega}(\check{\varsigma}_{\widetilde{m}\hat{n}}, \dot{\varrho}) = \left\{\begin{matrix} (r, u) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{y_r \overline{y_u}} \\ \left|\begin{matrix} \{(a, z) \in \mathfrak{I}_{ru} : \dot{\mu}\left(\mathfrak{x}^{\widetilde{m}\hat{n}}_{az} - \mathfrak{L}_{\widetilde{m}\hat{n}}, \dot{\varrho}\right) \leq 1 - \dfrac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{4} \text{ or} \\ \ddot{\nu}\left(\mathfrak{x}^{\widetilde{m}\hat{n}}_{az} - \mathfrak{L}_{\widetilde{m}\hat{n}}, \dot{\varrho}\right) \geq \dfrac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{4} \text{ and } \ddot{\omega}\left(\mathfrak{x}^{\widetilde{m}\hat{n}}_{az} - \mathfrak{L}_{\widetilde{m}\hat{n}}, \dot{\varrho}\right) \geq \dfrac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{4}\} \end{matrix}\right| < \check{\delta} \end{matrix}\right\}$$

belongs to $\mathfrak{F}(\mathcal{J}_2)$ and

$$\mathfrak{B}_{\mu,\nu,\omega}(\check{\varsigma}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}) = \left\{ \begin{array}{c} (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{y_r\overline{y_u}} \\ \left| \begin{array}{c} \dot{\mu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \leq 1 - \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4}\, or \\ \ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4} \\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4} \end{array} \right| < \check{\delta} \end{array} \right\}$$

belongs to $\mathfrak{F}(\mathcal{I}_2)$. Since $\mathfrak{A}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{\tilde{m}\hat{n}},\dot{\varrho}) \cap \mathfrak{B}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}) \in \mathfrak{F}(\mathcal{I}_2)$ and $\emptyset \notin \mathfrak{F}(\mathcal{I}_2)$, we can choose $(r,u) \in \mathfrak{A}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{\tilde{m}\hat{n}},\dot{\varrho}) \cap \mathfrak{B}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho})$. Then

$$\frac{1}{y_r\overline{y_u}} \left\| \left\{ \begin{array}{c} (a,z) \in \mathfrak{J}_{ru} : \dot{\mu}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) \leq 1 - \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}\, or \\ \ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4} \\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4} \\ \vee\, \dot{\mu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \leq 1 - \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4}\, or \\ \ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4} \\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}+1,\hat{n}+1}}{4} \end{array} \right\} \right\| \leq 2\check{\delta} < 1.$$

Since $y_r\overline{y_u} \to \infty$ and $\mathfrak{A}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{\tilde{m}\hat{n}},\dot{\varrho}) \cap \mathfrak{B}_{\dot{\mu},\ddot{\nu},\ddot{\omega}}(\check{\varsigma}_{m+1,n+1},\dot{\varrho}) \in \mathfrak{F}(\mathcal{I}_2)$ is finite, we can select the above r,u so that $y_r\overline{y_u} > 5$.

As a result, there must exist a $(a,z) \in \mathfrak{J}_{ru}$ for whatever we have simultaneously

$\dot{\mu}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) > 1 - \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}$ or $\ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) < \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}$ and $\ddot{\omega}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{L}_{\tilde{m}\hat{n}},\dot{\varrho}\right) < \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}$,

$\dot{\mu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) > 1 - \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}\, or\ \ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}$

$and\ \ddot{\omega}\left(\mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1} - \mathfrak{L}_{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \geq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}.$

For a given $\check{\varsigma}_{\tilde{m}\hat{n}} > 0$ chose $\frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{2}$ such that

$\left(1 - \frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}}\right) * \left(1 - \frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}}\right) > 1 - \check{\varsigma}_{\tilde{m}\hat{n}},\ \frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}}\ \Theta\ \frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}} < \check{\varsigma}_{\tilde{m}\hat{n}}$ and $\frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}}\ \Delta\ \frac{1}{2}\check{\varsigma}_{\tilde{m}\hat{n}} < \check{\varsigma}_{\tilde{m}\hat{n}}$.

Then it follows that

$$\ddot{\nu}\left(\mathfrak{L}_{\tilde{m}\hat{n}} - \mathfrak{x}_{az}^{\tilde{m}\hat{n}},\frac{\dot{\varrho}}{2}\right)\ \Theta\ \ddot{\nu}\left(\mathfrak{L}_{\tilde{m}+1,\hat{n}+1} - \mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1},\frac{\dot{\varrho}}{2}\right) \leq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}\ \Theta\ \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4} < \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{2}$$

and

$$\ddot{\nu}\left(\mathfrak{x}_{az}^{\tilde{m}\hat{n}} - \mathfrak{x}_{az}^{\tilde{m}+1,\hat{n}+1},\dot{\varrho}\right) \leq \sup_{\tilde{m},\hat{n}} \ddot{\nu}\left(\mathfrak{x} - \mathfrak{x}^{\tilde{m}\hat{n}},\frac{\dot{\varrho}}{2}\right)\ \Theta\ \sup_{\tilde{m},\hat{n}} \ddot{\nu}\left(\mathfrak{x} - \mathfrak{x}^{\tilde{m}+1,\hat{n}+1},\frac{\dot{\varrho}}{2}\right)$$

$$\leq \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4}\ \Theta\ \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{4} < \frac{\check{\varsigma}_{\tilde{m}\hat{n}}}{2}$$

Hence, we have

$$\ddot{v}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}+1,\hat{n}+1},\dot{\varrho}\right)\leq\left[\ddot{v}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{x}_{az}^{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\Theta\,\ddot{v}\left(\mathfrak{x}_{az}^{\widetilde{m}+1,\hat{n}+1}-\mathfrak{L}_{\widetilde{m}+1,\hat{n}+1},\frac{\dot{\varrho}}{3}\right)\Theta\,\ddot{v}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{x}_{az}^{\widetilde{m}+1,\hat{n}+1},\frac{\dot{\varrho}}{3}\right)\right]$$

$$\leq\frac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{2}\,\Theta\,\frac{\check{\varsigma}_{\widetilde{m}\hat{n}}}{2}<\check{\varsigma}_{\widetilde{m}\hat{n}}$$

and similarly $\dot{\mu}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}+1,\hat{n}+1},\dot{\varrho}\right)>1-\check{\varsigma}_{\widetilde{m}\hat{n}}$ *and* $\ddot{\omega}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}+1,\hat{n}+1},\dot{\varrho}\right)<\check{\varsigma}_{\widetilde{m}\hat{n}}$.
It implies that $\{\mathfrak{L}_{\widetilde{m}\hat{n}}\}_{\widetilde{m},\hat{n}\in\mathbb{N}}$ is a Cauchy sequence in Ξ and let $\mathfrak{L}_{\widetilde{m}\hat{n}}\to\mathfrak{L}\in\Xi$ as $\widetilde{m},\hat{n}\to\infty$.

We will demonstrate this $\mathfrak{x}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\mathfrak{L}_{\widetilde{m}\hat{n}}\left(I_\theta(\mathcal{J}_2)\right)$.

For any $\check{\varsigma}>0$ and $\dot{\varrho}>0$, select $(\widetilde{m},\hat{n})\in\mathbb{N}\times\mathbb{N}$ in such a way that

$$\check{\varsigma}_{\widetilde{m}\hat{n}}>\frac{1}{4}\check{\varsigma},\sup_{\widetilde{m},\hat{n}}\ddot{v}\left(\mathfrak{x}-\mathfrak{x}^{\widetilde{m}\hat{n}},\dot{\varrho}\right)<\frac{1}{4}\check{\varsigma},\qquad \ddot{v}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L},\dot{\varrho}\right)>1-\frac{1}{4}\check{\varsigma}\ or\ \frac{1}{4}\check{\varsigma},\ddot{v}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L},\dot{\varrho}\right)<1-\frac{1}{4}\check{\varsigma}.$$

Now since

$$\frac{1}{y_r\overline{y_u}}\left|\{(a,z)\in\mathfrak{I}_{ru}:\ \ddot{v}\left(\mathfrak{x}_{az}-\mathfrak{L},\dot{\varrho}\right)\geq\check{\varsigma}\}\right|\leq\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{array}{c}(a,z)\in\mathfrak{I}_{ru}:\ \ddot{v}\left(\mathfrak{x}_{az}-\mathfrak{x}_{az}^{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\Theta\\ \left[\ddot{v}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\Theta\,\ddot{v}\left(\mathfrak{L}_{\widetilde{m}\hat{n}}-\mathfrak{L},\frac{\dot{\varrho}}{3}\right)\right]\geq\check{\varsigma}\end{array}\right\}\right|$$

$$\leq\frac{1}{y_r\overline{y_u}}\left|\left\{(a,z)\in\mathfrak{I}_{ru}:\ \ddot{v}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\geq\frac{\check{\varsigma}}{2}\right\}\right|$$

and equivalently

$$\frac{1}{y_r\overline{y_u}}\left|\{(a,z)\in\mathfrak{I}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\mathfrak{L},\dot{\varrho}\right)\leq1-\check{\varsigma}\}\right|>\frac{1}{y_r\overline{y_u}}\left|\left\{(a,z)\in\mathfrak{I}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\leq1-\frac{\check{\varsigma}}{2}\right\}\right|$$

and

$$\frac{1}{y_r\overline{y_u}}\left|\{(a,z)\in\mathfrak{I}_{ru}:\ddot{\omega}\left(\mathfrak{x}_{az}-\mathfrak{L},\dot{\varrho}\right)\geq\check{\varsigma}\}\right|\leq\frac{1}{y_r\overline{y_u}}\left|\left\{(a,z)\in\mathfrak{I}_{ru}:\ddot{\omega}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\leq\frac{\check{\varsigma}}{2}\right\}\right|.$$

It follows that

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{array}{c}(a,z)\in\mathfrak{I}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\leq1-\check{\varsigma}\\ or\ \ddot{v}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\geq\check{\varsigma}\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}-\grave{\xi},\dot{\varrho}\right)\geq\check{\varsigma}\end{array}\right\}\right|\geq\breve{\delta}\right\}$$

$$\subset\left\{(r,u)\in\mathbb{N}\times\mathbb{N}\ :\frac{1}{y_r\overline{y_u}}\left|\left\{\begin{array}{c}(a,z)\in\mathfrak{I}_{ru}:\dot{\mu}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\leq1-\frac{\check{\varsigma}}{2}\\ or\ \ddot{v}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\geq\frac{\check{\varsigma}}{2}\ and\ \ddot{\omega}\left(\mathfrak{x}_{az}^{\widetilde{m}\hat{n}}-\mathfrak{L}_{\widetilde{m}\hat{n}},\frac{\dot{\varrho}}{3}\right)\geq\frac{\check{\varsigma}}{2}\end{array}\right\}\right|\geq\breve{\delta}\right\}$$

for any given $\breve{\delta}>0$. Hence we have $\mathfrak{x}\xrightarrow{(\dot{\mu},\ddot{v},\ddot{\omega})}\mathfrak{L}_{\widetilde{m}\hat{n}}\left(I_\theta(\mathcal{J}_2)\right)$..

4. Conclusion

We define the concept of $\mathcal{J}_2$-LSC additionally strong $\mathcal{J}_2$-LC in order to relate towards the *NN*, examine their relationship, and while certain observations regarding these. The research we conducted involving $\mathcal{J}_2$-statistical in addition $\mathcal{J}_2$-LSC about sequences in *NNS* give a technique for approaching convergence problems of fuzzy real number sequences.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Ranking and Analysis the Strategies of Crowd Management to Reduce the Risks of Crushes and Stampedes in Crowded Environments and Ensure the Safety of Passengers

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Abstract: Public gatherings, transit hubs, stadiums, and crowded retail malls are just a few examples of places where crowd management has become an urgent issue in recent years. Effective crowd management strategies have been required due to the increasing population, urbanization, and frequency of large-scale meetings. These strategies are used in dynamic, sometimes chaotic circumstances to protect people and facilitate their free movement. The purpose of this study is to analyze and rank various strategies for crowd management to reduce risks of crushes and stampedes, improve security, and facilitate smoother traffic flow. This study used the single-valued neutrosophic set to deal with uncertain and vague information in the evaluation process. There are various factors in ranking the various strategies. So, the concept of multi-criteria decision-making (MCDM) is used to deal with various criteria. The neutrosophic set integrated with the MCDM methodologies to rank various strategies. This study used the analytical hierarchy process (AHP) method to compute the weights of factors. Then the technique for order preference by similarity to the ideal solution (TOPSIS) method is used to rank the various strategies. An application was conducted to apply the proposed method. The outcome shows the safety and security factor is the heights important. The sensitivity analysis is applied to show the rank of strategies under various weights of factors. Finally, the comparative analysis is applied to show the robustness of the proposed method compared with other MCDM methods.

Keywords: Crowd Management; Neutrosophic Set; AHP; TOPSIS; Strategies; Risks.

1. Introduction

Public gatherings, transit hubs, stadiums, and crowded retail malls are just a few examples of places where crowd control has become an urgent issue in recent years. Effective crowd control measures have been required due to the increasing population, urbanization, and frequency of large-scale meetings. These methods are used in dynamic, sometimes chaotic circumstances to protect people and facilitate their free movement. The purpose of this study is to investigate and evaluate current approaches to crowd control to lessen hazards, improve security, and facilitate smoother traffic flow [1, 2].

Understanding crowd behavior and dynamics is crucial for effective crowd management. Researchers and practitioners can handle the issues given by various crowds by researching the nuances of crowd formation, movement, and response to stimuli. In addition, proper risk assessment and careful planning are essential for efficient crowd control. Event planners, facilities managers, and security staff may prepare thorough backup plans to enable rapid and effective reactions during crises by identifying possible hazards, weaknesses, and congested areas [3, 4].

Safety and order can't be maintained without using crowd control tactics. Crowds are managed via the use of physical barriers, staff, access control systems, and surveillance technology. To maximize the efficiency of crowd movement and lessen the likelihood of accidents, several crowd control strategies are used [5, 6]. Staff members who are well-prepared and have had enough training are also essential for managing large crowds successfully. To effectively manage a wide variety of crowd scenarios, security professionals, event workers, and volunteers should have thorough training in crowd psychology, conflict resolution, emergency response, and first aid [7, 8].

1.1 Crowd Management Challenges

There are a number of factors to consider and solutions to use when attempting to manage large crowds in crowded settings. Comprehensive crowd control strategies are of critical importance as populations rise and metropolitan areas become more congested. Public events, transit hubs, stadiums, and retail malls are just a few examples of the many different places where crowd management presents a unique set of issues. Stakeholders can assure the safety, security, and efficient movement of people in congested environments by recognizing and comprehending the difficulties they face [3].

Unpredictability of crowds presents the first difficulty in crowd management. Due to their fluid nature and sometimes complicated behavior patterns, crowds are notoriously difficult to predict. Crowds are notoriously difficult to control, and a thorough grasp of crowd psychology is essential for doing so, due to factors such as emotions, group dynamics, and external influences.

The possibility for congestion and overcrowding is another major obstacle. Overcrowding and capacity issues are common in high-density areas due to population growth. Threats to public safety, delays in responding to emergencies, and lower productivity are all possible outcomes. The dangers associated with overcrowding may be reduced by the use of crowd management measures such as controlling crowd density, avoiding bottlenecks, and allocating sufficient space [2].

When dealing with large groups of people, communication may be a real headache. It might be difficult to get the word out to a big group of individuals in a timely and precise manner, particularly under pressure. Confusion, fear, and compromised security are all possible outcomes of inefficient communication. Strong mechanisms that allow efficient two-way communication between organizers, security officials, and the audience are necessary to overcome communication issues and ensure that clear instructions, emergency warnings, and pertinent updates are properly transmitted.

Another difficulty with crowd control is that different crowds have different demographics. Audiences at various events have varying demographics, preferences, and risk profiles. A crowd at a music festival, for instance, is likely to behave differently from one at a sports event. In order to successfully adapt crowd management tactics and execute suitable measures to handle certain crowd dynamics, it is essential to understand and accommodate these varied characteristics.

Maintaining order and keeping people safe in densely populated areas is a very difficult task. Stampedes, terrorist attacks, and criminal acts are only some of the dangers that might arise in large crowds. Essential components of crowd management techniques to meet safety and security concerns include developing thorough risk assessment and management plans, employing sufficient security personnel, adopting surveillance technology, and creating efficient emergency response procedures [1].

So this paper aims to identify and rank the set of strategies to overcome these challenges and obtain safety and security.

There are many contributions to this paper:

- The first study ranks and analyses the crowd strategies to reduce the crushing and obtain safety and security.
- The first study uses the neutrosophic set to deal with uncertain data in the evaluation process.
- This study uses the multi-criterion decision-making (MCDM) methods to analyze and rank crowd strategies.
- There are six factors and ten strategies gathered in this study.
- The analytical hierarchy process (AHP) method is used to compute the weights of factors.

- The technique for order preference by similarity to the ideal solution (TOPSIS) method is used to rank several crowd strategies.

2. Related Work

This section introduces some previous work in crowd management and gives the limitations and outcomes of each work as shown in Table 1. From Table 1, there is no precious work introducing the neutrosophic set with crowd management. So, this paper integrated the neutrosophic set with the MCDM methods to rank and analyze the crowd strategies to reduce crowd crushes and give more safety and security.

Table 1. Previous work was in crowd management.

Year	Ref.	Techniques	Findings	Limitations
2019	[9]	Fuzzy Logic	The findings validate the use of fuzzy logic in emergencies. When using the directed selection criteria, people act naturally and with a clear focus on the intended outcome. Adjustments to velocity using fuzzy logic and the addition of a psychological impact component have both been shown to be effective in experiments. In addition, considering a person's proximity to exits, fuzzy Cellular Automata rules for intuitive exit selection have been included.	No validation of their models by the real data.
2021	[10]	Fuzzy System	The key contribution is the development of a fuzzy-based arousal and valence inference system. Because capturing the expressions of people in a throng is challenging. They used characteristics of a crowd (such as its enthalpy, variation in movement magnitude, confusion index, and density) to characterize its mood. The arousal fuzzy system takes as input variables the crowd's enthalpy and the variation in the magnitude of the crowd's movement. The inputs of the valence fuzzy inference system are the confusion index and the crowd density. In addition to emotion categorization, the suggested fuzzy system also provides arousal and valence ratings for the emotions of the audience as an output. The experimental findings confirm the viability of using this approach to assess the mood of a crowd. Two-feature fuzzy systems have been proven to be more flexible than single-feature systems. Crowd behavior and emotion are difficult to predict, making it challenging to find a universal solution that works in all situations.	A few crowd images, silent features, and emotion description models.

2020	[11]	Attentive Multi-stage CNN	Attentive Multi-stage CNN for Crowd Counting (AMCNN) is a unique network architecture presented by the researchers. Both a hierarchical density estimator (HDE) and an auxiliary count classifier (AUCC) are included in the AMCNN. The HDE takes a hierarchical approach to mine semantic characteristics from coarse to fine to solve the issue of size shifts and viewpoint distortions. The final density map is made using the collected composite characteristics. In addition, a soft attention mechanism is included in the AMCNN to differentiate between foreground and origins, which improves the quality of the density map.	The small size of the dataset.
2016	[12]	Fuzzy Logic	It is suggested to use a fuzzy logic method to characterize the behavior of crowds during an evacuation, factoring in the impact of potential attackers. The micro-level pedestrian and assailant models are first constructed, with each character's goals in mind during evacuation drills. There are three subgroups of pedestrians based on whether they have been targeted by violent criminals. Adjustable weighting factors, which are updated in real-time depending on the perceptions obtained from the intricate relationship with the surrounding situations, are used to integrate the suggestions of local obstacle-avoiding behavior, regional path-searching behavior, and global goal-seeking behavior to assess an individual's behaviors.	There are a small number of features.
2023	[13]	Fuzzy logic	They detailed a novel method for evaluating crowd health that broadens the range of possible interactions between people and motor vehicles. They employed fuzzy logic sorting to make the algorithm better at spotting outliers in large groups of people. A unique deep transfer learning (DTL) method is used to collect images from unmanned aerial vehicles to enhance making choices. The suggested combined model has a 98.5% accuracy percentage, good performance, and resilience to population behavior.	Their proposed model is not validated with other models.
2017	[14]	Fuzzy Theory	To examine the effects of communication on crowd dynamics, a fuzzy-theory-based	They haven't overcome the

			approach was provided. Human states are notoriously difficult to describe precisely, which is why fuzzy functions and rules were developed. Decision-making output values, like the rates and directions of pedestrian motion, are determined using reasonable inference. Good crowd dispersal phenomena are accomplished by modeling in four-way pedestrian settings. The simulation results under various scenarios show that the effective dispersal of the crowd cannot be guaranteed by the transmission of information. This is particularly relevant when considering the cohesiveness and efficiency of decision-making techniques in response to threat information in large groups. Increases in both the drift intensity at low density and the proportion of pedestrians selecting one of the furthest empty Von Neumann neighbors from the harmful origin as the drift route at high density were shown to be beneficial in dispersing the throng.	uncertain information in their proposed model.
2022	[15]	Heuristic Technique	Procedural methods are effective at solving a wide variety of issues. Nevertheless, these techniques aren't always feasible due to their slowness; sometimes it's just impossible to achieve an outcome using primitive methods. When the optimal answer cannot be guaranteed, a heuristic technique may be used to get close enough for practical purposes. The use of heuristic-based modeling helps clarify crowd behavior and enhance simulation dependability, and heuristic approaches are utilized for many real-world challenges, especially crowd management.	No focus in the time.
2022	[16]	Machine Learning	They researched automated machine learning support systems, which are useful for crowd control. Machine learning (ML) is a subset of AI that helps developers improve software's predictive abilities in ways that weren't originally designed for that purpose. To predict future output values, machine learning methods take in historical data as input. Given the significance of data, the next step towards fully autonomous agents must be the development of better ways for effectively	No model validation with other machine learning models.

			handling the now-ubiquitous crowd-powered content-gathering platforms. Apps for mobile devices, computers, the internet, and even online security all employ similar methods. They conclude that a machine-learning-based automated support system is necessary for efficient crowd control.	
2019	[17]	Reinforcement Learning	They developed a deep-reinforcement-learning-based smart routing method to ease network congestion and equalize network load for promoting smart city amenities with striking inequalities, thereby rendering the dispersed information and communication facilities thoroughly feasible and fulfilling the latency limitations of service demands from the crowd.	No model validation.

3. Methodology

In this section, we integrated the single-valued neutrosophic set with the MCDM methods. We used the AHP and TOPSIS methods. The AHP method is used to compute the weights of the criteria. The TOPSIS method is used to compute the rank of strategies. Figure 1 shows the proposed method.

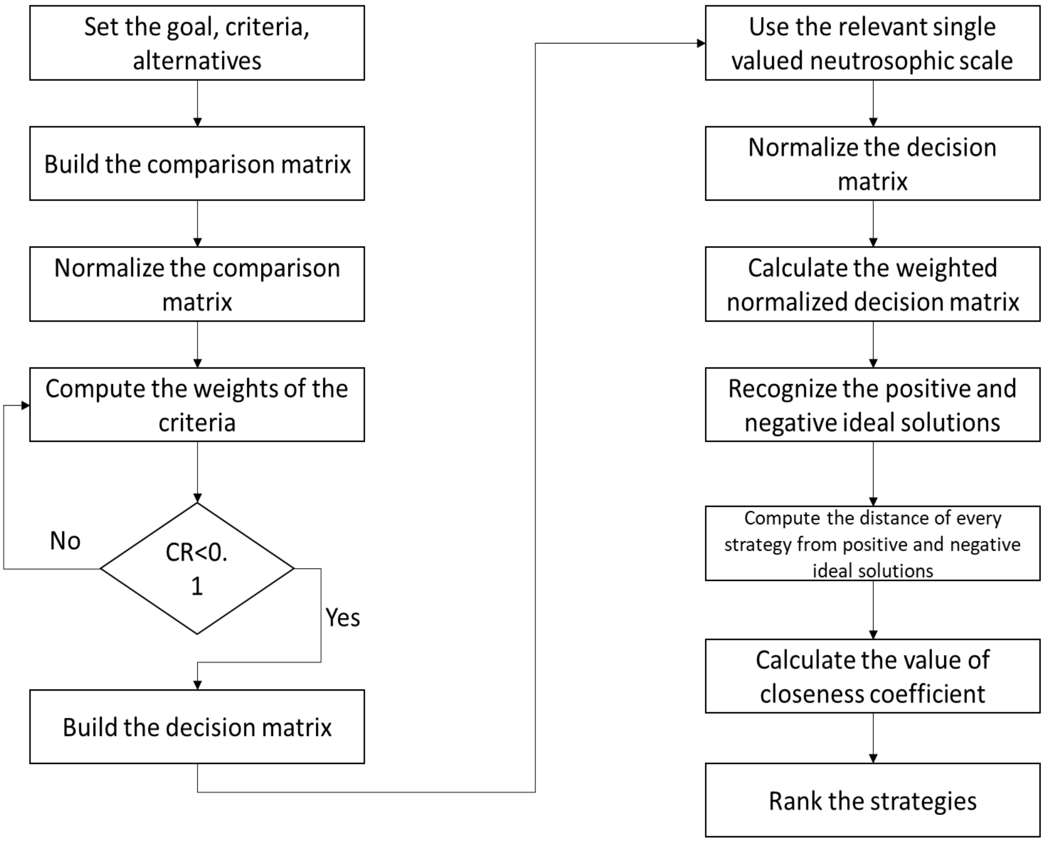

Figure 1. The framework of the proposed model.

3.1 Problem Definition

Public gatherings, transit hubs, stadiums, and crowded retail centers all provide unique crowd control difficulties. The issue at hand is how to manage large groups of people in a way that protects their health, safety, and happiness. The following critical challenges are becoming more important to handle as crowd sizes and complexity increase.

The danger of physical injury and injuries is elevated when there are huge numbers of people present. Accidents, falls, stampedes, and trampling events may occur as a result of overpopulation, inadequate infrastructure, poor crowd flow management, and insufficient safety measures. The challenge is coming up with plans and procedures to lessen these dangers and make the crowd a secure place for everyone.

Fires, terrorist attacks, and acts of violence are just a few examples of the types of events that may cause large crowds and provide a serious challenge to crowd control. Consequences from such incidents tend to multiply in high population concentrations because of the speed with which they may spread. In order to properly manage these circumstances and reduce the number of casualties among the crowd, we need to design thorough emergency response plans, perform risk assessments, and train employees.

Crowded places increase the likelihood that infectious illnesses may be spread among the crowd's participants, which poses a threat to public health. Infections may spread rapidly due to factors including close proximity, lack of sanitary facilities, poor ventilation, and restricted access to medical care. Public health concerns in large gatherings may be reduced with the help of improved sanitation and hygiene practices, as well as the installation of medical facilities and first-aid stations.

Concerns of a Social and Behavioral Nature, disorder, and illegal activity have been known to escalate in large crowds. Individuals may feel emboldened to participate in disruptive behavior in a crowded area, putting the safety and enjoyment of others at risk. The challenge is to find ways to keep the peace among the throng, resolve any conflicts that arise, and prevent any disruptive behavior from taking place.

The growing dependence on technology for crowd control raises concerns about the potential for disruptions and safety risks in the event of system failure or improper usage. Threats to crowd control activities might come from broken surveillance equipment, unstable communication lines, or malicious cyber activity. The challenge is to take precautions against cyberattacks, establish secure networks, and perform routine maintenance on all systems.

So, this paper used the set of criteria to rank crowd management strategies to reduce crowd crushes and obtain safety and security.

3.2 The AHP method

Saaty introduced AHP as a relative measuring strategy for qualitative and intangible criteria. It is a mathematical method that may also be used as a tool for decision analysis. It is an MCDM tool that puts complicated problems in a hierarchical sequence for easy analysis [18]–[23]. The following are only a few of the reasons why the AHP method was chosen for this study:

It provides support for dealing with complicated, unstructured, multi-attribute challenges.

It helps decision-makers break down complex problems into manageable chunks that are easier on the wallet.

It works effectively with both quantitative and qualitative information.

It presents complex choice issues in a hierarchical format.

A spreadsheet may be used to find the answer.

It allows us to ensure that our evaluation methods are consistent with one another.

Here is a quick rundown of what goes into an AHP analysis:

Step 1. Set the goal, criteria, and alternatives. This step identifies the goal of the study, the factors of crowd management, and the strategies to reduce the crowds.
Step 2. Build the comparison matrix. The comparison matrix is built between criteria.
Step 3. Normalize the comparison matrix. The normalization matrix is computed by computing the sum of every column, then dividing every value in pairwise comparison by the sum of each column.
Step 4. Compute the weights of the criteria. The weights of the criteria are computed by the mean value of every row in the normalization matrix.
Step 5. Compute the consistency ratio. The consistency ratio is computed by the consistency index and random index.

$$CR = \frac{(\lambda_{max} - n/\, n-1)}{RI} \tag{1}$$

Where n refers to the number of criteria.

3.3 The TOPSIS method

Hwang and Yoon devised this method. The goal of this approach is to return the largest possible deviation from the negative ideal solution and the smallest possible deviation from the positive ideal solution. Although it is a well-known technique in MCDM, it has several significant drawbacks. However, it falls short when dealing with topics that are, at best, nebulous [24]–[29].

Instead of utilizing numerical values, evaluating ratings and weights of the criterion using language variables is preferable. The neutrosophic set allows decision-makers to account for indeterminate, uncertain, or otherwise difficult-to-assess data and information [30]–[33].

Step 1. Use the relevant single-valued neutrosophic scale.
Step 2. Build the decision matrix. The decision matrix is built between criteria and alternatives. Then aggregate the multiple decision matrices into one matrix.
Step 3. Normalize the decision matrix. The decision matrix is normalized by using the beneficial and non-beneficial criteria.

$$a_j^* = \max a_{ij} \quad \text{beneficiale criteria} \tag{2}$$

$$a_j^- = \min a_{ij} \quad \text{non-beneficiale criteria} \tag{3}$$

The a_{ij} refers to the value in decision matrix $i = 1,2,3, \dots m$ $(alternatives); j = 1,2,3, \dots\dots n$ $(criteria)$

Step 4. Calculate the weighted normalized decision matrix.

$$H = [h_{ij}]_{m \times n} \tag{4}$$

$$h_{ij} = a_{ij} \times w_j \tag{5}$$

Where w_j refers to the weights of the criteria.

Step 5. Recognize the positive and negative ideal solutions.

$$S^+ = \{h_1^+, h_2^+, h_3^+ \dots\dots h_n^+\} \tag{6}$$

$$s_j^+ = \{\max(h_{ij})\} \tag{7}$$

$$S^- = \{h_1^-, h_2^-, h_3^- \dots\dots h_n^-\} \tag{8}$$

$$s_j^- = \{\max(h_{ij})\} \tag{9}$$

Step 6. Compute the distance of every strategy from positive and negative ideal solutions.

$$t_i^+ = \{\textstyle\sum_{j=1}^{n} (h_{ij} - h_{ij}^+)\}^{\frac{1}{2}} \tag{10}$$

$$t_i^- = \{\textstyle\sum_{j=1}^{n} (h_{ij} - h_{ij}^-)\}^{\frac{1}{2}} \tag{11}$$

Step 7. Calculate the value of the closeness coefficient.

$$F_i = \frac{t_i^-}{t_i^- + t_i^+} \tag{12}$$

Step 8. Rank the strategies.

The strategies are ranked based on the highest value of F_i.

4. Results and Discussion

This section aims to rank and analyze several strategies of the crowd to reduce risks of the crowd and achieve safety for passengers. This study used six factors and ten strategies. The factors of crowd management are ranked by using the AHP method. The strategies are ranked by using the MCDM methods. These methods are integrated with the single-valued neutrosophic sets to overcome the uncertain information.

Crush and stampede prevention in crowded areas calls for forethought, crowd management strategies, and clear lines of communication. The following are some measures that may be taken to prevent such potentially harmful events:

Safely managing large crowds requires knowing and sticking to the venue's maximum capacity. Keep an eye on the attendance and make sure it doesn't rise beyond the set limit. To keep an eye on crowd density and take appropriate measures if it rises to dangerous levels, crowd monitoring equipment or staff should be put into place.

The smooth flow of attendance may be ensured by designating and marking certain entrance and departure locations. Make sure the gates are large enough to prevent traffic jams at the access points. In the event of an emergency, the gathering should be dispersed as rapidly as possible.

To control the flow of people entering and exiting and to avoid congestion, it is important to establish clear and orderly lines or queues. Put up stanchions or barriers to direct the flow of people and ensure orderly lineups. Keep an eye on wait times and adjust as required to keep traffic moving smoothly.

Clear pathways should be marked out so that attendees may easily navigate the venue or event location. Make use of signs, floor markings, or electronic displays to direct traffic and avoid collisions. Avoid traffic jams and potential collisions by establishing designated routes or establishing one-way circulation patterns.

Make sure everyone in the audience knows what's going on by making announcements and posting signs that are easy to see. To ensure that critical information, instructions, and safety announcements reach everyone, use a variety of communication tools, including but not limited to public address systems, digital displays, and mobile apps.

Staff and crowd control employees with the necessary training to regulate crowd behavior and deal with emergencies should be deployed. They need to be familiar with methods of crowd management, methods of resolving conflicts, and emergency protocols. Put them to use in high-traffic or possible stampede situations.

Prepare for any kind of disaster by making sure you have a solid strategy in place and practicing it often. Create safe escape routes and meeting areas. Staff members should be prepared to handle large crowds in the event of an emergency by learning crowd control and evacuation procedures.

Always be on the lookout for any indicators of unease or danger within the crowd by keeping tabs on its size, movement, and demeanor. Make use of tools like video surveillance and crowd monitoring devices to get timely information. After the event is over, take stock and see where your crowd control might have been better.

Detect probable incidents or risky circumstances involving crowds by using early warning systems. Sensors, video analytics, and social media monitoring tools are all possible components of such infrastructures. Be ready to act quickly if you see any symptoms of discomfort or unusual crowd behavior.

Collaborate with Local Governments: Establish and execute crowd control strategies in close cooperation with local law enforcement and emergency services. Make sure that during times of crisis, everyone is on the same page in terms of communication, reaction, and backup plans.

The potential for accidents like crashes and stampedes may be mitigated in crowded settings by using these measures and keeping a proactive attitude toward crowd control.

The previous strategies can be evaluated by the following six factors of crowd management:

Overcrowding, stampedes, and other potentially disastrous events may be avoided with the use of crowd control methods. Creating cordons, barriers, and lineups helps direct traffic and regulate access to certain locations.

The safety and security of the audience must be prioritized above everything else. This includes the use of tools like luggage inspections, metal detectors, closed-circuit television, and trained security staff to spot and deal with any crises that may arise.

Maintaining open lines of communication is essential for successful crowd control. It is the responsibility of the event staff to advise the audience of any changes or announcements that may affect the enjoyment or safety of the event.

Managing a large gathering of people requires a well-trained and sufficiently staffed team. Crowd management, dispute resolution, disaster preparedness, and effective communication are all skills that event workers should have, and they should be taught to them by the event's organizers. The crew is prepared for every eventuality thanks to their extensive training and extensive expertise.

Being ready for and anticipating possible crises is an important part of managing large crowds. Making preparations for emergencies include mapping out escape routes and arranging ready access to medical care. Staff and guests should be trained and made acquainted with emergency protocols via regular exercises and rehearsals.

Understanding the dynamics and patterns of crowd behavior is essential for creating efficient crowd control plans. Crowd behavior may be affected by a wide variety of demographic, cultural, and personal factors. Crowd behavior may be better understood and anticipated with the use of data collected via analysis of prior occurrences.

4.1 The Results of the AHP Method

The AHP method is used to compute the weights of factors as:

Step 1. Set the goal, criteria, and alternatives. The goal of this study is to reduce crowd crushing and obtain the highest safety in crowd management by ranking strategies and analyzing the factors of crowd management. This study identified six factors and ten strategies.

Step 2. Build the comparison matrix. This study builds the pairwise comparison matrix between six factors from the opinions of the experts and decision-makers. This study used single-valued neutrosophic numbers to evaluate the criteria by building the pairwise comparison matrix.

Step 3. Normalize the comparison matrix. Then compute the normalized pairwise comparison matrix as shown in Table 2. Where CMF_1 refers to the first factor, and so on.

Table 2. The normalization pairwise comparison matrix by the AHP method.

	CMF_1	CMF_2	CMF_3	CMF_4	CMF_5	CMF_6
CMF_1	0.075376	0.019299	0.043127	0.074754	0.15296	0.104908
CMF_2	0.319389	0.081774	0.035045	0.030247	0.152867	0.096599
CMF_3	0.118702	0.158477	0.067917	0.042437	0.081632	0.05141
CMF_4	0.115963	0.310929	0.184056	0.115006	0.036766	0.18673
CMF_5	0.076477	0.08302	0.129119	0.485461	0.155195	0.151038
CMF_6	0.294092	0.346501	0.540737	0.252095	0.420581	0.409316

Step 4. Compute the weights of the criteria. Then compute the weights of the criteria as shown in Figure 2. The safety and security factor is the highest rank of all factors.

Step 5. Compute the consistency ratio. Then compute the consistency ratio by using Eq. (1). The consistency ratio is less than 0.1. Then the weights of factors are consistent. Then the data is ready to apply the TOPSIS method to rank the ten strategies.

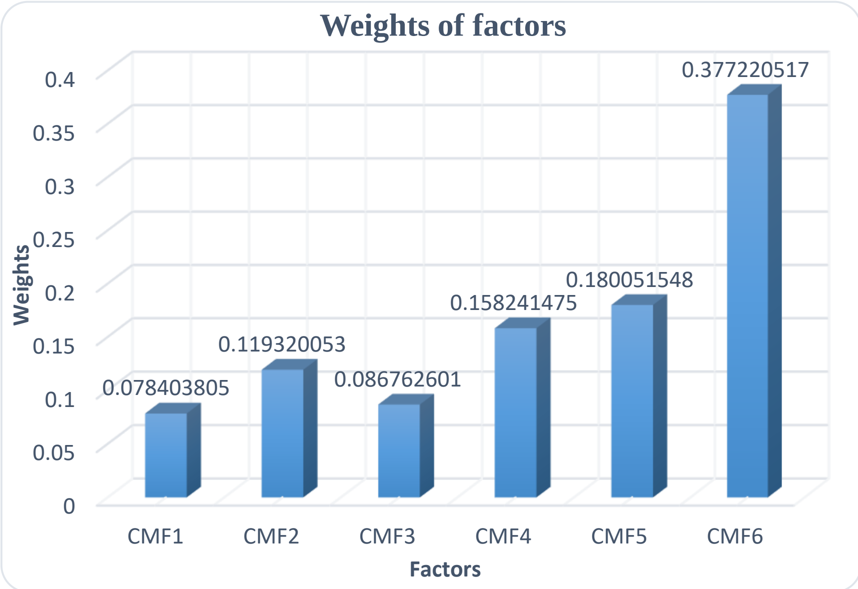

Figure 2. Weights of six factors.

4.2 The TOPSIS Method Results

The TOPSIS method is used to compute the rank of strategies as:

Step 1. Use the relevant single-valued neutrosophic scale. We used one single valued neutrosophic scale for the factors and strategies.

Step 2. Build the decision matrix. The decision matrix is built between the criteria and strategies by the experts and decision matrices. Then these matrices are combined into one matrix.

Step 3. Normalize the decision matrix. The normalization of the decision matrix is computed by using Eqs. (2 and 3) as shown in Table 3. by identifying the beneficial and non-beneficial criteria. All factors are beneficial factors. Where the CMS_1 refers to the first strategy and CMF_1 refers to the first factor.

Table 3. The normalization of decision matrix by the TOPSIS method.

	CMF_1	CMF_2	CMF_3	CMF_4	CMF_5	CMF_6
CMS_1	0.148622	0.44692	0.178969	0.150158	0.242999	0.415177
CMS_2	0.148622	0.221599	0.306528	0.235228	0.116009	0.345052
CMS_3	0.398004	0.604637	0.366766	0.134742	0.221506	0.237469
CMS_4	0.33125	0.221599	0.380734	0.292893	0.118262	0.270465
CMS_5	0.148622	0.313355	0.366766	0.300315	0.230516	0.45821
CMS_6	0.392336	0.221859	0.345474	0.360835	0.593185	0.299263
CMS_7	0.553553	0.221599	0.306528	0.300486	0.493695	0.408683
CMS_8	0.353921	0.204287	0.124309	0.486442	0.342583	0.167934
CMS_9	0.232379	0.221599	0.114463	0.450587	0.221975	0.237469
CMS_{10}	0.161217	0.221599	0.467067	0.260463	0.221506	0.167015

Step 4. Calculate the weighted normalized decision matrix. The weighted normalized decision matrix is computed by using Eqs. (4 and 5) as shown in Table 4. The weights of criteria are computed by the AHP method is multiplied by the value in the normalization decision matrix.

Table 4. The weighted normalization of the decision matrix by the TOPSIS method.

	CMF_1	CMF_2	CMF_3	CMF_4	CMF_5	CMF_6
CMS_1	0.011653	0.053327	0.015528	0.023761	0.043752	0.156613
CMS_2	0.011653	0.026441	0.026595	0.037223	0.020888	0.130161
CMS_3	0.031205	0.072145	0.031822	0.021322	0.039882	0.089578
CMS_4	0.025971	0.026441	0.033034	0.046348	0.021293	0.102025
CMS_5	0.011653	0.03739	0.031822	0.047522	0.041505	0.172846
CMS_6	0.030761	0.026472	0.029974	0.057099	0.106804	0.112888
CMS_7	0.043401	0.026441	0.026595	0.047549	0.088891	0.154164
CMS_8	0.027749	0.024375	0.010785	0.076975	0.061683	0.063348
CMS_9	0.018219	0.026441	0.009931	0.071302	0.039967	0.089578
CMS_{10}	0.01264	0.026441	0.040524	0.041216	0.039882	0.063002

Step 5. Recognize the positive and negative ideal solutions. The positive and negative ideal solutions are identified by using Eqs. (6-9). All factors are positive, so the positive ideal solution is computed.

Step 6. Compute the distance of every strategy from positive and negative ideal solutions. The distance from each strategy is computed by using Eqs. (10 and 11).

Step 7. Calculate the value of the closeness coefficient. The closeness coefficient is computed by using Eq. (12) as shown in Figure 3.

Step 8. Rank the strategies. The strategies are ranked based on the highest value of F_i.

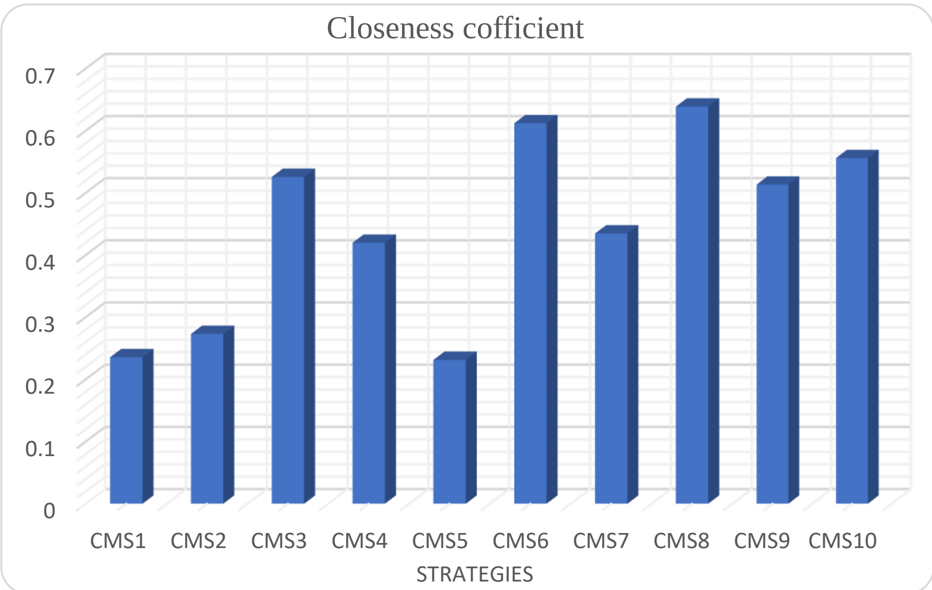

Figure 3. The closeness coefficient value.

4.3 The Sensitivity Analysis

In this sub-section, we changed the weights of factors to show the rank of strategies under different cases in weights. We change the weights of the criteria by increasing the factor by 50% and reducing the other factors by 50% to obtain a total weight of 100%. This study introduces the seven cases of changing the weights of factors as shown in Table 5. In the first case, the weights of factors are equal. Then the other factors we put the one factor with 50% weight and five factors are 50% weight. We applied these seven cases to the TOPSIS method to show the rank of strategies as shown in Figure 4.

Table 5. The seven cases in weights of factors.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
CMF_1	0.17	0.5	0.1	0.1	0.1	0.1	0.1
CMF_2	0.17	0.1	0.5	0.1	0.1	0.1	0.1
CMF_3	0.17	0.1	0.1	0.5	0.1	0.1	0.1
CMF_4	0.17	0.1	0.1	0.1	0.5	0.1	0.1
CMF_5	0.17	0.1	0.1	0.1	0.1	0.5	0.1
CMF_6	0.17	0.1	0.1	0.1	0.1	0.1	0.5

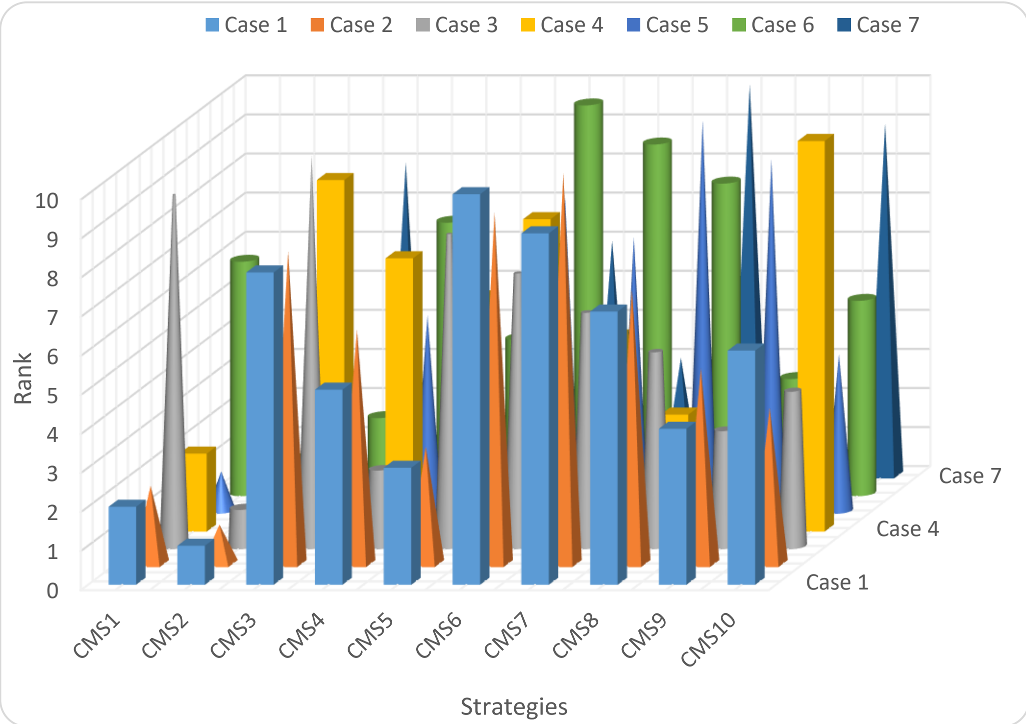

Figure 4. The sensitivity analysis ranking.

4.4 The Comparative Analysis

In this sub-section, we compare the rank of strategies by other MCDM methods to show the robustness of the proposed model. This study keeps the weights of factors by the AHP the same in all the MCDM methods. This study compared the proposed model by the single-valued neutrosophic VIKOR method, single-valued neutrosophic MABAC method, fuzzy TOPSIS method, crisp TOPSIS method, fuzzy VIKOR method, and fuzzy MABAC method. Table 6 shows a comparative analysis between the proposed method and other MCDM methods.

Table 6. The comparative analysis between the proposed method and other MCDM method.

	Single-valued neutrosophic VIKOR	Single-valued neutrosophic MABAC	Fuzzy TOPSIS	Fuzzy VIKOR	Fuzzy MABAC	Proposed method
CMS_1	8	9	9	8	9	9
CMS_2	9	8	8	5	8	8
CMS_3	5	4	4	9	4	4
CMS_4	7	7	7	7	7	7
CMS_5	10	10	10	10	10	10
CMS_6	2	6	2	2	6	2
CMS_7	6	2	6	6	1	6
CMS_8	1	1	1	1	2	1
CMS_9	4	5	5	3	5	5
CMS_{10}	3	3	3	4	3	3

Then compute the correlation between the proposed method and comparative MCDM methods. The correlation methods show the correlation between two variables. Figure 5 shows the correlation between the proposed method and other MCDM methods.

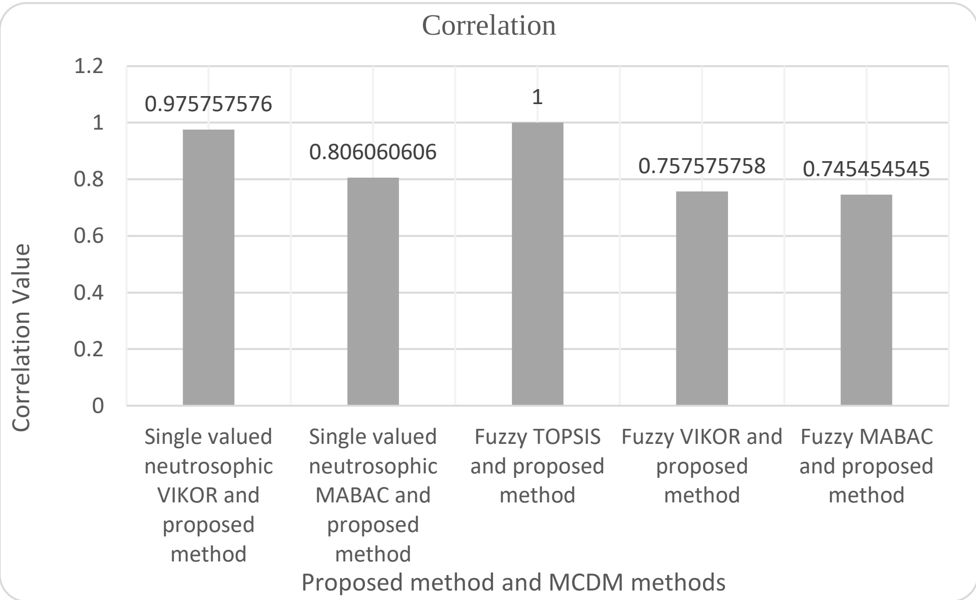

Figure 5. The correlation between the proposed method and other MCDM methods.

4.5 Managerial and Practical Implications

Discussion and management and practical implications according to the case application results are highlighted in this sub-section. The research takes into consideration a genuine situation where major difficulties with the crowd management strategies are recognized, and it then evaluates the suitable strategy selection for the effective deployment and security of the crowd management strategies ranking. Prioritization of strategies begins with an analysis of relevant criteria. To do this, we use the AHP approach, which is based on the single-valued neutrosophic set, to assign relative importance to the various criteria associated with putting into practice and maintaining the crowd management strategies. The strategies are ranked using the TOPSIS approach based on the single-valued neutrosophic set. As a result, we offer a method that uses the single-valued neutrosophic set with the AHP-TOPSIS together to help choose the best strategy of action for crowd management. Literature, studies, official papers, and expert views provide the basis for the suggested method's evaluation of the criteria and solutions for the crowd management strategies ranking. Challenges of crowd management are studied, and a plan of action is developed to give strategies to overcome them.

Collaboration and open lines of communication among all parties involved are essential for successful crowd management. Managers should encourage communication and coordination between event planners, venue managers, security staff, first responders, and local authorities. During both normal and emergency operations, it is crucial to have established lines of communication and systems for coordination in place to guarantee a synchronized response and the exchange of important information. Consistent get-togethers, drills, and team-building activities may help.

Managers should provide training and skill development opportunities for those working with large crowds. All employees should be trained and prepared to deal with any problems that may arise as a result of large crowds. Conflict resolution, crowd psychology, emergency responses, communication, and customer service should all be part of any appropriate training. A knowledgeable and prepared staff that can manage crowds and react to crises successfully may be maintained via regular training and exercises.

There is a lack of research on the use of MCDM to prioritize strategies for the efficient rank and analysis the crowd management strategies. Consequently, the suggested technique adds a lot to the theoretical and practical knowledge base of ranking crowd management strategies. The research has the following primary theoretical and practical implications:

- For the first time in the literature, we combine the AHP based on the single-valued neutrosophic numbers with TOPSIS methodology and apply it to the prioritization of crowd management strategies the reduce the crushes risks and obtain security and safety.
- The criteria for evaluating the methods to reduce the crushing risks and obtain the security and safety of ranking crowd management strategies are developed using research literature, reports, official documents, and expert comments. This adds to the existing literature a comprehensive framework for evaluating the difficulties.
- The proposed methodology in ranking crowd management strategies to reduce the crushing risks and obtain security and safety can be applied in Africa for example Egypt.

5. Conclusions

It is impossible to guarantee the safety, security, and efficiency of high-density settings without using appropriate crowd control strategies. Stakeholders can create safer and more organized spaces for crowds of all sizes by learning about crowd behavior, conducting thorough risk assessments, analyzing crowd flow patterns, implementing strong communication systems, deploying crowd control strategies, and providing comprehensive staff training. The purpose of this study is to investigate these strategies in depth to provide helpful information to professionals, academics, and policymakers working on crowd control. Also, ranking these strategies provides decreasing crowd crushing and increases safety and security. This study used six factors and ten strategies. Two MCDM methods are used in this paper. The AHP method is used to compute the weights of six factors. Then the TOPSIS method was used to rank the ten strategies. The single valued neutrosophic set was used to deal with uncertain information. The sensitivity analysis shows the proposal is robust. This study also compared the proposed model with various MCDM methods like VIKOR and MABAC. In future studies, the proposed model can be used in various industries like welding selection, energy selection, and others. Also, various MCDM methods can be applied to this problem like entropy to compute the weights of factors, VIKOR, and MABAC method to rank strategies. Researchers must be more careful in their selection of experts if they want reliable results. Therefore, it is necessary to establish a set of standards by which to choose competent professionals. The reliability of the expert evaluations may also be enhanced by increasing the number of academic experts that participate. In addition, we suggest that future integrated decision-making models make use of an uncertain possibility programming framework, a precise algorithm, and a double normalized approach.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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