NEUTROSOPHIC SYSTEMS WITH APPLICATIONS

AN INTERNATIONAL JOURNAL ON INFORMATICS, DECISION SCIENCE, INTELLIGENT SYSTEMS APPLICATIONS

ISSN (ONLINE):2993-7159
ISSN (PRINT): 2993-7140

VOLUME 9
2023

Editors-in-Chief
Florentin Smarandache
Mohamed Abdel-Basset
Said Broumi

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications

An International Journal on Informatics, Decision Science, Intelligent Systems Applications

Information for Authors and Subscribers

*"***Neutrosophic Systems with Applications***"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.
The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.
Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.
This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e., notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.
Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.
In a classical way <A>, <neutA>, <antiA> are disjointed two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.
Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of *]-0, 1+[*.
Neutrosophic Probability is a generalization of the classical probability and imprecise probability.
Neutrosophic Statistics is a generalization of classical statistics.
What distinguishes neutrosophic from other fields is the <neutA>, which means neither <A> nor <antiA>.
<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.
All submissions should be designed in MS Word format using our template file on the journal website.
A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science: http://fs.unm.edu/ScienceLibrary.htm.
To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief.
This journal is a non-commercial, academic edition. It is printed from private donations.
Information about the neutrosophic you get from the UNM website:
http://fs.unm.edu/neutrosophy.htm.
The home page of the journal is accessed on
https://nswajournal.org/

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Neutrosophic Systems with Applications
An International Journal on Informatics, Decision Science, Intelligent Systems Applications
ISSN (online): 2993-7159
ISSN (print): 2993-7140

Contents

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.66

Covering Properties via Neutrosophic b-open Sets

Sudeep Dey [1,2,*] and **Gautam Chandra Ray** [2]

[1] Department of Mathematics, Science College, Kokrajhar, Assam, India; sudeep.dey.1976@gmail.com.
[2] Department of Mathematics, Central Institute of Technology, Kokrajhar, Assam, India; gautomofcit@gmail.com.

* Correspondence: sudeep.dey.1976@gmail.com.

Abstract: The purpose of this article is to study some covering properties in neutrosophic topological spaces using neutrosophic b-open sets. We define neutrosophic b-open cover, neutrosophic b-compactness, neutrosophic countably b-compactness neutrosophic b-Lindelöfness, neutrosophic local b-compactness and study various properties entangled with them. We study some covering properties involving neutrosophic continuous, neutrosophic b-continuous and neutrosophic b^*-continuous functions. Lastly, we define neutrosophic base, neutrosophic subbase, neutrosophic second countability via neutrosophic b-open sets and investigate some properties.

Keywords: Neutrosophic b-open cover; Neutrosophic b-compact space; Neutrosophic countably b-compact space; Neutrosophic local b-compact space; Neutrosophic b-base.

1. Introduction

In 1965, Zadeh [30] introduced the concept of a fuzzy set. K. Atanassov [1], in 1986, extended this notion to intuitionistic fuzzy set. After that, the idea of a neutrosophic set was developed and studied by Florentin Smarandache [20-22]. Later, the theory was studied and taken ahead by many researchers [9,12,26,28]. It had been proved by Smarandache [22] that a neutrosophic set was a generalized form of an intuitionistic fuzzy set. Various applications [4,5,15,29] in different fields were done in a neutrosophic environment.

In the year 1968, C. L. Chang [7] created the notion of a fuzzy topological space and then, in 1997, D. Coker [8] gave the idea of intuitionistic fuzzy topological space. In the year 2012, Salama & Alblowi [23] introduced neutrosophic topological space as a generalization of intuitionistic fuzzy topological space. Afterwards, many studies were done by the researchers [2,3,6,11,16-19,24,25,27] to develop various aspects of neutrosophic topological spaces. The concept of neutrosophic b-open sets was given by Ebenanjar *et al*.[14]. Recently Dey & Ray [10] studied compactness in neutrosophic topological spaces. But compactness via neutrosophic b-open sets has not bee studied so far. In this write-up, we study covering properties using neutrosophic b-open sets.

The article is organized by stating some basic concepts in section 2. In section 3, we define neutrosophic b-open covering, neutrosophic b-compactness, neutrosophic countably b-compactness and neutrosophic b-Lindelöfness and study various properties associated with them. In section 4, we define neutrosophic local b-compactness and try to establish some properties. We define neutrosophic b-base, neutrosophic b-subbase, neutrosophic b-second countability and investigate some covering properties in section 5 and lastly, in section 6, we confer a conclusion.

2. Preliminaries

In this section, we state some basic concepts which will be helpful in the later sections.

2.1. Definition: [20] Let X be the universe of discourse. A neutrosophic set A over X is defined as $A = \{\langle x, \mathrm{T}_A(x), \mathrm{I}_A(x), \mathrm{F}_A(x)\rangle : x \in X\}$, where the functions $\mathrm{T}_A, \mathrm{I}_A, \mathrm{F}_A$ are real standard or non-standard subsets of $]^-0, 1^+[$, i.e., $\mathrm{T}_A: X \to]^-0, 1^+[$, $\mathrm{I}_A: X \to]^-0, 1^+[$, $\mathrm{F}_A: X \to]^-0, 1^+[$ and $^-0 \le \mathrm{T}_A(x) + \mathrm{I}_A(x) + \mathrm{T}_A(x) \le 3^+$.

The neutrosophic set A is characterized by the truth-membership function T_A, indeterminacy-membership function I_A, falsehood-membership function F_A.

2.2. Definition: [28] Let X be the universe of discourse. A single valued neutrosophic set A over X is defined as $A = \{\langle x, \mathrm{T}_A(x), \mathrm{I}_A(x), \mathrm{F}_A(x)\rangle : x \in X\}$, where $\mathrm{T}_A, \mathrm{I}_A, \mathrm{F}_A$ are functions from X to [0,1] and $0 \le \mathrm{T}_A(x) + \mathrm{I}_A(x) + \mathrm{T}_A(x) \le 3$.

The set of all single valued neutrosophic sets over X is denoted by $\mathcal{N}(X)$.

Throughout this article, a neutrosophic set (NS, for short) will mean a single-valued neutrosophic set.

2.3. Definition: [16] Let $A, B \in \mathcal{N}(X)$. Then

i) (Inclusion): If $\mathrm{T}_A(x) \le \mathrm{T}_B(x), \mathrm{I}_A(x) \ge \mathrm{I}_B(x), \mathrm{F}_A(x) \ge \mathrm{F}_B(x)$ for all $x \in X$ then A is said to be a neutrosophic subset of B and which is denoted by $A \subseteq B$.

ii) (Equality): If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

iii) (Intersection): The intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B = \{\langle x, \mathrm{T}_A(x) \wedge \mathrm{T}_B(x), \mathrm{I}_A(x) \vee \mathrm{I}_B(x), \mathrm{F}_A(x) \vee \mathrm{F}_B(x)\rangle : x \in X\}$.

iv) (Union): The union of A and B, denoted by $A \cup B$, is defined as $A \cup B = \{\langle x, \mathrm{T}_A(x) \vee \mathrm{T}_B(x), \mathrm{I}_A(x) \wedge \mathrm{I}_B(x), \mathrm{F}_A(x) \wedge \mathrm{F}_B(x)\rangle : x \in X\}$.

v) (Complement): The complement of the NS A, denoted by A^c, is defined as $A^c = \{\langle x, \mathrm{F}_A(x), 1 - \mathrm{I}_A(x), \mathrm{T}_A(x)\rangle : x \in X\}$

vi) (Universal Set): If $\mathrm{T}_A(x) = 1, \mathrm{I}_A(x) = 0, \mathrm{F}_A(x) = 0$ for all $x \in X$ then A is said to be neutrosophic universal set and which is denoted by $\tilde{X}$.

vii) (Empty Set): If $\mathrm{T}_A(x) = 0, \mathrm{I}_A(x) = 1, \mathrm{F}_A(x) = 1$ for all $x \in X$ then A is said to be neutrosophic empty set and which is denoted by $\tilde{\emptyset}$.

2.4. Definition: [18] Let $\mathcal{N}(X)$ be the set of all neutrosophic sets over X. An NS $P = \{\langle x, \mathrm{T}_A(x), \mathrm{I}_A(x), \mathrm{F}_A(x)\rangle : x \in X\}$ is called a neutrosophic point (NP, for short) iff for any element $y \in X$, $\mathrm{T}_P(y) = \alpha, \mathrm{I}_P(y) = \beta, \mathrm{F}_P(y) = \gamma$ for $y = x$ and $\mathrm{T}_P(y) = 0, \mathrm{I}_P(y) = 1, \mathrm{F}_P(y) = 1$ for $y \ne x$, where $0 < \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1$. A neutrosophic point $P = \{\langle x, \mathrm{T}_A(x), \mathrm{I}_A(x), \mathrm{F}_A(x)\rangle : x \in X\}$ will be denoted by $x_{\alpha,\beta,\gamma}$. For the NP $x_{\alpha,\beta,\gamma}$, x will be called its support. The complement of the NP $x_{\alpha,\beta,\gamma}$ will be denoted by $x^c_{\alpha,\beta,\gamma}$ or $\left(x_{\alpha,\beta,\gamma}\right)^c$.

2.5. Definition: [16] Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a neutrosophic topology on X if

i) $\tilde{\emptyset}$ and $\tilde{X}$ belong to τ.

ii) An arbitrary union of neutrosophic sets in τ is in τ.

iii) The intersection of any two neutrosophic sets in τ is in τ.

If τ is a neutrosophic topology on X then the pair (X, τ) is called a neutrosophic topological space (NTS, for short) over X. The members of τ are called neutrosophic open sets in X. If for a neutrosophic set A, $A^c \in \tau$ then A is said to be a neutrosophic closed set in X.

2.6. Definition:[14] Let (X, τ) be an NTS and G be a NS over X. Then G is called a

i) Neutrosophic b-open (NBO, for short) set iff $G \subseteq \left[int\left(cl(G)\right)\right] \cup \left[cl\left(int(G)\right)\right]$.

ii) Neutrosophic b-closed (NBC, for short) set iff $G \supseteq \left[int\left(cl(G)\right)\right] \cup \left[cl\left(int(G)\right)\right]$.

If G is an NBO (resp. NBC) set in (X, τ) then we shall also say that G is a τ-NBO (resp. τ-NBC) set.

2.7. Theorem: [14] Let (X,τ) be an NTS.

i) If $G \in \mathcal{N}(X)$ then G is an NBO set iff G^c is an NBC set.
ii) If $G \in \mathcal{N}(X)$ then G is an NBC set iff G^c is an NBO set.

2.8. Theorem: [13] Let (X,τ) be an NTS and $A \in \mathcal{N}(X)$. Then

i) Every neutrosophic open set in an NTS is an NBO set.
ii) Every neutrosophic closed set in an NTS is an NBC set.

2.9. Definition: [27] Let f be a function from an NTS (X,τ) to the NTS (Y,σ). Then

i) f is called a neutrosophic open function if $f(G) \in \sigma$ for all $G \in \tau$
ii) f is called a neutrosophic continuous function if $f^{-1}(G) \in \tau$ for all $G \in \sigma$.

2.10. Definition: [13] Let f be a function from an NTS (X,τ) to the NTS (Y,σ). Then f is called a neutrosophic

i) b-open function if $f(G)$ is an NBO set in Y for every neutrosophic open set G in X.
ii) b-continuous function if $f^{-1}(G)$ is an NBO set in X for every σ-open NS G in Y.
iii) b^*-continuous function if $f^{-1}(G)$ is an NBO set in X for every NBO set G in Y.

2.11. Proposition: [13] Let $(Y,\tau\mid_Y)$ be a neutrosophic subspace of the NTS (X,τ). Then

i) $G\mid_Y$ is a $\tau\mid_Y$-NBO set in Y for every τ-NBO set G in X.
ii) $G\mid_Y$ is a $\tau\mid_Y$-NBC set in Y for every τ-NBC set G in X.

2.12. Definition: [10] Let (X,τ) be an NTS. A collection $\{G_\lambda : \lambda \in \Delta\}$ of neutrosophic sets of X is said to have the finite intersection property (FIP, in short) iff every finite subcollection $\{G_{\lambda_k} : k = 1,2,\cdots,n\}$ of $\{G_\lambda : \lambda \in \Delta\}$ satisfies the condition $\bigcap_{k=1}^{n} G_{\lambda_k} \neq \widetilde{\emptyset}$, where Δ is an index set.

*For neutrosophic function and its properties, please see [25].

3. Neutrosophic b-compactness

3.1. Definition: Let (X,τ) be an NTS and $A \in \mathcal{N}(X)$. A collection $C = \{G_i : i \in \Delta\}$ of NBO sets of X is called a neutrosophic b-open cover (NBOC, in short) of A iff $A \subseteq \cup_{i\in\Delta} G_i$. In particular, C is said to be an NBOC of X iff $\tilde{X} = \cup_{i\in\Delta} G_i$.

Let C be an NBOC of the NS A and $C' \subseteq C$. Then C' is called a neutrosophic b-open subcover (NBOSC, in short) of C if C' is also a NBOC of A.

An NBOC C of an NS A is said to be countable (resp. finite) if C consists of a countable (resp. finite) number of NBO sets.

3.2. Definition: An NS A in an NTS (X,τ) is said to be a neutrosophic b-compact set iff every NBOC of A has a finite NBOSC.

An NS A in an NTS (X,τ) is said to be a neutrosophic b-Lindelöf (resp. neutrosophic countably b-compact) set iff every NBOC (resp. countable NBOC) of A has a countable(resp. finite) NBOSC.

An NTS (X,τ) is said to be a neutrosophic b-compact space iff every NBOC of X has a finite NBOSC.

An NTS (X, τ) is said to be a neutrosophic b-Lindelöf (neutrosophic countably b-compact) space iff every NBOC (countable NBOC) of X has a countable(finite) NBOSC.

3.3. Proposition: Every neutrosophic b-compact space is a neutrosophic countably b-compact space.

Proof: Obvious.

3.4. Proposition: In an NTS, every neutrosophic b-compact set is a neutrosophic compact set.

Proof: Let A be a neutrosophic b-compact set of an NTS (X, τ). Let $C = \{G_i : i \in \Delta\}$ be an NOC of A. Since every neutrosophic open set is an NBO set[by 2.9], so G_i is an NBO set for each $i \in \Delta$. Therefore C is an NBOC of A. Since A is b-compact, so there exists a finite subcollection $\{G_i^1, G_i^2, \ldots, G_i^m\}$, say, of C such that $A \subseteq G_i^1 \cup G_i^2 \cup \ldots \cup G_i^m$. Thus the NOC C of A has a finite NOSC $\{G_i^1, G_i^2, \ldots, G_i^m\}$. Hence A is a neutrosophic compact set.

3.5. Example : Converse of the prop. 3.4 is not true. We establish it by the following example.

Let $X = \{a, b\}$, $B = \{\langle a, 0,1,1\rangle, \langle b, 1,0,0\rangle\}$, $G_n = \{\langle a, 0,1,1\rangle, \langle b, \frac{n}{n+1}, \frac{1}{n}, \frac{1}{n+1}\rangle\}, n \in \mathrm{N} = \{1,2,3, \cdots\}$ and $\tau = \{\tilde{X}, \tilde{\emptyset}, B\}$. Clearly (X, τ) is an NTS and G_n is an NBO set for each $n \in \mathrm{N}$. Obviously B is a neutrosophic compact set. We observe that $\{G_n : n \in \mathrm{N}\}$ is an NBOC of B but it has no NBOSC. Therefore B is not a neutrosophic b-compact set.

3.6. Proposition: Every neutrosophic b-compact space is a neutrosophic compact space.

Proof: Obvious from prop. 3.4.

3.7. Remark: Converse of prop. 3.6 is not true. We establish it by the following example.

Let us consider the NTS (N, τ), where $\tau = \{\tilde{\emptyset}, \tilde{\mathrm{N}}\}, \mathrm{N} = \{1, 2, 3, \cdots\}$. Clearly (N, τ) is a neutrosophic compact space. We show that (N, τ) is not a neutrosophic b-compact space. For $n \in \mathrm{N}$, we define $G_n = \{\langle x, \mathrm{T}_{G_n}(x), \mathrm{I}_{G_n}(x), \mathrm{F}_{G_n}(x) : x \in \mathrm{N}\}$, where $\mathrm{T}_{G_n}(x) = 1, \mathrm{I}_{G_n}(x) = 0, \mathrm{F}_{G_n}(x) = 0$ if $x = n$ and $\mathrm{T}_{G_n}(x) = 0, \mathrm{I}_{G_n}(x) = 1, \mathrm{F}_{G_n}(x) = 1$ if $x \neq n$. Clearly, for each $n \in \mathrm{N}$, G_n is an NBO set in (N, τ). Obviously the collection $C = \{G_n : n \in \mathrm{N}\}$ is an NBOC of N but it has no finite NBOSC. Therefore (N, τ) is not a neutrosophic b-compact space. Thus (N, τ) is a neutrosophic compact space but not a neutrosophic b-compact space.

3.8. Proposition: In an NTS, union of two neutrosophic b-compact sets is neutrosophic b-compact.

Proof: Let A and B be two neutrosophic b-compact sets of an NTS (X, τ). Let $C = \{G_i : i \in \Delta\}$ be an NBOC of $A \cup B$. Then $A \cup B \subseteq \cup_{i \in \Delta} G_i$. Since $A \subseteq A \cup B$, so C is an NBOC of A. Again since A is neutrosophic b-compact, so there exists a finite subcollection $\{G_i^1, G_i^2, \ldots, G_i^m\}$ of C such that $A \subseteq G_i^1 \cup G_i^2 \cup \ldots \cup G_i^m$. Similarly, since B is neutrosophic b-compact, so there exists a finite subcollection $\{H_i^1, H_i^2, \ldots, H_i^n\}$ of C such that $B \subseteq H_i^1 \cup H_i^2 \cup \ldots \cup H_i^n$. Therefore $A \cup B \subseteq G_i^1 \cup G_i^2 \cup \ldots \cup G_i^m \cup H_i^1 \cup H_i^2 \cup \ldots \cup H_i^n$. Thus there exists a finite subcollection $\{G_i^1, G_i^2, \ldots, G_i^m, H_i^1, H_i^2, \ldots, H_i^n\}$ of C such that $\cup B \subseteq G_i^1 \cup G_i^2 \cup \ldots \cup G_i^m \cup H_i^1 \cup H_i^2 \cup \ldots \cup H_i^n$. Therefore $A \cup B$ is neutrosophic b-compact. Hence proved.

3.9. Proposition: In an NTS, finite union of neutrosophic b-compact sets is neutrosophic b-compact.

Proof: Immediate from the prop. 3.8.

3.10. Proposition: In an NTS, union of a neutrosophic b-compact set and a neutrosophic compact set is a neutrosophic compact set.

Proof: Obvious.

3.11. Definition: Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ). Then the set of all NBO sets $G|_Y$ in Y for which G is an NBO set in X will be denoted by $NBO(Y)$, i.e., $NBO(Y) = \{G|_Y \subseteq Y : G|_Y$ is an NBO set in Y and $G \subseteq X$ is an NBO set in $X\}$.

3.12. Proposition: Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. Then A is neutrosophic b-compact in X iff every cover of A by the sets in $NBO(Y)$ has a finite subcover.

Proof: Necessary part: Let $C = \{G_i|_Y : i \in \Delta\}$ be a cover of A, where $G_i|_Y \in NBO(Y)$ for each $i \in \Delta$. Then $A \subseteq \cup_{i \in \Delta} G_i|_Y \Rightarrow A \subseteq \cup_{i \in \Delta} G_i$. Clearly G_i is an NBO set in X [by 3.11] for each $G_i|_Y \in C$ and so, $C^* = \{G_i : G_i|_Y \in NBO(Y)\}$ is an NBOC of A in X. Since A is b-compact in X, so there exists a finite subcollection $\{G_{i_k} : k = 1,2,3, \ldots, n\}$ of C^* such that $A \subseteq \cup_{k=1}^n G_{i_k} \Rightarrow A \subseteq \left(\cup_{k=1}^n G_{i_k}\right)|_Y \Rightarrow A \subseteq \cup_{k=1}^n \left(G_{i_k}|_Y\right)$. Thus the cover C of A has a finite subcover $\{G_{i_k} : k = 1,2,3, \ldots, n\}$.

Sufficient part: Let $B = \{G_i: i \in \Delta\}$ be an NBOC of A in X, where G_i is an NBO set in X for each $i \in \Delta$. Then $A \subseteq \cup_{i\in\Delta} G_i \Rightarrow A \subseteq (\cup_{i\in\Delta} G_i) |_Y \Rightarrow A \subseteq \cup_{i\in\Delta} (G_i |_Y)$. Since $G_i |_Y \in NBO(Y)$ for each $G_i \in B$ [by 2.12], so $B^* = \{G_i |_Y: i \in \Delta\}$ is a cover of A by the NBO sets in $NBO(Y)$. Therefore, by hypothesis, there exists a finite subcollection $\{G_{i_k} |_Y: k = 1,2,3,\ldots,n\}$ of B^* such that $A \subseteq \cup_{k=1}^{n} \left(G_{i_k} |_Y\right) \Rightarrow A \subseteq \left(\cup_{k=1}^{n} G_{i_k}\right) |_Y \Rightarrow A \subseteq \cup_{k=1}^{n} G_{i_k}$. Thus the NBOC B of A has a finite NBOSC $\{G_{i_k}: k = 1,2,3,\ldots,n\}$. Therefore, A is neutrosophic b-compact in X.

3.13. Proposition: Let $(Y, \tau |_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. Then A is neutrosophic countably b-compact in X iff every countable cover of A by the sets in $NBO(Y)$ has a finite subcover.

Proof: Obvious from the prop. 3.12.

3.14. Proposition: Let $(Y, \tau |_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. Then A is neutrosophic b-Lindelöf in X iff every cover of A by the sets in $NBO(Y)$ has a countable subcover.

Proof: Obvious from the prop. 3.12.

3.15. Proposition: Let $(Y, \tau |_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. If A is neutrosophic b-compact in X then A is neutrosophic compact in Y.

Proof: Let $C = \{G_i |_Y: i \in \Delta\}$ be an NOC of A in Y, where $G_i |_Y \in \tau |_Y$ for each $i \in \Delta$. Then $A \subseteq \cup_{i\in\Delta} (G_i |_Y) \Rightarrow A \subseteq \cup_{i\in\Delta} G_i$. Obviously $G_i \in \tau$ and so, G_i is an NBO set in X for each $i \in \Delta$. Therefore, $C^* = \{G_i: G_i |_Y \in C\}$ is an NBOC of A in X. Since A is b-compact in X, so there exists a finite subcollection $\{G_{i_k}: k = 1,2,3,\ldots,n\}$ of C^* such that $A \subseteq \cup_{k=1}^{n} G_{i_k} \Rightarrow A \subseteq \left(\cup_{k=1}^{n} G_{i_k}\right) |_Y \Rightarrow A \subseteq \cup_{k=1}^{n} \left(G_{i_k} |_Y\right)$. Thus the NOC C of A has a finite NOSC $\{G_{i_k}: k = 1,2,3,\ldots,n\}$. Therefore A is neutrosophic compact in Y.

3.16. Proposition: Let $(Y, \tau |_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. If A is b-compact in Y then A is b-compact in X.

Proof: Obvious.

3.17. Proposition: If G is an NBC subset of a neutrosophic b-compact space (X, τ) such that $G \cap G^c = \widetilde{\emptyset}$ then G is a neutrosophic b-compact.

Proof: Let $C = \{H_i: i \in \Delta\}$ be an NBOC of G. Then $A \subseteq \cup_{i\in\Delta} H_i$. Since G^c is an NBO set and since $G \cap G^c = \widetilde{\emptyset}$, i.e., $G \cup G^c = \widetilde{X}$, so $D = \{H_i: i \in \Delta\} \cup \{G^c\}$ is an NBOC of X. As X is neutrosophic b-compact, so there exists a finite subcollection $D' = \{H_{i_1}, H_{i_2}, \ldots, H_{i_n}\} \cup \{G^c\}$ of D such that $X \subseteq H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_n} \cup G^c$. Therefore $G \subseteq H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_n} \cup G^c$. But $G \cap G^c = \widetilde{\emptyset}$, so $G \subseteq H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_n}$. Thus the NBOC C of G has a finite NBOSC $\{H_{i_1}, H_{i_2}, \ldots, H_{i_n}\}$. Hence G is a neutrosophic b-compact set.

3.18. Proposition: If G is an NBC subset of a neutrosophic b-compact space (X, τ) such that $G \cap G^c = \widetilde{\emptyset}$ then G is neutrosophic compact.

Proof: Immediate from the prop. 3.17 as b-compactness implies compactness.

3.19. Proposition: If G is a neutrosophic closed subset of a neutrosophic b-compact space (X, τ) such that $G \cap G^c = \widetilde{\emptyset}$ then G is neutrosophic b-compact.

Proof: Immediate from the prop. 3.17 as every neutrosophic closed set is an NBC set.

3.20. Proposition: If G is a neutrosophic closed subset of a neutrosophic b-compact space (X, τ) such that $G \cap G^c = \widetilde{\emptyset}$ then G is neutrosophic compact.

Proof: Immediate from the prop. 3.19.

3.21. Proposition: Let (X, τ) be an NTS. An NS $A = \{\langle x, \mathrm{T}_A(x), \mathrm{I}_A(x), \mathrm{F}_A(x)\rangle: x \in X\}$ over X is neutrosophic b-compact iff for every collection $C = \{G_\lambda: \lambda \in \Delta\}$ of NBO sets of X satisfying $\mathrm{T}_A(x) \leq \vee_{\lambda\in\Delta} \mathrm{T}_{G_\lambda}(x)$, $1 - \mathrm{I}_A(x) \leq \vee_{\lambda\in\Delta} \left(1 - \mathrm{I}_{G_\lambda}(x)\right)$ and $1 - \mathrm{F}_A(x) \leq \vee_{\lambda\in\Delta} \left(1 - \mathrm{F}_{G_\lambda}(x)\right)$, there exists a finite subcollection $\{G_{\lambda_k}: k = 1,2,3,\ldots,n\}$ such that $\mathrm{T}_A(x) \leq \vee_{k=1}^{n} \mathrm{T}_{G_{\lambda_k}}(x)$, $1 - \mathrm{I}_A(x) \leq \vee_{k=1}^{n} \left(1 - \mathrm{I}_{G_{\lambda_k}}(x)\right)$ and $1 - \mathrm{F}_A(x) \leq \vee_{k=1}^{n} \left(1 - \mathrm{F}_{G_{\lambda_k}}(x)\right)$.

Proof: Necessary Part: Let $C = \{G_\lambda: \lambda \in \Delta\}$ be any collection of NBO sets of X satisfying $\mathrm{T}_A(x) \leq \vee_{\lambda\in\Delta} \mathrm{T}_{G_\lambda}(x)$, $1 - \mathrm{I}_A(x) \leq \vee_{\lambda\in\Delta} \left(1 - \mathrm{I}_{G_\lambda}(x)\right)$ and $1 - \mathrm{F}_A(x) \leq \vee_{\lambda\in\Delta} \left(1 - \mathrm{F}_{G_\lambda}(x)\right)$. Now $1 - \mathrm{I}_A(x) \leq$

$\vee_{\lambda\in\Delta}\left(1-\mathrm{I}_{G_\lambda}(x)\right) \Rightarrow 1-\mathrm{I}_A(x) \leq 1-\mathrm{I}_{G_\beta}(x)$ for some $\in \Delta \Rightarrow \mathrm{I}_A(x) \geq \mathrm{I}_{G_\beta}(x) \Rightarrow \mathrm{I}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{I}_{G_\lambda}(x)$. Similarly $1-\mathrm{F}_A(x) \leq \vee_{\lambda\in\Delta}\left(1-\mathrm{F}_{G_\lambda}(x)\right) \Rightarrow \mathrm{F}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{F}_{G_\lambda}(x)$. Therefore $A \subseteq \cup_{\lambda\in\Delta} G_\lambda$, i.e., C is an NBOC of A. Since A is neutrosophic b-compact, so C has a finite NBOSC $\{G_{\lambda_k}: k = 1,2,3,\cdots,n\}$, say. Therefore $A \subseteq \cup_{k=1}^{n} G_{\lambda_k}$. Then $\mathrm{T}_A(x) \leq \vee_{k=1}^{n}\mathrm{T}_{G_{\lambda_k}}(x)$, $\mathrm{I}_A(x) \geq \wedge_{k=1}^{n}\mathrm{I}_{G_{\lambda_k}}(x)$ and $\mathrm{F}_A(x) \geq \wedge_{k=1}^{n}\mathrm{F}_{G_{\lambda_k}}(x)$. Now $\mathrm{I}_A(x) \geq \wedge_{k=1}^{n}\mathrm{I}_{G_{\lambda_k}}(x) \Rightarrow \mathrm{I}_A(x) \geq \mathrm{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n \Rightarrow 1-\mathrm{I}_A(x) \leq 1-\mathrm{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n \Rightarrow 1-\mathrm{I}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{I}_{G_{\lambda_k}}(x)\right)$. Similarly $\mathrm{F}_A(x) \geq \wedge_{k=1}^{n}\mathrm{F}_{G_{\lambda_k}}(x) \Rightarrow 1-\mathrm{F}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{F}_{G_{\lambda_k}}(x)\right)$. Thus $\mathrm{T}_A(x) \leq \vee_{k=1}^{n}\mathrm{T}_{G_{\lambda_k}}(x)$, $1-\mathrm{I}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{I}_{G_{\lambda_k}}(x)\right)$ and $1-\mathrm{F}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{F}_{G_{\lambda_k}}(x)\right)$.

Sufficient Part: Let $C = \{G_\lambda: \lambda \in \Delta\}$ be an NBOC of A. Then $A \subseteq \cup_{\lambda\in\Delta} G_\lambda$, i.e., $\mathrm{T}_A(x) \leq \vee_{\lambda\in\Delta}\mathrm{T}_{G_\lambda}(x)$, $\mathrm{I}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{I}_{G_\lambda}(x)$ and $\mathrm{F}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{F}_{G_\lambda}(x)$. Now $\mathrm{I}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{I}_{G_\lambda}(x) \Rightarrow \mathrm{I}_A(x) \geq \mathrm{I}_{G_\alpha}(x)$ for some $\in \Delta$ $\Rightarrow 1-\mathrm{I}_A(x) \leq 1-\mathrm{I}_{G_\alpha}(x) \Rightarrow 1-\mathrm{I}_A(x) \leq \vee_{\lambda\in\Delta}\left(1-\mathrm{I}_{G_\lambda}(x)\right)$. Similarly $\mathrm{F}_A(x) \geq \wedge_{\lambda\in\Delta}\mathrm{F}_{G_\lambda}(x) \Rightarrow 1-\mathrm{F}_A(x) \leq \vee_{\lambda\in\Delta}\left(1-\mathrm{F}_{G_\lambda}(x)\right)$. Thus the collection C satisfies the condition $\mathrm{T}_A(x) \leq \vee_{\lambda\in\Delta}\mathrm{T}_{G_\lambda}(x)$, $1-\mathrm{I}_A(x) \leq \vee_{\lambda\in\Delta}\left(1-\mathrm{I}_{G_\lambda}(x)\right)$ and $1-\mathrm{F}_A(x) \leq \vee_{\lambda\in\Delta}\left(1-\mathrm{F}_{G_\lambda}(x)\right)$. By the hypothesis, there exists a finite subcollection $\{G_{\lambda_k}: k = 1,2,3,\ldots,n\}$ such that $\mathrm{T}_A(x) \leq \vee_{k=1}^{n}\mathrm{T}_{G_{\lambda_k}}(x)$, $1-\mathrm{I}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{I}_{G_{\lambda_k}}(x)\right)$ and $1-\mathrm{F}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{F}_{G_{\lambda_k}}(x)\right)$. Now $1-\mathrm{I}_A(x) \leq \vee_{k=1}^{n}\left(1-\mathrm{I}_{G_{\lambda_k}}(x)\right) \Rightarrow 1-\mathrm{I}_A(x) \leq 1-\mathrm{I}_{G_{\lambda_m}}(x)$ for some $, 1 \leq m \leq n \Rightarrow \mathrm{I}_A(x) \geq \mathrm{I}_{G_{\lambda_m}}(x) \Rightarrow \mathrm{I}_A(x) \geq \wedge_{k=1}^{n}\mathrm{I}_{G_{\lambda_k}}(x)$. Similarly, we shall have $\mathrm{F}_A(x) \geq \wedge_{k=1}^{n}\mathrm{F}_{G_{\lambda_k}}(x)$. Therefore $A \subseteq \cup_{k=1}^{n} G_{\lambda_k}$, i.e., the NBOC C of A has a finite NBOSC $\{G_{\lambda_k}: k = 1,2,3,\cdots,n\}$. Therefore, A is neutrosophic b-compact set.

Hence proved.

3.22. Proposition: Let (X,τ) be an NTS. Then X is neutrosophic b-compact iff for every collection $C = \{G_\lambda: \lambda \in \Delta\}$ of NBO sets of X satisfying $\vee_{\lambda\in\Delta}\mathrm{T}_{G_\lambda}(x) = 1$, $\vee_{\lambda\in\Delta}\left(1-\mathrm{I}_{G_\lambda}(x)\right) = 1$ and $\vee_{\lambda\in\Delta}\left(1-\mathrm{F}_{G_\lambda}(x)\right) = 1$, there exists a finite subcollection $\{G_{\lambda_k}: k = 1,2,3,\ldots,n\}$ such that $\vee_{k=1}^{n}\mathrm{T}_{G_{\lambda_k}}(x) = 1$, $\vee_{k=1}^{n}\left(1-\mathrm{I}_{G_{\lambda_k}}(x)\right) = 1$ and $\vee_{k=1}^{n}\left(1-\mathrm{F}_{G_{\lambda_k}}(x)\right) = 1$.

Proof: Immediate from the prop. 3.21.

3.23. Proposition: An NTS (X,τ) is neutrosophic b-compact iff every collection of NBC sets with FIP has a non-empty intersection.

Proof: Necessary part: Let $A = \{N_i: i \in \Delta\}$ be an arbitrary collection of NBC sets with FIP. We show that $\cap_{i\in\Delta} N_i \neq \widetilde{\emptyset}$. On the contrary, suppose that $\cap_{i\in\Delta} N_i = \widetilde{\emptyset}$. Then $(\cap_{i\in\Delta} N_i)^c = \left(\widetilde{\emptyset}\right)^c \Rightarrow \cup_{i\in\Delta} N_i^c = \tilde{X}$. Therefore $B = \{N_i^c: N_i \in A\}$ is an NBOC of X and so, B has a finite NBOSC $\{N_{i_1}^c, N_{i_2}^c, \ldots, N_{i_k}^c\}$, say. Then $\cup_{j=1}^{k} N_{i_j}^c = \tilde{X} \Rightarrow \cap_{j=1}^{k} N_{i_j} = \widetilde{\emptyset}$, which is a contradiction as A has FIP. Therefore $\cap_{i\in\Delta} N_i \neq \widetilde{\emptyset}$.

Sufficient part: Suppose that X is not neutrosophic b-compact. Then there exists an NBOC $C = \{G_i: i \in \Delta\}$ of X which has no finite NBOSC. Then for every finite subcollection $\{G_{i_1}, G_{i_2}, \ldots, G_{i_k}\}$ of C, we have $\cup_{j=1}^{k} G_{i_j} \neq \tilde{X} \Rightarrow \cap_{j=1}^{k} G_{i_j}^c \neq \widetilde{\emptyset}$. Therefore, $\{G_i^c: G_i \in C\}$ is a collection of NBC sets having the FIP. By the assumption, $\cap_{i\in\Delta} G_i^c \neq \widetilde{\emptyset} \Rightarrow \cup_{i\in\Delta} G_i \neq \tilde{X}$. This shows that C is not an NBOC of X, which is a contradiction. Therefore, the NBOC C of X must have a finite NBOSC. Therefore X is neutrosophic b-compact.

Hence proved.

3.24. Proposition: Let f be a neutrosophic b-open function from an NTS (X,τ) to the NTS (Y,σ) and $A \in \mathcal{N}(Y)$. If A is neutrosophic b-compact in Y then $f^{-1}(A)$ is neutrosophic compact in X.

Proof: Let $B = \{G_\lambda: \lambda \in \Delta\}$ be an NOC of $f^{-1}(A)$. Then $f^{-1}(A) \subseteq \cup_{\lambda\in\Delta} G_\lambda \Rightarrow A \subseteq f(\cup_{\lambda\in\Delta} G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f(G_\lambda)$. Since G_λ is τ-open set, so $f(G_\lambda)$ is σ-NBO set for each $\lambda \in \Delta$ as f is a b-open function. Therefore, $C = \{f(G_\lambda): G_\lambda \in B\}$ is an NBOC of A. Since A is neutrosophic b-compact, so C has a finite NBOSC $\{f(G_{\lambda_1}), f(G_{\lambda_2}), f(G_{\lambda_3}), \ldots, f(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f(G_{\lambda_i}) \Rightarrow A \subseteq$

$f(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f^{-1}(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NOC B of $f^{-1}(A)$ has a finite NOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots, G_{\lambda_n}\}$. Therefore $f^{-1}(A)$ is neutrosophic compact in X. Hence proved.

3.25. Proposition: Let f be a neutrosophic b-open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic b-compact (resp. neutrosophic countably b-compact, neutrosophic b-Lindelöf) then (X, τ) is neutrosophic compact (resp. neutrosophic countably compact, neutrosophic Lindelöf).

Proof: Immediate from the prop. 3.24 as f is onto.

3.26. Proposition: Let f be a neutrosophic open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic b-compact (resp. neutrosophic countably b-compact, neutrosophic b-Lindelöf) then (X, τ) is neutrosophic compact (resp. neutrosophic countably compact, neutrosophic Lindelöf).

Proof: Obvious as every neutrosophic open set is an NBO set.

3.27. Proposition: Let f be a neutrosophic b-continuous function from an NTS (X, τ) to the NTS (Y, σ). If A is neutrosophic b-compact set in X then $f(A)$ is neutrosophic compact set in Y.

Proof: Let $B = \{G_\lambda : \lambda \in \Delta\}$ be an NOC of $f(A)$. Then $f(A) \subseteq \cup_{\lambda\in\Delta} G_\lambda \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) \Rightarrow f^{-1}(f(A)) \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda)$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NBO set in X as f is b-continuous. Therefore $C = \{f^{-1}(G_\lambda) : G_\lambda \in B\}$ is an NBOC of A. Since A is neutrosophic b-compact, so C has a finite NBOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \ldots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow A \subseteq f^{-1}(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NOC B of $f(A)$ has a finite NOSC. Therefore $f(A)$ is neutrosophic compact. Hence proved.

3.28. Proposition: Let f be a neutrosophic continuous function from an NTS (X, τ) to the NTS (Y, σ). If f is neutrosophic b-compact in X then $f(A)$ is neutrosophic compact in Y.

Proof: Obvious from the prop. 3.27 as every neutrosophic open set is an NBO set.

3.29. Proposition: Let f be a neutrosophic b-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic b-compact then (Y, σ) is neutrosophic compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $B = \{G_\lambda : \lambda \in \Delta\}$ be an NOC of Y. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow \cup_{\lambda\in\Delta} G_\lambda = f(\tilde{X}) \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = \tilde{X} \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NBO set in X as f is b-continuous. Therefore $C = \{f^{-1}(G_\lambda) : G_\lambda \in B\}$ is an NBOC of X. Since X is b-compact, so C has a finite NBOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \ldots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f^{-1}(\cup_{i=1}^{n} G_{\lambda_i}) = \tilde{X} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$. Thus the NOC B of Y has a finite NOSC. Therefore Y is neutrosophic compact. Hence proved.

3.30. Proposition: Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic b-compact then (Y, σ) is neutrosophic compact.

Proof: Obvious from the prop. 3.29 as every neutrosophic open set is an NBO set.

3.31. Proposition: Let f be a neutrosophic b-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic countably b-compact then Y is neutrosophic countably compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $B = \{G_\lambda : \lambda \in \Delta\}$ be a countable NOC of Y. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow \cup_{\lambda\in\Delta} G_\lambda = f(\tilde{X}) \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = \tilde{X} \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NBO set in X as f is b-continuous. Therefore $C = \{f^{-1}(G_\lambda) : G_\lambda \in B\}$ is an NBOC of X. Obviously C is countable as B is countable. Again since X is neutrosophic countably b-compact, so C has a finite NBOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \ldots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f^{-1}(\cup_{i=1}^{n} G_{\lambda_i}) = \tilde{X} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$. Thus the countable NOC B of Y has a finite NOSC. Hence Y is neutrosophic countably compact.

3.32. Proposition: Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic countably b-compact then Y is neutrosophic countably compact.

Proof: Immediate from the prop. 3.31 as every neutrosophic open set is an NBO set.

3.33. Proposition: Let f be a neutrosophic b-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic b-Lindelöf then Y is neutrosophic Lindelöf.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $C = \{G_i : i \in \Delta\}$ be an NOC of Y. Then $\cup_{i\in\Delta} G_i = \tilde{Y} \Rightarrow \cup_{i\in\Delta} G_i = f(\tilde{X}) \Rightarrow f^{-1}(\cup_{i\in\Delta} G_i) = \tilde{X} \Rightarrow \cup_{i\in\Delta} f^{-1}(G_i) = \tilde{X} \Rightarrow \{f^{-1}(G_i) : G_i \in C\}$ is an NBOC of X. Since X

is neutrosophic b- Lindelöf, so $\{f^{-1}(G_i): G_i \in C\}$ has a countable NBOSC $B = \{f^{-1}(G_{i_k}): k = 1,2,3, \dots\}$, say. Therefore, $\tilde{X} = f^{-1}(G_{\lambda_1}) \cup f^{-1}(G_{\lambda_2}) \cup f^{-1}(G_{\lambda_3}) \cup \dots$. This gives $\tilde{X} = f^{-1}(G_{\lambda_1} \cup G_{\lambda_2} \cup G_{\lambda_3} \cup \dots) \Rightarrow f(\tilde{X}) = G_{\lambda_1} \cup G_{\lambda_2} \cup G_{\lambda_3} \cup \dots \Rightarrow \tilde{Y} = G_{\lambda_1} \cup G_{\lambda_2} \cup G_{\lambda_3} \cup \dots \Rightarrow \{G_{i_k}: k = 1,2,3, \dots\}$ is an NOC of Y. Since B is countable so, $\{G_{i_k}: k = 1,2,3, \dots\}$ is also countable. Therefore, the NOC C of Y has a countable NOSC $\{G_{i_k}: k = 1,2,3, \dots\}$ and so, Y is neutrosophic Lindelöf.

3.34. Proposition: Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic b-Lindelöf then Y is neutrosophic Lindelöf.

Proof: Immediate from the prop. 3.33 as every neutrosophic open set is an NBO set.

3.35. Definition: Let f be a neutrosophic function from an NTS (X, τ) to the NTS (Y, σ). Then f is called a neutrosophic b^*-open function if $f(G)$ is an NBO set in Y for every NBO set G in X.

3.36. Proposition: Let f be a neutrosophic b^*-open function from an NTS (X, τ) to the NTS (Y, σ) and $A \in \mathcal{N}(Y)$. If A is neutrosophic b-compact in Y then $f^{-1}(A)$ is neutrosophic b-compact in X.

Proof: Let $B = \{G_i: i \in \Delta\}$ be an NBOC of $f^{-1}(A)$. Then $f^{-1}(A) \subseteq \cup_{i\in\Delta} G_i \Rightarrow A \subseteq f(\cup_{i\in\Delta} G_i) \Rightarrow A \subseteq \cup_{i\in\Delta} f(G_i)$. Since G_i is a τ-NBO set, so $f(G_i)$ is a σ-NBO set for each $i \in \Delta$ as f is a neutrosophic b^*-open function. Therefore, $C = \{f(G_i): G_i \in B\}$ is an NBOC of A. Since A is neutrosophic b-compact, so C has a finite NBOSC $\{f(G_{\lambda_1}), f(G_{\lambda_2}), f(G_{\lambda_3}), \dots, f(G_{\lambda_n})\}$, say. Therefore, $A \subseteq \cup_{i=1}^{n} f(G_{\lambda_i}) \Rightarrow A \subseteq f(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f^{-1}(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NBOC B of $f^{-1}(A)$ has a finite NBOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \dots, G_{\lambda_n}\}$. Therefore $f^{-1}(A)$ is neutrosophic b-compact in X. Hence proved.

3.37. Proposition: Let f be a neutrosophic b^*-open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic b-compact in then (X, τ) is also neutrosophic b-compact.

Proof: Immediate from the prop. 3.36 as f is onto.

3.38. Proposition: Let f be a neutrosophic b^*-open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic countably b-compact (neutrosophic b-Lindelöf) then (X, τ) is also neutrosophic countably b-compact (neutrosophic b-Lindelöf).

Proof: Obvious.

3.39. Definition: Let f be a neutrosophic function from an NTS (X, τ) to the NTS (Y, σ). Then f is called a neutrosophic b^*-continuous function if $f^{-1}(G)$ is an NBO set in X for every NBO set G in Y.

3.40. Proposition: Let f be a neutrosophic b^*-continuous function from an NTS (X, τ) to the NTS (Y, σ). If A is neutrosophic b-compact in X then $f(A)$ is also neutrosophic b-compact in Y.

Proof: Let $B = \{G_\lambda: \lambda \in \Delta\}$ be an NBOC of $f(A)$. Then $f(A) \subseteq \cup_{\lambda\in\Delta} G_\lambda \Rightarrow A \subseteq f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda)$. Since G_λ is σ-NBO set in Y, so $f^{-1}(G_\lambda)$ is τ-NBO set in X as f is neutrosophic b^*-continuous function. Therefore $C = \{f^{-1}(G_\lambda): G_\lambda \in B\}$ is an NBOC of A. Since A is neutrosophic b-compact in X, so C has a finite NBOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \dots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow A \subseteq f^{-1}(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NBOC B of $f(A)$ has a finite NBOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \dots, G_{\lambda_n}\}$.. Therefore $f(A)$ is neutrosophic b-compact.

3.41. Proposition: Let f be a neutrosophic b^*-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic b-compact then (Y, σ) is also neutrosophic b-compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $B = \{G_\lambda: \lambda \in \Delta\}$ be an NBOC of Y. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow \cup_{\lambda\in\Delta} G_\lambda = f(\tilde{X}) \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = \tilde{X} \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is σ-NBO set in Y, so $f^{-1}(G_\lambda)$ is τ-NBO set in X as f is neutrosophic b^*-continuous function. Therefore, $C = \{f^{-1}(G_\lambda): G_\lambda \in B\}$ is an NBOC of X . Since X is neutrosophic b -compact, so C has a finite NBOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \dots, f^{-1}(G_{\lambda_n})\}$, say. Therefore, $\tilde{X} = \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow \tilde{X} = f^{-1}(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f(\tilde{X}) = \cup_{i=1}^{n} G_{\lambda_i} \Rightarrow \tilde{Y} = \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NBOC B of Y has a finite NBOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \dots, G_{\lambda_n}\}$. Therefore Y is neutrosophic b-compact.

3.42. Proposition: Let f be a neutrosophic b^*-continuous function from an NTS (X,τ) onto the NTS (Y,σ). If (X,τ) is neutrosophic countably b-compact (resp. neutrosophic b-Lindelöf) then (Y,σ) is also neutrosophic countably b-compact (resp. neutrosophic b-Lindelöf).
Proof: Obvious.

4. Neutrosophic local b-compactness

4.1. Definition: An NTS (X,τ) is said to be a neutrosophic locally b-compact space iff for every NP $x_{\alpha,\beta,\gamma}$ in X, there exists an NBO set G in X such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic b-compact in X.

4.2. Proposition: Every neutrosophic b-compact space is a neutrosophic locally b-compact space.
Proof: Let (X,τ) be a neutrosophic b-compact space and let $x_{\alpha,\beta,\gamma}$ be an NP in X. Since X is neutrosophic b-compact and since $\tilde{X}$ is an NBO set such that $x_{\alpha,\beta,\gamma} \in \tilde{X}$, so, (X,τ) is a neutrosophic locally b-compact space.

4.3. Proposition: Let f be a neutrosophic b^*-open and b^*-continuous function from an NTS space (X,τ) to the NTS (Y,τ). If (Y,τ) neutrosophic locally b-compact then (X,τ) is also a neutrosophic locally b-compact space.
Proof: Let $x_{\alpha,\beta,\gamma}$ be any NP in X. Also let $y_{p,q,r}$ be the NP in Y such that $f(x_{\alpha,\beta,\gamma}) = y_{p,q,r}$. Since $y_{p,q,r} \in Y$ and Y neutrosophic locally b-compact, so there exists a σ-NBO set G such that $y_{p,q,r} \in G$ and G is neutrosophic b-compact in Y. Now $y_{p,q,r} \in G \Rightarrow f(x_{\alpha,\beta,\gamma}) \in G \Rightarrow x_{\alpha,\beta,\gamma} \in f^{-1}(G)$. Since f is neutrosophic b^*-open and G is neutrosophic b-compact in Y, so by the prop. 3.36, $f^{-1}(G)$ is neutrosophic b-compact in X. Again since f is a neutrosophic b^*-continuous function, so $f^{-1}(G)$ is a τ-NBO set. Thus for any any NP $x_{\alpha,\beta,\gamma}$ in X, there exists a τ-NBO set $f^{-1}(G)$ such that $x_{\alpha,\beta,\gamma} \in f^{-1}(G)$ and $f^{-1}(G)$ is neutrosophic b-compact in X. Therefore (X,τ) is neutrosophic locally b-compact space.

4.4. Proposition: Let f be a neutrosophic b^*-open and b^*-continuous function from an NTS space (X,τ) onto the NTS (Y,τ). If (X,τ) neutrosophic locally b-compact then (Y,σ) is also a neutrosophic locally b-compact space.
Proof: Let $y_{p,q,r}$ be any NP in Y. Since f is onto, so there exists an NP $x_{\alpha,\beta,\gamma}$ in X such that $f(x_{\alpha,\beta,\gamma}) = y_{p,q,r}$. Since $x_{\alpha,\beta,\gamma} \in X$ and X neutrosophic locally b-compact, so there exists a τ-NBO set G such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic b-compact in X. Now $x_{\alpha,\beta,\gamma} \in G \Rightarrow f(x_{\alpha,\beta,\gamma}) \in f(G) \Rightarrow y_{p,q,r} \in f(G)$. Since f is neutrosophic b^*-continuous and G is neutrosophic b-compact in X, so by 3.40, $f(G)$ is neutrosophic b-compact in Y. Again since f is a neutrosophic b^*-open function, so $f(G)$ is a σ-NBO set. Thus for any any NP $y_{p,q,r}$ in Y, there exists a σ-NBO set $f(G)$ such that $y_{p,q,r} \in f(G)$ and $f(G)$ is neutrosophic b-compact in Y. Therefore (Y,σ) is neutrosophic locally b-compact space.

5. Covering properties via neutrosophic b-base

5.1. Definition : Let (X,τ) be an NTS and $NBO(X)$ be the collection of all NBO sets in X. A subcollection B of $NBO(X)$ is called a neutrosophic b-base (Nb-base, for short) for X iff for each $A \in NBO(X)$, there exists a subcollection $\{A_i : i \in \Delta\}$ of B such that $A = \cup \{A_i : i \in \Delta\}$, where Δ is an index set.
A subcollection B_* of $NBO(X)$ is called a neutrosophic b-subbase (Nb-subbase, for short) for X iff the finite intersection of members of B_* forms a neutrosophic b-base for X.

5.2. Definition: An NTS (X,τ) is said to be a neutrosophic b-second countable or neutrosophic $b-C_{II}$ space iff X has a countable neutrosophic b-base.

5.3. Proposition: Let B be an Nb-base for an NTS (X,τ). Then X is neutrosophic b-compact iff every NBOC of X by the members of B has a finite NBOSC.
Proof: Necessary Part: Obvious.
Sufficient Part : Let $B = \{B_\alpha : \alpha \in \Delta\}$ be the Nb-base. Also let $C = \{G_\lambda : \lambda \in \Delta\}$ be an NBOC of X. Then each member G_λ of C is the union of some members of B and the totality of such members of B is

evidently an NBOC of X. By the hypothesis, this collection of members of B has a finite NBOSC $D = \{B_{\alpha_j}: j = 1,2,3,\cdots,n\}$, say. Clearly for each B_{α_j} in D, there is a G_{λ_j} in C such that $B_{\alpha_j} \subseteq G_{\lambda_j}$. Therefore the finite subcollection $\{G_{\lambda_j}: j = 1,2,3,\cdots,n\}$ of C is an NBOC of X, i.e., the NBOC C of X has a finite NBOSC. Therefore X is neutrosophic b-compact.

5.4. Proposition: Let (X,τ) be a neutrosophic countably b-compact space. If X is neutrosophic $b - C_{II}$ then X neutrosophic b-compact.

Proof: Let $D = \{A_i: i \in \Delta\}$ be any NBOC of X. Since X is neutrosophic $b - C_{II}$, so there exists a countable Nb-base $B = \{B_n: n = 1,2,3,\cdots\}$ for X. Then each $A_i \in D$ can be expressed as a countable union of members of B, i.e., for each $A_i \in D$, we have $A_i = \cup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k} \in B$ and i_0 may be infinity. Clearly $B_0 = \{B_{n_k}\}$ is an NBOC of X. Also B_0 is countable as $B_0 \subseteq B$. Therefore, B_0 is a countable NBOC of X. Since X is countably b-compact, so B_0 has a finite NBOSC B', say. Since by construction, each member of B' is contained in one member A_i of D, so these A_i's form a finite NBOC of X. Thus the NBOC D of X has a finite NBOSC. Therefore X is neutrosophic b-compact. Hence Proved.

5.5. Remark: From the propositions 3.3 and 5.4, it is clear that if an NTS (X,τ) is neutrosophic $b - C_{II}$ then neutrosophic b-compactness and neutrosophic countably b-compactness are equivalent.

5.6. Proposition: If an NTS (X,τ) is neutrosophic $b - C_{II}$ then it is neutrosophic b-Lindelöf.

Proof: Let $D = \{A_i: i \in \Delta\}$ be any NBOC of X. Since X is neutrosophic $b - C_{II}$, so there exists a countable Nb-base $B = \{B_n: n = 1,2,3,\cdots\}$ for X. Then each $A_i \in D$ can be expressed as the countable union of members of B, i.e., for each $A_i \in D$, we have $A_i = \cup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k} \in B$ and i_0 may be infinity. Let $B_0 = \{B_{n_k}\}$. Then B_0 is an NBOC of X. Also B_0 is countable as $B_0 \subseteq B$. Therefore, B_0 is a countable NBOC of X. By construction, each member of B_0 is contained in one member A_i of D. So, these A_i's of D form a countable NBOSC of X. Thus the NBOC D of X has a countable NBOSC. Therefore X is neutrosophic b- Lindelöf.

5.7. Proposition: Let β be an Nb-subbase of an NTS (X,τ). Then X is neutrosophic b-compact iff for every collection of NBC sets taken from β^c having the FIP, there is a non-empty intersection.

Proof: Necessary part: Immediate from the prop. 3.23.

Sufficient Part: On the contrary, let us suppose that X is not b-compact. Then by the prop. 3.23, there exists a collection $C = \{G_i: i \in I\}$ of NBC sets of X having FIP such that $\cap_{i\in I} G_i = \tilde{\emptyset}$. The collection $F = \{C\}$ of all such collections C can be arranged in an order by using the classical inclusion($\subseteq$) and therefore, the collection F will have an upper bound. By Zorn's lemma, there will be a maximal collection of all these collections C. Let $P = \{K_j: j \in J\}$ be the maximal collection. Clearly, this collection P has the following properties:

(i) $\tilde{\emptyset} \notin P$ (ii) $A \in P, A \subseteq B \Rightarrow B \in P$ (iii) $A, B \in P \Rightarrow A \cap B \in P$ (iv) $\cap(P \cap \beta^c) = \tilde{\emptyset}$.

Clearly the property (iv) creates a contradiction to the hypothesis. Therefore X is neutrosophic b-compact.

Hence proved.

6. Conclusions

In this article, we have defined neutrosophic b-open cover with the help of neutrosophic b-open sets and then we have defined neutrosophic b-compact, neutrosophic countably b-compact, neutrosophic b-Lindelöf spaces and investigated various covering properties. We have proved that every neutrosophic b-compact space is a neutrosophic compact space but the converse is not true. We have shown that if f is a neutrosophic continuous or a b-continuous function from a neutrosophic b-compact (resp. countably b-compact, b-Lindelöf) space (X,τ) onto a neutrosophic topological space (Y,σ) then (Y,σ) is a neutrosophic compact (resp. countably compact, Lindelöf) space. In 3.41 (resp. 3.42), we have established that neutrosophic b-compactness (resp. countably b-compactness, b-Lindelöfness) is preserved under a neutrosophic b^*-continuous function. We have then defined and studied a few properties of neutrosophic local b-compactness. At last, in section 5, we have defined neutrosophic b-base, b-subbase, neutrosophic b-C_{II} and investigated some properties. We have set up that if a neutrosophic topological space is neutrosophic b-C_{II} then neutrosophic b-compactness and neutrosophic countably b-compactness are equivalent. In 5.7, we have stated and proved "Alexander subbase lemma" in case of a neutrosophic b-compact space. Hope that the findings in this article will assist the research fraternity to move forward for the development of different aspects of neutrosophic topology.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.
2. Arar, M. (2020). About Neutrosophic Countably Compactness. Neutrosophic Sets and Systems, 36(1), 246-255.
3. Arokiarani, I. ; Dhavaseelan, R. ; Jafari, S. ; Parimala, M. (2017). On Some New Notions and Functions in Neutrosophic Topological Space. Neutrosophic Sets and Systems, 16, 16-19.
4. Abdel-Basset, M. ; Gamal, A. ; Chakrabortty, R.K. ; Ryan, M.J. (2020). Evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach: A case study. International Journal of Hydrogen Energy, DOI : 10.1016/j.ijhydene.2020.10.232.
5. Abdel-Basset, M. ; Mohamed, R. ; Smarandache, F. ; Elhoseny, M. (2021). A New Decision-Making Model Based on Plithogenic Set for Supplier Selection. Computers, Materials & Continua, 66(3), 2751-2769.
6. Bageerathi, K. ; Puvaneswary, P. (2019). Neutrosophic Feebly Connectedness and Compactness. IOSR Journal of Polymer and Textile Engineering, 6(3), 7-13.
7. Chang, C.L. (1968). Fuzzy Topological Space. J. of Mathematical Analysis and Application, 24, 182-193.
8. Coker, D. (1997). An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, 88, 81-89.
9. Deli, I. ; Broumi, S. (2015). Neutrosophic soft relations and some properties. Ann. Fuzzy Math. Inform., 9, 169-182.
10. Dey, S. ; Ray, G.C. (2022). Covering properties in Neutrosophic Topological Spaces. Neutrosophic Sets and Systems, 51, 525-537.
11. Dey, S ; Ray, G.C. (2023). Separation Axioms in Neutrosophic Topological Spaces. Neutrosophic Systems with Applications, 2, 38-54. DOI: https://doi.org/10.5281/zenodo.8195851
12. Dey, S ; Ray, G.C. (2023). Properties of Redefined Neutrosophic Composite Relation. Neutrosophic Systems with Applications, 7, 1-12. DOI: https://doi.org/10.5281/zenodo.8207037
13. Dey, S ; Ray, G.C. On neutrosophic b-open sets and b-separation axioms (Under review).
14. Ebenanjar, P.E. ; Immaculate, H.J. ; Wilfred, C B. (2018). On Neutrosophic $ b $-open sets in Neutrosophic topological space. J.Phys.: Conf. Ser.1139 012062, International Conference on Applied and Computational Mathematics 2018, Tamilnadu, India, 10 August, IOP Publishing, doi:10.1088/1742-6596/1139/1/012062.
15. Guo, Y. ; Cheng, H.D. (2009). New neutrosophic approach to image segmentation. Pattern Recognition, 42, 587-595.
16. Karatas, S. ; Kuru, C. (2016). Neutrosophic Topology. Neutrosophic Sets and Systems, 13(1), 90-95.
17. Ozturk, T.Y. ; Benek, A. ; Ozkan, A. (2020). Neutrosophic soft compact spaces. Afrika Matematika, doi.org/10.1007/s13370-020-00827-9.

18. Ray, G.C. ; Dey, S. (2021). Neutrosophic point and its neighbourhood structure. Neutrosophic Sets and Systems, 43, 156-168.
19. Ray, G.C. ; Dey, S. (2021). Relation of Quasi-coincidence for Neutrosophic Sets. Neutrosophic Sets and Systems, 46, 402-415.
20. Smarandache, F. (1999). A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM.
21. Smarandache, F. (2002). Neutrosophy and neutrosophic logic. First international conference on neutrosophy, neutrosophic logic, set, probability, and statistics, University of New Mexico, Gallup, NM 87301, USA .
22. Smarandache, F. (2005). Neutrosophic set - a generalization of the intuitionistic fuzzy set. International Journal of Pure and Applied Mathematics, 24(3), 287-297.
23. Salama, A.A. ; Alblowi, S. (2012). Neutrosophic set and Neutrosophic Topological Spaces. IOSR Journal of Mathematics, 3(4), 31-35.
24. Salama, A.A. ; Alagamy, H. (2013). Neutrosophic filters. Int. J. of Comp. Sc. Eng. and I.T. Research, 3(1), 307-312.
25. Salama, A. A. ; Smarandache, F. ; Kroumov, V. (2014). Closed sets and Neutrosophic Continuous Functions. Neutrosophic Sets and Systems, 4, 4-8.
26. Salma, A.A. ; Smarandache, F. (2015). Neutrosophic Set Theory. The Educational Publisher 415 Columbus, Ohio.
27. Senyurt, S. ; Kaya, G. (2017). On Neutrosophic Continuity. Ordu University Journal of Science and Technology, 7(2), 330-339.
28. Wang, H. ; Smarandache, F. ; Zhang, Y.Q. ; Sunderraman, R. (2010). Single valued neutrosophic sets. Multispace Multistruct, 4, 410-413.
29. Ye, S. ; Fu, J. ; Ye, J. (2014). Medical diagnosis using distance based similarity measures of single valued neutrosophic multisets. Neutrosophic Sets and Systems, 7, 47-52.
30. Zadeh, L.A. (1965). Fuzzy sets. Inform. and Control, 8, 338-353.

Received: May 22, 2023. Accepted: Aug 25, 2023

NSWA Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.65

Toward Energy Transformation: Intelligent Decision-Making Model Based on Uncertainty Neutrosophic Theory

Mona Mohamed [1,*] and Nissreen El-Saber [2]

[1] Higher Technological Institute, 10th of Ramadan City 44629, Egypt; mona.fouad@hti.edu.eg.
[2] Faculty of Computers and Informatics, Zagazig University, Zagazig 44519, Egypt; naelsaber@fci.zu.edu.eg.

* Correspondence: mona.fouad@hti.edu.eg.

Abstract: There has been an increasing tendency for the generation of energy from diverse renewable resources because of the application of contemporary pollution mitigation and justification regulations. Precisely a consequence, choosing the best renewable energy source might be considered a challenging issue given the complexity of the future conditions in any society. Environmental, economic, social, and technical aspects are merely some of the factors that are taken into consideration while evaluating renewable energy sources (RnESs). The suitable RES selection problem, which relies on ambiguous and imprecise data, is also influenced by a variety of circumstances. Hence, this study constructs multi-stages intelligent decision-making model (MsIDMM) based on multi-criteria decision making (MCDM) with support with neutrosophic theory especially, interval valued neutrosophic sets to rank the sources of renewable energy. Ultimately, combinative distance-based assessment (CODAS) method under interval-valued neutrosophic sets is used to rank the sources of renewable energy.

Keywords: Renewable Energy; CODAS; MCDM; Interval Valued Neutrosophic Set; Sustainability.

1. Introduction

Given the ongoing rise in energy demand and the potential depletion of fossil fuels, academics and energy producers alike should focus more on the sustainability of renewable energy sources (RnESs) [1]. Arguably based on [2] the most serious issues the world is currently facing are the enormous and rapid growth of the population reaching 9 Billion by 2050, innovation, growth, and cultural advancement, which is related to the enormous demand and excessive consumption patterns of energy, water, and food resources compared to the generation of energy and the limited natural land, water, materials, and fuels resources. Therefore, changes in energy usage have a substantial impact on economic activity and the determination of income [3]. Hence sustainable Energy (SusE) is crucial for a nation's economic and social development as well as for improving people's quality of life [4].

In order to ensure that everyone has access to cheap, dependable, sustainable, and contemporary energy, one of the global goals of the 2030 Agenda for Sustainable Development (SusD) is to promote SE [4]. The goals of SusD are threatened by conventional energy-generating techniques such as those that rely on fossil fuels. According to [5] utilizing fossil fuels not only harms the environment and produces harmful pollutants, not only damage the environment and emit hazardous gases, but also their energy sources are not sustainable. From the researchers' point of view in [6], the problem raised in [5] can be controlled by offering substitute and cleaner sources, and the nation's level of living may raise. The solution of [6] is represented in RnESs which play an essential role in guaranteeing the cleanest possible energy that is sustainable. In a similar vein, [7] emphasized that to fulfill the energy demand, combat climate change, and fulfill the need for clean and sustainable development, the continued growth of RnESs has become a crucial strategy.

Making the best choice for a renewable energy source would benefit sustainability in other aspects as social, and environmental in addition to the economic one. Contrariwise [8], the wrong choice of RnES might have negative effects on aspects of sustainability as the environment and the economy as exhibited in Figure 1.

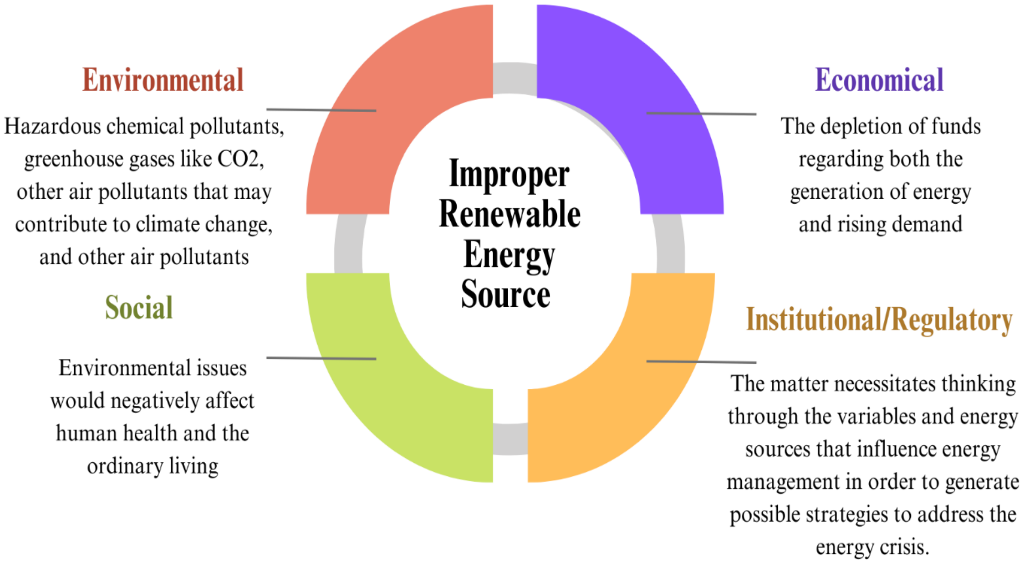


Figure 1. Consequences of adopting an undesirable renewable energy source.

Wherefore, selecting optimal or suitable RnES is a critical process. In order to reduce environmental pollution, usage of traditional resources, and improve economic growth, [6] affirmed that the selection process need to be strategically chosen.

For the purpose of planning for sustainable energy, [9] indicated that there are a number of aspects including environmental, social, economic, technological, and institutional considerations, should be utilized as benchmarks. Consequently [10] contributed multi-criteria decision-making (MCDM) methods in choosing an optimal RnES candidate. Nevertheless, there is a further perspective on the application of these methods. For instance, [6] MCDM methods are insufficient for handling the ambiguous information that frequently arises during energy planning procedures. So, scholars as Zadeh [11] resorted to using Fuzzy Sets (FSs). Its adaptability and efficiency in resolving circumstances where the information at hand is ambiguous or insufficient are remarkable. Ditto the generalization of (FSs) is Intuitionistic fuzzy sets (IFSs) which take into consideration measurements of truth and non-truth otherwise FSs which concerns truth.

Nonetheless, herein the study is followed suit Smarandache [12] through volunteering neutrosophic theory. This is a result of having a significant aptitude for developing approaches using vague and erratic information. The neutrosophic theory is distinguished by three separate membership functions that represent the roles of truth, indeterminacy, and falsity.

Interval valued neutrosophic theory is inspired by neutrosophic theory. Therefore, in this study SVNS combined with MCDM methods, especially CODAS method for handling the multi-criteria RnESs to choose optimal one.

This study is organized into a set of sections; each one plays a certain role. Whereas the motives on where the study was based are illustrated in Section 2. Through prior studies we aggregated essential sources of renewable energy in Section 3. These sources need to be analyzed and evaluated, hence we constructed hybrid model for evaluating these sources in Section 4. After that we are applying this model to verify it through evaluating 6 sources based on 22 criteria. Finally, we exhibit synopsis for the study.

2. Motivations of the Study

This section represents motivations for conducting this study. These motivations are illustrated through set of aspects as following:

- **Societal Aspect:** According to [13], utilization of fossil fuels or improper RnESs leads to environment problems; i.e. global warming is caused by greenhouse gas (GHG) emissions such as carbon dioxide, methane oxide, and nitrous oxide in the atmosphere. Which in turn affects human life and threatens health.
- **Technical/Practical Aspect:** Selecting suitable and optimal RnES is vital to lessen the hazards associated with selecting renewable energy incorrectly, which jeopardizes sustainability. Therefore, it is important to utilize flexible and efficient techniques which can analyze various alternatives of RnESs based on a set of criteria. Herein, the study uses MCDM methods to rank the sources of renewable energy with support of neutrosophic theory especially Interval valued neutrosophic to strength CODAS of MCDM to generate robust hybrid intelligent model.
- **Experimental Aspect:** We are applying constructed hybrid intelligent framework for ranking six renewable energy sources as alternatives based on 22 criteria. Herein, the utilized alternatives **(**A_n) are solar energy, wind energy, hydro energy, biomass energy, geothermal energy, and wave energy.

3. Essential Principles of Renewable Energy Sources

This section exhibits the different RnESs based on prior studies which related to our interested scope. For instance, [14]-[15] exhibited set of RnESs where aggregated in Figure 2.

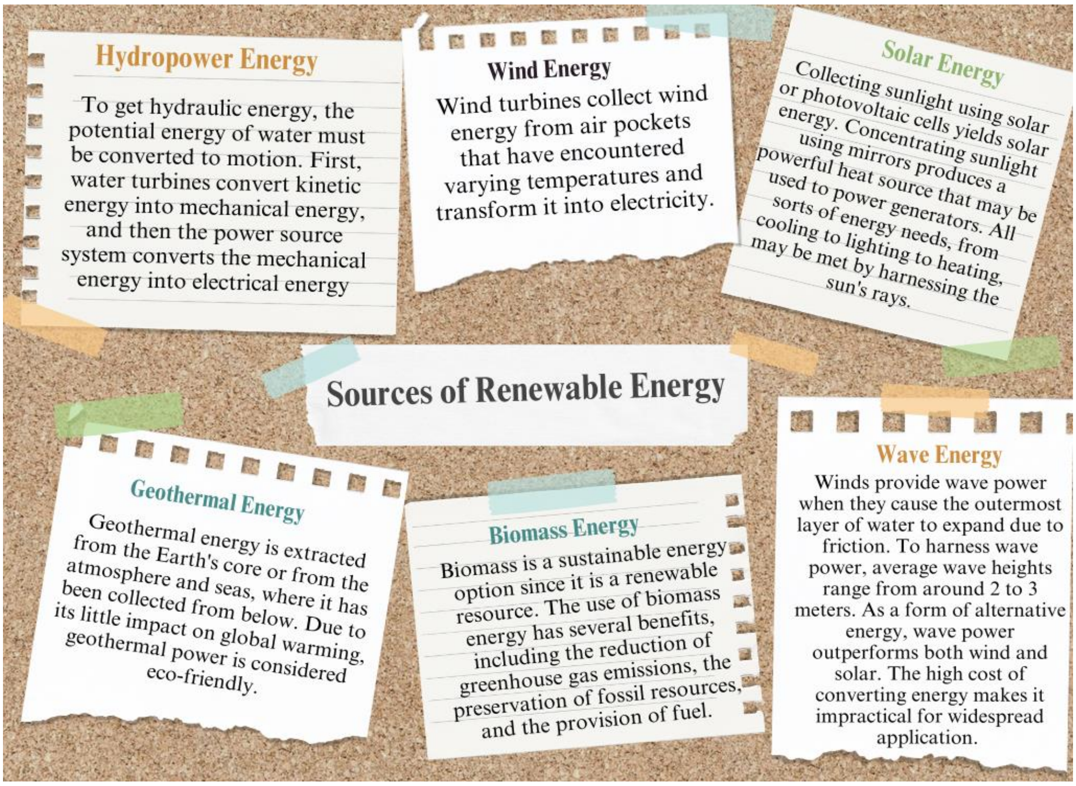


Figure 2. Various sources of renewable energy.

4. Multi-stages Intelligent Decision-Making Model (MsIDMM)

In our constructed model, we use two elements to determine which options are preferable. Both the Euclidean and Taxicab distances of options to the cost-ideal are used as indicators of attractiveness, the greater the distance means the more desirable the option. The CODAS method is integrated with the neutrosophic method to deal with vague data. We evaluate the criteria and alternatives according to [16]- [21].

In this study, the first stage included determining 22 criteria which contribute to selection process for RnESs. After that the second stage is evaluating the determined criteria by formed expert panel. Third stage represented in analyzing process for expert's evaluations through MsIDMM based on neutrosophic theory combined with MCDM methods. The result of MsIDMM is ranking and selecting optimal RnES. Figure 3 summarizes the stages of model.

Step 4.1 *Determine The Criteria of Renewable Energy*

In the first stage, the process aim is established, and the relevant criteria for assessing the options are selected.

Step 4.2 Formulate the Matrix Between Criteria and Resources of Renewable Energy.

The matrix is built by the criteria $i = 1,2,3 \dots . m; j = 1,2,3 \dots n$, and the element of matrix is k_{ij}.

Step 4.3 Normalize the Decision Matrix.

$$r_{ij} = \begin{cases} \frac{k_{ij}}{\max_i k_{ij}} \\ \frac{\min_i k_{ij}}{k_{ij}} \end{cases} \quad (1)$$

Step 4.4 Determine the Weighted Normalized Matrix.

$$q_{ij} = e_j r_{ij} \quad (2)$$

Where e_j refers to the weights of criteria.

Step 4.5 Compute the Point of Cost Ideal Solution.

$$cq_j = \min_i q_{ij} \quad (3)$$

Step 4.6 Compute the Taxicab and Euclidean Distances.

$$A_i = \sum_{j=1}^{m} \left| q_{ij} - cq_j \right| \quad (4)$$

$$D_i = \sqrt{\sum_{j=1}^{m} \left(q_{ij} - cq_j \right)^2} \quad (5)$$

Step 4.7 Compute the Matrix of Comparative Assessment.

$$Ass_{iy} = \left(D_i - D_y \right) + \left(\alpha \left(D_i - D_y \right) \times \left(A_i - A_y \right) \right) \quad (6)$$

Where $y = 1,2,3, \dots n$, and α refers to the function of threshold.

$$\alpha(k) = \begin{cases} 1, if \ |k| \geq \beta \\ 0, if \ |k| < \beta \end{cases} \quad (7)$$

Where $\beta \ between \ 0.01 \ and \ 0.05$ refers to the expert's threshold.

Step 4.8 Compute the Evaluation Score.

$$U_i = \sum_{y=1}^{n} Ass_{iy} \quad (8)$$

Step 4.9 Rank the Sources of Renewable Energy

The renewable energy resources are ranked according to the Maximum value of U_i

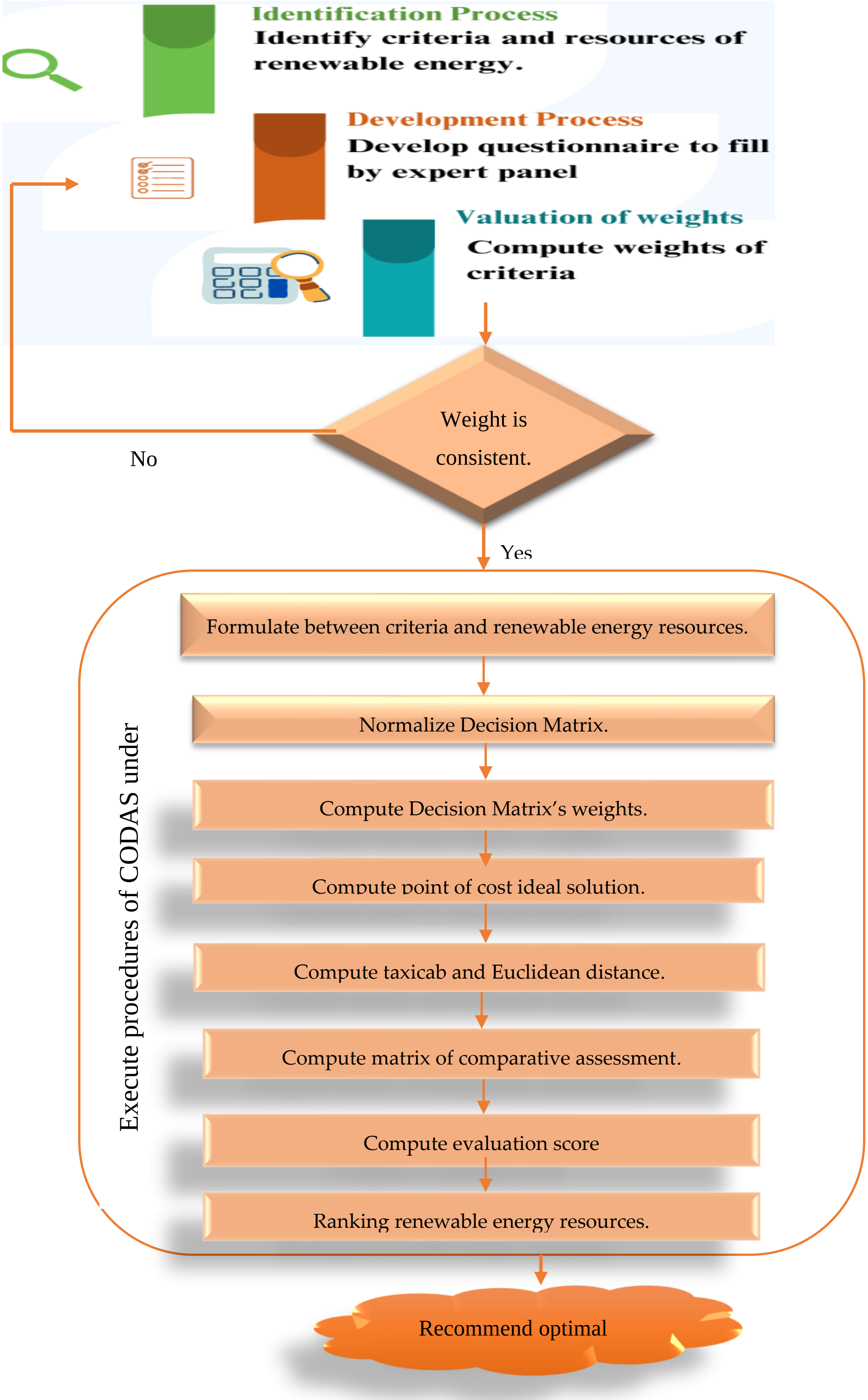


Figure 3. Various stages of Multi-stages Intelligent Decision-Making Model.

5. Validation of Renewable Energy Resources based on MsIDMM

Herein, the study validates the constructed MsIDMM for assessing determined RnESs. It computes the weights of criteria for determined RnESs highlighted by earlier studies as [14] ,[5]. Whereas there are 22 criteria. There are six renewable energy sources like solar, wind, hydro, biomass, geothermal, and wave energy. Figure 4 reveals the utilized criteria and alternatives.

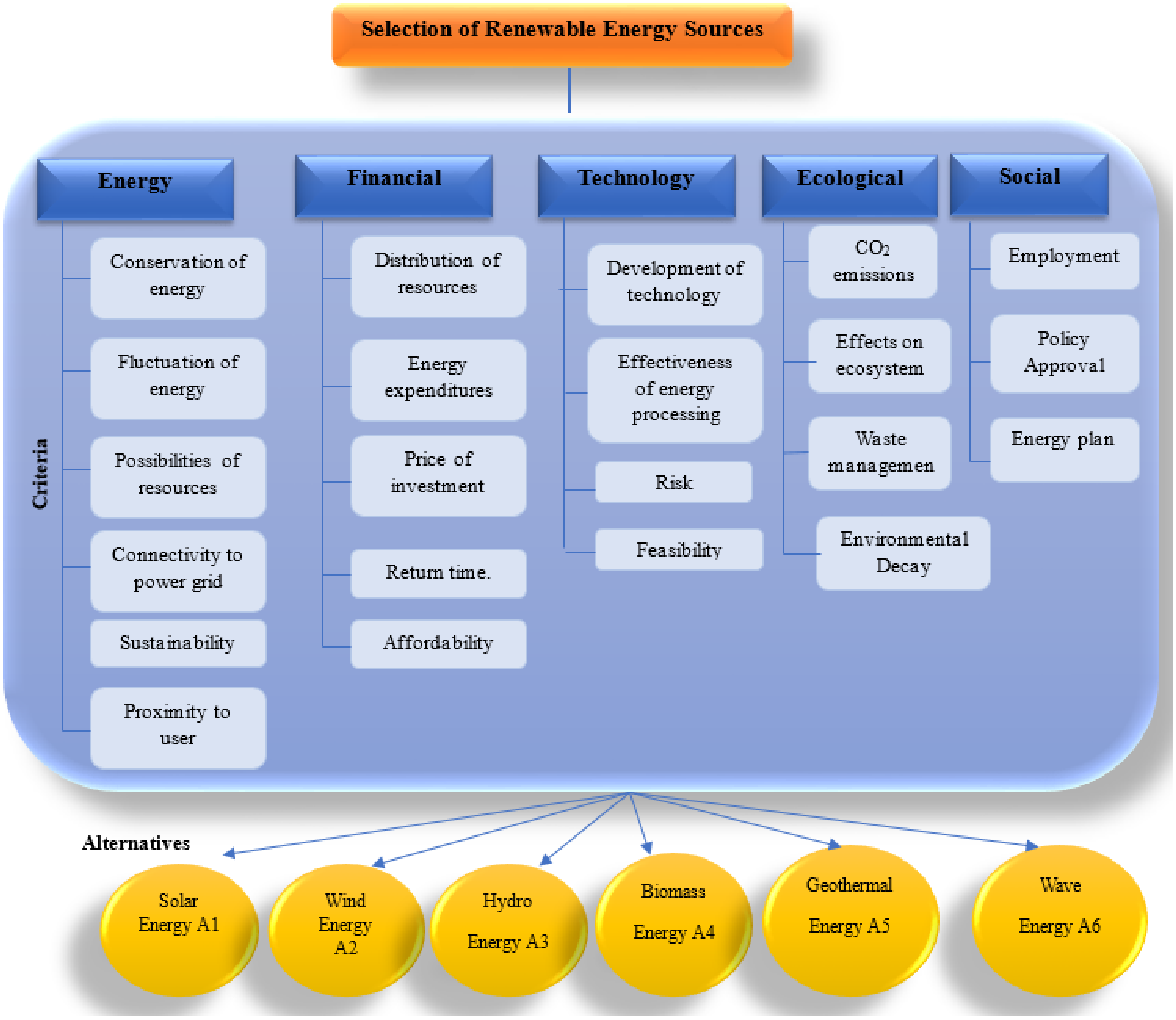


Figure 4. Selection alternatives of renewable energy sources based on criteria.

Decision makers (DMs) and experts evaluate the renewable energy criteria and sources to create decision matrix between criteria and alternatives by using interval valued neutrosophic numbers. Then the decision matrix is normalized as shown in Table 1 by using Eq. (1). After that the weights of renewable energy criteria are computed. Then the weights of the criteria are multiplied by the normalization matrix using Eq. (3).

Then the point of cost for the ideal solution is computed using Eq. (3). Then the taxicab is computed as shown in Table 2 and Euclidean distances using Eqs. (4-5) as shown in Table 3. After that the matrix of comparative assessment is computed using Eqs. (6-7). Then the evaluation score is computed using Eq. (8) as shown in Figure 5. The second renewable energy source is the best and the third renewable energy source is the worst.

Table 1. The normalization matrix between renewable energy criteria and sources.

	RNC_1	RNC_2	RNC_3	RNC_4	RNC_5	RNC_6	RNC_7	RNC_8	RNC_9	RNC_{10}	RNC_{11}
RNA_1	0.623306	0.422764	1	0.467681	0.369408	0.566038	0.659218	0.639566	1	0.125102	1
RNA_2	0.642005	1	0.590862	0.721166	0.532468	0.72119	0.414763	0.996201	0.203188	0.186068	0.654023
RNA_3	0.642005	0.639566	0.383178	0.494297	0.369408	0.192453	0.996201	0.996201	0.548755	0.517518	0.272299
RNA_4	0.642005	0.97561	0.383178	0.324461	0.532468	0.34688	0.987448	0.907692	0.550847	0.26886	0.793103
RNA_5	0.642005	0.639566	0.383178	0.878327	0.950938	1	0.311798	1	0.501931	1	0.309195
RNA_6	1	0.639566	0.590862	1	1	0.342525	1	0.395973	0.147895	0.175193	0.413793
	RNC_{12}	RNC_{13}	RNC_{14}	RNC_{15}	$RNsub_{16}$	$RNsub_{17}$	RNC_{18}	RNC_{19}	RNC_{20}	RNC_{21}	RNC_{22}
RNA_1	1	0.669412	0.42907	0.51236	0.639326	1	1	0.99803	1	0.639295	0.246002
RNA_2	0.800203	0.302235	0.146512	0.26618	0.265169	0.713561	0.339398	0.516524	0.338109	1	0.245067
RNA_3	0.240264	0.305447	0.298721	0.26618	0.265169	0.467681	0.339398	0.52309	0.338109	1	0.133956
RNA_4	0.84787	1	1	1	1	0.340938	0.338109	0.516524	0.338109	0.885908	1
RNA_5	0.272819	0.536471	0.277907	0.265169	0.640225	0.299113	0.528653	1	0.339398	0.642005	0.265836
RNA_6	0.462475	0.654118	0.42907	0.51236	0.414607	0.300253	0.654585	0.807617	0.815186	0.642005	0.369678

Table 2. The taxicab distance from cost ideal solution.

	RNC_1	RNC_2	RNC_3	RNC_4	RNC_5	RNC_6	RNC_7	RNC_8	RNC_9	RNC_{10}	RNC_{11}
RNA_1	0	0	0.053351	0.004747	0	0.013086	0.011171	0.008073	0.009949	0	0.056863
RNA_2	0.000386	0.008088	0.017963	0.013148	0.003749	0.018521	0.003311	0.019893	0.000646	0.005366	0.029828
RNA_3	0.000386	0.003038	0	0.005629	0	0	0.022007	0.019893	0.00468	0.034541	0
RNA_4	0.000386	0.007746	0	0	0.003749	0.005409	0.021725	0.01696	0.004705	0.012654	0.040696
RNA_5	0.000386	0.003038	0	0.018356	0.013371	0.028287	0	0.020019	0.004134	0.077009	0.002883
RNA_6	0.007782	0.003038	0.017963	0.022389	0.014499	0.005257	0.022129	0	0	0.004409	0.011056
	RNC_{12}	RNC_{13}	RNC_{14}	RNC_{15}	RNC_{16}	RNC_{17}	RNC_{18}	RNC_{19}	RNC_{20}	RNC_{21}	RNC_{22}
RNA_1	0.067282	0.018765	0.009365	0.010124	0.019122	0.049669	0.041495	0.019721	0.041495	0	0.002384
RNA_2	0.049588	0	0	4.14E-05	0	0.02937	8.08E-05	0	0	0.007643	0.002364
RNA_3	0	0.000164	0.005045	4.14E-05	0	0.011946	8.08E-05	0.000269	0	0.007643	0
RNA_4	0.053809	0.03566	0.028287	0.030096	0.037554	0.002964	0	0	0	0.005225	0.018427
RNA_5	0.002883	0.011971	0.004355	0	0.019167	0	0.011946	0.019801	8.08E-05	5.74E-05	0.002806
RNA_6	0.019679	0.017983	0.009365	0.010124	0.007637	8.08E-05	0.01984	0.011922	0.029909	5.74E-05	0.005016

Table 3. The Euclidean distance from cost ideal solution.

	RNC_1	RNC_2	RNC_3	RNC_4	RNC_5	RNC_6	RNC_7	RNC_8	RNC_9	RNC_{10}	RNC_{11}
RNA_1	0	0	0.053351	0.004747	0	0.013086	0.011171	0.008073	0.009949	0	0.056863
RNA_2	0.000386	0.008088	0.017963	0.013148	0.003749	0.018521	0.003311	0.019893	0.000646	0.005366	0.029828
RNA_3	0.000386	0.003038	0	0.005629	0	0	0.022007	0.019893	0.00468	0.034541	0
RNA_4	0.000386	0.007746	0	0	0.003749	0.005409	0.021725	0.01696	0.004705	0.012654	0.040696
RNA_5	0.000386	0.003038	0	0.018356	0.013371	0.028287	0	0.020019	0.004134	0.077009	0.002883
RNA_6	0.007782	0.003038	0.017963	0.022389	0.014499	0.005257	0.022129	0	0	0.004409	0.011056
	RNC_{12}	RNC_{13}	RNC_{14}	RNC_{15}	RNC_{16}	RNC_{17}	RNC_{18}	RNC_{19}	RNC_{20}	RNC_{21}	RNC_{22}
RNA_1	0.067282	0.018765	0.009365	0.010124	0.019122	0.049669	0.041495	0.019721	0.041495	0	0.002384
RNA_2	0.049588	0	0	4.14E-05	0	0.02937	8.08E-05	0	0	0.007643	0.002364
RNA_3	0	0.000164	0.005045	4.14E-05	0	0.011946	8.08E-05	0.000269	0	0.007643	0
RNA_4	0.053809	0.03566	0.028287	0.030096	0.037554	0.002964	0	0	0	0.005225	0.018427
RNA_5	0.002883	0.011971	0.004355	0	0.019167	0	0.011946	0.019801	8.08E-05	5.74E-05	0.002806
RNA_6	0.019679	0.017983	0.009365	0.010124	0.007637	8.08E-05	0.01984	0.011922	0.029909	5.74E-05	0.005016

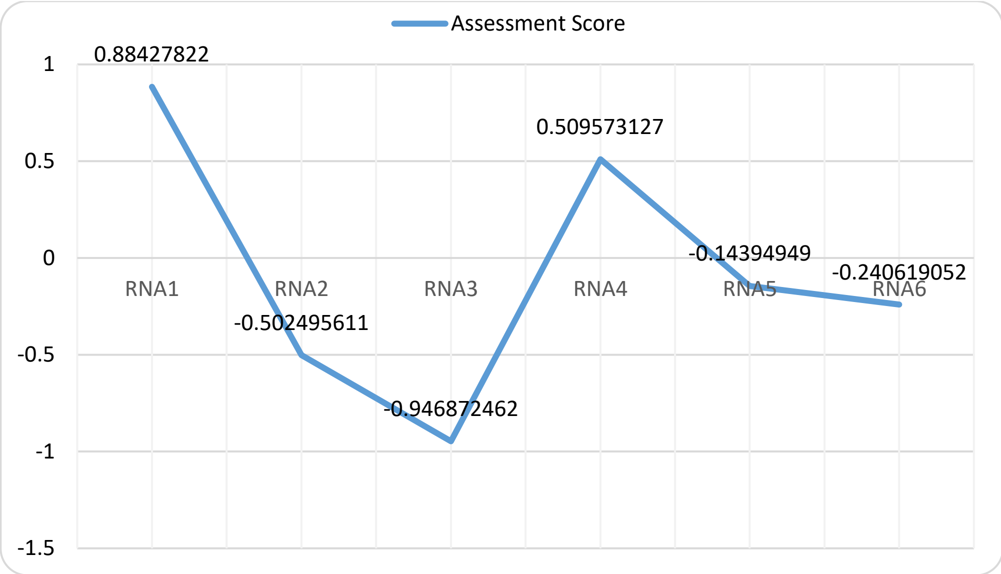


Figure 5. Evaluation score of each renewable energy sources.

6. Conclusions

There is no doubt that the quick advancement of energy is enticing owing to the growth in population and production firms, as well as the rise in air pollution and greenhouse gas emissions, which has resulted in significant advancements in renewable energies and the technologies that are related to them.

The overall objectives of this study are fulfilling two aims. Firstly, MCDM methods (i.e., CODAS) have been strengthened by neutrosophic theory as supporter in uncertainty situations and incomplete data. Secondly, hybrid techniques of CODAS based Interval value neutrosohic have been employed for analyzing RnESs alternatives based on a set of determined criteria from earlier studies. For achieving such objective, we constructed MsIDMM.

DMs are formed and volunteered for rating determined 6 alternatives of RnESs which being in wind energy, solar energy, hydro energy, biomass energy, geothermal energy, and wave energy. While 22 criteria are determined based on conducted survey for prior studies. These criteria have been rated by DMs. Consequently, MsIDMM analyzes DMS' rating of 6 alternatives and 22 criteria in order to produce the optimum solution that overcomes all environmental and local challenges in real-time application. Finally, the optimal and suitable RnES is obtained by constructed MsIDMM to sustain sustainability and its aspects. According to evaluation score for 6 RnESs in Figure 5, solar energy (A1) is the most appropriate and sustainable one with score value 0.88 followed by biomass energy (A4) with score value 0.509. Otherwise, hydro energy is the worst and least sustainable renewable energy resource with a score value -0.946.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Y. Torul Yürek, M. Bulut, B. Özyörük, and E. Özcan, "Evaluation of the hybrid renewable energy sources using sustainability index under uncertainty," Sustain. Energy, Grids Networks, vol. 28, p. 100527, 2021, doi: 10.1016/j.segan.2021.100527.
2. C. Ghenai, M. Albawab, and M. Bettayeb, "Sustainability indicators for renewable energy systems using multi-criteria decision-making model and extended SWARA/ARAS hybrid method," Renew. Energy, vol. 146, pp. 580–597, 2020, doi: 10.1016/j.renene.2019.06.157.
3. C. J. Cleveland, R. Costanza, C. A. S. Hall, and R. Kaufmann, "Energy and the US economy: a biophysical perspective," Science (80-.)., vol. 225, no. 4665, pp. 890–897, 1984.
4. N. Rahim, L. Abdullah, and B. Yusoff, "A border approximation area approach considering bipolar neutrosophic linguistic variable for sustainable energy selection," Sustain., vol. 12, no. 10, 2020, doi: 10.3390/SU12103971.
5. M. Abdel-Basset, A. Gamal, R. K. Chakrabortty, and M. J. Ryan, "Evaluation approach for sustainable renewable energy systems under uncertain environment: A case study," Renew. Energy, vol. 168, pp. 1073–1095, 2021, doi: 10.1016/j.renene.2020.12.124.
6. P. Rani, J. Ali, R. Krishankumar, A. R. Mishra, F. Cavallaro, and K. S. Ravichandran, "An integrated single-valued neutrosophic combined compromise solution methodology for renewable energy resource selection problem," Energies, vol. 14, no. 15, 2021, doi: 10.3390/en14154594.
7. T. Chen, Y. Wang, J. Wang, L. Li, and P.-F. Cheng, "Multistage decision framework for the selection of renewable energy sources based on prospect theory and PROMETHEE," Int. J. Fuzzy Syst., vol. 22, pp. 1535–1551, 2020.
8. Ü. Şengül, M. Eren, S. E. Shiraz, V. Gezder, and A. B. Şengül, "Fuzzy TOPSIS method for ranking renewable energy supply systems in Turkey," Renew. energy, vol. 75, pp. 617–625, 2015.
9. S. K. Saraswat and A. K. Digalwar, "Evaluation of energy alternatives for sustainable development of energy sector in India: An integrated Shannon's entropy fuzzy multi-criteria decision approach," Renew. Energy, vol. 171, pp. 58–74, 2021.
10. Y. Wu, C. Xu, and T. Zhang, "Evaluation of renewable power sources using a fuzzy MCDM based on cumulative prospect theory: A case in China," Energy, vol. 147, pp. 1227–1239, 2018.
11. L. A. Zadeh, "Fuzzy sets," Inf. Control, vol. 8, no. 3, pp. 338–353, 1965.
12. F. Smarandache, "A unifying field in Logics: Neutrosophic Logic.," in Philosophy, American Research Press, 1999, pp. 1–141.
13. T. F. Stocker et al., "Climate Change 2013: The physical science basis. contribution of working group I to the fifth assessment report of IPCC the intergovernmental panel on climate change," 2014.
14. Neha and J. Rambeer, "Renewable Energy Sources: A Review," J. Phys. Conf. Ser., vol. 1979, no. 1, 2021, doi: 10.1088/1742-6596/1979/1/012023.
15. W. O. Sarkodie, E. A. Ofosu, and B. C. Ampimah, "Decision optimization techniques for evaluating renewable energy resources for power generation in Ghana: MCDM approach," Energy Reports, vol. 8, pp. 13504–13513, 2022.
16. I. Sahmutoglu, A. Taskin, and E. Ayyildiz, "Assembly area risk assessment methodology for post-flood evacuation by integrated neutrosophic AHP-CODAS," Nat. Hazards, vol. 116, no. 1, pp. 1071–1103, 2023.
17. Q. Wang, "Research on teaching quality evaluation of college english based on the CODAS method under interval-valued intuitionistic fuzzy information," J. Intell. Fuzzy Syst., vol. 41, no. 1, pp. 1499–1508, 2021.

Received: Apr 30, 2023. Accepted: Aug 31, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.51

An Analysis of Obesity in School Children during the Pandemic COVID-19 Using Plithogenic Single Valued Fuzzy Sets

S.P.Priyadharshini [1,*] and F. Nirmala Irudayam [2]

[1] Research Scholar, Nirmala College for Women, India; priyadharshini125@gmail.com.
[2] Assistant Professor, Nirmala College for Women, India; nirmalairudayam@ymail.com.

* Correspondence: priyadharshini125@gmail.com.

Abstract: The objective of this research is to examine the perception that school children with obesity, when excluded from organized academic performance and constrained to their residences during the coronavirus epidemic 2019 will reveal negative consequences in health behaviors. To meet the objective, the concept of Plithogenic Single valued fuzzy sets (PSFS) and their aggregation operators were introduced. Based on the proposed theory, an analysis is presented with the case study to highlight its practicality and preciseness.

Keywords: Fuzzy Set; Plithogenic Set; Plithogenic Single Valued Fuzzy Set; PSFS Operators.

1. Introduction

Global health analysts predict that school closures have worsened the epidemic of childhood obesity rates due to the COVID-19 pandemic. Analysts believe that school shutdowns associated with COVID-19 will double out-of-school time for several children worldwide the previous year and could increase hazard factors involved with a summer break for gaining weight [1-3].

Plithogeny which was introduced in 2017 by Florentine Smarandache [4, 5, 6] is the origination, existence, development, growth, and emergence of various entities from technologies and organic combinations of old objects that are conflicting and/or neutral and/or non-contradictory. A plithogenic set P is a set whose members are characterized by one or more attributes and there may be several values for each attribute. Moreover, it is the generalization of Crisp, Fuzzy, Intuitionistic fuzzy, and Neutrosophic sets.

In this research work, we study how the Plithogenic Single valued fuzzy sets (PSFS) [7-10] and their aggregation operators help in analyzing the main factors for an increase in obesity among school children during the pandemic COVID-19 lockdown with the analyst's fuzzy degree.

The uniqueness of this technique is its effectiveness, as the learner does not have to engage with complex operators based on lengthy calculations. The proposed method also has a realistic approach to the need for a broad spectrum that can penetrate alterations according to the need for the social structure provided.

2. Plithogenic Single valued fuzzy sets and its Operators

Definition 3.1: Let U be a universal set and P is the subset and $x \in P$ be an element. P is called a Plithogenic set which has the form $(P, \mathrm{A}, \Lambda, D_F, C_F)$ where A is the attribute Values, Λ is the set of all attributes values that helps in solving an application, D_F is the degree of appurtenance and C_F is the dissimilarity degree.

Let us assume two Analyst A & B each evaluating the PSFS degree of appurtenance of Λ of x to the Plithogenic set P with some given constraints

$D^F{}_A(\lambda) = \alpha \in [0,1]$ *and* $D^F{}_B(\lambda) = \beta \in [0,1]$.
Also $\wedge_f$ *be the fuzzy* τ_{norm} *and* $\vee_f$ *be the fuzzy* τ_{conorm} *correspondingly*

3.1.1 PSFS Intersection

$$\alpha \wedge_p \beta = C_O * (\alpha \vee_f \beta) + (1 - C_O) * (\alpha \wedge_f \beta) \quad (1)$$

3.1.2 PSFS Union

$$\alpha \vee_p \beta = C_O * (\alpha \wedge_f \beta) + (1 - C_O) * (\alpha \vee_f \beta) \quad (2)$$

3.1.3 PSFS Negation

Denying the attribute Value
$\neg_p(\lambda) = anti(\lambda)$, i.e. the opposite of λ, where $anti(\lambda) \in \Lambda$ **or** $anti(\lambda) \in \mathrm{Re}\ fined\,\Lambda(refined\ set\ of\ \Lambda)$.

So we get $D^f{}_X(anti(\lambda)) = x$.

Results:

(i) When more emphasis is allocated to $\tau_{norm}(\alpha,\beta) = \alpha \wedge_f \beta$ when compared to $\tau_{conorm}(\alpha,\beta) = \alpha \vee_f \beta$ for $C(\lambda_d,\lambda) = C_O \in [0, 0.5)$ is called an accurate plithogenic intersection.
(ii) When more emphasis is allocated to $\tau_{conorm}(\alpha,\beta) = \alpha \vee_f \beta$ when compared to $\tau_{norm}(\alpha,\beta) = \alpha \wedge_f \beta$ for $C(\lambda_d,\lambda) = C_O \in [0, 0.5)$ is called an accurate plithogenic union.
(iii) When more emphasis is allocated to $\tau_{norm}(\alpha,\beta) = \alpha \wedge_f \beta$ when compared to $\tau_{conorm}(\alpha,\beta) = \alpha \vee_f \beta$ for $C(\lambda_d,\lambda) = C_O \in (0.5, 1]$ is called an inaccurate plithogenic union.
(iv) When more emphasis is allocated to $\tau_{conorm}(\alpha,\beta) = \alpha \vee_f \beta$ when compared to $\tau_{norm}(\alpha,\beta) = \alpha \wedge_f \beta$ for $C(\lambda_d,\lambda) = C_O \in (0.5, 1]$ is called an inaccurate plithogenic intersection.
(v) $\tau_{conorm}(\alpha,\beta) = \alpha \wedge_f \beta$ and $\tau_{norm}(\alpha,\beta) = \alpha \vee_f \beta$ has allocated the same emphasis 0.5 for $C(\lambda_d,\lambda) = C_O \in 0.5$

3. Proposed Method to Find the Optimum Solution Using PSFS Operators.

Step 1: Classify the problem with the attributes and its corresponding values of attribute.
Step 2: Find the dissimilarity degree according to the Experts X and Y fuzzy degrees.
Step 3: Compute the optimum solution using Eq. (1).
Note: We have used the intersection operator. But the alternative is free for the reader to work with other operators also.

4. Application

Consider the primary attribute "Reason for obesity in school children during lockdown" which has the attribute values.
Food Habits- whose refined values are- less vegetable intake, sugary drinks, junk food and meat consumption which is represented by $\{g_1, g_2, g_3, g_4\}$.

Screen time - whose refined values are-mobile, television and computer which is symbolized by $\{t_1, t_2, t_3\}$.

Sleeping pattern - whose refined values are- increase in day time sleep and decrease in night time sleep which is denoted by $\{h_1, h_2\}$.

Sports- whose refined values are- More Indoor games and lack of outdoor games which is signified by $\{r_1, r_2\}$.

The multi attribute of dimension 4 is,

$R_4 = \{(g_i, t_j, h_k, r_l), \text{ for all } 1 \le i \le 4, 1 \le j \le 3, 1 \le k \le 2, 1 \le l \le 2\}$

The dominant attribute values are g_3, t_1, h_1, r_2 respectively for each corresponding uni-dimensional attribute.

The unit dimensional attribute contradiction degrees are:

$$C(g_1, g_2) = \frac{1}{3}, C(g_2, g_3) = \frac{2}{3}, C(g_1, g_3) = 1,$$

$$C(t_1, t_2) = \frac{1}{2}, C(t_1, t_3) = 1$$

$$C(h_1, h_2) = 1 \text{ and } C(l_1, l_2) = 1 \text{ .}$$

Let us use $fuzzy\, \tau_{norm} = a \wedge_F b = ab$ & $fuzzy\ \tau_{conorm} = a \vee_F b = a + b - ab$

- ***Four-dimensional PSFS Intersection***

Let $x_A = \{d_A(x, g_i, t_j, h_k, r_l) \text{ for all } 1 \le i \le 4, 1 \le j \le 3, 1 \le k \le 2, 1 \le l \le 2\}$

and $x_B = \{d_B(x, g_i, t_j, h_k, r_l) \text{ for all } 1 \le i \le 4, 1 \le j \le 3, 1 \le k \le 2, 1 \le l \le 2\}$

Then

$$\begin{aligned}
& x_A(g_i, t_j, h_k, r_l) \wedge_p x_B(g_i, t_j, h_k, r_l) = \\
& \{c(g_D, g_i) * [d_A(x, g_D) \vee_f d_B(x, g_i) + (1 - c(g_D, g_i)) * [d_A(x, g_D) \wedge_f d_B(x, g_i)] 1 \le i \le 4; \\
& c(t_D, t_j) * [d_A(x, t_D) \vee_f d_B(x, t_j) + (1 - c(t_D, t_j)) * [d_A(x, t_D) \wedge_f d_B(x, t_j)] 1 \le j \le 3; \\
& c(h_D, h_k) * [d_A(x, h_D) \vee_f d_B(x, h_k) + (1 - c(h_D, h_k)) * [d_A(x, h_D) \wedge_f d_B(x, h_k)] 1 \le k \le 2; \\
& c(r_D, r_l) * [d_A(x, r_D) \vee_f d_B(x, r_l) + (1 - c(r_D, r_l)) * [d_A(x, r_D) \wedge_f d_B(x, r_l)] 1 \le l \le 2\}.
\end{aligned}$$

According to Analyst (A & B) fuzzy degrees the following Table 1 and Figure 1 represents the optimum solution.

Table 1. Analysis Table for obesity in school children during pandemic Covid-19 lockdown using PSFS.

Attribute	**Food Habits**				**Screen time**			**Sleeping Pattern**		**Sports**	
values of Attribute	Lack of Vegetable intake	Sugary drinks	Junk food	Meat consumption	Mobile	Television	Computer	More Day time sleep	Less Night time sleep	More of Indoor games	Lack of Outdoor games
Dissimilarity degree	0	1/3	2/3	1	0	1/2	1	0	1	0	1
Analyst A Fuzzy degree	0.4	0.4	0.8	0.6	0.8	0.6	0.5	0.7	0.8	0.3	0.8
Analyst B Fuzzy degree	0.5	0.7	0.9	0.7	0.6	0.5	0.7	0.6	0.7	0.4	0.9
$x_A \wedge_p x_B$	0.7	0.7	0.8	0.4	0.9	0.6	0.4	0.9	0.6	0.6	0.7

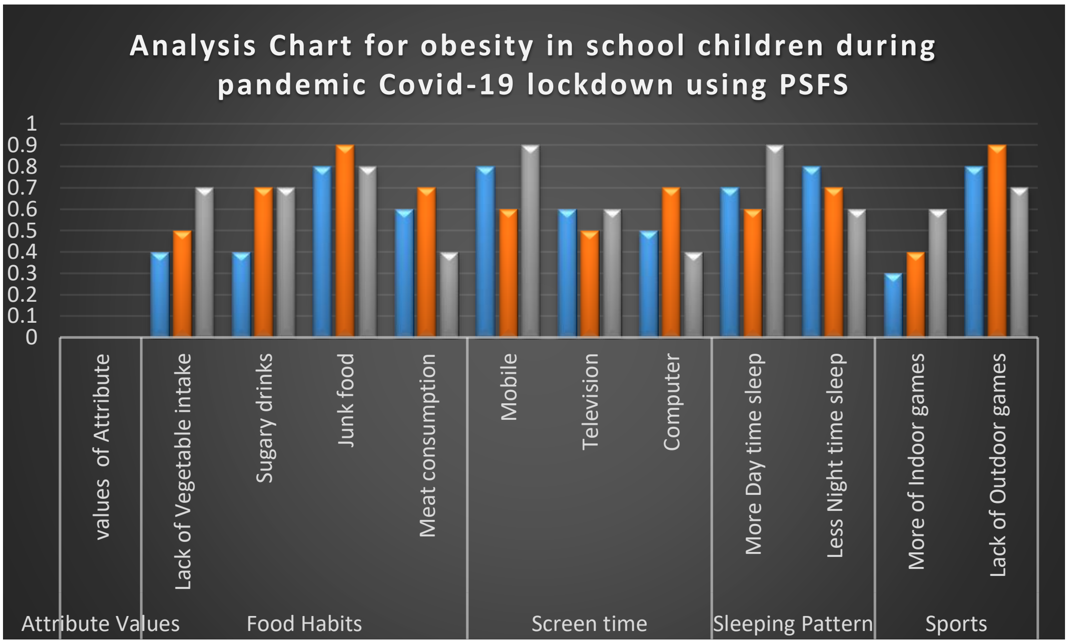


Figure 1. Analysis chart for obesity in school children during the pandemic Covid-19 lockdown using PSFS.

5. Conclusion

Based on the fuzzy degrees of Analyst's (A & B) it is clearly shown that the major reasons for the obesity in children during Covid-19 lockdown is the consumption of more junk food and the time spending on using mobile phones, more day time sleep along with the lack of outdoor sports which reduces all their physical activities and in turn results in the obesity. In future, we can extend this PSFS concept to interval valued and also learn its applications in decision making.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Y. Deng, F.T. Chan, Y. Wu and D. Wang, A new linguistic MCDM method based on multiple-criteria data fusion, Expert systems with Applications, 2011, 38(6),6985-6993.
2. N. Martin and F. Smarandache, Plithogenic Cognitive Maps in Decision Making, International Journal of Neutrosophic Science, 2020, 9(1), 09-21.
3. P.K. Singh, Plithogenic set for multi-variable data analysis, International Journal of Neutrosophic Science, 2020, 1(2), 81-89.
4. F. Smarandache, Plithogeny, PlithogenicSet, Logic, Probability, and Statistics, 141 pages, Pons Editions,Brussels,Belgium,2017, arXiv.org (Cornell University), Computer Science-Artificial Intelligence,03Bxx:abstract:https://arxiv.org/abs/1808.03948,book:https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf,HarvardSAO/NASA. ADS: http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948.
5. F. Smarandache, Plithogenic set, an extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophicsets-Revisited, Neutrosophic Sets and Systems, **2018**, 21, 153-166.
6. F. Smarandache, Physical Plithogenic sets, 71st Annual Gaseous Electronics conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5-9, **2018**.
7. I. B. Turksen, Interval-valued strict preference with Zadeh triples, Fuzzy Sets and Systems, **1996**, 78, 183–195.
8. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy sets and Systems, **1968**, 20,191-210.
9. B. Turksen, Interval-valued fuzzy sets and compensatory, Fuzzy Sets and Systems, **1992**, 51, 295–307.
10. L.A Zadeh, "Fuzzy sets", Inform and Control, **1965**, 8, 338-353.

Received: Apr 25, 2023. Accepted: Aug 28, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.43

A Neutrosophic Approach for B-Spline Curve by Using Interpolation Method

Siti Nur Idara Binti Rosli [1] and Mohammad Izat Emir Bin Zulkifly [2,*]

[1,2] Department of Mathematic, Faculty of Sciences, University of Technology Malaysia (UTM) 81310, Johor Bahru, Malaysia.

Emails: idara-1999@graduate.utm.my; izatemir@utm.my.

* Correspondence: izatemir@utm.my.

Abstract: This study introduces a B-spline curve interpolation model based on the neutrosophic set technique. To begin, the neutrosophic notion is used to define the neutrosophic control point relation. After that, the neutrosophic control point is combined with the B-spline basis function. Besides, the neutrosophic B-spline curve interpolation model is illustrated using the interpolation method. Furthermore, an example and the methods are shown for creating the right curve.

Keywords: Neutrosophic Set; Curve of B-Spline; Method of Interpolation; Neutrosophic Control Points.

1. Introduction

The link between a curve created from control polygon vertices and the curve is technically dependent i on some interpolation or approximation approach [1]. Basis function determines this scheme. Bézier curves are generated via the Bernstein basis. Piegl and Tiller [1] also noted that two Bernstein basis features restrict curve flexibility. The curve's polynomial order depends on the number of polygon vertices supplied. A four-vertex, three-span polygon defines a cubic curve. Six-vertex polygons always generate fifth-degree curves. Decrease the number of vertices to decrease the curve degree, and vice versa.

According to Piegl and Tiller [1], the second limiting feature of the Bernstein foundation is globality. For all parameter values along the curve, the blending function is nonzero. Because each point is formed by mixing all control vertices, a change in one control vertex affects the entire Bézier curve. This avoids local alterations to the curve. The end slopes of a Bézier curve are defined by the orientations of the first and last polygon spans, so changing the center vertex of a five-point polygon has no effect. The Bernstein basis modifies the curvature of the curve worldwide. A lack of local control could be problematic. As a result, Bernstein basis is a subset of B-spline basis. This foundation does not operate on a global scale [1]. Since each vertex has a basis function, B-spline curves are non-global. Thus, each vertex only influences curve shape in the parameter range where its basis function is nonzero . The B-spline basis allows user to change the order of the Basis functions and the degree of the curve without changing the control polygon vertices . B-splines were invented by Schoenberg [2]. Cox [3] and De Boor [4] each created their own definition of recursive numerical computing. The B-spline basis was used by Gordon and Riesenfeld [5] to define curves.

Data points, according to Hoschek and Lasser [6] require curves. Data analysis and representation are complicated by noise and ambiguity. This problem is addressed by fuzzy set theory and geometric modelling. Tuohy and Patrikalakis [7] proposed using regionally scattered geophysical data to rebuild ambiguous surfaces. Their technique has been extended to describe volume data with periodic B-spline volume function [8,9]. Based on Tuohy and Patrikalakis [7], they developed enclosing or gap specific B-spline geometry [10] to describe underwater geophysical data and sensor measurement error. Tuohy and Patrikalakis [11] depicted functions with uncertainty defining an observed geophysical parameter using interval B-spline.

Anile et al. extended the techniques presented in [12] to data modelling and data reduction difficulties [13-15]. Anile et al. [16] improved on the modelling of Patrikalakis et al. [10]. They begin by reducing a big data set to fuzzy integers with suitable membership functions . They created fuzzy B-splines to interpret fuzzy data and rapid algorithms to calculate spline alpha values. Anile and Spinella [17] created the fuzzy B-splines methodology and used fuzzy arithmetic concepts to uncertain sparse data caused by measurement mistakes, data reduction issues, and modelling flaws. Using rigorous procedures, fuzzy B-splines that fit uncertain sparse data were generated and examined.

To address uncertainty problems, Wahab et al. [18] employ fuzzy numbers and Zadeh's [19] fuzzy set theory. The ideas of instability in data, fuzzy numbers, implementation of control measures, B-spline, and Bézier were employed. In CAGD, the approaches are used to construct fuzzy Bézier and B-spline curves. Each crisper control point is composed of left and right vague control points, each of which has a different degree of similarity to the initial control points (crisp control points). Its membership function is left and right continuous in a closed interval at each alpha value.

Fuzzy set theory (FST) only takes into account membership data, but not non-membership data and uncertainty. In 1986, Krasimir Atanassov expanded FST to include truth, falsehood, and uncertainty degrees [20]. It is best to accept ambiguity. As FST only accepts full membership data, Intuitionistic Fuzzy Sets (IFS) can be used when the data for categorization and processing is insufficient [21]. Florentin Smarandache, on the other hand, proposed mathematical theory, and neutrosophy advocates equality [22]. Neutrophil sets might be members, non-members, or undecided. Transdisciplinary challenges are addressed and described using Neutrosophic Set (NS) approaches. A true, incorrect, or ambiguous NS theory element can exist. This allows for more nuanced doubt and ambiguity, for as when two statements contradict each other. Geometric modelling has been employed by certain academics to build neutrosophic set procedures [23,24].

The Neutrosophic B-spline Curve Interpolation (NB-SCI) Model will be the primary focus of this project's creation of a geometric model that can deal with uncertainty data. The Neutrosophic Control Point Relation (NCPR) must be determined using neutrosophic set theories and the qualities it holds before generating the B-spline interpolation. These control points, together with the B-spline basis function, are used to build NB-SCI models, which are subsequently displayed using an interpolation method. The following section shows how to use the format of this paper. The initial part of this paper gave background information on the issue. Section 2 introduces the reader to the basic concept of Neutrosophic Set (NS), followed by Neutrosophic Point Relation (NPR) and Neutrosophic Control Point Relation (NCPR). The third section discusses how to calculate the NB-SCI using NCPR. The fourth section includes a numerical example, a graphical representation of NB-SCA, and the model-creation algorithm. The fifth and final segment will conclude the probe.

2. Preliminaries

The intuitionistic set in fuzzy systems can accommodate imperfect information but not indeterminate or inconsistent information [25]. A NS has three membership functions. With the addition of the "indeterminacy" parameter to the NS specification [25], there are three sorts of membership functions: a membership function (denoted by the letter T), an indeterminacy membership function (denoted by the letter I), and a non-membership function (denoted by the letter F).

Definition 1: [22] Let Z be the main of conversation, with element in Z denoted as z. The NS is an item in the form below and $\hat{N}$ denoted as NS.

$$\hat{N} = \{\left\langle z : T_{\hat{N}(z)}, I_{\hat{N}(z)}, F_{\hat{N}(z)} \right\rangle | \ z \in Z\} \quad (1)$$

where, the degrees $T, I, F : Z \to]^{-}0,1^{+}[$ the meaning of accordingly, the degree to which an element is a member of the truth, the degree to which it is indeterminate, and the degree to which it is a member of the false $z \in Z$ to the set Z with the condition;

$$0^{-} \leq T_{\hat{N}}(z) + I_{\hat{N}}(z) + F_{\hat{N}}(z) \leq 3^{+} \quad (2)$$

There is no restriction to values of $T_{\hat{N}}(z), I_{\hat{N}}(z)$ and $F_{\hat{N}}(z)$

NS will pick a value from either one of the actual standard subsets or one of the non-standard subsets of $]^{-}0,1^{+}[$. The actual value of the interval $[0,1]$, on the other hand, $]^{-}0,1^{+}[$ will be utilized in technical applications since its utilization in real data such as the resolution of scientific challenges, will be physically impossible. As a direct consequence of this, membership value utilization is increased.

$$\begin{aligned} &\hat{N} = \{\left\langle z : T_{\hat{N}(z)}, I_{\hat{N}(z)}, F_{\hat{N}(z)} \right\rangle | \ z \in Z\} \text{ and} \\ &T_{\hat{N}}(z), I_{\hat{N}}(z), F_{\hat{N}}(z) \in [0,1] \end{aligned} \quad (3)$$

There is no restriction on the sum of $T_{\hat{N}}(z), I_{\hat{N}}(z), F_{\hat{N}}(z)$ .Therefore,

$$0 \leq T_{\hat{N}}(z) + I_{\hat{N}}(z) + F_{\hat{N}}(z) \leq 3 \quad (4)$$

Definition 2: [23, 24] Let $\hat{N} = \{\left\langle z : T_{\hat{N}(z)}, I_{\hat{N}(z)}, F_{\hat{N}(z)} \right\rangle | \ z \in Z\}$ and $\hat{M} = \{\left\langle y : T_{\hat{M}(y)}, I_{\hat{M}(y)}, F_{\hat{M}(y)} \right\rangle | \ y \in Y\}$ be neutrosophic elements. Thus, $NR = \{\left\langle (z,y) : T_{(z,y)}, I_{(z,y)}, F_{(z,y)} \right\rangle | \ z \in \hat{N}, y \in \hat{M}\}$ is a Neutrosophic Relation (NR) on $\hat{N}$ and $\hat{M}$.

Definition 3: [23,24] NS of $\hat{N}$ in space Z is Neutrosophic Point (NP) and $\hat{N} = \{\hat{N}_i\}$ where $i = 0,...,n$ is a collection of NPs where the existences $T_{\hat{N}} : Z \to [0,1]$ as truth degree, $I_{\hat{N}} : Z \to [0,1]$ as indeterminacy degree and $F_{\hat{N}} : Z \to [0,1]$ as false degree with

$$\begin{aligned} T_{\hat{N}}(\hat{N}) &= \begin{cases} 0 & \text{if } \hat{N}_i \notin \hat{N} \\ a \in (0,1) & \text{if } \hat{N}_i \,\hat{\in}\, \hat{N} \\ 1 & \text{if } \hat{N}_i \in \hat{N} \end{cases} \\ I_{\hat{N}}(\hat{N}) &= \begin{cases} 0 & \text{if } \hat{N}_i \notin \hat{N} \\ b \in (0,1) & \text{if } \hat{N}_i \,\hat{\in}\, \hat{N} \\ 1 & \text{if } \hat{N}_i \in \hat{N} \end{cases} \\ F_{\hat{N}}(\hat{N}) &= \begin{cases} 0 & \text{if } \hat{N}_i \notin \hat{N} \\ c \in (0,1) & \text{if } \hat{N}_i \,\hat{\in}\, \hat{N} \\ 1 & \text{if } \hat{N}_i \in \hat{N} \end{cases} \end{aligned} \quad (5)$$

2.1 Neutrosophic Point Relation (NPR)

The concept of the NS, which was discussed in the previous section, serves as the cornerstone for NPR. If is a group of Euclid eternal space points and then, the following is how NPR is described:

Definition 4: Let N,M be a grouping of elements in global area that are part of a set that is not null and $N,M,O \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$, then the term "NPR" refers to

$$\hat{R} = \left\{ \begin{array}{l} \left\langle \left(n_i,m_j\right), T_R(n_i,m_j), I_R(n_i,m_j), F_R(n_i,m_j) \right\rangle \\ \mid T_R(n_i,m_j), I_R(n_i,m_j), F_R(n_i,m_j) \in I \end{array} \right\} \tag{6}$$

Where $\left(n_i,m_j\right)$ is a set of ordered positions and $\left(n_i,m_j\right) \in N\times M$ while $T_R(n_i,m_j), I_R(n_i,m_j), F_R(n_i,m_j)$ are the truth membership, the indeterminacy membership, and the false membership that follows the condition of the neutrosophic set which is respectively, $0 \le T_{\hat{N}}(z) + I_{\hat{N}}(z) + F_{\hat{N}}(z) \le 3$.

2.2 Neutrosophic Control Point Relation (NCPR)

The geometry of a spline only be determined by all of the data required to form the curve. The word "control point" relates to this. The control point is essential in the design, control, and production of smooth curves. In this section, the neutrosophic control point relationship (NCPR) is defined by first employing the concept of fuzzy control point from the research published in Wahab et al. [26] in the following:

Definition 5: Let $\hat{R}$ be a NPR, then NCPR is viewed as a group of points $n+1$ that denotes a locations and coordinates and is used to describe the curve and is indicated by

$$\begin{aligned} \hat{P}_i^T &= \left\{ \hat{p}_0^T, \hat{p}_1^T, ..., \hat{p}_n^T \right\} \\ \hat{P}_i^I &= \left\{ \hat{p}_0^I, \hat{p}_1^I, ..., \hat{p}_n^I \right\} \\ \hat{P}_i^F &= \left\{ \hat{p}_0^F, \hat{p}_1^F, ..., \hat{p}_n^F \right\} \end{aligned} \tag{7}$$

Where $\hat{p}_i^T$, $\hat{p}_i^I$ and $\hat{p}_i^F$ are NCP for membership truth, indeterminacy and i is one less than n.

3. Neutrosophic B-Spline Curve Interpolation (NB-SCI)

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

The NB-SCI is defined as follows after combining NCPR with a B-spline basis function:

Definition 6: Let $\hat{P}_i^T = \left\{ \hat{p}_0^T, \hat{p}_1^T, ..., \hat{p}_n^T \right\}, \hat{P}_i^I = \left\{ \hat{p}_0^I, \hat{p}_1^I, ..., \hat{p}_n^I \right\}, \hat{P}_i^F = \left\{ \hat{p}_0^F, \hat{p}_1^F, ..., \hat{p}_n^F \right\}$ where $i = 1,2,...,n+1$ is NCPR and NB-SCI denoted by BSC with the vector along curve as parameter t. As a result of combining, it with the blending function, NB-SCI is described as

$$BSC(t) = \sum_{i=1}^{n+1} \hat{P}_i N_i^k(t) \tag{8}$$

Where $t_{\min} \le t \le t_{\max}$ and $2 \le k \le n+1$ when $\hat{P}_i$ are the position vector of $n+1$ as control polygon vertices and N_i^k as the B-spline basis function. The $N_i^k(t)$ is describe as

$$N_i^1(t) = \begin{cases} 1 & if \quad t_i \le t < t_{i+1} \\ 0 & otherwise \end{cases} \tag{9}$$

and

$$N_i^k(t) = \frac{(t - t_i)}{t_{i+k-1} - t_i} N_i^{k-1}(t) + \frac{(t_{i+k} - t)}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t) \tag{10}$$

The parametric function NB-SCI in (8) is defined as follows and is made up of three curves: a member curve, a non-member curve, and an indeterminacy curve.

$$BSC(t)^T = \sum_{i=1}^{n+1} \hat{P}_i^T N_i^k(t) \tag{11}$$

$$BSC(t)^F = \sum_{i=1}^{n+1} \hat{P}_i^F N_i^k(t) \tag{12}$$

$$BSC(t)^I = \sum_{i=1}^{n+1} \hat{P}_i^I N_i^k(t) \tag{13}$$

Assuming the data points are in the NB-SCI range, then the data point should be (8). For each data point indicated by M_j, equation (8) has been modified as follows:

$$\begin{aligned} &\hat{Q}_1(t_1) = M_1^k(t_1)\hat{S}_1 + M_2^k(t_1)\hat{S}_2 + \ldots + M_{n+1}^k(t_1)\hat{S}_{n+1} \\ &\hat{Q}_2(t_2) = M_1^k(t_2)\hat{S}_1 + M_2^k(t_2)\hat{S}_2 + \ldots + M_{n+1}^k(t_2)\hat{S}_{n+1} \\ &\vdots \\ &\hat{Q}_j(t_j) = M_1^k(t_j)\hat{S}_1 + M_2^k(t_j)\hat{S}_2 + \ldots + M_{n+1}^k(t_j)\hat{S}_{n+1} \end{aligned} \tag{14}$$

When $2 \le k \le n+1 \le j$. Equation (14) is expressed as a matrix as

$$\left[\hat{Q}\right] = [M]\left[\hat{S}\right] \tag{15}$$

where

$$\begin{aligned} &\left[\hat{Q}\right]^T = \left[\hat{Q}_1(t_1) \quad \hat{Q}_2(t_2) \quad \cdots \quad \hat{Q}_j(t_j)\right] \\ &[M] = \begin{bmatrix} M_1^k(t_1) & \cdots & \cdots & M_{n+1}^k(t_1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ M_1^k(t_j) & \cdots & \cdots & M_{n+1}^k(t_j) \end{bmatrix} \\ &\left[\hat{S}\right]^T = \left[\hat{S}_1 \quad \hat{S}_2 \quad \cdots \quad \hat{S}_j\right] \end{aligned} \tag{16}$$

The measurement of data points along NB-SCI is the metric value t_j for each output. The parametric value on data point to l for data point is j as follows.

$$t_1 = 0$$
$$\frac{t_l}{t_{\max}} = \frac{\sum_{r=2}^{l}\left|\hat{Q}_r - \hat{Q}_{r-1}\right|}{\sum_{r=2}^{j}\left|\hat{Q}_r - \hat{Q}_{r-1}\right|}; l \geq 2 \tag{17}$$

The greatest parameter is indicated by $t_{\max}$, which is usually considered as the greatest value for the knot vector. If $2 \leq k \leq n+1 = j$, then $[M]$ is a square matrix, and the control polygon is derived immediately using an inverse matrix, such as

$$\left[\hat{Q}\right] = [M]^{-1}\left[\hat{S}\right] \qquad 2 \leq k \leq n+1 = j \tag{18}$$

As a result, NB-SCI can be acquired using (18).

3.1. Properties of Neutrosophic B-Spline Curve Interpolation (NB-SCI)

Since a B-spline basis is utilized to define a B-spline curve, numerous features, in addition to those already described, are easily understood:

- For any parameter value t, the sum of the B-spline basis functions is [4, 5]

$$\sum_{i=1}^{n+1} N_i^k(t) \equiv 1 \tag{19}$$

- For all values of parameters, each basis functional is either positive or zero. Thus, $N_i^k \geq 0$
- Each basis function, $k=1$ with the exception of first-order basis functions with, has a single highest value.
- The highest order of the curve matches the number of control polygon vertices. The highest value is one degree less.
- The curve demonstrates the variation-diminishing characteristic. As a result, the curve does not oscillate more frequently around any straight line than its control polygon.
- In general, the curve follows the shape of the control polygon.
- Any affine transformation is applied to the curve by transforming the control polygon vertices, which transforms the curve.
- The control polygon's convex hull contains the curve.

4. Numerical Example and Visualization

This section will go over the application of NB-SCAI and visualization. The examples will only use a numerical example at random and will employ an interpolation method. A NB-SCI will be shown that consists of NCPR with a degree of polynomial of four $n=4$.

4.1. Application of Neutrosophic B-Spline Curve Interpolation (NB-SCI)

To illustrate NB-SCA, let's consider NB-SCA with five neutrosophic control point relation as in Table 1.

Table 1. The NCPRs

NCPR $\hat{P}_i$	Truth Membershi $\hat{P}_i^T$	False Membership $\hat{P}_i^F$	Indeterminacy Membership $\hat{P}_i^I$
$\hat{P}_0 = (2,2)$	0.7	0.4	0.2
$\hat{P}_2 = (7,8)$	0.5	0.5	0.3
$\hat{P}_3 = (11,13)$	0.8	0.3	0.2
$\hat{P}_4 = (17,18)$	0.6	0.2	0.5
$\hat{P}_5 = (25,23)$	0.3	0.4	0.6

From Figure 1 through Figure 3, the planned interpolation curve is presented on its own with its matching data points (black dots) and NCP (red dots) utilizing (18). A neutrosophic control polygon connects the control points and is made up of truth degree, false degree, and indeterminacy control polygons. Figures 1-3 are also known as "truth membership," "false membership," and "indeterminacy B-spline curve interpolation." The NCP and controlling polygon governed the curve and ensured that the data points were interpolated.

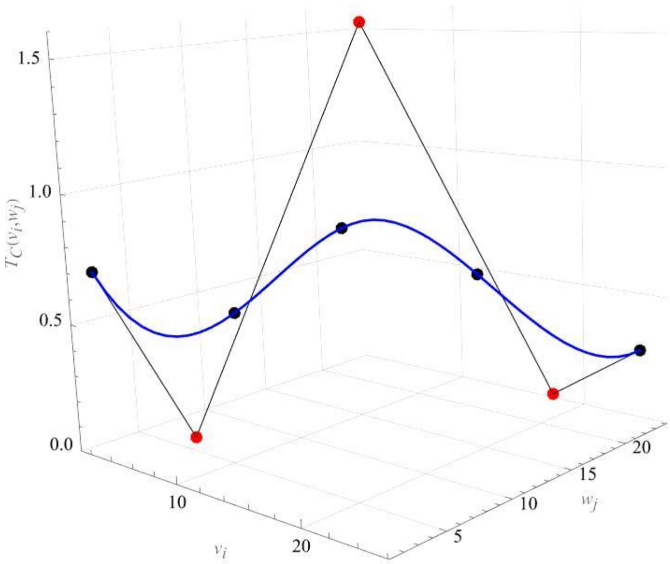


Figure 1. NB-SCI for Truth Membership with its Data Points, NCPs and Control Polygon.

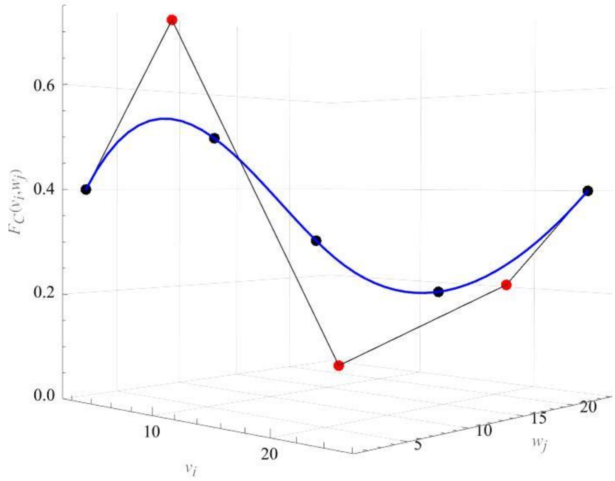


Figure 2. NB-SCI for False Membership with its Data Points, NCPs and Control Polygon

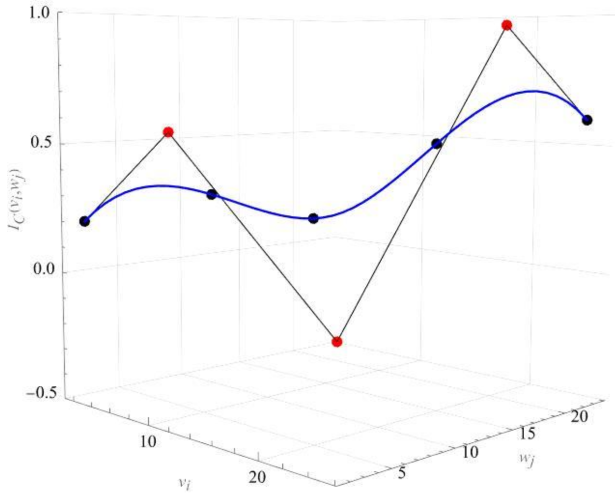


Figure 3. NB-SCI for Indeterminacy Membership with Data Points, NCPs and its Control Polygon

Figures 4 through 6 depict NB-SCI as true membership, false membership, and indeterminacy curves with data points and connected data points, separately.

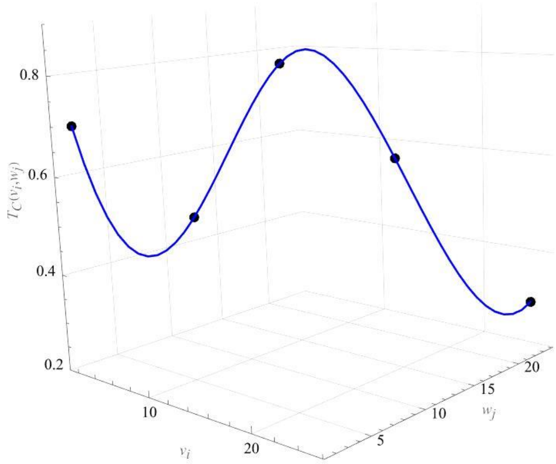


Figure 4. NB-SCI for Truth Membership with its Data Points

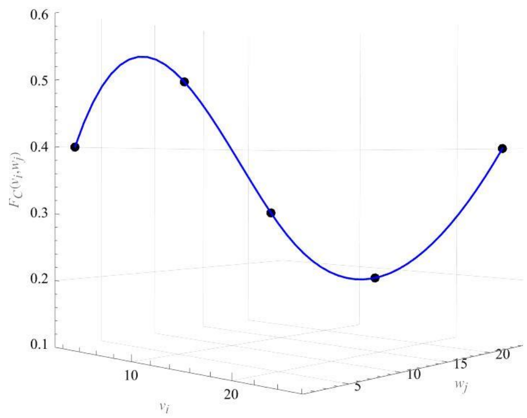


Figure 5. NB-SCI for False Membership with its Data Points

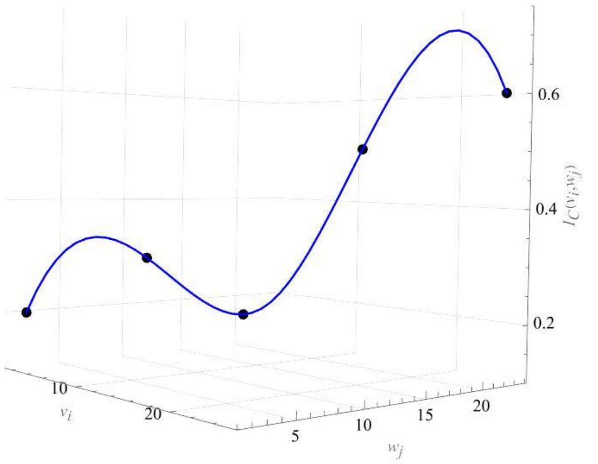


Figure 6. NB-SCI for Indeterminacy Membership with its Data Points

Figures 7 and 8 depicted NB-SCI from various points of view. Figure 7 depicted NB-SCI using data points, NCPs, and control polygons. Finally, Figure 8 depicts NB-SCI with data points. The NB-SCI for blue curve represents truth membership, green curve represents false membership and pink curve represents indeterminacy membership. All the memberships are demonstrated in an axis as Figures 7 and 8 shown.

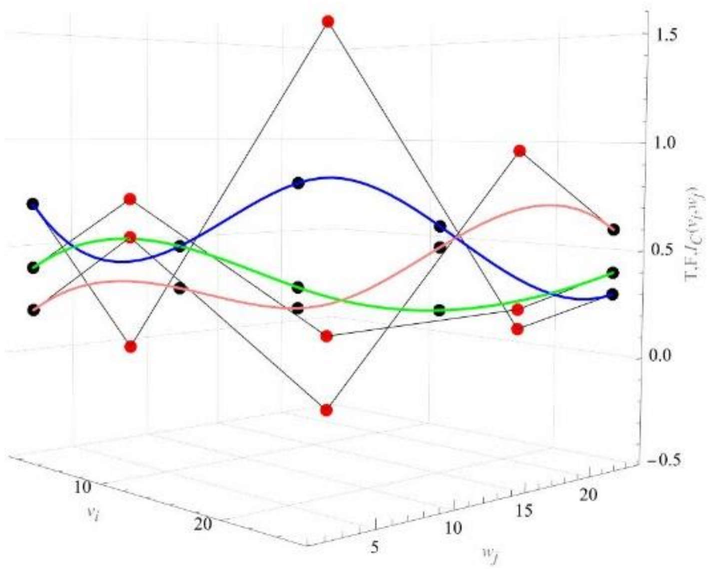


Figure 7. NB-SCI for All Membership with its Data Points, NCPs and Control Polygons.

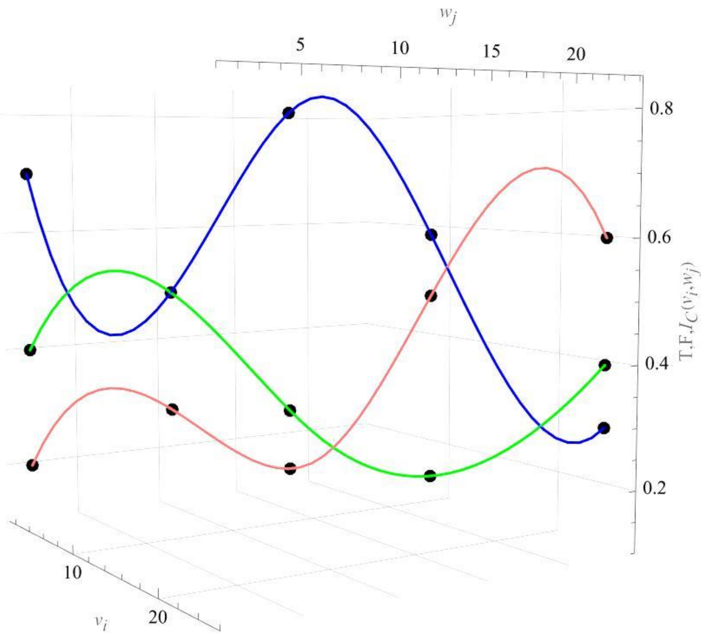


Figure 7. NB-SCI for All Membership with its Data Points only.

Following that, the algorithm for obtaining NB-SCI is summarized below:

Step 1: The knot vector and the neutrosophic data point relation are computed using $\hat{Q} = \left\{ \hat{Q}_j \right\}_{j=1}^{n+1}$ and $k = \left\{ k_j \right\}_{j=1}^{n+1}$.

Step 2: Determine the parametric value along the neutrosophic B-spline curves that corresponds to each NCPR by using (17).

Step 3:

1. Calculate the chord lengths between each point.

$$\left|\hat{Q}_2 - \hat{Q}_1\right|, \left|\hat{Q}_3 - \hat{Q}_2\right|, \cdots, \left|\hat{Q}_r - \hat{Q}_{r-1}\right|$$

2. The parameter is computed as.

$$\sum_{r=2}^{r}\left(\hat{Q}_r - \hat{Q}_{r-1}\right) \quad \text{and} \quad t_1, \frac{t_2}{t_{\max}}, \ldots, \frac{t_l}{t_{\max}}$$

Step 3: Determine the B-spline basis function based on the knot vector in Step 1 by creating the $[M]$ matrix using (15) and (16).

Step 4: Following that, NCPR can be obtained by using (18).

Step 5: Lastly, the NCPR is combined with the B-spline basis function as shown in (8) - (13) to produce NB-SCI.

5. Conclusions

This paper provides an introduction to NB-SCI as well as some of its characteristics. NB-SCI is an extremely useful methodology that has the potential to be implemented in a broad variety of business sectors, such as real civil engineering concepts, shipbuilding, designs for architecture, aerospace, manufacture and a great deal more besides. Due to the availability of truth degree, false degree, and indeterminacy degree, the neutrosophic approach may solve a greater variety of challenges. This neutrosophic set approach, when combined with tools based on the B-spline, can construct a continuously differentiable smooth curve that is capable of providing a comprehensive description of any explored subject. This technique can be made more effective by utilizing the surface of interpolation or approximation for B-spline and NURBS.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. L. Piegl and W. Tiller. The NURBS Book (Springer-Verlag Berlin Heidelberg, Germany, 1995).
2. Schoenberg, I.J., Contributions to the problem of approximation of equidistant data by analytic functions, Q. Appl. Math., Vol. 4, pp. 45-99; pp 112-141, 1946
3. Cox, M.G., The numerical evaluation of B-splines, National Physical Laboratory DNAC 4, August 1971.
4. De Boor, C., On calculation with B-splines. Jour. Approx Theory, Vol. 6, pp. 50-62, 1972
5. Gordon, W.J., and Riesenfeld, R.F., Bernstein-Bézier methods for the computer aided design of free-form curves and surfaces, J. ACM, Vol. 21, pp. 293-310, 1974.
6. J. Hoschek, and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A. K. Peters, CRC Press, Wellesley, MA, 1993.

7. S. T. Tuohy & N. M. Patrikalakis, "Representation of Geophysical Maps with Uncertainty", in N. M. Thalmann and D. Thalmann, editors, *Communication with Virtual Worlds*, *Proceedings of CG International '93*, pp. 179–192. Springer, Tokyo, June 1993.
8. S. T. Tuohy & N. M. Patrikalakis, "Nonlinear Representation of Geophysical Data with Uncertainty", Design Laboratory Memorandum 94–2, MIT, Department of Ocean Engineering, Cambridge, MA, March 1994.
9. J. W. Yoon, "A virtual environment for the Visualization of Geophysical Ocean Data Sets", M.S Thesis, Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, August, 1994.
10. N. M. Patrikalakis, C. Chryssostomidis, S. T. Tuohy, J.G. Bellingham, J. J. Leonard, J. W. Bales, B. A Moran and J. W. Yoon, "Virtual Environments for Ocean Exploration and Visualization", Massachusetts Institutes of Technology (MIT) Design Laboratory Memorandum 95–1, January 3, 1995.
11. S. T. Tuohy & N. M. Patrikalakis, "Non-linear Data Representation for Ocean Exploration and Visualization", *Journal of Visualization and Computer Animation*, vol. 7, no. 3, pp. 125–139, 1996.
12. A. M. Anile, S. Deodato and G. Privitera, "Implementing Fuzzy Arithmetic", *Fuzzy Sets and Systems*, vol. 72, no. 2, pp. 239–250, 1995.
13. A. M. Anile, B. Faldicieno, G. Gallo, S. Spinello and M. Spagnuolo, "Approximation of Bathymetric Data with FBS", in *SIAM CAGD 97 Conference*, Nashville, TN, (1997).
14. G. Gallo and S. Spinello, "Pixels Classification in Noisy Digital Pictures using Fuzzy Arithmetic", in *Proceedings WSCG'98*, Pilsen, The Czech Republic, 1998.
15. G. Gallo and M. Spagnuolo, "Uncertainty Coding and Controlled Data Reduction using FBS", in *Proceedings CGI98*, Hannover, Germany, 1998.
16. A. M. Anile, B. Faldicieno, G. Gallo, M. Spagnuolo and S. Spinello, "Modeling Uncertain Data with Fuzzy B-Splines", *Fuzzy Sets and Systems*, vol. 113, no. 3, pp. 397–410, 2000.
17. A. M. Anile & S. Spinella, "Modeling Uncertain Sparse Data with Fuzzy B-splines", *Reliable Computing*, vol. 10, no. 5, pp. 335–355, 2004.
18. A. F. Wahab, J. M. Ali, A. A. Majid and A. O. M Tap, "Fuzzy Set in Geometric Modeling", in *Proceedings International Conference on Computer Graphics, Imaging and Visualization*, CGIV, Penang, pp. 227–232, (2004).
19. L. A. Zadeh, "Fuzzy Sets", *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
20. K. Atanassov, Intuitionistic fuzzy sets. Fuzzy Sets Systems, (1986), 20,87–96.
21. N. Z Zaidi, and M. I. E Zulkifly, Intuitionistic Fuzzy Bézier Curve Approximation Model for Uncertainty Data. Proceeding of Sciences and Mathematics Faculty of Sciences UTM, (2021), 3, 42-53.
22. F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. Infinite Study, (2005).
23. F. Tas, and S. Topal, Bezier Curve Modeling for Neutrosophic Data Problem. Neutrosophic set and system University of New Mexico, (2017).
24. S. Topal, and F. Tas, Bézier Surface Modeling for Neutrosophic Data Problems. Neutrosophic set and system University of New Mexico, (2018), (19), 19-23.
25. S. Alias and D. Mohamad, A Review of the Neutrosophic Set and Its Development. Discovering Mathematics. (2017). Vol. 3(2). Pp. 61-69.
26. A. F. Wahab, J. M. Ali, A. A. Majid, "Fuzzy Geometric Modeling", in *Proceedings Sixth International Conference on Computer Graphics, Imaging and Visualization*, *CGIV*, Tianjin, China, pp. 276–280, 11–14 August 2009.

Received: Mar 22, 2023. Accepted: Aug 25, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.67

Graphical Method for Solving Neutrosophical Nonlinear Programming Models

Maissam Jdid [1,*] and Florentin Smarandache [2]

[1] Faculty member, Damascus University, Faculty of Science, Department of Mathematics, Syria; maissam.jdid66@damascusuniversity.edu.sy.

[2] University of New Mexico, Mathematics, Physics and Natural Sciences Division 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu.

* Correspondence: maissam.jdid66@damascusuniversity.edu.sy.

Abstract: An important method for finding the optimal solution for linear and nonlinear models is the graphical method, which is used if the linear or nonlinear mathematical model contains one, two, or three variables. The models that contain only two variables are among the most models for which the optimal solution has been obtained graphically, whether these models are linear or non-linear in references and research that are concerned with the science of operations research, when the data of the issue under study is classical data. In this research, we will present a study through, which we present the graphical method for solving Neutrosophical nonlinear models in the following case: A nonlinear programming issue, the objective function is a nonlinear function, and the constraints are linear functions. Note that we can use the same method if **(i)** the objective function follower is a linear follower and the constraints are nonlinear; **(ii)** the objective function is a non-linear follower and the constraints are non-linear. In the three cases, the nonlinear models are neutrosophic, and as we know, the mathematical model is a nonlinear model if any of the components of the objective function or the constraints are nonlinear expressions, and the nonlinear expressions may be in both. At the left end of the constraints are neutrosophic values, at least one or all of them. Then, the possible solutions to the neutrosophic nonlinear programming problem are the set of rays $NX \in R^n$that fulfills all the constraints. As for the region of possible solutions, it is the region that contains all the rays that fulfill the constraints. The optimal solution is the beam that fulfills all constraints and at which the function reaches a maximum or minimum value, depending on the nature of the issue under study (noting that it is not necessary to be alone).

Keywords: Nonlinear Models; Neutrosophic Logic; Neutrosophic Nonlinear Models; Graphical Method.

1. Introduction

Problems of mathematical examples search for maximizing or minimizing a certain quantity that we call the objective function, and this quantity depends on a number of decision variables, as these variables may be independent of each other or linked to each other through a set of constraints. Studying the methods of solving nonlinear programming problems that we encounter in many practical issues, for example when we want to determine the cost of producing or purchasing goods, as well as the cost of storing manufactured or unprocessed materials – and so on. It led to the creation of a basic structure used to find these solutions from these methods, the graphical solution method that was presented in many references using classical data, and due to the great interest in the research that was published in many international journals, which dealt with some topics of operations research using the concepts of science neutrosophic [1-11] The science that laid the foundations of the American scientist and mathematical philosopher Florentin Smarandache, which explains the stages of its development. What was mentioned in the research [1], we will present in this research the graphic method used to find the optimal solution for nonlinear neutrosophic models, models that take data neutrosophic values of indefinite values A complete determination is not certain, and in reality it belongs to any neighborhood of the classical values and is given as follows: It is $Na = \text{a} \pm \varepsilon$ where ε is the indeterminacy and takes one of the forms $\varepsilon = [\lambda_1, \lambda_2\,]$ or $\varepsilon = \{\lambda_1\,, \lambda_2\}$ or $\varepsilon \in [\lambda_1, \lambda_2\,]$ -- otherwise, which is any neighborhood of the value a that we get during data collection.

2. Discussion

The importance of nonlinear models comes from the fact that many practical issues lead to nonlinear models, which prompted many researchers and scholars to search for ways to solve these models. Many methods were presented that helped the great development of computer science to find them and were presented according to classical logic, i.e. data were specific values. Appropriate for the time period in which they were collected, and since the purpose of any study of such issues is to develop plans for the course of work in the future, the decision makers faced great difficulty because of the instability of the conditions surrounding the work environment and in order to control all conditions and provide ideal decisions for the issues that turn into models In two previous researches, we presented a formulation of some concepts of nonlinear programming, and one of the ways to solve it is the method of Lagrangian multiplication for models constrained by equal constraints using the concepts of neutrosophic science [12,13].

The neutrosophic mathematical model [12]:

In the problem of examples where the objective and constraints are in the form of neutrosophic mathematical functions, then the neutrosophic mathematical model is written in the following form:

$$Nf = Nf(x_1\,, x_2, --, x_n) \to (Max\,)or\ (Min)$$

According to the following restrictions:

$$Ng_i(x_1\,, x_2, --, x_n)\begin{pmatrix}\leq\\ \geq\\ =\end{pmatrix}Nb_i \quad ; i = 1,2, ---, m$$

$$x_1\,, x_2, --, x_n \geq 0$$

In this model, the examples of the variables in the objective function and in the constraints are neutrosophic values, as well as the other side of the relations that represent the constraints.

Based on the information provided in the reference [14]:

The graphic method to find the optimal solution for nonlinear problems:

This method is suitable for simple problems that contain only two variables, it is impractical for problems that contain more than one variables or in which the objective function is complex in addition to the presence of restrictions that we cannot express in simple forms, so to find the optimal solution for a nonlinear model in a graphic way we represent the constraints among the coordinate axes, we define the common solution area for these constraints, so that it is the area of the accepted solutions for the mathematical model, then we represent the objective function in order to determine the optimal solution.

We have the following example using classic values [14]:

$$Maxf(x_1\,, x_2) = (x_1 -\ 2)^2 + (x_2 -\ 3)^2$$

$$x_1 + 2x_2 \leq 12$$

$$x_1 + x_2 \leq 9$$

$$x_1\,, x_2 \geq 0$$

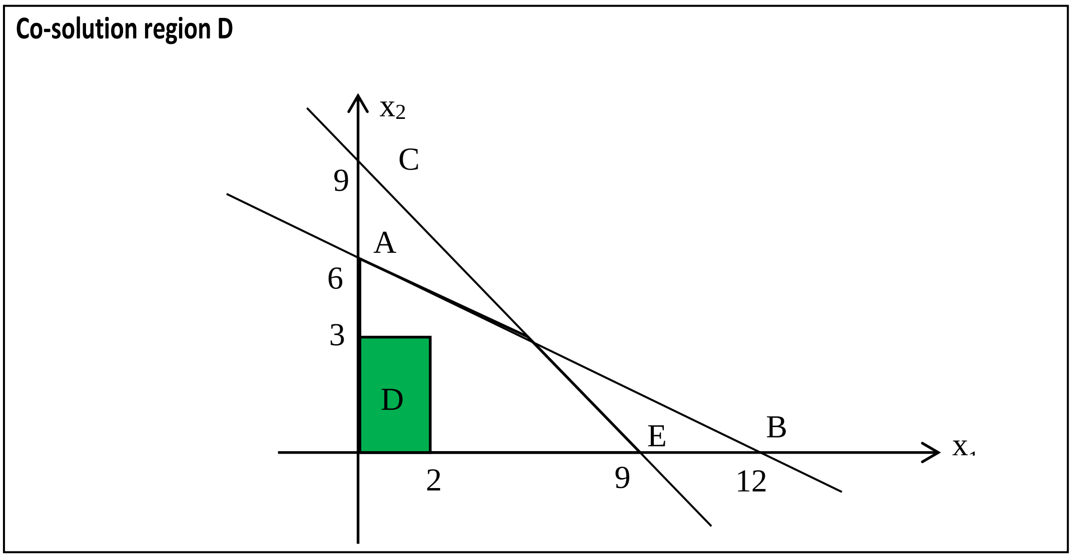


Figure 1. The first figure represents the solution to Example No (1).

From the Figure 1, it is clear $f^* = Minf = (2-2)^2 + (3-3)^2 = 0$
Whereas, the smallest value reached by the function is in the center $M(2\,,3)$
As for if it is required to find $Maxf$ it will be on point $E(9\,,0)$ and therefore

$$f^* = Maxf = (9-2)^2 + (0-3)^2 = 58$$

Neutrosophical formula for the previous example:
Finding the optimal solution for a nonlinear programming problem, where the objective function is a nonlinear function, and the constraints are linear functions:
Find the minimum value of the function:

$$MinNf(x_1\,,x_2) = (x_1 - 2)^2 + (x_2 - 3)^2$$

Within the restrictions:

$$x_1 + 2x_2 \leq 12 + \varepsilon_1$$

$$x_1 + x_2 \leq 9 + \varepsilon_2$$

$$x_1\,,x_2 \geq 0$$

Where ε_1 and ε_2 It is the indeterminacy and we take it as follows.
Then the restrictions will look like this:
In this example, we take $\varepsilon_1 \in [0\,,3]$ and $\varepsilon_2 \in [0\,,2]$, and then the problem is written as follows:

$$x_1 + 2x_2 \in [12\,,15]$$

$$x_1 + x_2 \in [9\,,11]$$

$$x_1\,,x_2 \geq 0$$

Clarification: The second term of the constraints expresses the available potentials taken as Neutrosophical values.
We need to find the vector $NX^* = (x_1^*\,,x_2^*)$ So that the inequality is fulfilled:

$$Nf(NX^*) \leq Nf(X)$$

$$\forall NX \in D$$

The solution:

1. We define the solution area D This is done by representing the constraints in the coordinate plane ox_1x_2

First constraint:

$$x_1 + 2x_2 \in [12 , 15]$$

We draw the straight line represented by the equation $x_1 + 2x_2 \in [12 , 15]$

$$x_1 = 0 \Rightarrow 2x_2 \in [12 , 15] \Rightarrow x_2 \in [6 , 7.5]$$

$$A(0 , [6 , 7.5])$$

$$x_2 = 0 \Rightarrow x_1 \in [12 , 15] \Rightarrow$$

$$B([12 , 15] , 0)$$

We define the region where the first constraint is satisfied:
We know that the line represented by equation $x_1 + 2x_2 \in [12 , 15]$ the plane defined by the first quadrant is divided into two halves of a plane. We take a point, not on the specificity, from one of the two halves of the plane, and let the point be $(0 , 0)$ we substitute in the constraint, we note that it achieves the inequality that represents the first constraint, that is, half of the plane to which this point belongs is half of the solution plane.

The second constraint:

$$x_1 + x_2 \in [9 , 11]$$

We draw the straight line represented by the equation $x_1 + x_2 \in [9 , 11]$

$$x_1 = 0 \Rightarrow x_2 \in [9 , 11]$$

$$C(0 , [9 , 11])$$

$$x_2 = 0 \Rightarrow x_1 \in [9 , 11] \Rightarrow$$

$$E([9 , 11], 0)$$

We define the region where the second constraint is satisfied:
The line represented by equation $x_1 + x_2 \in [9 , 11]$, the plane defined by the first quarter is divided into two halves of a plane. We take a point, not to be determined, from one of the two halves of the plane, let the point be $(0 , 0)$ and substitute in the constraint. We note that it achieves the inequality representing the second constraint, meaning that half of the plane to which this point belongs is half of the solution plane.
To find the optimal solution, we draw the objective function, which is a circle with a point center $M(2 , 3)$ radius $r = \sqrt{Nf}$.
From the figure, it is clear that $Min\,Nf = 0$ the minimum value reached by the objective function is at the center of the circle, i.e. at the point $M(2 , 3)$.
If required, find the maximum value of the function:

$$Maxf(x_1 , x_2) = (x_1 - 2)^2 + (x_2 - 3)^2$$

Within the restrictions:

$$x_1 + 2x_2 \leq 12 + \varepsilon_1$$

$$x_1 + x_2 \leq 9 + \varepsilon_2$$

$$x_1 , x_2 \geq 0$$

In this case, we know that the optimal solution is located on the vertices of the common solution region, i.e. on the vertices of the polygon $OADE$, we have the coordinates of these points $O(0 , 0)$ ، $A(0 , [6 , 7.5])$ ، $E([9 , 11], 0)$ for the coordinates of the point D , and we determine it from the study of the intersection of the two lines represented by the following equations:

$$x_1 + 2x_2 \in [12\,,15]$$

$$x_1 + x_2 \in [9\,,11]$$

Solving the two equations, we get $D([6\,,7],[3\,,4])$ then we calculate the value of the function at these points

$$(O(0\,,0)\,) = (0 - \;2)^2 + (0 - \;3)^2 = 13$$

$$f\big(A(0\,,[6\,,7.5]\,)\big) = (0 - \;2)^2 + ([6\,,7.5] - \;3)^2 = 4 + ([3\,,4.5]\,)^2 \in [13\,,24.25]$$

$$f\big(E([9\,,11],0\,)\big) = ([9\,,11] - \;2)^2 + (0 - \;3)^2 = ([7\,,9]\,)^2 + 9 \in [58\,,90]$$

$$f\big(D([6\,,7],[3\,,4]\;)\big) = ([6\,,7] - \;2)^2 + ([3\,,4] - \;3)^2 \in [19\,,24]$$

The Maximum value the function takes at point $E([9\,,11],0\,)$ and is

$$R = \sqrt{[58\,,90]} \in [7.6\,,9.5]$$

Therefore, the optimal solution is a circle centered at point $M(2\,,3)$ and whose radius is one of the domain values $[7.6\,,9.5]$.

For clarification, we draw the Figure 2, for one of the values we find:

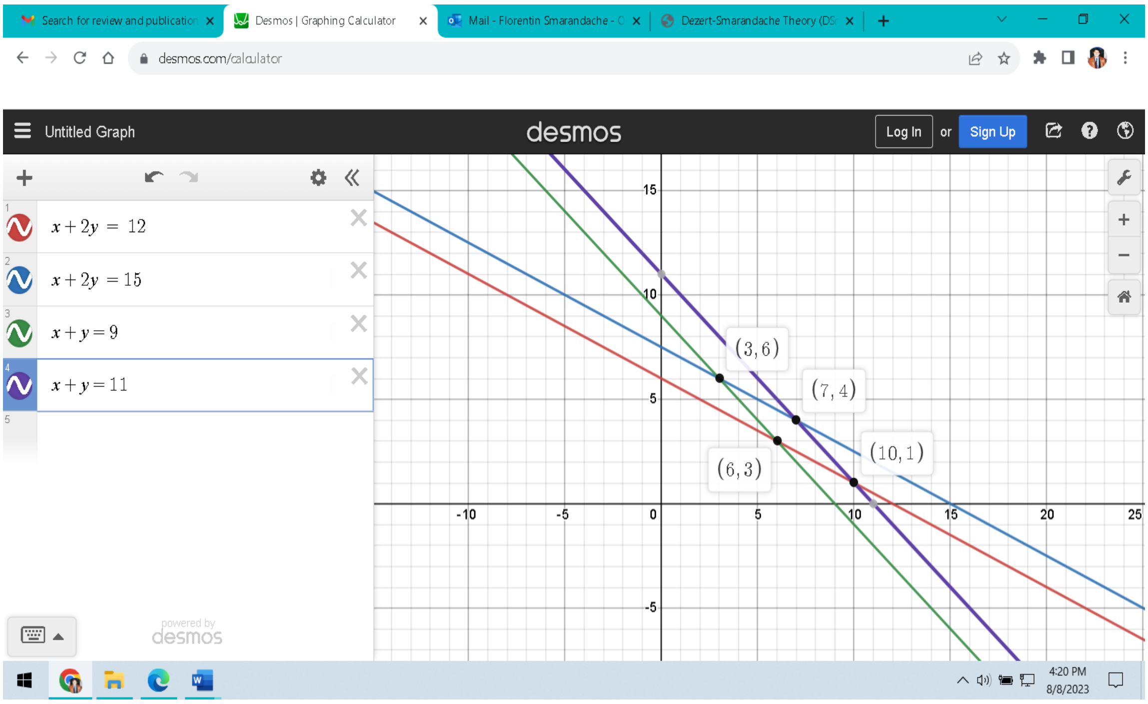


Figure 2. Determination of the joint solution region of neutrosophic constraints.

For the Neutrosophical objective function, it represents a set of circles whose center $M(2\,,3)$ and radius are one of the domain values $[7.6\,,9.5]$.

The following Figure 3 shows one of these circles:

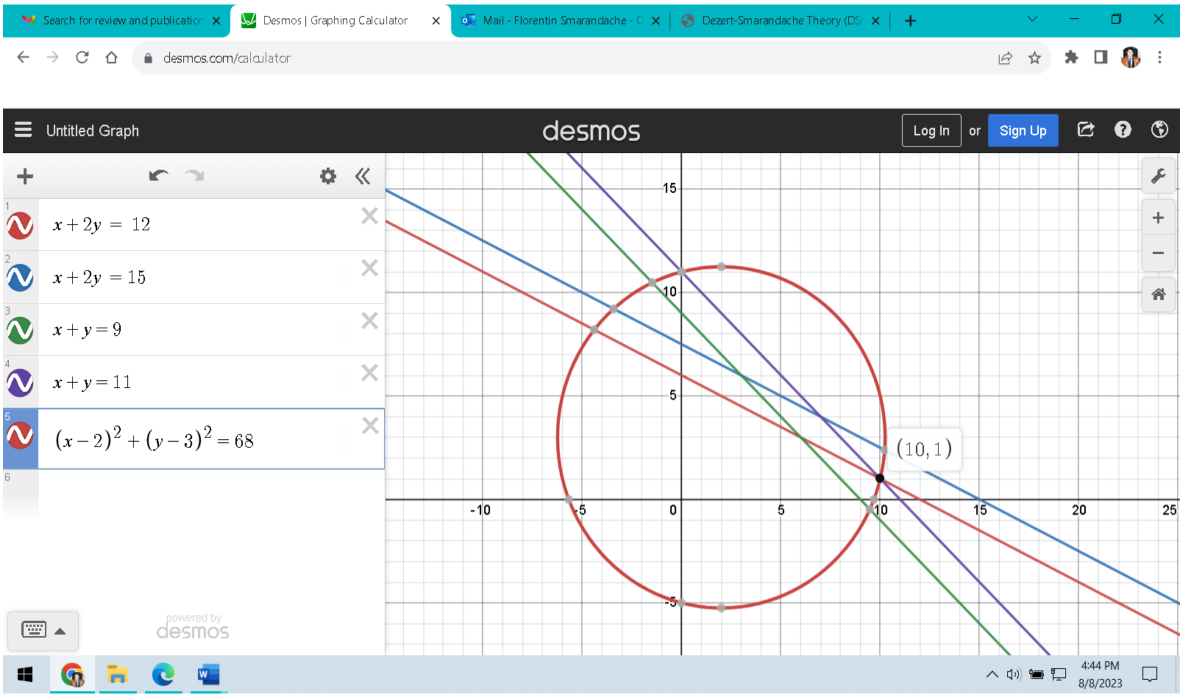


Figure 3. The graphical representation of one of the solutions of the example neutrosophical values.

We note that for both sides of the domain in both constraints is the optimal solution $f^* = Maxf = 68 \in [58\,,90]$, wear $[58\,,90]$, the field that represents the maximum optimal solution of the neutrosophic model.

3. Conclusions

The graphical method is one of the important methods for finding the optimal solution for linear and nonlinear models. Therefore, it was necessary to present this study, which explains the difference between dealing with classical values and Neutrosophical values, and as we noticed from the results of the solution in the example, Neutrosophical values give us optimal solutions that are close to the optimal solution in the case of classical values, that is, they are in line with the conditions surrounding a work environment The system that this mathematical model represents, so it provides a safe environment that protects the systems from falling into losses and making greatest profits from them.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Florentin Smarandache , Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
2. Maissam jdid ,Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
3. Maissam jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS),Vol. 20, No.1,2023
4. Mohammed Alshikho, Maissam Jdid, Said Broumi , Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19 ,Journal Prospects for Applied Mathematics and Data Analysis ,Vol 01, No,02 USA,2023
5. Mohammed Alshikho, Maissam Jdid, Said Broumi ,A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, , Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
6. Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model $\mathrm{M}\backslash M\backslash 1$, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .1, 2023
7. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .2, 2023
8. Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global,2023
9. Maissam Jdid, Florentin Smarandache , Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023
10. Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables , Neutrosophic Sets and Systems ,NSS,Vol.56,2023
11. Maissam Jdid, Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, , Journal of Neutrosophic Systems with Applications, Vol. 3, 2023
12. Maissam Jdid, Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
13. Maissam Jdid, Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints, Journal of Neutrosophic Systems with Applications, Vol. 6, 2023
14. Al Hamid .Mohammed Dabbas, Mathematical programming, Aleppo University, Syria, 2010. (Arabic version).

Received: May 10, 2023. Accepted: Aug 31, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.63

Fixed Point Results in Neutrosophic Rectangular Extended b-Metric Spaces

Naeem Saleem [1,2,*], Umar Ishtiaq [3], Khaleel Ahmad [4], Salvatore Sessa [5,*], and Ferdinando Di Martino [5]

[1] Department of Mathematics, University of Management and Technology, Lahore, Pakistan; naeem.saleem@umt.edu.pk.
[2] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa.
[3] Office of research, innovation and commercialization, University of management and technology, Lahore, Pakistan; umar.ishtiaq@umt.edu.pk.
[4] Department of Math & Stats, International Islamic University Islamabad, Pakistan; s2023265011@umt.edu.pk.
[5] Dipartimento di Architettura, Università Dinapoli Federico II, Via Toledo 403, 80121 Napoli, Italy; sessa@unina.it; fdimarti@unina.it.

* Correspondence: naeem.saleem@umt.edu.pk; sessa@unina.it.

Abstract: In this manuscript, we establish the notion of neutrosophic rectangular extended b-metric spaces and derive some fixed point results for contraction mappings. Also, we provide several non-trivial examples. Our results are more generalized with respect to the existing ones in the literature. At the end of the paper, we provide an application to non-linear fractional differential equations to test the validity of the results.

Keywords: Neutrosophic Metric Spaces; Fixed Point; Graphical View; Non-Linear Fractional Differential Equations.

1. Introduction

In 1965, Zadeh [1] developed the "fuzzy notion" to contrast imprecise terms. Fuzzy sets (FSs) presented in [1] and metric spaces presented in [2] are combined to establish the concept of fuzzy metric spaces (FMSs), in which membership function is used. The notion of FMSs first introduced by Kramosil and Michalak [3] in 1975 and then George and Veeramani [4, 5] updated in 1994. Garbiec [6] established the fuzzy version of the Banach fixed point result. The notion of FSs only deals with membership functions, so there is a gap that FSs did not deal with non-membership functions. Atanassov [7] filled this gap to establish the concept of intuitionistic fuzzy sets (IFSs), in which, he used both degrees, the degree of membership and the degree of non-membership. But, there is still a gap that IFSs did not deals with naturalness. Smarandache [30] filled this gap to propose the concept of neutrosophic sets (NSs), as a generalization of IFSs. By combining the concepts of NSs and metric spaces, Kirişci and Simsek [32] presented the notion of neutrosophic metric spaces (NMSs).

Fuzzy rectangular metric spaces and fuzzy rectangular b-metric spaces (FRBMSs) were introduced by Mehmood et al. [9], who also demonstrated the Banach contraction principle in the context of FRBMSs. The concept of orthogonal FMSs was developed by Hezarjaribi [10], who also demonstrated several fixed point theorems. The authors in [11–14, 33-38] established several interesting fixed point results. Park and Jeong [15] established fixed point results for fuzzy mappings. An intuitionistic fuzzy b-metric space was presented by Konwar [16]. The authors in [17–18,] demonstrated a number of fixed point results for in the context of an IFMS. Nice work was done on fractional differential equations by the authors in [19–20]. Several fixed point results were proven by Javed et al. [21] in the setting of fuzzy b-metric-like spaces. Uddin et al. [22] presented a number of fixed point theorems for contraction mappings in the context of orthogonal controlled fuzzy metric spaces. Numerous algebraic structures have been used by mathematicians to apply several novel fuzzy set models [23–27, 32-35]. The idea of pentagonal controlled FMSs was recently given by Aftab et al. [28], who also demonstrated various fixed point theorems. Kattan et al. [29] established some fixed point results in a generalization of an IFMS.

Jeyaraman et al. [39] proved common fixed point theorems in intuitionistic generalized fuzzy cone metric spaces. Ishtiaq et al. [40] derived several a fixed point results in the context of generalized neutrosophic cone metric spaces. Gupta et al. [41] examined the uniqueness of solution by employing CLR-property on V-fuzzy metric spaces. Chauhan et al. [42] examined the existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces. Gupta et al. [43] solved some fixed point theorems for contraction mappings and investigate the xistence of fixed points for J-ψ-fuzzy contractions in fuzzy metric spaces endowed with graph.

In this manuscript, we aim to introduce the concept of neutrosophic rectangular extended b-metric spaces (NREBMSs) and to establish several fixed point results for contraction mappings. Also, we provide some non-trivial examples and an application to non-linear fractional differential equations to show the validity of results herein. An open problem is also raised after the conclusion section.

2. Preliminaries

In this section, we provide some basic notions that are helpful for readers to understand the main results.

Definition 2.1: [8] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm (CTN) if it satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $\hbar * 1 = \hbar$ for all $\hbar \in [0,1]$;

(iv) $\hbar * \ell \le c * d$ whenever $\hbar \le c$ and $\ell \le d$, for all $\hbar, \ell, c, d \in [0,1]$.

Example 2.1: [8] $\hbar * \ell = \hbar\ell$ and $\hbar * \ell = \min\{\hbar, \ell\}$ are CTN.

Definition 2.2: [8] A binary operation $\circ$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-conorm (CTCN) if it meets the below assertions:

T1. $\circ$ is associative and commutative;

T2. $\circ$ is continuous;

T3. $\hbar \circ 0 = 0$, for all $\hbar \in [0, 1]$;

T4. $\hbar \circ \ell \le c \circ d$ whenever $\hbar \le c$ and $\ell \le d$, for all $\hbar, \ell, c, d \in [0,1]$.

Example 2.2: [8] $\hbar \circ \ell = \max\{\hbar, \ell\}$ is CTCN.

Definition 2.3: [4] Let Σ is nonempty set, $\mathbb{K}$ is a FS on $\Sigma \times \Sigma \times (0, +\infty)$, and $*$ is a CTN. Then a triplet $(\Sigma, \mathbb{K}, *)$ is known as FMS, if it verifies the following conditions, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau > 0$:

F1. $\mathbb{K}(\varkappa, \vartheta, \sigma) > 0$;

F2. $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

F3. $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

F4. $\mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, z, \tau) \le \mathbb{K}(\varkappa, z, \sigma + \tau)$;

F5. $\mathbb{K}(\varkappa, \vartheta, .): (0, + + \infty) \to (0,1]$ is continuous.

Definition 2.4: [9] Let Σ is nonempty set, $\mathbb{K}$ is a FS on $\Sigma \times \Sigma \times [0, +\infty)$, and $*$ is a CTN. Then $(\Sigma, \mathbb{K}, *, \ell)$ is known as FRBMS, if it verifies the following conditions, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w \geq 0$:

S1. $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

S2. $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

S3. $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

S4. $\mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathrm{u}, \tau) * \mathbb{K}(\mathrm{u}, z, w) \leq \mathbb{K}(\varkappa, z, \ell(\sigma + \tau + w))$;

S5. $\mathbb{K}(\varkappa, \vartheta, .): (0, +\infty) \to (0,1]$ is left continuous and $\lim_{\sigma \to +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

Definition 2.5: [32] Let Σ be a non-empty set and $\mathbb{K}, \Pi, \Delta$ are NSs on $\Sigma \times \Sigma \times [0, +\infty)$. Suppose $\psi: \Sigma \times \Sigma \to [1, +\infty)$ be a function, $*$ and $\circ$ are CTN and CTCN respectively. Then, a six tuple $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is known as NMS, if the following conditions are satisfying, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w > 0$,

(N1) $\mathbb{K}(\varkappa, \vartheta, \sigma) + \Pi(\varkappa, \vartheta, \sigma) + \Delta(\varkappa, \vartheta, \sigma) \leq 3$;

(N2) $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

(N3) $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

(N4) $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

(N5) $\mathbb{K}(\varkappa, z, \sigma + \tau) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, z, \tau)$;

(N6) $\mathbb{K}(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

(N7) $\Pi(\varkappa, \vartheta, 0) = 1$;

(N8) $\Pi(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(N9) $\Pi(\varkappa, \vartheta, \sigma) = \Pi(\vartheta, \varkappa, \sigma)$;

(N10) $\Pi(\varkappa, z, \sigma + \tau) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, z, \tau)$;

(N11) $\Pi(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$.

(N12) $\Delta(\varkappa, \vartheta, 0) = 1$;

(N13) $\Delta(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(N14) $\Delta(\varkappa, \vartheta, \sigma) = \Delta(\vartheta, \varkappa, \sigma)$;

(N15) $\Delta(\varkappa, z, \sigma + \tau) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, z, \tau)$;

(N16) $\Delta(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma \to +\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$.

Then $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is called an NMS.

3. Main Section

In this section, we introduce the concept of NREBMS and establish some fixed point results.

Definition 3.1: Let Σ be a non-empty set and $\mathbb{K}, \Pi, \Delta$ are NSs on $\Sigma \times \Sigma \times [0, +\infty)$. Suppose $\psi: \Sigma \times \Sigma \to [1, +\infty)$ be a function, $*$ and $\circ$ are CTN and CTCN respectively. Then, a six tuple $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is known as NREBMS, if the following conditions are satisfying, for all $\varkappa, \vartheta, z \in \Sigma$ and $\sigma, \tau, w > 0$,

(NRE1) $\mathbb{K}(\varkappa, \vartheta, \sigma) + \Pi(\varkappa, \vartheta, \sigma) + \Delta(\varkappa, \vartheta, \sigma) \leq 3$;

(NRE2) $\mathbb{K}(\varkappa, \vartheta, 0) = 0$;

(NRE3) $\mathbb{K}(\varkappa, \vartheta, \sigma) = 1$ if and only if $\varkappa = \vartheta$;

(NRE4) $\mathbb{K}(\varkappa, \vartheta, \sigma) = \mathbb{K}(\vartheta, \varkappa, \sigma)$;

(NRE5) $\mathbb{K}\big(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)\big) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathfrak{u}, \tau) * \mathbb{K}(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E}\backslash\{\varkappa, z\}$;

(NRE6) $\mathbb{K}(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma\to+\infty} \mathbb{K}(\varkappa, \vartheta, \sigma) = 1$.

(NRE7) $\Pi(\varkappa, \vartheta, 0) = 1$;

(NRE8) $\Pi(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(NRE9) $\Pi(\varkappa, \vartheta, \sigma) = \Pi(\vartheta, \varkappa, \sigma)$;

(NRE10) $\Pi\big(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)\big) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathfrak{u}, \tau) \circ \Pi(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E}\backslash\{\varkappa, z\}$;

(NRE11) $\Pi(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma\to+\infty} \Pi(\varkappa, \vartheta, \sigma) = 0$.

(NRE12) $\Delta(\varkappa, \vartheta, 0) = 1$;

(NRE13) $\Delta(\varkappa, \vartheta, \sigma) = 0$ if and only if $\varkappa = \vartheta$;

(NRE14) $\Delta(\varkappa, \vartheta, \sigma) = \Delta(\vartheta, \varkappa, \sigma)$;

(NRE15) $\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)\big) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathfrak{u}, \tau) \circ \Delta(\mathfrak{u}, z, w), \forall$ distinct $\vartheta, \mathfrak{u} \in \mathcal{E}\backslash\{\varkappa, z\}$;

(NRE16) $\Delta(\varkappa, \vartheta, .): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\sigma\to+\infty} \Delta(\varkappa, \vartheta, \sigma) = 0$.

Example 3.1: Let $(\mathcal{E}, d)$ be a rectangular metric space, define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$ and define $\mathbb{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$\mathbb{K}(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + d(\varkappa, \vartheta)},$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)} \text{ and } \Delta(\varkappa, \vartheta, \sigma) = \frac{d(\varkappa, \vartheta)}{\sigma} \text{ for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0,$$

with CTN $\hbar * \ell = \min\{\hbar, \ell\}$ and CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$. Then $(\mathcal{E}, \mathbb{K}, \Pi, \Delta, *, \circ)$ is an NREBMS.

Proof: Properties (NRE1)-(NRE4), (NRE6)-(NRE9), (NRE11)-(NRE14) and (NRE16) are easy obvious. Here, we prove (NRE5), (NRE10) and (NRE15).

(NRE5) $\mathbb{K}\big(\varkappa, z, \psi(\varkappa, z)(\sigma+\tau+w)\big) \geq \mathbb{K}(\varkappa, \vartheta, \sigma) * \mathbb{K}(\vartheta, \mathfrak{u}, \tau) * \mathbb{K}(\mathfrak{u}, z, w)$ for all distinct $\vartheta, \mathfrak{u} \in \mathcal{E}\backslash\{\varkappa, z\}$.

Suppose that

$$\mathbb{K}(\varkappa, \vartheta, \sigma) \leq \mathbb{K}(\vartheta, \mathfrak{u}, \tau)$$

and

$$\mathbb{K}(\varkappa, \vartheta, \sigma) \leq \mathbb{K}(\mathfrak{u}, z, w),$$

which implies that

$$\frac{\sigma}{\sigma + d(\varkappa, \vartheta)} \leq \frac{\tau}{\tau + d(\vartheta, \mathfrak{u})}$$

and

$$\frac{\sigma}{\sigma + d(\varkappa, \vartheta)} \leq \frac{w}{w + d(\mathfrak{u}, z)}.$$

So, we obtain

$$\sigma d(\vartheta, \mathrm{u}) \leq \tau d(\varkappa, \vartheta) \text{ and } \sigma d(\mathrm{u}, z) \leq w d(\varkappa, \vartheta).$$

This implies

$$(\tau + w) d(\varkappa, \vartheta) \geq \sigma[d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)] \qquad (1)$$

Now, observe that

$$\mathrm{K}(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \geq \mathrm{K}(\varkappa, \vartheta, \sigma)$$
$$\Leftrightarrow \frac{\psi(\varkappa, z)(\sigma + \tau + w)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$
$$\Leftrightarrow \frac{\psi(\varkappa, z)(\sigma + \tau + w)}{\psi(\varkappa, z)(\sigma + \tau + w) + \psi(\varkappa, z)[d(\varkappa, \vartheta) + d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)]} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$
$$\Leftrightarrow \frac{\sigma + \tau + w}{\sigma + \tau + w + d(\varkappa, \vartheta) + d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)} \geq \frac{\sigma}{\sigma + d(\varkappa, \vartheta)}$$
$$\Leftrightarrow (\tau + w) d(\varkappa, \vartheta) \geq \sigma[d(\vartheta, \mathrm{u}) + d(\mathrm{u}, z)].$$

Hence,

$$\mathrm{K}(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \geq \mathrm{K}(\varkappa, \vartheta, \sigma) * \mathrm{K}(\vartheta, \mathrm{u}, \tau) * \mathrm{K}(\mathrm{u}, z, w).$$

(NRE10) $\Pi(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathrm{u}, \tau) \circ \Pi(\mathrm{u}, z, w)$ for all distinct $\vartheta, \mathrm{u} \in \Sigma \backslash \{\varkappa, z\}$.

Recall that

$$d(\varkappa, z) = d(\varkappa, z) \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Therefore,

$$d(\varkappa, z) \leq [\sigma + \tau + w + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Define $\psi: \Sigma \times \Sigma \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$. Then

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{d(\mathrm{u}, z)}\right\}.$$

Also observe the fact that

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{\tau + d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{w + d(\mathrm{u}, z)}\right\}.$$

This implies

$$\frac{d(\varkappa, z)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \leq \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma + d(\varkappa, \vartheta)}, \frac{d(\vartheta, \mathrm{u})}{\tau + d(\vartheta, \mathrm{u})}, \frac{d(\mathrm{u}, z)}{w + d(\mathrm{u}, z)}\right\}.$$

Then

$$\Pi(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \leq \max\{\Pi(\varkappa, \vartheta, \sigma), \Pi(\vartheta, \mathrm{u}, \tau), \Pi(\mathrm{u}, z, w)\}.$$

Hence,

$$\Pi(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \leq \Pi(\varkappa, \vartheta, \sigma) \circ \Pi(\vartheta, \mathrm{u}, \tau) \circ \Pi(\mathrm{u}, z, w).$$

(NRE15) $\Delta(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathrm{u}, \tau) \circ \Delta(\mathrm{u}, z, w)$ for all distinct $\vartheta, \mathrm{u} \in \Sigma \backslash \{\varkappa, z\}$.

Observe that,

$$d(\varkappa, z) \leq [\sigma + \tau + w + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

Define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa + \vartheta$. Then

$$d(\varkappa, z) \leq [\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)] \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

This implies

$$\frac{d(\varkappa, z)}{\psi(\varkappa, z)(\sigma + \tau + w) + d(\varkappa, z)} \leq \max\left\{\frac{d(\varkappa, \vartheta)}{\sigma}, \frac{d(\vartheta, \mathrm{u})}{\tau}, \frac{d(\mathrm{u}, z)}{w}\right\}.$$

Then

$$\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \max\{\Delta(\varkappa, \vartheta, \sigma), \Delta(\vartheta, \mathrm{u}, \tau), \Delta(\mathrm{u}, z, w)\}.$$

Hence

$$\Delta\big(\varkappa, z, \psi(\varkappa, z)(\sigma + \tau + w)\big) \leq \Delta(\varkappa, \vartheta, \sigma) \circ \Delta(\vartheta, \mathrm{u}, \tau) \circ \Delta(\mathrm{u}, z, w).$$

Therefore, $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a NREBMS.

Remark 3.1: The above example is not a NMS. But, if we let $\psi = 1$, then it is NMS.

Example 3.2: Let $\mathcal{E} = [0,1]$ and define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \varkappa^2 + \vartheta^3$ and $\mathrm{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ \dfrac{\sigma}{\sigma + \max\{\varkappa, \vartheta\}^p}, & \text{otherwise} \end{cases}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{\max\{\varkappa, \vartheta\}^p}{\sigma + \max\{\varkappa, \vartheta\}^p}, & \text{otherwise} \end{cases}$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{\max\{\varkappa, \vartheta\}^p}{\sigma}, & \text{otherwise} \end{cases} \quad \text{for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0.$$

Then $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Example 3.3: Let $\mathcal{E} = [0, +\infty)$ and define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = 1 + \frac{\varkappa}{\vartheta}$ and $\mathrm{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \begin{cases} 1, & \text{if } \varkappa = \vartheta \\ \dfrac{\sigma}{\sigma + (\varkappa + \vartheta)^p}, & \text{otherwise} \end{cases}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{(\varkappa + \vartheta)^p}{\sigma + (\varkappa + \vartheta)^p}, & \text{otherwise} \end{cases}$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \begin{cases} 0, & \text{if } \varkappa = \vartheta \\ \dfrac{(\varkappa + \vartheta)^p}{\sigma}, & \text{otherwise} \end{cases} \quad \text{for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0.$$

Then $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Example 3.4: Let $\mathcal{E} = [0, +\infty)$ and define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if} \quad \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and $\mathrm{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0.$$

Then $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is an NREBMS with CTN $\hbar * \ell = \hbar \cdot \ell$, CTCN $\hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$.

Remark 3.2: The above examples 3.3 and 3.4 are also NREBMSs if we take $\hbar * \ell = \min\{\hbar, \ell\}$, and $\hbar \circ \ell = \max\{\hbar, \ell\}$.

Definition 3.2: Suppose $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a NREBMS and assume $\{\varkappa_n\}$ be a sequence in $\mathcal{E}$. Then

- $\{\varkappa_n\}$ is said to be a convergent sequence if there exists $\varkappa \in \mathcal{E}$ such that

$$\lim_{n \to +\infty} \mathrm{K}(\varkappa_n, \varkappa, \sigma) = 1, \text{for all } \sigma > 0$$

$$\lim_{n \to +\infty} \Pi(\varkappa_n, \varkappa, \sigma) = 0, \text{ for all } \sigma > 0.$$

and

$$\lim_{n \to +\infty} \Delta(\varkappa_n, \varkappa, \sigma) = 0, \text{ for all } \sigma > 0.$$

- $\{\varkappa_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \to +\infty} \mathrm{K}(\varkappa_n, \varkappa_{n+q}, \sigma) = 1 \text{ for all } \sigma > 0$$

$$\lim_{n \to +\infty} \Pi(\varkappa_n, \varkappa_{n+q}, \sigma) = 0 \text{ for all } \sigma > 0.$$

and

$$\lim_{n \to +\infty} \Delta(\varkappa_n, \varkappa_{n+q}, \sigma) = 0 \text{ for all } \sigma > 0.$$

- The NREBMS $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is called complete, if every Cauchy sequence is convergent in $\mathcal{E}$.

Example 3.5: Let $\mathcal{E} = [0, +\infty)$ and define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if} \quad \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and

$\mathrm{K}, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$\mathrm{K}(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa,\vartheta,\sigma)=\frac{|\varkappa-\vartheta|^p}{\sigma+|\varkappa-\vartheta|^p}\ \text{ for all } \varkappa,\vartheta\in\mathcal{E} \text{ and } \sigma>0,$$

then $(\mathcal{E},\mathrm{K},\Pi,\Delta,*,\circ)$ is an NREBMS with CTN $\hbar*\ell=\hbar\cdot\ell$, CTCN $\hbar\circ\ell=\max\{\hbar,\ell\}$ and $p\geq 1$.

Let $\{\varkappa_n\}=\frac{1}{n}$ for all $n\in\{1,2,3,\cdots\}$ be a sequence in $\mathcal{E}$, then $\{\varkappa_n\}$ converges to 0. Now

$$\lim_{n\to+\infty}\mathrm{K}(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\sigma}{\sigma+\left(\frac{1}{n}\right)^p}=1,$$

$$\lim_{n\to+\infty}\Pi(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n}\right)^p}{\sigma+\left(\frac{1}{n}\right)^p}=0,$$

and

$$\lim_{n\to+\infty}\Delta(\varkappa_n,0,\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n}\right)^p}{\sigma}=0.$$

That is, the sequence $\{\varkappa_n\}$ is convergent.

Example 3.6: Consider the preceding example and a sequence $\varkappa_n=\frac{1}{n}$ for all $n\in\{1,2,3,\cdots,\}$. Then for all $q\in\{1,2,3,\cdots\}$, we get

$$\lim_{n\to+\infty}\mathrm{K}(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\sigma}{\sigma+\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}=1,$$

$$\lim_{n\to+\infty}\Pi(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}{\sigma+\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}=0,$$

and

$$\lim_{n\to+\infty}\Delta(\varkappa_n,\varkappa_{n+q},\sigma)=\lim_{n\to+\infty}\frac{\left(\frac{1}{n+1}\right)^p+\cdots+\left(\frac{1}{n+q}\right)^p}{\sigma}=0.$$

That is, the sequence $\{\varkappa_n\}$ is Cauchy.

Lemma 3.1: Let $\{\varkappa_n\}$ be a Cauchy sequence in a NREBMS $(\mathcal{E},\mathrm{K},\Pi,\Delta,*,\circ)$ such that $\varkappa_n\neq\varkappa_m$, whenever $n\neq m$ for all $m,n\in\mathbb{N}$. Then $\{\varkappa_n\}$ converges to at most one point in $\mathcal{E}$.

Lemma 3.2: Let $\varkappa$ and ϑ be any two points in a NREBMS $(\mathcal{E},\mathrm{K},\Pi,\Delta,*,\circ)$. If for any $\eta\in(0,1)$, we have

$$\mathrm{K}(\varkappa,\vartheta,\eta\sigma)\geq\mathrm{K}(\varkappa,\vartheta,\sigma),\Pi(\varkappa,\vartheta,\eta\sigma)\leq\Pi(\varkappa,\vartheta,\sigma)\ \text{ and }\ \Delta(\varkappa,\vartheta,\eta\sigma)\leq\Delta(\varkappa,\vartheta,\sigma),$$

then $\varkappa=\vartheta$.

Theorem 3.1: Let $(\mathcal{E},\mathrm{K},\Pi,\Delta,*,\circ)$ be a complete NREBMS such that

$$\lim_{\sigma\to+\infty}\mathrm{K}(\varkappa,\vartheta,\sigma)=1,\ \lim_{\sigma\to+\infty}\Pi(\varkappa,\vartheta,\sigma)=0,\ \text{and}\ \lim_{\sigma\to+\infty}\Delta(\varkappa,\vartheta,\sigma)=0,\ \text{ for all } \varkappa,\vartheta\in\mathcal{E}.\quad(2)$$

Let $\xi:\mathcal{E}\to\mathcal{E}$ be a mapping satisfying

$$\mathrm{K}(\xi\varkappa,\xi\vartheta,\eta\sigma)\geq\mathrm{K}(\varkappa,\vartheta,\sigma),\qquad\Pi(\xi\varkappa,\xi\vartheta,\eta\sigma)\leq\Pi(\varkappa,\vartheta,\sigma),$$
$$\text{and }\Delta(\xi\varkappa,\xi\vartheta,\eta\sigma)\leq\Delta(\varkappa,\vartheta,\sigma)\qquad(3)$$

for all $\varkappa,\vartheta\in\mathcal{E},\eta\in(0,1)$. Then ξ has a unique fixed point $\mathfrak{u}\in\mathcal{E}$.

Proof: Let $\varkappa_0 \in \Sigma$ be an arbitrary point and let $n \in \mathbb{N}$ then begin an iterative process such that $\varkappa_{n+1} = \xi\varkappa_n$. Continuously, applying an inequality (3), we deduce that

$$\mathbb{K}(\varkappa_n, \varkappa_{n+1}, \sigma) \geq \mathbb{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right), \Pi(\varkappa_n, \varkappa_{n+1}, \sigma) \leq \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right) \text{ and } \Delta(\varkappa_n, \varkappa_{n+1}, \sigma)$$

$$\leq \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{\eta^n}\right). \qquad (4)$$

Since, $(\Sigma, \mathbb{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS, then for the sequence $\{\varkappa_n\}$, writing $\sigma = \frac{\sigma}{3} + \frac{\sigma}{3} + \frac{\sigma}{3}$ and using the rectangular inequality given in (NRE5), (NRE10) and (NRE15) on $\mathbb{K}(\varkappa_n, \varkappa_{n+p}, \sigma), \Pi(\varkappa_n, \varkappa_{n+p}, \sigma)$ and $\Delta(\varkappa_n, \varkappa_{n+p}, \sigma)$, we have the following cases.

Case 1: If p is odd, then $p = 2m + 1$ where $m \in \{1,2,3, \dots\}$. So, we have

$$\mathbb{K}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathbb{K}\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Using (4) in the above inequalities, we deduce

$$\mathrm{K}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \,\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathrm{K}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^m\eta^{n+m}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right),$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)\circ \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^m\eta^{n+m}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right),$$

and

$$\Delta(\varkappa_n,\varkappa_{n+2m+1},\sigma) \leq \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)\circ \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\eta^{n+2m}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)\circ \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3 \eta^{n+2} \psi(\varkappa_n, \varkappa_{n+2m+1}) \psi(\varkappa_{n+2}, \varkappa_{n+3}) \psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^m \eta^{n+m} \psi(\varkappa_n, \varkappa_{n+2m+1}) \psi(\varkappa_{n+2}, \varkappa_{n+3}) \psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Case 2: If p is even, then $p = 2m; m \in \{1,2,3, \cdots\}$. So, the we have

$$\mathbb{K}(\varkappa_n, \varkappa_{n+2m}, \sigma) \geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right) * \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) * \mathbb{K}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3}) \psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \mathbb{K}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3}) \psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\mathbb{K}\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^{m-1} \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3}) \psi(\varkappa_{n+4}, \varkappa_{n+5}) \cdots \psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2 \psi(\varkappa_n, \varkappa_{n+2m}) \psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m-2}, \varkappa_{n+2m}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right).$$

Using (4) in the above inequalities, we deduce

$$\mathbb{K}(\varkappa_n,\varkappa_{n+2m},\sigma)\geq \mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\eta^{n+2m-2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\geq \mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

$$\Pi(\varkappa_n,\varkappa_{n+2m},\sigma)\leq \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\,\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\cdots\circ$$

$$\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\eta^{n+2m-2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_0,1,\frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n,\varkappa_{n+2m},\sigma) \le \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\eta^{n+3}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+4}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\eta^{n+5}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\eta^{n+2m-2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\le \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_0, 1, \frac{\sigma}{(3\eta)\eta^n\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^n\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^2\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^3\eta^{n+2}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3\eta)^{m-1}\eta^{n+m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right).$$

Therefore, from $\lim_{\sigma\to+\infty} \mathrm{K}(\varkappa,\vartheta,\sigma) = 1$, $\lim_{\sigma\to+\infty} \Pi(\varkappa,\vartheta,\sigma) = 0$ and $\lim_{\sigma\to+\infty} \Delta(\varkappa,\vartheta,\sigma) = 0$, and cases (1),(2), we

get

$$\lim_{n\to+\infty} Ꝃ(\varkappa_n, \varkappa_{n+p}, \sigma) = 1, \lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa_{n+p}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \varDelta(\varkappa_n, \varkappa_{n+p}, \sigma) = 0.$$

That is, a sequence $\{\varkappa_n\}$ is Cauchy. Therefore, $(\Sigma, Ꝃ, \Pi, \varDelta, *, \circ)$ is a complete NREBMS, so there exists $\mathfrak{u} \in \Sigma$, and we have

$$\lim_{n\to+\infty} Ꝃ(\varkappa_n, \mathfrak{u}, \sigma) = 1, \lim_{n\to+\infty} \Pi(\varkappa_n, \mathfrak{u}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \varDelta(\varkappa_n, \mathfrak{u}, \sigma) = 0, \text{for all } \sigma > 0 \text{ and } q \geq 1.$$

Now, we show the existence of a fixed point $\mathfrak{u}$.

$$Ꝃ(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \geq Ꝃ\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq Ꝃ\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq Ꝃ\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * Ꝃ\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 1 * 1 * 1 = 1, \text{as } n \to +\infty,$$

$$\Pi(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0, \text{as } n \to +\infty$$

and

$$\varDelta(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \varDelta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \varDelta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \varDelta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\eta\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \varDelta\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\eta\ell\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0 \text{ as } n \to +\infty.$$

Uniqueness: Suppose $v \neq \mathfrak{u}$, be another fixed point, then

$$Ꝃ(v, \mathfrak{u}, \sigma) = Ꝃ(\xi v, \xi\mathfrak{u}, \sigma) \geq Ꝃ\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = Ꝃ\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\geq Ꝃ\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \geq \cdots \geq Ꝃ\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 1 \text{ as } n \to +\infty,$$

$$\Pi(v, \mathfrak{u}, \sigma) = \Pi(\xi v, \xi\mathfrak{u}, \sigma) \leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = \Pi\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \leq \cdots \leq \Pi\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 0 \text{ as } n \to +\infty,$$

and

$$\varDelta(v, \mathfrak{u}, \sigma) = \varDelta(\xi v, \xi\mathfrak{u}, \sigma) \leq \varDelta\left(v, \mathfrak{u}, \frac{\sigma}{\eta}\right) = \varDelta\left(\xi v, \xi\mathfrak{u}, \frac{\sigma}{\eta}\right)$$

$$\leq \Delta\left(v, \mathfrak{u}, \frac{\sigma}{\eta^2}\right) \leq \cdots \leq \Delta\left(v, \mathfrak{u}, \frac{\sigma}{\eta^n}\right) \to 0 \text{ as } n \to +\infty,$$

Hence, $\mathfrak{u} = v$.

Definition 3.3: Suppose $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS. A mapping $\xi: \Sigma \to \Sigma$ is known as neutrosophic rectangular contraction, if

$$\frac{1}{\mathrm{K}(\xi\varkappa, \xi\vartheta, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa, \vartheta, \sigma)} - 1\right], \qquad \Pi(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Pi(\varkappa, \vartheta, \sigma)$$

$$\text{and } \Delta(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Delta(\varkappa, \vartheta, \sigma) \qquad (5)$$

for all $\varkappa, \vartheta \in \Sigma, \eta \in (0,1)$ and $\sigma > 0$.

Theorem 3.2: Suppose $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS, such that

$$\lim_{\sigma\to+\infty} \mathrm{K}(\varkappa, \vartheta, \sigma) = 1, \lim_{\sigma\to+\infty} \Pi(\varkappa, \vartheta, \sigma) = , \text{and } \lim_{\sigma\to+\infty} \Delta(\varkappa, \vartheta, \sigma) = 0 \quad \text{for all } \varkappa, \vartheta \in \Sigma. \qquad (6)$$

Let $\xi: \Sigma \to \Sigma$ be a Neutrosophic rectangular contraction. Then ξ has a unique fixed point $\mathfrak{u} \in \Sigma$.

Proof: Assume $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ be a complete NREBMS, let an arbitrary point $\varkappa_0 \in \Sigma$, and define a sequence $\{\varkappa_n\}$ in Σ by

$$\varkappa_1 = \xi\varkappa_0 \;, \; \varkappa_2 = \xi^2\varkappa_0 = \xi\varkappa_1, \ldots, \varkappa_n = \xi^n\varkappa_0 = \xi\varkappa_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $\varkappa_n = \varkappa_{n-1}$ for some $n \in \mathbb{N}$ then $\varkappa_n$ is a fixed point of ξ. We suppose that $\varkappa_n \neq \varkappa_{n-1}$ for all $n \in \mathbb{N}$. For $\sigma > 0$ and $n \in \mathbb{N}$, utilizing (5), we get

$$\frac{1}{\mathrm{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} - 1 = \frac{1}{\mathrm{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - 1\right].$$

That is,

$$\frac{1}{\mathrm{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} \leq \frac{\eta}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} + (1-\eta), \forall\, \sigma > 0,$$

$$= \frac{\eta}{\mathrm{K}(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)} + (1-\eta) \leq \frac{\eta^2}{\mathrm{K}(\varkappa_{n-2}, \varkappa_{n-1}, \sigma)} + \eta(1-\eta) + (1-\eta).$$

Continuing this way, we get

$$\frac{1}{\mathrm{K}(\varkappa_n, \varkappa_{n+1}, \sigma)} \leq \frac{\eta^n}{\mathrm{K}(\varkappa_0, \varkappa_1, \sigma)} + \eta^{n-1}(1-\eta) + \eta^{n-2}(1-\eta) + \cdots + \eta(1-\eta) + (1-\eta)$$

$$\leq \frac{\eta^n}{\mathrm{K}(\varkappa_0, \varkappa_1, \sigma)} + (\eta^{n-1} + \eta^{n-2} + \cdots + 1)(1-\eta)$$

$$\leq \frac{\eta^n}{\mathrm{K}(\varkappa_0, \varkappa_1, \sigma)} + (1-\eta^n).$$

We have,

$$\frac{1}{\frac{\eta^n}{\mathrm{K}(\varkappa_0, \varkappa_1, \sigma)} + (1-\eta^n)} \leq \mathrm{K}(\varkappa_n, \varkappa_{n+1}, \sigma), \forall\, \sigma > 0, n \in \mathbb{N}. \qquad (7)$$

$$\Pi(\varkappa_n, \varkappa_{n+1}, \sigma) = \Pi(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma) \leq \eta\Pi(\varkappa_{n-1}, \varkappa_n, \sigma) = \eta\Pi(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)$$
$$\leq \eta^2\Pi(\varkappa_{n-2}, \varkappa_{n-1}, \sigma) \leq \cdots \leq \eta^n\Pi(\varkappa_0, \varkappa_1, \sigma) \qquad (8)$$

and

$$\Delta(\varkappa_n, \varkappa_{n+1}, \sigma) = \Delta(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma) \leq \eta\Delta(\varkappa_{n-1}, \varkappa_n, \sigma) = \eta\Delta(\xi\varkappa_{n-2}, \xi\varkappa_{n-1}, \sigma)$$
$$\leq \eta^2\Delta(\varkappa_{n-2}, \varkappa_{n-1}, \sigma) \leq \cdots \leq \eta^n\Delta(\varkappa_0, \varkappa_1, \sigma). \qquad (9)$$

Since $(\mathcal{E}, \text{Ꝁ}, \Pi, \Delta, *, \circ)$ is a complete NREBMS for the sequence $\{\varkappa_n\}$, writing $\sigma = \frac{\sigma}{3} + \frac{\sigma}{3} + \frac{\sigma}{3}$ and using the rectangular inequalities given in (N5), (N10) and (N15) on $\text{Ꝁ}(\varkappa_n, \varkappa_{n+p}, \sigma), \Pi(\varkappa_n, \varkappa_{n+p}, \sigma)$ and $\Delta(\varkappa_n, \varkappa_{n+p}, \sigma)$, in the following cases.

Case 1: If p is odd, then $p = 2m + 1$ where $m \in \{1,2,3, \dots\}$. So, we have

$$\text{Ꝁ}(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \geq \text{Ꝁ}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \text{Ꝁ}\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\geq \text{Ꝁ}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \text{Ꝁ}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\geq \text{Ꝁ}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) * \text{Ꝁ}\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \text{Ꝁ}\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$* \cdots *$$

$$\text{Ꝁ}\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\Pi(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Pi\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right),$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+2}, \varkappa_{n+2m+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\leq \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \Delta\left(\varkappa_{n+1}, \varkappa_{n+2}, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2}, \varkappa_{n+3}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3}, \varkappa_{n+4}, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4}, \varkappa_{n+5}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5}, \varkappa_{n+6}, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m}, \varkappa_{n+2m+1}, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

By using (7), (8) and (9) in the above inequalities, we have

$$\begin{aligned}
&\mathbb{K}(\varkappa_n,\varkappa_{n+2m+1},\sigma)\\
&\geq \frac{1}{\frac{\eta^n}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^n)} * \frac{1}{\frac{\eta^{n+1}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^{n+1})}\\
&* \frac{1}{\frac{\eta^{n+2}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-\eta^{n+2})}\\
&* \frac{1}{\frac{\eta^{n+3}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-\eta^{n+3})}\\
&* \cdots *\\
&\frac{1}{\frac{\eta^{n+2m}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)}+(1-\eta^{n+2m})},\\
&\geq \frac{1}{\frac{\eta^n}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-\eta^n)} * \frac{1}{\frac{(\eta)\eta^{n}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)}+(1-(\eta)\eta^{n})}\\
&* \frac{1}{\frac{(\eta)^2\eta^{n}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_2,\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^{n})}\\
&* \frac{1}{\frac{(\eta)^2\eta^{n+1}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)}+(1-(\eta)^2\eta^{n+1})}\\
&* \cdots *\\
&\frac{1}{\frac{(\eta)^m\eta^{n+m}}{\mathbb{K}\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)}+(1-(\eta)^m\eta^{n+m})},
\end{aligned}$$

$$\begin{aligned}
\Pi(\varkappa_n,\varkappa_{n+2m+1},\sigma) &\leq \eta^n\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)\circ\eta^{n+1}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m+1})}\right)\\
&\circ\eta^{n+2}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)\\
&\circ\eta^{n+3}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)\\
&\circ\eta^{n+4}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)\\
&\circ\eta^{n+5}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)\\
&\circ\cdots\circ\\
&\eta^{n+2m}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m+1})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m},\varkappa_{n+2m+1})}\right)
\end{aligned}$$

$$\leq \eta^n \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta(\eta^n) \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^2(\eta^n)\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^m(\eta^{n+m})\Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

and

$$\Delta(\varkappa_n, \varkappa_{n+2m+1}, \sigma) \leq \eta^n \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta^{n+1}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^{n+2}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m}\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)$$

$$\leq \eta^n \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m+1})}\right) \circ \eta(\eta^n) \Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m+1})}\right)$$

$$\circ\, \eta^2(\eta^n)\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^m(\eta^{n+m})\Delta\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right).$$

Case 2: If p is even, then $p = 2m; m \in \{1,2,3,\cdots\}$. So, we have

$$\mathrm{K}(\varkappa_n,\varkappa_{n+2m},\sigma) \geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+2m},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)*\mathrm{K}\left(\varkappa_{n+4},\varkappa_{n+2m},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\geq \mathrm{K}\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)*\mathrm{K}\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+4},\varkappa_{n+5},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\,\mathrm{K}\left(\varkappa_{n+5},\varkappa_{n+6},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$*\cdots*$$

$$\mathrm{K}\left(\varkappa_{n+2m-2},\varkappa_{n+2m},\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

$$\Pi(\varkappa_n,\varkappa_{n+2m},\sigma) \leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+2},\varkappa_{n+2m},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+4},\varkappa_{n+2m},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\leq \Pi\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\Pi\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\,\Pi\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+4},\varkappa_{n+5},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Pi\left(\varkappa_{n+5},\varkappa_{n+6},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Pi\left(\varkappa_{n+2m-2},\varkappa_{n+2m},\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right),$$

and

$$\Delta(\varkappa_n,\varkappa_{n+2m},\sigma) \le \Delta\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+2},\varkappa_{n+2m},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\le \Delta\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4},\varkappa_{n+2m},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\le \Delta\left(\varkappa_n,\varkappa_{n+1},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right) \circ \Delta\left(\varkappa_{n+1},\varkappa_{n+2},\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+2},\varkappa_{n+3},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+3},\varkappa_{n+4},\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+4},\varkappa_{n+5},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \Delta\left(\varkappa_{n+5},\varkappa_{n+6},\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\Delta\left(\varkappa_{n+2m-2},\varkappa_{n+2m},\frac{\sigma}{(3)^m\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right).$$

By using (7A), (8A) and (9A) in the above inequalities, we have

$$\text{Ҡ}(\varkappa_n, \varkappa_{n+2m}, \sigma) \geq \frac{1}{\frac{\eta^n}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)} + (1-\eta^n)} * \frac{1}{\frac{\eta^{n+1}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)} + (1-\eta^{n+1})}$$

$$* \frac{1}{\frac{\eta^{n+2}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_2, \varkappa_{n+3})}\right)} + (1-\eta^{n+2})}$$

$$* \frac{1}{\frac{\eta^{n+3}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)} + (1-\eta^{n+3})}$$

$$* \cdots *$$

$$\frac{1}{\frac{\eta^{n+2m-2}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^{m-1}\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)} + (1-\eta^{n+2m-2})}$$

$$\geq \frac{1}{\frac{\eta^n}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)} + (1-\eta^n)} * \frac{1}{\frac{(\eta)\eta^n}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)} + (1-(\eta)\eta^n)}$$

$$* \frac{1}{\frac{(\eta)^2\eta^n}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_2, \varkappa_{n+3})}\right)} + (1-(\eta)^2\eta^n)}$$

$$* \frac{1}{\frac{(\eta)^2\eta^{n+1}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)} + (1-(\eta)^2\eta^{n+1})}$$

$$* \cdots *$$

$$\frac{1}{\frac{(\eta)^{m-1}\eta^{n+m-1}}{\text{Ҡ}\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^{m-1}\psi(\varkappa_n, \varkappa_{n+2m+1})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m}, \varkappa_{n+2m+1})}\right)} + (1-(\eta)^{m-1}\eta^{n+m-1})}$$

$$\Pi(\varkappa_n, \varkappa_{n+2m}, \sigma) \leq \eta^n \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ \eta^{n+1} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \eta^{n+2} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^2\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^3\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m-2} \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{(3)^m\psi(\varkappa_n, \varkappa_{n+2m})\psi(\varkappa_{n+2}, \varkappa_{n+3})\psi(\varkappa_{n+4}, \varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2}, \varkappa_{n+2m})}\right)$$

$$\leq \eta^n \Pi\left(\varkappa_0, \varkappa_1, \frac{\sigma}{3\psi(\varkappa_n, \varkappa_{n+2m})}\right) \circ (\eta)\eta^n \Pi\left(\varkappa_0, 1, \frac{\sigma}{(3)\psi(\varkappa_n, \varkappa_{n+2m})}\right)$$

$$\circ\, \eta^2(\eta^n)\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$(\eta)^{m-1}\eta^{n+m-1}\Pi\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right).$$

and

$$\Delta(\varkappa_n,\varkappa_{n+2m},\sigma) \le \eta^n\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\,\eta^{n+1}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \eta^{n+2}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+3}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^{n+4}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \eta^{n+5}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$\eta^{n+2m-2}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right)$$

$$\le \eta^n\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{3\psi(\varkappa_n,\varkappa_{n+2m})}\right)\circ\,(\eta)\eta^n\Delta\left(\varkappa_0,1,\frac{\sigma}{(3)\psi(\varkappa_n,\varkappa_{n+2m})}\right)$$

$$\circ\, \eta^2(\eta^n)\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^2(\eta^{n+1})\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^2\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})}\right)$$

$$\circ\, \eta^3(\eta^{n+1})\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3\eta)^3\eta^{n+1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ\, \eta^3(\eta^{n+2})\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^3\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})}\right)$$

$$\circ \cdots \circ$$

$$(\eta)^{m-1}\eta^{n+m-1}\Delta\left(\varkappa_0,\varkappa_1,\frac{\sigma}{(3)^{m-1}\psi(\varkappa_n,\varkappa_{n+2m})\psi(\varkappa_{n+2},\varkappa_{n+3})\psi(\varkappa_{n+4},\varkappa_{n+5})\cdots\psi(\varkappa_{n+2m-2},\varkappa_{n+2m})}\right).$$

Therefore, from $\lim_{\sigma\to+\infty} Ꝁ(\varkappa,\vartheta,\sigma)=1$, $\lim_{\sigma\to+\infty}\Pi(\varkappa,\vartheta,\sigma)=0$ and $\lim_{\sigma\to+\infty}\Delta(\varkappa,\vartheta,\sigma)=0$, and Cases (1), (2),

we get

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \varkappa_{n+p}, \sigma) = 1,\ \lim_{n\to+\infty} \Pi(\varkappa_n, \varkappa_{n+p}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \varkappa_{n+p}, \sigma) = 0.$$

Hence $\{\varkappa_n\}$ is a Cauchy sequence. Since, $(\mathcal{E}, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS, so there exists $\mathfrak{u} \in \mathcal{E}$ such that

$$\lim_{n\to+\infty} \mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma) = 1\ \lim_{n\to+\infty} \Pi(\varkappa_n, \mathfrak{u}, \sigma) = 0 \text{ and } \lim_{n\to+\infty} \Delta(\varkappa_n, \mathfrak{u}, \sigma) = 0, \qquad \text{for all } \sigma > 0 \text{ and } q \geq 1.$$

Now, we show the existence of a fixed point $\mathfrak{u}$. Utilizing (5), we have

$$\frac{1}{\mathrm{K}(\xi\varkappa_n, \xi\mathfrak{u}, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} - 1\right] = \frac{\eta}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} - \eta,$$

$$\frac{1}{\frac{\eta}{\mathrm{K}(\varkappa_n, \mathfrak{u}, \sigma)} + 1 - \eta} \leq \mathrm{K}(\xi\varkappa_n, \xi\mathfrak{u}, \sigma),$$

and

$$\frac{1}{\mathrm{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma)} - 1 \leq \eta\left[\frac{1}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - 1\right] = \frac{\eta}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} - \eta,$$

$$\frac{1}{\frac{\eta}{\mathrm{K}(\varkappa_{n-1}, \varkappa_n, \sigma)} + 1 - \eta} \leq \mathrm{K}(\xi\varkappa_{n-1}, \xi\varkappa_n, \sigma).$$

Using the above inequalities, we deduce

$$\mathrm{K}(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \mathrm{K}\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\geq \mathrm{K}\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) * \frac{1}{\frac{\eta}{\mathrm{K}\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)} + 1 - \eta} * \frac{1}{\frac{\eta}{\mathrm{K}\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)} + 1 - \eta}$$

$$\to 1 * 1 * 1 = 1 \text{ as } n \to +\infty$$

$$\Pi(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Pi\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Pi\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Pi\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Pi\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0 \text{ as } n \to +\infty$$

and

$$\Delta(\mathfrak{u}, \xi\mathfrak{u}, \sigma) \leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_n, \varkappa_{n+1}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\varkappa_{n+1}, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_{n-1}, \xi\varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \Delta\left(\xi\varkappa_n, \xi\mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\leq \Delta\left(\mathfrak{u}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Delta\left(\varkappa_{n-1}, \varkappa_n, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right) \circ \eta\Delta\left(\varkappa_n, \mathfrak{u}, \frac{\sigma}{3\psi(\mathfrak{u}, \xi\mathfrak{u})}\right)$$

$$\to 0 \circ 0 \circ 0 = 0 \text{ as } n \to +\infty.$$

That is, $\xi \mathfrak{u} = \mathfrak{u}$.

Uniqueness: Suppose $v \neq \mathfrak{u}$ be another fixed point of ξ, such that $Ꝁ(\mathfrak{u}, v, t) < 1$ for some $\sigma > 0$, and utilizing (5), we have

$$\frac{1}{Ꝁ(\mathfrak{u}, v, \sigma)} - 1 = \frac{1}{Ꝁ(\xi\mathfrak{u}, \xi v, \sigma)} - 1$$

$$\leq \eta \left[\frac{1}{Ꝁ(\mathfrak{u}, v, \sigma)} - 1\right] < \frac{1}{Ꝁ(\mathfrak{u}, v, \sigma)} - 1$$

a contradiction,

$$\Pi(\mathfrak{u}, v, \sigma) = \Pi(\xi\mathfrak{u}, \xi v, \sigma) \leq \eta\Pi(\mathfrak{u}, v, \sigma)$$

and

$$\Delta(\mathfrak{u}, v, \sigma) = \Delta(\xi\mathfrak{u}, \xi v, \sigma) \leq \eta\Delta(\mathfrak{u}, v, \sigma)$$

a contradiction. Therefore, we must have $Ꝁ(\mathfrak{u}, v, \sigma) = 1, \Pi(\mathfrak{u}, v, \sigma) = 0$ and $\Delta(\mathfrak{u}, v, \sigma) = 0$ for all $\sigma > 0$, and hence, $\mathfrak{u} = v$.

Example 3.8: Let $\mathcal{E} = [0,1]$. Define $\psi: \mathcal{E} \times \mathcal{E} \to [1, +\infty)$ by $\psi(\varkappa, \vartheta) = \begin{cases} 1, & \text{if} \quad \varkappa = \vartheta \\ 1 + \varkappa + \vartheta, & \text{otherwise} \end{cases}$ and define $Ꝁ, \Pi, \Delta: \mathcal{E} \times \mathcal{E} \times [0, +\infty) \to [0,1]$ by

$$Ꝁ(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p},$$

and

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \mathcal{E} \text{ and } \sigma > 0.$$

Defined by $\hbar * \ell = \hbar \cdot \ell, \hbar \circ \ell = \max\{\hbar, \ell\}$ and $p \geq 1$, then $(\mathcal{E}, Ꝁ, \Pi, \Delta, *, \circ)$ is a complete NREBMS.

Define $\xi: \mathcal{E} \to \mathcal{E}$ by $\xi(\varkappa) = \sqrt[p]{\eta}\varkappa$. Then

$$Ꝁ(\xi\varkappa, \xi\vartheta, \eta\sigma) = Ꝁ\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\eta\sigma}{\eta\sigma + \left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}$$

$$= \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p} = Ꝁ(\varkappa, \vartheta, \sigma)$$

$$\Pi(\xi\varkappa, \xi\vartheta, \eta\sigma) = \Pi\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}{\eta\sigma + \left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}$$

$$= \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p} = \Pi(\varkappa, \vartheta, \sigma), \text{ and}$$

$$\Delta(\xi\varkappa, \xi\vartheta, \eta\sigma) = \Delta\left(\sqrt[p]{\eta}\varkappa, \sqrt[p]{\eta}\vartheta, \eta\sigma\right) = \frac{\left|\sqrt[p]{\eta}\varkappa - \sqrt[p]{\eta}\vartheta\right|^p}{\eta\sigma}$$

$$= \frac{|\varkappa - \vartheta|^p}{\sigma} = \Delta(\varkappa, \vartheta, \sigma).$$

Also, contraction conditions of Theorem 3.2,

$\frac{1}{Ꝁ(\xi\varkappa, \xi\vartheta, \sigma)} - 1 \leq \eta \left[\frac{1}{Ꝁ(\varkappa, \vartheta, \sigma)} - 1\right], \Pi(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Pi(\varkappa, \vartheta, \sigma)$ and $\Delta(\xi\varkappa, \xi\vartheta, \sigma) \leq \eta\Delta(\varkappa, \vartheta, \sigma)$ are satisfied.

Consequently, all of the assumptions of Theorems 3.1 and 3.2 are satisfied, and 0 is a unique fixed point.

4. Application to Nonlinear Fractional Differential Equation

Theorem 3.1 is used in this section to determine a solution's existence and uniqueness in nonlinear fractional differential equation (see [19]) given by

$$D_c^\alpha \varkappa(\varrho) = \psi(\varrho, \varkappa(\varrho)) \qquad (\varrho \in (0,1), \alpha \in (1,2]),$$

with boundary conditions

$$\varkappa(0) = 0, \varkappa'(0) = I\varkappa(\varrho) \qquad \varrho \in (0,1),$$

Where D_c^α means caputo fractional derivative of order α, defined by

$$D_c^\alpha \psi(\varrho) = \frac{1}{\Gamma(n-\alpha)} \int_0^\varrho (\varrho - \varpi)^{n-\alpha-1} \psi^n(\varpi) d\varpi \qquad (n-1 < \alpha < n, \; n = [\alpha] + 1),$$

and $\psi: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. We suppose that $\Sigma = C([0,1], \mathbb{R})$, from $[0,1]$ into $\mathbb{R}$ with supremum $|\varkappa| = \sup_{\theta \in [0,1]} |\varkappa(\varrho)|$.

The Riemann-Liouville fractional integral of order α (see [20]) is given by

$$I^\alpha \psi(\varrho) = \frac{1}{\Gamma(\alpha)} \int_0^\varrho (\varrho - \varpi)^{\alpha-1} \psi(\varpi) d\varpi \qquad (\alpha > 0)$$

We first provide an acceptable form for a nonlinear fractional differential equation before investigating the existence of a solution. Now, we suppose the following fractional differential equation

$$D_c^\alpha \varkappa(\varrho) = \psi(\varrho, \varkappa(\varrho)) \qquad (\varrho \in (0,1), \alpha \in (1,2]), \qquad (10)$$

with the boundary conditions

$$\varkappa(0) = 0, \qquad \varkappa'(0) = I\varkappa(\varrho) \qquad (\varrho \in (0,1)),$$

where

i. $\psi: [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
ii. $\varkappa(\varrho): [0,1] \to \mathbb{R}$ is continuous,

and satisfying the following condition

$$|\psi(\varrho, \varkappa) - \psi(\varrho, \vartheta)| \leq Л L |\varkappa - \vartheta|,$$

for all $\varrho \in [0,1]$ and L is a constant with $LЛ < 1$ where

$$Л = \frac{1}{\Gamma(\alpha+1)} + \frac{2\vartheta^{\alpha+1}\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+1)}.$$

Then the equation (10) has a unique solution.

Proof: Suppose that

$$Ꝁ(\varkappa, \vartheta, \sigma) = \frac{\sigma}{\sigma + |\varkappa - \vartheta|^p}$$

$$\Pi(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma + |\varkappa - \vartheta|^p} \text{ , and}$$

$$\Delta(\varkappa, \vartheta, \sigma) = \frac{|\varkappa - \vartheta|^p}{\sigma} \text{ for all } \varkappa, \vartheta \in \Sigma \text{ and } \sigma > 0,$$

defined by $\hbar * \ell = \hbar \cdot \ell$, and $\hbar \circ \ell = \max\{\hbar, \ell\}$. Let $|\varkappa - \vartheta| = \sup_{\varrho\in[0,1]} |\varkappa(\varrho) - \vartheta(\varrho)|$, for all $\varkappa, \vartheta \in \Sigma$. Then $(\Sigma, \mathrm{K}, \Pi, \Delta, *, \circ)$ is a complete NREBMS. We define a mapping $\xi: \Sigma \to \Sigma$ by

$$\xi\varkappa(\varrho) = \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi\big(\varpi, \varkappa(\varpi)\big)d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi\big(m, \varkappa(m)\big)dm\right)d\varpi \qquad (11)$$

for all $\varrho \in [0,1]$. Equation (10) has a solution $\varkappa \in \Sigma$ iff $\varkappa(\varrho) = \xi\varkappa(\varrho)$ for all $\varrho \in [0,1]$. Now

$$\left.\begin{aligned} \mathrm{K}(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{\sigma}{\sigma + |\varkappa(\varrho) - \vartheta(\varrho)|^p} \\ \Pi(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{|\varkappa(\varrho) - \vartheta(\varrho)|^p}{\sigma + |\varkappa(\varrho) - \vartheta(\varrho)|^p} \\ \Delta(\varkappa(\varrho), \vartheta(\varrho), \sigma) &= \frac{|\varkappa(\varrho) - \vartheta(\varrho)|^p}{\sigma} \end{aligned}\right\} \qquad (12)$$

$$|\xi\varkappa(\varrho) - \xi\vartheta(\varrho)| = \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}\psi\big(\varpi, \varkappa(\varpi)\big)d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi\big(m, \varkappa(m)\big)dm\right)d\varpi|\Bigg|$$

$$-\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}\psi\big(\varpi, \vartheta(\varpi)\big)d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi\big(m, \vartheta(m)\big)dm\right)d\varpi\Bigg|.$$

That is,

$$|\xi\varkappa(\varrho) - \xi\vartheta(\varrho)| = \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}\psi\big(\varpi, \varkappa(\varpi)\big)d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi\big(m, \varkappa(m)\big)dm\right)d\varpi$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}\psi\big(\varpi, \vartheta(\varpi)\big)d\varpi$$

$$-\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi\big(m, \vartheta(m)\big)dm\right)d\varpi\Bigg|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho-\varpi)^{\alpha-1}\left|\psi\big(\varpi,\varkappa(\varpi)\big)-\psi\big(\varpi,\vartheta(\varpi)\big)\right|d\varpi$$

$$+\frac{2\varrho}{(2-\vartheta^2)\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi-m)^{\alpha-1}\left|\psi\big(m,\varkappa(m)\big)-\psi\big(m,\vartheta(m)\big)\right|dm\right)d\varpi$$

$$\leq \frac{L|\varkappa-\vartheta|}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho-\varpi)^{\alpha-1}d\varpi+\frac{2L|\varkappa-\vartheta|}{\Gamma(\alpha)}\int_0^{\vartheta}\left(\int_0^{\varpi}(\varpi-m)^{\alpha-1}dm\right)d\varpi$$

$$\leq \frac{L|\varkappa-\vartheta|}{\Gamma(\alpha+1)}+\frac{2\vartheta^{\alpha+1}L|\varkappa-\vartheta|\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+2)}$$

$$\leq L|\varkappa-\vartheta|\left(\frac{1}{\Gamma(\alpha+1)}+\frac{2\vartheta^{\alpha+1}\Gamma(\alpha)}{(2-\vartheta^2)\Gamma(\alpha+2)}\right)=L\text{Л}|\varkappa-\vartheta|.$$

Utilizing $L\text{Л}<1$ and (12), we have

$$\text{Ҟ}(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{\eta\sigma}{\eta\sigma+|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}\geq\frac{\eta\sigma}{\eta\sigma+L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}$$

$$\geq\frac{\sigma}{\sigma+|\varkappa(\varrho)-\vartheta(\varrho)|^p}=\text{Ҟ}(\varkappa(\varrho),\vartheta(\varrho),\sigma)$$

$$\Pi(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}{\eta\sigma+|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}\leq\frac{L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\eta\sigma+L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}$$

$\leq \frac{|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\sigma+|\varkappa(\varrho)-\vartheta(\varrho)|^p}=\Pi(\varkappa(\varrho),\vartheta(\varrho),\sigma)$ and

$$\Delta(\xi\varkappa(\varrho),\xi\vartheta(\varrho),\eta\sigma)=\frac{|\xi\varkappa(\varrho)-\xi\vartheta(\varrho)|^p}{\eta\sigma}\leq\frac{L\text{Л}|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\eta\sigma}$$

$$\leq\frac{|\varkappa(\varrho)-\vartheta(\varrho)|^p}{\sigma}=\Delta(\varkappa(\varrho),\vartheta(\varrho),\sigma).$$

As a result, the conditions of Theorem 3.1 are all met. This shows that ξ has unique solution.

5. Conclusion

In this manuscript, we introduced the notion of NREBMS and provided some non-trivial examples of defined space. Several fixed point results for contraction mappings are established with examples. Also, we provided an application to non-linear fractional differential equations to support the validity of main result. This is extendable in several more generalized spaces including neutrosophic rectangular controlled metric spaces, graphical neutrosophic metric spaces, neutrosophic rectangular double controlled metric-like spaces and Hausdorff neutrosophic rectangular metric spaces. Also, this work is extendable by increasing the number of self-mappings.

6. Open Problem

How to prove Theorem 3.1 and Theorem 3.2 (proved in this paper) in the context of graphical neutrosophic extended b-metric spaces?

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. L. A. Zadeh, "Fuzzy sets" Information and Control, vol. 8, no. 3, pp. 338–353, 1965.
2. E. Kreyszig, "Introductory functional analysis with applications" pp. 3-4.
3. I. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces" Kybernetika, vol. 11, no. 5, pp. 336–344, 1975.
4. A. George and P. Veeramani, "On some results in fuzzy metric spaces," Fuzzy Sets and Systems, vol. 64, no. 3, pp. 395–399, 1994.
5. A. George, P. Veeramani, "On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems" vol. 90, pp. 365–368, 1997.
6. M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy sets and Systems*, 27(1988), 385-389.
7. K. Atanassov, Intuitionistic fuzzy sets. Fuzzy sets and systems. Vol. 20, pp. 87–96, 1986.
8. J. H. Park, Intuitionistic fuzzy metric spaces. Chaos Solitons Fract., vol. 22, pp. 1039–1046, 2004.
9. F. Mehmood, R. Ali, N. Hussain, Contractions in fuzzy rectangular b-metric spaces with application. Journal of Intelligent and Fuzzy Systems, 2021.
10. M. Hezarjaribi, "Fixed point results in orthogonal fuzzy metric spaces" Jordan journal of mathematics and statistics vol. 11, no.4, pp. 295-308, 2018.
11. Y. Zhou, "Fixed point in fuzzy metric spaces, Bio-Inspired Computing-Theories and applications" pp. 659-663.
12. R. Vasuki, P. Veeramani, "Fixed Point Theorems and Cauchy Sequences in Fuzzy Metric Spaces, Fuzzy Sets and Systems" vol. 135, pp. 415–417, 2003.
13. Y. Shen, D. Qiu, W. Chen, "Fixed Point Theorems in Fuzzy Metric Spaces". Appl. Math. Lett. 25, pp. 138–141, 2012.
14. D. Wardowski, "Fuzzy Contractive Mappings and Fixed Points in Fuzzy Metric Spaces, Fuzzy Sets and Systems" vol. 222, pp. 108–114, 2013.
15. J.Y. Park, J.U. Jeong, "Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems" vol. 87, pp. 111-116, 1997.
16. N. Konwar, Extension of fixed results in intuitionistic fuzzy b-metric spaces, *Journal of Intelligent & Fuzzy Systems*, 39 (2020), 7831-7841.
17. C. Alaca, D. Turkoglu, C. Yildiz, Fixed points in intuitionistic fuzzy metric spaces, *Chaos Solitons and Fractals*, 29(2006), 1073–1078.
18. A. Mohamad, Fixed-point theorems in intuitionistic fuzzy metric spaces, *Chaos Solitons and Fractals*, 34 (2007), 1689–1695.
19. D. Baleanu, S. Rezapour and H. Mohammadi, Some existence results on nonlinear fractional differential equations, Philos. Trans. A 371(1990) (2013), 1–7.
20. W. Sudsutad and J. Tariboon, Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions, Adv. Difference Eqn. 2012:93 (2012), 1–10.
21. K. Javed, F. Uddin, H. Aydi, M. Arshad, U. Ishtiaq, H. Alsamir, On fuzzy b-metric-like spaces, Journal of Function Spaces, 2021, 1-9.
22. F. Uddin, K. Javed, H. Aydi, U. Ishtiaq, M. Arshad, Control Fuzzy Metric Spaces via Orthogonality with an Application, Journal of Mathematics, 2021, 1-12.
23. M. Gulzar, G. Abbas, F. Dilawar, Algebraic Properties of ω-Q-fuzzy subgroups, International Journal of Mathematics and Computer Science, 15(2020), no. 1, 265–274.

24. M. Gulzar, F. Dilawar, D. Alghazzawi and M. H. Mateen, A note on complex fuzzy subfield, Indonesian Journal of Electrical Engineering and Computer Science, Vol. 21, No. 2, February 2021, pp. 1048-1056.
25. S. Bhunia, G. Ghorai, M.A. Kutbi, M. Gulzar and M.A. Alam, On the Algebraic Characteristics of Fuzzy Sub e-Groups, Journal of Function Spaces, ID5253346, 2021.
26. M. Gulzar, M.H. Mateen, D. Alghazzawi and N. Kausar, A novel application of complex intuitionistic fuzzy sets in group theory, IEEE Access, 8(2020), pp.196075-196085.
27. M. Gulzar, D. Alghazzawi, M. H. Mateen and N. Kausar, "A Certain Class of t-Intuitionistic Fuzzy Subgroups," in IEEE Access, vol. 8, pp. 163260-163268, 2020.
28. A. Hussain, U. Ishtiaq, K. Ahmed, H. Al-Sulami, on pentagonal controlled fuzzy metric spaces with an application to dynamic market equilibrium, Journal of function spaces, (2021).
29. D. Kattan , A. O. Alzanbaqi, and S. Islam, Contraction Mappings in Intuitionistic Fuzzy Rectangular Extended B-Metric Spaces, Mathematical Problems in Engineering,21(2022).
30. F. Smarandache, (2005) Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter J Pure Appl Math, 24, 287–297.
31. V.B. Shakila and M. Jeyaraman, Fixed Point Theorems Of Contractive Mappings In Neutrosophic B - Metric Spaces, journal of Algebraic statistic, 13(2022), 1330-1342.
32. M. Kirişci, N. Simsek, Neutrosophic metric spaces, Mathematical Sciences, 14(2020), 241–248.
33. N. Saleem, J. Vujaković, W. U. Baloch, and S. Radenović, (2019). Coincidence point results for multivalued Suzuki type mappings using θ-contraction in b-metric spaces. Mathematics, 7(11), 1017.
34. N. Saleem, M. Zhou, S. Bashir, and S. M. Husnine, (2021). Some new generalizations of F-contraction type mappings that weaken certain conditions on Caputo fractional type differential equations. Aims Math, 6, 12718-12742.
35. M. Zhou, N. Saleem, X. Liu, A. Fulga, and A. F. Roldán López de Hierro, (2021). A new approach to proinov-type fixed-point results in non-archimedean fuzzy metric spaces. Mathematics, 9(23), 3001.
36. M. Abbas, F. Lael, and N. Saleem, (2020). Fuzzy b-Metric Spaces: Fixed Point Results for ψ-Contraction Correspondences and Their Application. Axioms, 9(2), 36.
37. N. Saleem, I. Habib, and M. D. L. Sen, (2020). Some new results on coincidence points for multivalued Suzuki-type mappings in fairly complete spaces. Computation, 8(1), 17.
38. H. Alolaiyan, N. Saleem, and M. Abbas, (2018). A natural selection of a graphic contraction transformation in fuzzy metric spaces. J. Nonlinear Sci. Appl, 11, 218-227.
39. M. Jeyaraman, M. Suganthi, and W. Shatanawi. Common fixed point theorems in intuitionistic generalized fuzzy cone metric spaces. *Mathematics* 8, no. 8 (2020): 1212.
40. U. Ishtiaq, M. Asif, A. Hussain, K. Ahmad, I. Saleem, and H. Al Sulami. Extension of a Unique Solution in Generalized Neutrosophic Cone Metric Spaces. *Symmetry* 15, no. 1 (2022): 94.
41. V. Gupta, W. Shatanawi, and A. Kanwar. Coupled fixed point theorems employing CLR-property on V-fuzzy metric spaces. *Mathematics* 8, no. 3 (2020): 404.
42. S. Chauhan, W. Shatanawi, S. Kumar, and S. Radenovic. Existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces. *J. Nonlinear Sci. Appl* 7, no. 1 (2014): 28-41.
43. V. Gupta, V. Shatanawi, and M. Verma. Existence of fixed points for J-ψ-fuzzy contractions in fuzzy metric spaces endowed with graph. *J Anal* 27, 417–427 (2019).

Received: Jun 01, 2023. Accepted: Sep 01, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.52

A New Approach for the Statistical Convergence over Non-Archimedean Fields in Neutrosophic Normed Spaces

Jeyaraman. M [1,*] and Iswariya. S [2]

[1] P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; jeya.math@gmail.com.

[2] Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; iiswariya1234@gmail.com.

* Correspondence: jeya.math@gmail.com.

Abstract: The goal of the research involves elaborating on the topics of statistical convergence, including statistical Cauchy sequences within non-Archimedean Neutrosophic normed spaces, as well as achieving specific useful conclusions. The present research shows how, within a non-Archimedean field, certain sections of statistically convergent sequences that could not be true often become true. Likewise, we created statistically complete and statistically continuous spaces for such regions that demonstrated certain essential facts. κ indicates a complete field of non-Archimedean and non-trivially valued research.

Keywords: Neutrosophic Normed Spaces; Non-Archimedean Fields; Statistically Cauchy Sequence; Statistically Convergent.

1. Introduction

Zadeh [16] became the initial one person who creates the fuzzy set using a membership function. Many later researchers were adapted this idea to classical set theory. Atanassov [1] introduced an Intuitionistic Fuzzy (IF) set theory. Saadati along with Park proposed the notion of IF normed space. The study of analysis through fields of Non-Archimedean (NA) is referred to as NA analysis. Suja and Srinivasan [15] newly created statistically convergent along with statistically Cauchy sequences within NA fields. Eghbali and Ganji [3] investigated NAL-fuzzy normed spaces for extended statistical convergence. The research shows that statistical convergence exists in Non-Archimedean Neutrosophic Normed Spaces (NA-NNS) and confirms that key properties of statistical convergence from real sequences are still valid in NA fields [2,4-5,8-14]. The research article concentrates primarily upon the analysis of sequences in the field of NA κ.

In 1998, Smarandache [12] developed the ideas of neutrosophic logic in addition to the Neutrosophic Set [NS]. Kirisci and Simsek establish the Neutrosophic Metric Space [NMS] suggestion which is associated with membership, non-membership and neutralness. Jeyaraman, Ramachandran and Shakila [7] established approximate fixed point theorems in 2022 regarding weak contractions on Neutrosophic Normed Spaces (NNS). Statistical Δ^m convergence in NNS was recently presented by Jeyaraman and Jenifer [6].

A sequence $\bar{\bar{\mathfrak{x}}}= \{v_{p}\}$ is said to have been statistically convergent towards a limit $\mathfrak{L}$when for any $\tilde{\omega}>0$, $\lim_{n\to\infty}\frac{1}{n}\{p \leq n : |v_{p} - \mathfrak{L}| \geq \tilde{\omega}\} = 0$.

In that case above, we put $stat - \lim_{p\to\infty} v_{p} = \mathfrak{L}$.

Example 1.1. Consider to define the $\bar{\bar{\mathfrak{x}}}= \{v_{p}\}$ sequence by

$$v_{p} = \begin{cases} \frac{p-1}{p^2}, & p \text{ is a perfect square.} \\ 0, & \text{otherwise;} \end{cases}$$

Selecting the NA valuation to be 2-adic, the sequence terms become (0,0,0,1,0,0,0,0,1/8,0,0,....).
As a result, it converges to zero statistically.
A sequence of Statistically Cauchy (SC) when for all $\tilde{\omega}>0$, then existing a range $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\{i \leq n : n \in \mathbb{N} : |\bar{\bar{\mathfrak{x}}}_{i+1} - \bar{\bar{\mathfrak{x}}}_{i}| \geq \tilde{\omega}\} = 0$$

Consider that κ to be NA fields. A valuation on $\kappa$is referred with the NA if it meets these three given axioms: [1]
(i) $|\bar{\bar{\mathfrak{x}}}| \geq 0$ and$|\bar{\bar{\mathfrak{x}}}| = 0$ iff $\bar{\bar{\mathfrak{x}}}= 0$,
(ii)$|\bar{\bar{\mathfrak{x}}}\breve{\mathfrak{y}}| = |\bar{\bar{\mathfrak{x}}}||\breve{\mathfrak{y}}|$,
(iii) $|\bar{\bar{\mathfrak{x}}} + \breve{\mathfrak{y}}| \leq \max[|\bar{\bar{\mathfrak{x}}}|, |\breve{\mathfrak{y}}|]$ for every $\bar{\bar{\mathfrak{x}}}, \breve{\mathfrak{y}} \in \kappa$ (Ultrametric Inequality).

2. Preliminaries

Here, we will go through the notations along with definitions which will be utilized throughout this article in order to ensure a general understanding of the terminology and symbols used.

Definition 2.1. The 7-tuple $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$is said to be a *NA-NNS*, if $*$ acts as a continuous t-norm, $\diamond$ and $\star$acts as a t-co norms which are continuous, Ξ become a vector space over a field κ and then $\tilde{\varsigma}, \dot{\varphi}, \psi$ are fuzzy sets functions on $\Xi \times \mathbb{R}$ to [0, 1], for all $v, \hbar \in \Xi$ and $\acute{f}$, $\grave{t} \in$ κ.

(cn1) $\tilde{\varsigma}(v, \acute{f}) + \dot{\varphi}(v, \acute{f}) + \psi(v, \acute{f}) \leq 3$
(cn2) $0 \leq \tilde{\varsigma}(v, \acute{f}) \leq 1; 0 \leq \dot{\varphi}(v, \acute{f}) \leq 1$ *and* $0 \leq \psi(v, \acute{f}) \leq 1$;
(cn3) $\tilde{\varsigma}(v, \acute{f}) > 0$;
(cn4) $\tilde{\varsigma}(v, \acute{f}) = 1 \Leftrightarrow v = 0$,
(cn5) $\tilde{\varsigma}(\ddot{\gamma} v, \acute{f}) = \tilde{\varsigma}\left(v, \frac{\acute{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \neq 0$;
(cn6) $\tilde{\varsigma}(v + \hbar, \max\{\acute{f} + \grave{t}\}) \geq \tilde{\varsigma}(v, \acute{f}) * \tilde{\varsigma}(\hbar, \grave{t})$,
(cn7) $\tilde{\varsigma}(v, .)$: $(0, \infty) \to [0,1]$ and it is continuous,
(cn8) $\lim_{\acute{f}\to\infty} \tilde{\varsigma}(v, \acute{f}) = 1$ and $\lim_{\acute{f}\to\infty} \tilde{\varsigma}(v, \acute{f}) = 0$;
(cn9)$\dot{\varphi}(v, \acute{f}) < 1$;
(cn10) $\dot{\varphi}(v, \acute{f}) = 0 \Leftrightarrow v = 0$,
(cn11) $\dot{\varphi}(\ddot{\gamma} v, \acute{f}) = \dot{\varphi}\left(v, \frac{\acute{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \neq 0$;

(cn12) $\dot{\varphi}(\mathscr{v}+\hbar, \max\{\not{f}+\mathfrak{t}\}) \le \dot{\varphi}(\mathscr{v},\not{f}) \diamond \dot{\varphi}(\hbar,\mathfrak{t})$,
(cn13) $\dot{\varphi}(\mathscr{v},.) : (0,\infty) \to [0,1]$ and it is continuous;
(cn14) $\lim_{\not{f}\to\infty} \dot{\varphi}(\mathscr{v},\not{f}) = 0$ and $\lim_{\not{f}\to\infty} \dot{\varphi}(\mathscr{v},\not{f}) = 1$;
(cn15) $\psi(\mathscr{v},\not{f}) < 1$,
(cn16) $\psi(\mathscr{v},\not{f}) = 0 \Leftrightarrow \mathscr{v} = 0$,
(cn17) $\psi(\ddot{\gamma}\mathscr{v},\not{f}) = \psi\left(\mathscr{v},\frac{\not{f}}{|\ddot{\gamma}|}\right)$, for all $\ddot{\gamma} \in \mathbb{R}$ and $\ddot{\gamma} \neq 0$,
(cn18) $\psi(\mathscr{v}+\hbar, \max\{\not{f}+\mathfrak{t}\}) \le \psi(\mathscr{v},\not{f}) \star \psi(\hbar,\mathfrak{t})$,
(cn19) $\psi(\mathscr{v},.) : (0,\infty) \to [0,1]$ is continuous and
(cn20) $\lim_{\not{f}\to\infty} \psi(\mathscr{v},\not{f}) = 0$ and $\lim_{\not{f}\to\infty} \psi(\mathscr{v},\not{f}) = 1$.

Here, $(\tilde{\varsigma},\dot{\varphi},\psi)$ is known as a *NA-NNS.*

A sequence $\{\mathscr{v}_{\wp}\}$ is referred to be convergent in *NA-NNS* $(\mathfrak{V},\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ or simply $(\tilde{\varsigma},\dot{\varphi},\psi)$-convergent to $\bar{\bar{\mathfrak{x}}}\in\Xi$ if for all $\not{f}>0$ and $\widetilde{\omega}>0$, then there exist $\wp_0\in\mathbb{N}$ so that $\wp \ge \wp_0$,

$$\tilde{\varsigma}(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) > 1-\widetilde{\omega}, \dot{\varphi}(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) < \widetilde{\omega} \text{ and } \psi(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) < \widetilde{\omega}$$

In this case, we write $(\tilde{\varsigma},\dot{\varphi},\psi)-\lim \mathscr{v}_{\wp}=\bar{\bar{\mathfrak{x}}}$.

Example 2.2 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA* normed space, $\mathscr{v}*\hbar=\mathscr{v}\hbar$, $\mathscr{v}\diamond\hbar=$ *min* $\{\mathscr{v}+\hbar, 1\}$ and $\mathscr{v}\star\hbar=$ *min* $\{\mathscr{v}+\hbar, 1\}$ for all $\mathscr{v},\hbar\in[0,1]$. For every $\bar{\bar{\mathfrak{x}}}\in\Xi$, every $\hat{\xi}>0$ and $\wp= 1, 2,\ldots$.Consider the following form,

$$\tilde{\varsigma}_{\wp}(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \begin{cases} \dfrac{\hat{\xi}}{\hat{\xi}+\wp\|\bar{\bar{\mathfrak{x}}}\|}, & if \quad \hat{\xi}>0 \\ 0, & \hat{\xi}\le 0; \end{cases}$$

$$\dot{\varphi}_{\wp}(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \begin{cases} \dfrac{\wp\|\bar{\bar{\mathfrak{x}}}\|}{\hat{\xi}+\wp\|\bar{\bar{\mathfrak{x}}}\|}, & if \quad \hat{\xi}>0 \\ 0, & \hat{\xi}\le 0; \end{cases}$$

$$\psi_{\wp}(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \begin{cases} \dfrac{\wp\|\bar{\bar{\mathfrak{x}}}\|}{\hat{\xi}}, & if \quad \hat{\xi}>0 \\ 0, & \hat{\xi}\le 0; \end{cases}$$

Then $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ which is a *NA-NNS*.

Definition 2.3 A $\{\mathscr{v}_{\wp}\}$ sequence in a *NA-NNS*$(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ is said to be a statistically convergent towards a limit $\bar{\bar{\mathfrak{x}}}\in\Xi$ relate with the *NA* fuzzy norm $(\tilde{\varsigma},\dot{\varphi},\psi)$ when for each $\widetilde{\omega}>0$ and $\not{f}>0$,

$$\lim_{n}\frac{1}{n}\left|\{\wp\le n : \tilde{\varsigma}(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) \le 1-\widetilde{\omega} \text{ or } \dot{\varphi}(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) \ge \widetilde{\omega} \text{ and } \psi(\mathscr{v}_{\wp}-\bar{\bar{\mathfrak{x}}},\not{f}) \ge \widetilde{\omega}\}\right| = 0.$$

In this case, we write $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{\wp}\mathscr{v}_{\wp}=\bar{\bar{\mathfrak{x}}}$ where $\bar{\bar{\mathfrak{x}}}$ is the $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-limit$.

Example 2.4 Let $(Q_P,|.|)$ indicate the p-adic numbers space in the standard norm, and consider $\mathscr{v}*\hbar=\mathscr{v}\hbar$, $\mathscr{v}\diamond\hbar=$ *min* $\{\mathscr{v}+\hbar, 1\}$ and $\mathscr{v}\star\hbar=$ *min* $\{\mathscr{v}+\hbar, 1\}$ for every $\mathscr{v},\hbar\in[0, 1]$. For every $\bar{\bar{\mathfrak{x}}}\in Q_P$ and all $\hat{\xi}>0$, let $\tilde{\varsigma}_0(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \frac{\hat{\xi}}{\hat{\xi}+|\bar{\bar{\mathfrak{x}}}|}$, $\dot{\varphi}_0(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \frac{|\bar{\bar{\mathfrak{x}}}|}{\hat{\xi}+|\bar{\bar{\mathfrak{x}}}|}$ and $\psi_0(\bar{\bar{\mathfrak{x}}},\hat{\xi}) = \frac{|\bar{\bar{\mathfrak{x}}}|}{\hat{\xi}}$. In this case observe that $(Q_P,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ is a *NA-NNS*.

Define a sequence $\bar{\bar{x}}=\{v_p\}$ the terms of which are provided by

$$v_p=\begin{cases}1, & if\ p=m^2 (m\in\mathbb{N})\\ 0, & otherwise;\end{cases}$$

Then for every $0<\tilde{\omega}<1$ and for any $\hat{\xi}>0$, let $\mathfrak{K}_n(\tilde{\omega},\hat{\xi})=p\le n:\tilde{\varsigma}_0(v_p,\hat{\xi})\le 1-\tilde{\omega}$ or $\dot{\varphi}_0(v_p,\hat{\xi})\ge \tilde{\omega}\ and\ \psi_0(v_p,\hat{\xi})\ge\tilde{\omega}$.

Since

$$p_n(\tilde{\omega},\hat{\xi})=\{p\le n:\frac{\hat{\xi}}{\hat{\xi}+|v_p|}\le 1-\tilde{\omega}\text{ or }\frac{|v_p|}{\hat{\xi}+|v_p|}\ge\tilde{\omega}\ and\ \frac{|v_p|}{\hat{\xi}}\ge\tilde{\omega}\}$$

$$=\left\{p\le n:|v_p|\ge\frac{\tilde{\omega}\hat{\xi}}{1-\tilde{\omega}}>0\right\}=\{p\le n:|v_p|=1\}=\{p\le n:p=m^2\ and\ m\in\mathbb{N}\}.$$

We have,

$$\frac{1}{n}|p_n(\tilde{\omega},\hat{\xi})|=\frac{1}{n}\{p\le n:p=m^2\ and\ m\in\mathbb{N}\}\le\frac{\sqrt{n}}{n}.$$

This yields that

$$\lim_n\frac{1}{n}|p_n(\tilde{\omega},\hat{\xi})|=0.$$

Hence by the above definition, $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-lim\, v_p=0$.

3. Statistical Convergence on Neutrosophic Normed Spaces

Here, that portion having the goal is to determine theorems concerning convergence and statistical convergence within the context of *NNS* over *NA* fields κ.

Lemma 3.1 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA-NNS*. After that the given statements is equivalent for all $\tilde{\omega}>0$ and $\acute{f}>0$

$(i)\ Stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_n v_p=\bar{\bar{x}}$.

$$(ii)\ \lim_n\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})\le 1-\tilde{\omega}\}\right|=\lim_n\frac{1}{n}\left|\{p\le n:\dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})\ge\tilde{\omega}\}\right|$$

$$=\lim_n\frac{1}{n}\left|\{p\le n:\psi(v_p-\bar{\bar{x}},\acute{f})\ge\tilde{\omega}\}\right|=0.$$

$$(iii)\ \lim_n\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})>1-\tilde{\omega},\dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})<\tilde{\omega}\text{ and }\psi(v_p-\bar{\bar{x}},\acute{f})<\tilde{\omega}\}\right|=1.$$

$$(iv)\ \lim_n\frac{1}{n}\left|\{p\le n:\ \tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})>1-\tilde{\omega}\}\right|=\lim_n\frac{1}{n}\left|\{p\le n:\dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})<\tilde{\omega}\}\right|$$

$$=\lim_n\frac{1}{n}\left|\{p\le n:\psi(v_p-\bar{\bar{x}},\acute{f})<\tilde{\omega}\}\right|=1.$$

$(v)\ stat-lim\,\tilde{\varsigma}(v_p-\bar{\bar{x}},\acute{f})=1, stat-lim\,\dot{\varphi}(v_p-\bar{\bar{x}},\acute{f})=0\ and\ \ stat-lim\,\psi(v_p-\bar{\bar{x}},\acute{f})=0$

Theorem 3.2 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be a *NA-NNS*. If a $\{v_p\}$ sequence is statistically convergent with respect to the *NN*$(\tilde{\varsigma},\dot{\varphi},\psi)$, then $stat_{\tilde{\varsigma},\dot{\varphi},\psi}$ −limit is unique.

Proof: Assume that $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_p v_p=\bar{\bar{x}}_1$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_p v_p=\bar{\bar{x}}_2$. Consider a given $\tilde{\omega}>0$, select $\hat{\xi}>0$ so that we have $(1-\hat{\xi})*(1-\hat{\xi})>1-\tilde{\omega},\hat{\xi}\diamond\hat{\xi}<\tilde{\omega}$ and $\hat{\xi}\star\hat{\xi}<\tilde{\omega}$. After that for any $\acute{f}>0$, define the sets given below:

$\wp_{\tilde{\varsigma},1}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \tilde{\varsigma}(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) \leq 1 - \hat{\xi}\}$, $\wp_{\tilde{\varsigma},2}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \tilde{\varsigma}(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) \leq 1 - \hat{\xi}\}$,
$\wp_{\dot{\varphi},1}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \dot{\varphi}(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) \geq \hat{\xi}\}$, $\wp_{\dot{\varphi},2}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \dot{\varphi}(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) \geq \hat{\xi}\}$ and
$\wp_{\psi,1}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \psi(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) \geq \hat{\xi}\}$, $\wp_{\psi,2}(\hat{\xi},\acute{\phi}) := \{\wp \in \mathbb{N} : \psi(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) \geq \hat{\xi}\}$.

Since $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp} v_{\wp} = \bar{\bar{x}}_1$, we have

$$\lim_{n} \frac{1}{n} \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{\phi})\} = \lim_{n} \frac{1}{n} \{\wp_{r,1}(\tilde{\omega},\acute{\phi})\} = 0 \; for \; all \; \acute{\phi} > 0.$$

Furthermore, using $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{\wp} v_{\wp} = \bar{\bar{x}}_2$, we get

$$\lim_{n} \frac{1}{n} \{\wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{\phi})\} = \lim_{n} \frac{1}{n} \{\wp_{r,2}(\tilde{\omega},\acute{\phi})\} = 0 \; for \; all \; \acute{\phi} > 0.$$

Now let,

$$\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{\phi}) := \{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{\phi})\} \cap \{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{\phi})\} \cap \{\wp_{\psi,1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\psi,2}(\tilde{\omega},\acute{\phi})\}.$$

If $\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\tilde{\omega},\acute{\phi}) = \wp_{\tilde{\varsigma},\dot{\varphi},\psi}$, $\{\wp_{\tilde{\varsigma},1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\tilde{\varsigma},2}(\tilde{\omega},\acute{\phi})\} = \wp_{\tilde{\varsigma}}$, $\{\wp_{\dot{\varphi},1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\dot{\varphi},2}(\tilde{\omega},\acute{\phi})\}$and $\{\wp_{\psi,1}(\tilde{\omega},\acute{\phi}) \cup \wp_{\psi 2}(\tilde{\omega},\acute{\phi})\} = \wp_{\psi}, then \wp_{\tilde{\varsigma},\dot{\varphi},\psi} = \wp_{\tilde{\varsigma}} \cap \wp_{\dot{\varphi}} \cap \wp_{\psi}$.

Then observe that, $\lim_{n} \frac{1}{n} \{\wp_{\tilde{\varsigma},\dot{\varphi},\psi}\} = 0$. which implies, $\lim_{n} \frac{1}{n} \{\wp^{C}_{\tilde{\varsigma},\dot{\varphi},\psi}\} = 1$.

If $\wp \in \wp^{C}_{\tilde{\varsigma},\dot{\varphi},\psi}$, then there are three possibilities to consider:
Then we have to select the initial part which is $\wp \in \{\wp^{C}_{\tilde{\varsigma}}\}$, the second part which is $\wp \in \{\wp^{C}_{\dot{\varphi}}\}$ and the later is $\wp \in \{\wp^{C}_{\psi}\}$.

We first consider that $\wp \in \{\wp^{C}_{\tilde{\varsigma}}\}$, then we have,

$$\begin{aligned}\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) = \tilde{\varsigma}(\bar{\bar{x}}_1 - v_{\wp} + v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) &\geq \tilde{\varsigma}(\bar{\bar{x}}_1 - v_{\wp}, \acute{\phi}) * \tilde{\varsigma}(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) \\ &= \tilde{\varsigma}(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) * \tilde{\varsigma}(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) \\ &> (1 - \hat{\xi}) * (1 - \hat{\xi}).\end{aligned}$$

Since $(1 - \hat{\xi}) * (1 - \hat{\xi}) > 1 - \tilde{\omega}$, it follows that

$$\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) > 1 - \tilde{\omega}.$$

Since $\tilde{\omega} > 0$ was arbitrary, $\tilde{\varsigma}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) > 1$ for all $\acute{\phi} > 0$, which given $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
On the second hand, if $\wp \in \{\wp^{C}_{\dot{\varphi}}\}$, then we may write that,

$$\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) \leq \dot{\varphi}(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) \diamond \dot{\varphi}(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) < \hat{\xi} \diamond \hat{\xi}.$$

Now using the fact$\hat{\xi} \diamond \hat{\xi} < \tilde{\omega}$, we see that$\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) < \tilde{\omega}$.
Again, since $\tilde{\omega} > 0$ was arbitrary, we have $\dot{\varphi}(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) = 0$ for all $\acute{\phi} > 0$.
This implies $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
And on the other side, if $\wp \in \{\wp^{C}_{\psi}\}$, then we put

$$\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) \leq \psi(v_{\wp} - \bar{\bar{x}}_1, \acute{\phi}) \star \psi(v_{\wp} - \bar{\bar{x}}_2, \acute{\phi}) < \hat{\xi} \star \hat{\xi}.$$

By using$\hat{\xi} \star \hat{\xi} < \tilde{\omega}$, we get $\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) < \tilde{\omega}$.
Hence $\tilde{\omega} > 0$ is arbitrary, $\psi(\bar{\bar{x}}_1 - \bar{\bar{x}}_2, \acute{\phi}) = 0$ for every $\acute{\phi} > 0$, which implies $\bar{\bar{x}}_1 = \bar{\bar{x}}_2$.
Therefore, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} -$ limit is unique.

Theorem 3.3 If a sequence $\{v_{\wp}\}$ in a *NA-NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ is $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ −convergent to $\bar{\bar{x}} \in \Xi$, then this is $stat_{\tilde{\varsigma},\dot{\varphi},\psi}$ −convergent towards $\bar{\bar{x}} \in \Xi$.

Proof: Since $\{v_p\}$ is $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ −convergent towards $\bar{\bar{x}} \in \Xi$, for all $\tilde{\omega} > 0$ and $f > 0$, then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\tilde{\varsigma}(v_p - \bar{\bar{x}}, f) > 1 - \tilde{\omega}, \dot{\varphi}(v_p - \bar{\bar{x}}, f) < \tilde{\omega} \textit{ and } \psi(v_p - \bar{\bar{x}}, f) < \tilde{\omega}.$$

This given the set $\{p \in \mathbb{N} : \tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \text{ and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega}\}$ has at the most a finite number of terms.

$$\text{i.e.,} \lim_n \frac{1}{n} \left| \begin{Bmatrix} p \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \\ \textit{and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

i.e., $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - lim v_p = \bar{\bar{x}}$.

Note: It is interesting to note that the converse of this, which is not true classically, is true in a *NA-NNS* as shown below.

Let $\{v_p\}$ be $stat_{\tilde{\varsigma},\dot{\varphi},\psi} -$ convergent towards $\bar{\bar{x}} \in \Xi$. Then for all $\tilde{\omega} > 0$ and $f > 0$,

$$\lim_n \frac{1}{n} \left| \{p \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \textit{ and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega}\} \right| = 0.$$

Now to prove that $\{v_p\}$ is $(\tilde{\varsigma}, \dot{\varphi}, \psi) -$ convergent to $v \in \Xi$. i.e., to prove that for all $\tilde{\omega} > 0$ and $f > 0$ therefore $n_0 \in \mathbb{N}$ exists in that way and for every $n \geq n_0$,

$$\tilde{\varsigma}(v_p - \bar{\bar{x}}, f) > 1 - \tilde{\omega} \textit{ and } \dot{\varphi}(v_p - \bar{\bar{x}}, f) < \tilde{\omega} \text{ and } \psi(v_p - \bar{\bar{x}}, f) < \tilde{\omega}.$$

Let us assume the contrary that,

$$\tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \text{ and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega}.$$

This implies that the set

$$\{p \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \text{ and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega}\}$$

has infinitely many terms.

ie, $\lim_n \frac{1}{n} \left| \{p \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, f) \leq 1 - \tilde{\omega} \text{ or } \dot{\varphi}(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega} \text{ and } \psi(v_p - \bar{\bar{x}}, f) \geq \tilde{\omega}\} \right| \neq 0$ which is a contradiction. Therefore, $\{v_p\}$ is $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ −convergent to $v \in \Xi$.

Theorem 3.4 Let $\{v_p\}$ and $\{h_p\}$ be sequences in a *NA-NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ so that $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} v_p = v$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} h_p = h$, where $v, h \in \Xi$. Then we have $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} (v_p + h_p) = v + h$.

Proof: Let $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} v_p = v$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{n\to\infty} h_p = h$, choose $\hat{\xi} > 0$ such that $(1 - \hat{\xi}) * (1 - \hat{\xi}) > 1 - \tilde{\omega}, \hat{\xi} \diamond \hat{\xi} < \tilde{\omega}$ and $\hat{\xi} \star \hat{\xi} < \tilde{\omega}$ for a given $\tilde{\omega} > 0$. Then, for $f > 0$, define

$p_{\tilde{\varsigma},1}(\hat{\xi}, f) := \{p \in \mathbb{N} : \tilde{\varsigma}(v_p - v, f) \leq 1 - \hat{\xi}\}$,
$p_{\tilde{\varsigma},2}(\hat{\xi}, f) := \{p \in \mathbb{N} : \tilde{\varsigma}(h_p - h, f) \leq 1 - \hat{\xi}\}$,
$p_{\dot{\varphi},1}(\hat{\xi}, f) := \{p \in \mathbb{N} : \dot{\varphi}(v_p - v, f) \geq \hat{\xi}\}$,
$p_{\dot{\varphi},2}(\hat{\xi}, f) := \{p \in \mathbb{N} : \dot{\varphi}(h_p - h, f) \geq \hat{\xi}\}$ and
$p_{\psi,1}(\hat{\xi}, f) := \{p \in \mathbb{N} : \psi(v_p - v, f) \geq \hat{\xi}\}$,
$p_{\psi,2}(\hat{\xi}, f) := \{p \in \mathbb{N} : \psi(h_p - h, f) \geq \hat{\xi}\}$.

Since $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} h_p = h$,

$$\lim_n \frac{1}{n} \{p_{\tilde{\varsigma},1}(\tilde{\omega}, f)\} = \lim_n \frac{1}{n} \{p_{\dot{\varphi},1}(\tilde{\omega}, f)\} = \lim_n \frac{1}{n} \{p_{\psi,1}(\tilde{\omega}, f)\} = 0,$$

$$\lim_{n}\frac{1}{n}\{\wp_{\tilde{\varsigma},2}(\widetilde{\omega},\not{f})\}=\lim_{n}\frac{1}{n}\{\wp_{\dot{\varphi},2}(\widetilde{\omega},\not{f})\}=\lim_{n}\frac{1}{n}\{\wp_{\psi,2}(\widetilde{\omega},\not{f})\}=0.$$

Now let,

$\wp_{\tilde{\varsigma},\dot{\varphi},\psi}(\widetilde{\omega},\not{f}):=\{\wp_{\tilde{\varsigma},1}(\widetilde{\omega},\not{f})\cup\wp_{\tilde{\varsigma},2}(\widetilde{\omega},\not{f})\}\cap\{\wp_{\dot{\varphi},1}(\widetilde{\omega},\not{f})\cup\wp_{\dot{\varphi},2}(\widetilde{\omega},\not{f})\}\cap\{\wp_{\psi,1}(\widetilde{\omega},\not{f})\cup\wp_{\psi,2}(\widetilde{\omega},\not{f})\}$
i.e., if $\mathfrak{K}=\mathfrak{K}_{\tilde{\varsigma},\dot{\varphi},\psi}(\widetilde{\omega},\not{f}),\mathfrak{K}_1=\{\wp_{\tilde{\varsigma},1}(\widetilde{\omega},\not{f})\cup\wp_{\tilde{\varsigma},2}(\widetilde{\omega},\not{f})\},\mathfrak{K}_2=\{\wp_{\dot{\varphi},1}(\widetilde{\omega},\not{f})\cup\wp_{\dot{\varphi},2}(\widetilde{\omega},\not{f})\}$
$\mathfrak{K}_3=\{\wp_{\psi,1}(\widetilde{\omega},\not{f})\cup\wp_{\psi,2}(\widetilde{\omega},\not{f})\}$. Then $\mathfrak{K}=\mathfrak{K}_1\cap\mathfrak{K}_2\cap\mathfrak{K}_3$.
Since $\mathfrak{K}^C$ is a non-empty set. Consider $\wp\in\mathfrak{K}^C$, then we have three possible cases. The former is $\wp\in\mathfrak{K}_1^C$, the second is $\wp\in\mathfrak{K}_2^C$ and the later is $\wp\in\mathfrak{K}_3^C$. First consider, $\wp\in\mathfrak{K}_1^C$, then we have,

$$\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})>1-\hat{\xi}\ and\ \tilde{\varsigma}(\hbar_{\wp}-\hbar,\not{f})>1-\hat{\xi}.$$

Now, we have,

$$\tilde{\varsigma}(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})>\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})*\tilde{\varsigma}(\hbar_{\wp}-\hbar,\not{f})$$
$$>(1-\hat{\xi})*(1-\hat{\xi}).$$

Since $(1-\hat{\xi})*(1-\hat{\xi})>1-\widetilde{\omega}$, it follows that $\tilde{\varsigma}(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})>1-\widetilde{\omega}$.
Since $\widetilde{\omega}$ is arbitrary, $\tilde{\varsigma}(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})=1$ for all $\not{f}>0$,
which yields, $\tilde{\varsigma}(\upsilon_{\wp}+\hbar_{\wp}-(\upsilon+\hbar),\not{f})=1$.
Similarly, if $\wp\in\mathfrak{K}_2^C$ then,
$\dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})<\hat{\xi}\ and\ \dot{\varphi}(\hbar_{\wp}-\hbar,\not{f})<\hat{\xi}$.
$\Rightarrow\ \dot{\varphi}(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})\le\dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})\diamond\dot{\varphi}(\hbar_{\wp}-\hbar,\not{f})<\hat{\xi}<\hat{\xi}\diamond\hat{\xi}<\widetilde{\omega}$.
Since $\widetilde{\omega}$ is arbitrary, $\dot{\varphi}(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})=0$, for all $\not{f}>0$

$$\Rightarrow\ \dot{\varphi}(\upsilon_{\wp}+\hbar_{\wp}-(\upsilon+\hbar),\not{f})=0$$

And if $\wp\in\mathfrak{K}_3^C$ then,
$\psi(\upsilon_{\wp}-\upsilon,\not{f})<\hat{\xi}\ and\ \psi(\hbar_{\wp}-\hbar,\not{f})<\hat{\xi}$
$\Rightarrow\ \psi(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})\le\psi(\upsilon_{\wp}-\upsilon,\not{f})\star\psi(\hbar_{\wp}-\hbar,\not{f})<\hat{\xi}<\hat{\xi}\star\hat{\xi}<\widetilde{\omega}$.
Since $\widetilde{\omega}$ is arbitrary, $\psi(\upsilon_{\wp}+\hbar_{\wp}-\upsilon-\hbar,\not{f})=0$, for all $\not{f}>0$

$$\Rightarrow\ \psi(\upsilon_{\wp}+\hbar_{\wp}-(\upsilon+\hbar),\not{f})=0.$$

Thus, $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{\wp\to\infty}(\upsilon_{\wp}+\hbar_{\wp})=\upsilon+\hbar$.

Theorem 3.5 Let $(\Xi,\tilde{\varsigma},\dot{\varphi},\psi,*,\diamond,\star)$ be an *NA- NNS* over κ. If $\lim_{\wp\to\infty}\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})=1$, $\lim_{\wp\to\infty}\dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})=1$ and $\lim_{\wp\to\infty}\psi(\upsilon_{\wp}-\upsilon,\not{f})=1$ then $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{\wp\to\infty}\upsilon_{\wp}=\upsilon$.

Proof: Let $\lim_{\wp\to\infty}\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})=1$, $\lim_{\wp\to\infty}\dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})=1$ and $\lim_{\wp\to\infty}\psi(\upsilon_{\wp}-\upsilon,\not{f})=1$. Then for all $\hat{\xi}>0$ and $\widetilde{\omega}>0$, that is a number $\wp_0\in\mathbb{N}$ in that way, $\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})>1-\widetilde{\omega},\dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})<\widetilde{\omega}$ and $\psi(\upsilon_{\wp}-\upsilon,\not{f})<\widetilde{\omega}$ for every $\wp\ge\wp_0$. Hence the set, $\{\wp\in\mathbb{N}:\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})\le1-\widetilde{\omega},\ \dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})\ge\widetilde{\omega}\ and\ \psi(\upsilon_{\wp}-\upsilon,\not{f})\ge\widetilde{\omega}\}$ has a finite number of terms.

So, $\lim_{n}\frac{1}{n}\left|\left\{\begin{matrix}\wp\le n:\tilde{\varsigma}(\upsilon_{\wp}-\upsilon,\not{f})\le1-\widetilde{\omega}\ or\ \dot{\varphi}(\upsilon_{\wp}-\upsilon,\not{f})\ge\widetilde{\omega}\\ and\ \psi(\upsilon_{\wp}-\upsilon,\not{f})\ge\widetilde{\omega}\end{matrix}\right\}\right|=0.$

Thus, $stat_{\tilde{\varsigma},\dot{\varphi},\psi}-\lim_{\wp\to\infty}\upsilon_{\wp}=\upsilon$.

4. Statistically Cauchy Sequences on NNS

Definition 4.1 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$be a *NA-NNS* over κ. Then, a $\{v_p\}$ sequence is referred to be SC when for each $\tilde{\omega} > 0$ and $\acute{f} > 0$ therefore $\mathbb{N}$ exists in which case for every $p,\ m \geq \mathbb{N}$,

$$\lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p, m \leq n : \tilde{\varsigma}(v_p - v_m, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_p - v_m, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - v_m, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

Definition 4.2 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NNS*. A sequence $\{v_p\}$ is refer as a Cauchy sequence when for each$\tilde{\omega} > 0$ and $\acute{f} > 0$, that is a number$p_0 \in \mathbb{N}$ exist that way, for every $p, m \geq p_0$,

$$\tilde{\varsigma}(v_p - v_m, \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(v_p - v_m, \acute{f}) < \tilde{\omega} \ and\ \psi(v_p - v_m, \acute{f}) < \tilde{\omega}.$$

Theorem 4.3 Every Cauchy sequence with respect to $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ in *NA- NNS*$(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ over κ is SC.

Proof: If $\{v_p\}$ is a Cauchy sequence with relate to $(\tilde{\varsigma}, \dot{\varphi}, \psi)$, then there exists$p_0 \in \mathbb{N}$ for all $\tilde{\omega} > 0$ and $\acute{f} > 0$ and let t be an arbitrary constant, we have

$$\tilde{\varsigma}(v_{p+t} - v_p, \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}\ (v_{p+t} - v_p, \acute{f}) < \tilde{\omega} \ and\ \psi(v_{p+t} - v_p, \acute{f}) < \tilde{\omega}.$$

The number of terms in the set $\begin{Bmatrix} p \in \mathbb{N} : \tilde{\varsigma}(v_{p+t} - v_p, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_{p+t} - v_p, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_{p+t} - v_p, \acute{f}) \geq \tilde{\omega} \end{Bmatrix}$ is limited.

So

$$\lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p + t, p \leq n : \tilde{\varsigma}(v_{p+t} - v_p, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_{p+t} - v_p, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_{p+t} - v_p, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

Theorem 4.4 If a statistically convergent sequence in a *NA- NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ over κ, then it is *SC*.

Proof: If the sequence $\{v_p\}$ is statistically convergent to $\bar{\bar{x}}$then,

$$\lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_p - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

Now, we get

$$\lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p, m \leq n : \tilde{\varsigma}(v_p - v_m, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_p - v_m, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - v_m, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right|$$

$$= \lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p, m \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, \acute{f}) * \tilde{\varsigma}(v_m - \bar{\bar{x}}, \acute{f}) \leq 1 - \tilde{\omega} \\ or\ \dot{\varphi}(v_p - \bar{\bar{x}}, \acute{f}) \diamond \dot{\varphi}(v_m - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - \bar{\bar{x}}, \acute{f}) \star \psi(v_m - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0.$$

5. Statistically complete and statistically continuous on NNS

A *NA- NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ is said to be complete if all$(\tilde{\varsigma}, \dot{\varphi}, \psi)$-Cauchy is $(\tilde{\varsigma}, \dot{\varphi}, \psi)$- convergent.

Definition 5.1 A *NA- NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ over κ is said to be statistically complete when all *SC* sequence with respect to $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ is statistically convergent in relate with the $(\tilde{\varsigma}, \dot{\varphi}, \psi)$.

Theorem 5.2 Every *NA-NNS* $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ over κ is statistically complete with relate to $(\tilde{\varsigma}, \dot{\varphi}, \psi)$.

Proof: Let $\{v_p\}$ be *SC*. If it is not statistically convergent to$\bar{\bar{x}} \in \Xi$, then we get,

$$\lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p, m \leq n : \tilde{\varsigma}(v_p - v_m, \acute{f}) \leq 1 - \tilde{\omega} \ or\ \dot{\varphi}(v_p - v_m, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - v_m, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right|$$

$$= \lim_{n} \frac{1}{n} \left| \begin{Bmatrix} p, m \leq n : \tilde{\varsigma}(v_p - \bar{\bar{x}}, \acute{f}) * \tilde{\varsigma}(v_m - \bar{\bar{x}}, \acute{f}) \leq 1 - \tilde{\omega} \\ or\ \dot{\varphi}(v_p - \bar{\bar{x}}, \acute{f}) \diamond \dot{\varphi}(v_m - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \\ and\ \psi(v_p - \bar{\bar{x}}, \acute{f}) \star \psi(v_m - \bar{\bar{x}}, \acute{f}) \geq \tilde{\omega} \end{Bmatrix} \right| = 0$$

which is contradiction.

Definition 5.3 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NNS* over κ. A map $\jmath$: $\Xi \to \Xi$ is called $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ continuousat a point $v \in \Xi$, when the sequence with convergence in the *NA-NNS* implies that the sequence $\jmath(v_p)$ to $\jmath(v)$ convergence in the *NA- NNS*.

Definition 5.4 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NNS* over κ. A map $\jmath$: $\Xi \to \Xi$ is called statistically continuous at a point $v \in$, when $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$ implies that $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \jmath(v_p) = \jmath(v)$.

Theorem 5.5 Let $(\Xi, \tilde{\varsigma}, \dot{\varphi}, \psi, *, \diamond, \star)$ be a *NA- NN* space over κ. If $\jmath$: $\Xi \to \Xi$ is continuous in relate to the $(\tilde{\varsigma}, \dot{\varphi}, \psi)$, then this is statistically continuous.

Proof: Let $\{v_p\} \in \Xi$ and $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$. Then for every $\tilde{\omega} > 0$ and $\acute{f} > 0$, the inequality,

$\tilde{\varsigma}(v_p - v, \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(v_p - v, \acute{f}) < \tilde{\omega}$ *and* $\psi(v_p - v, \acute{f}) < \tilde{\omega}$ implies that $\tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) > 1 - \tilde{\omega}, \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) < \tilde{\omega}$ *and* $\psi(\jmath(v_p) - \jmath(v), \acute{f}) < \tilde{\omega}$. Since $\jmath$ is continuous in relate to the $(\tilde{\varsigma}, \dot{\varphi}, \psi)$ at $v \in \Xi$. Thus,

$$\left\{\begin{matrix} p \in \mathbb{N} : \tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) \leq 1 - \tilde{\omega} \ or \ \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) \geq \tilde{\omega} \\ and \ \psi(\jmath(v_p) - \jmath(v), \acute{f}) \geq \tilde{\omega} \end{matrix}\right\}$$
$$\subset \left\{\begin{matrix} p \in \mathbb{N} : \tilde{\varsigma}(v_p - v, \acute{f}) \leq 1 - \tilde{\omega} \ and \ \dot{\varphi}(v_p - v, \acute{f}) \geq \tilde{\omega} \\ and \ \psi(v_p - v, \acute{f}) \geq \tilde{\omega} \end{matrix}\right\}$$

Since, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} v_p = v$.

We have

$$\lim_n \frac{1}{n} \left| \left\{\begin{matrix} p \leq n : \tilde{\varsigma}(v_p - v, \acute{f}) \leq 1 - \tilde{\omega} \ or \ \dot{\varphi}(v_p - v, \acute{f}) \geq \tilde{\omega} \\ and \ \psi(v_p - v, \acute{f}) \geq \tilde{\omega} \end{matrix}\right\} \right| = 0.$$

This implies that,

$$\lim_n \frac{1}{n} \left| \left\{\begin{matrix} p \leq n : \tilde{\varsigma}(\jmath(v_p) - \jmath(v), \acute{f}) \leq 1 - \tilde{\omega} \ or \ \dot{\varphi}(\jmath(v_p) - \jmath(v), \acute{f}) \geq \tilde{\omega} \\ and \ \psi(\jmath(v_p) - \jmath(v), \acute{f}) \geq \tilde{\omega} \end{matrix}\right\} \right| = 0.$$

This means that, $stat_{\tilde{\varsigma},\dot{\varphi},\psi} - \lim_{p\to\infty} \jmath(v_p) = \jmath(v)$.

Hence, $\jmath$ is statistically continuous.

6. Conclusions

The NA fields were extended from Archimedean fields with the established outcomes. In this article, we prove certain including relations involving statistical convergence along with *SC* sequences on the *NNS* regarding NA fields.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Atanassov K. T., "Intuitionistic fuzzy sets," Fuzzy Sets and Systems vol. 20, pp. 87-96, 1986.
2. Bilalov B. T., Nazarova T.Y., "On statistical convergence in metric spaces," Journal of mathematics research, vol. 7, no. 1, pp. 37-43, 2015.
3. Eghbali N., Ganji M., "Generalized statistical convergence in the Non-Archimedean L-fuzzy normed spaces," Azerbaijam Journal of Mathematics, vol. 6, no. 1, 2016.
4. Jeyaraman M., "Generalized Hyers-Ulam-Rassias Stability in Neutrosophic Normed Spaces," Octogon Mathematical Magazine, vol. 30, no. 2, pp. 773 – 792, 2022.
5. Jeyaraman M., Mangayarkkarasi AN., Jeyanthi V., Pandiselvi R., "Hyers-Ulam-RassiasStability for Functional Equation in Neutrosophic Normed Spaces," International Journal of Neutrosophic Science (IJNS), vol. 18, no. 1, pp. 127-143, 2022.
6. Jeyaraman M., Jenifer P., "Statistical Δ^m-Convergence in Neutrosophic Normed Spaces,"Journal of Computational Mathematica, vol. 7, no. 1, pp. 46-60, 2023.
7. Jeyaraman M., Ramachandran A., Shakila VB., "Approximate fixed point theorems for weak contractions on neutrosophic normed spaces," Journal of Computational Mathematica, vol. 6, no. 1, pp.134-158, 2022.
8. Karakaya V., Simsek N., Erturk M., Gursoy F., "Statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces," Abstract and Applied Analysis, vol. 2012, 19pages, 2012.
9. Mangayarkkarasi AN., Jeyaraman M., Jeyanthi V., "On Stability of a cubic FunctionalEquation in Neutrosophic Normed Spaces,"Advances and Applications in MathematicalSciences, vol. 21, no. 4, pp. 1975 – 1988, 2022.
10. Mohiuddine S.A., Danish Lohani Q. M., "On generalized statistical convergence in intuitionistic fuzzy normed space," Choas, Solitons and Fractals, vol. 42, pp.1731-1737, 2009.
11. Mohiuddine S. A., Sevli H., Cancan M., "Statistical convergence of double sequences infuzzy normed spaces," Filomat, vol. 26, no.4, pp. 673-681, 2012.
12. Smarandache F., "Neutrosophy. Neutrosophic Probability, Set, and Logic," Pro QuestInformation & Learning, Ann Arbor, Michigan, USA, 1998.
13. Smarandache F., "Neutrosophic set a generalization of the intuitionistic Fuzzy sets," Inter JPure Appl Math, vol. 24, pp. 287297, 2005.
14. Sowndrarajan S., Jeyaraman M., "FlorentinSmarandache: Fixed Point theorems in Neutrosophic metric spaces," Neutrosophic Sets a System, vol. 36, pp. 251-268, 2020.
15. Suja K., Srinivasan V., "On Statistically Convergent and Statistically Cauchy Sequences in Non-Archimedean Fields," Journal of Advances in Mathematics, vol. 6, no. 3, pp.1038-1043, 2014.
16. Zadeh L. A., "Fuzzy Sets," Inform and Control, vol. 8, pp.338-353, 1965.

Received: Apr 02, 2023. Accepted: Aug 30, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.53

Trigonometric Similarity Measures of Pythagorean Neutrosophic Hypersoft Sets

Ramya G [1,*] and Francina Shalini A [2]

[1] Research Scholar, Nirmala College for Women, Coimbatore, India; ramyasrisai11@gmail.com.
[2] Associate Professor, Nirmala College for Women, Coimbatore, India; francshalu@gmail.com.

* Correspondence: ramyasrisai11@gmail.com.

Abstract: In our daily life, most problems stem from wrong decisions. Similarity measures (SMs) are very helpful in making good decisions. In this paper, distinct similarities of Pythagorean Neutrosophic Hypersoft Sets (PNHSSs) and its properties are presented. Finally, the proposed SMs applied for converting plastic waste into energy source problems. Comparing various suggested similarities makes decision-making simple, easy and accurate.

Keywords: PNHSS; SM; Tangent Similarity Measure; Cotangent Similarity Measure; Cosine Similarity Measure.

1. Introduction

Recently, many ideas have been introduced to deal with ambiguity and uncertainty (UC). Fuzzy set (FS) theory [1, 2], Intuitionistic fuzzy set (IFS) [3] serve different means when dealing with inconsistent data. However, all of the above theories fail to address the conflicting information that exists in belief systems. In 1998, Smarandache proposed neutrosophic set (NS) [4] theory as a generalization of the theories mentioned above. He considered truth, ambiguity and falsehood separately. Later, Yager [5] was decided to introduce the novel idea of Pythagorean fuzzy sets (PFSs). PFSs have a limitation that their square sum is less than or equal to 1. To overcome unconstrained ambiguity, Molodtsov [6] proposed the concept of soft set (SS) as a new mathematical method. PFSS is derived from the combination of PFS and SS. Smarandache [7] introduced a new technique for dealing with UC. He generalized the SS to the hypersoft set (HSS) by turning the function into a multi-decision function.

In section 2, the basic definitions of Pythagorean Neutrosophic Hypersoft Sets (PNHSSs) are presented. In section 3, six Tangent Similarity Measure (TSM) for PNHSSs are presented. In section 4, given resources were used to determine the accuracy of the similarity measurements.

2. Preliminaries

Definition 2.1: [8] Let $\tilde{\tilde{\Delta}}$ be the universe and $\mathcal{P}\left(\tilde{\tilde{\Delta}}\right)$ be a power set of $\tilde{\tilde{\Delta}}$. Consider $\dddot{\tilde{\mathfrak{A}}}_{\mathbf{1}}, \dddot{\tilde{\mathfrak{A}}}_{\mathbf{2}}, \ldots, \dddot{\tilde{\mathfrak{A}}}_{\widehat{\kappa}}$ for $\widehat{\boldsymbol{\kappa}} \geq 1$ be $\widehat{\boldsymbol{\kappa}}$ well - defined attributes and attributive values are $\dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}}, \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}}, \ldots, \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}}$ with $\dddot{\tilde{\mathfrak{G}}}_{\hat{\mathfrak{i}}} \cap \dddot{\tilde{\mathfrak{G}}}_{\hat{\mathfrak{j}}} = \emptyset$, for $\hat{\mathfrak{i}} \neq \hat{\mathfrak{j}}$, $\hat{\mathfrak{i}}, \hat{\mathfrak{j}} \,\dot{\in}\, \{\mathbf{1}, \mathbf{2}, \ldots, \widehat{\boldsymbol{\kappa}}\}$ and their relation $\dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}} \times \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}}, \ldots \times \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}} = \dddot{\tilde{\aleph}}$, $\left(\boldsymbol{\eta}, \dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}} \times \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}} \times \ldots \times \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}}\right)$ is said to be PNHSS over $\tilde{\tilde{\Delta}}$ where $\boldsymbol{\eta}: \dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}} \times \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}} \times \ldots \times \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}} \rightarrow \mathcal{P}\left(\tilde{\tilde{\Delta}}\right)$ and $\boldsymbol{\eta}\left(\dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}} \times \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}} \times \ldots \times \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}}\right) = \left\{\left(\dddot{\tilde{\aleph}}, < \breve{\varkappa}, \acute{\mathbb{T}}_{\boldsymbol{\eta}(\dddot{\tilde{\aleph}})}\left(\breve{\varkappa}\right), \acute{\mathbb{I}}_{\boldsymbol{\eta}(\dddot{\tilde{\aleph}})}\left(\breve{\varkappa}\right), \acute{\mathbb{F}}_{\boldsymbol{\eta}(\dddot{\tilde{\aleph}})}\left(\breve{\varkappa}\right) >\right) : \breve{\varkappa} \,\dot{\in}\, \tilde{\tilde{\Delta}}, \dddot{\tilde{\aleph}} \,\dot{\in}\, \dddot{\tilde{\mathfrak{G}}}_{\mathbf{1}} \times \dddot{\tilde{\mathfrak{G}}}_{\mathbf{2}} \times \ldots \times \dddot{\tilde{\mathfrak{G}}}_{\widehat{\kappa}}\right\}$ where $\acute{\mathbb{T}}$ is the truthiness, $\acute{\mathbb{I}}$ is the indeterminacy and $\acute{\mathbb{F}}$ is the falseness such that $\acute{\mathbb{T}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$, $\acute{\mathbb{I}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right), \acute{\mathbb{F}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) \dot{\in} [\mathbf{0}, \mathbf{1}]$ also $\mathbf{0} \leq \left(\acute{\mathbb{T}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)\right)^{2} + \left(\acute{\mathbb{I}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)\right)^{2} + \left(\acute{\mathbb{F}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)\right)^{2} \leq \mathbf{2}$.

3. Trigonometric Similarity Measures for Pythagorean Neutrosophic Hypersoft Sets

Definition 3.1: Let $\ddot{\mathcal{W}} = \left(\ddot{\aleph}, < \underset{\smile}{\varkappa}, \acute{\mathcal{T}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right), \acute{\mathcal{J}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right), \acute{\mathcal{F}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) >: \underset{\smile}{\varkappa} \dot{\in} \tilde{\Delta}\right)$;
$\ddot{\mathcal{V}} = \left(\ddot{\aleph}, < \underset{\smile}{\varkappa}, \acute{\mathcal{T}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right), \acute{\mathcal{J}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right), \acute{\mathcal{F}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) >: \underset{\smile}{\varkappa} \dot{\in} \tilde{\Delta}\right)$ be PNHSSs. The TSMs between $\ddot{\mathcal{W}}$, $\ddot{\mathcal{V}}$ is,

$\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) =$

$$\left(\ddot{\aleph}, < \underset{\smile}{\varkappa}, \frac{1}{\acute{n}}\sum_{i=1}^{n}\left[1 - tan\left(\frac{\pi\left(\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| + \left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| + \left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right|\right)}{12}\right)\right] >\right) \quad (1)$$

Proposition 1:
The TSM $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right)$ satisfies the following properties:

(1) $0 \leq \acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) \leq 1$

(2) $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = 1$ iff $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$

(3) $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = \acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{V}}, \ddot{\mathcal{W}}\right)$

(4) If $\ddot{\mathfrak{Q}}$ is a PNHSS set and $\ddot{\mathcal{W}} \subset \ddot{\mathcal{V}} \subset \ddot{\mathfrak{Q}}$ then $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathfrak{Q}}\right) \leq \acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right)$ and $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathfrak{Q}}\right) \leq \acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{V}}, \ddot{\mathfrak{Q}}\right)$.

Proof:

(1) Since tangent values of PNHSSs are in the interval [0, 1]. Hence $0 \leq \acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) \leq 1$.

(2) If $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$, then $\acute{\mathcal{T}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{T}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$, $\acute{\mathcal{J}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{J}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$, $\acute{\mathcal{F}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{F}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$. Hence $\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$, $\left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$, $\left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$. Thus $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = 1$. Conversely, if $\acute{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = 1$ then $\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$, $\left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$, $\left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}\left(\underset{\smile}{\varkappa}\right)\right| = 0$. Since, tan (0) = 0. $\acute{\mathcal{T}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{T}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$, $\acute{\mathcal{J}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{J}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$, $\acute{\mathcal{F}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right) = \acute{\mathcal{F}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\underset{\smile}{\varkappa}\right)$. Hence $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$.

(3) Proof is Straightforward.

(4) If $\ddot{\mathcal{W}} \subset \ddot{\mathcal{V}} \subset \ddot{\mathfrak{Q}}$, $\acute{\mathcal{T}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \leq \acute{\mathcal{T}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \leq \acute{\mathcal{T}}^{\ddot{\mathfrak{Q}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa})$, $\acute{\mathcal{J}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \geq \acute{\mathcal{J}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \geq \acute{\mathcal{J}}^{\ddot{\mathfrak{Q}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa})$,

$\acute{\mathcal{F}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \geq \acute{\mathcal{F}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) \geq \acute{\mathcal{F}}^{\ddot{\mathfrak{Q}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa})$.

$$\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \leq \left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|,$$

$$\left|\acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \leq \left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|,$$

$$\left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \geq \left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|,$$

$$\left|\acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \geq \left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|,$$

$$\left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \geq \left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|,$$

$$\left|\acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \geq \left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathfrak{Q}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|.$$

Thus, $\grave{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathfrak{Q}}\right) \leq \grave{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right)$; $\grave{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathfrak{Q}}\right) \leq \grave{\mathbb{T}}_{PNHSS}\left(\ddot{\mathcal{V}}\ \ddot{\mathfrak{Q}}\right)$.

Definition 3.2:

Let $\ddot{\mathcal{W}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \acute{\mathcal{T}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}), \acute{\mathcal{J}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}), \acute{\mathcal{F}}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) >: \breve{\varkappa} \dot{\in} \tilde{\Delta}\right)$;

$\ddot{\mathcal{V}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \acute{\mathcal{T}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}), \acute{\mathcal{J}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}), \acute{\mathcal{F}}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}(\breve{\varkappa}) >: \breve{\varkappa} \dot{\in} \tilde{\Delta}\right)$ be PNHSSs. The Cotangent Similarity Measure (CTSM) based on the co-tangent function between $\ddot{\mathcal{W}}$, $\ddot{\mathcal{V}}$ is, $CT^1{}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) =$

$$\left(\ddot{\aleph}, < \breve{\varkappa},\ \frac{1}{\acute{n}}\ \sum_{i=1}^{n} cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \vee \left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \vee \left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|\right)\right] \quad (2)$$

$$CT^2{}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = \left(\ddot{\aleph}, < \breve{\varkappa},\ \frac{1}{\acute{n}}\ \sum_{i=1}^{n} cot\left[\frac{\pi}{4} + \frac{\pi}{12}\left(\left|\acute{\mathcal{T}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{T}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \vee \left|\acute{\mathcal{J}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{J}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right| \vee \left|\acute{\mathcal{F}}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa}) - \acute{\mathcal{F}}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right|\right)\right] \quad (3)$$

Here $\vee$ denotes Max Operator.

Proposition 2:

The CTSMs $CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right)$ satisfies,

(1) $0 \leq CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) \leq 1$

(2) $CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}}, \ddot{\mathcal{V}}\right) = 1$ iff $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$

(3) $CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) = CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{V}},\ddot{\mathcal{W}}\right)$

(4) If $\ddot{\mathfrak{Q}}$ is a PNHSS set and $\ddot{\mathcal{W}} \subset \ddot{\mathcal{V}} \subset \ddot{\mathfrak{Q}}$ then $CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathfrak{Q}}\right) \leq CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right)$;

$CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathfrak{Q}}\right) \leq CT^{1,2}_{PNHSS}\left(\ddot{\mathcal{V}},\ddot{\mathfrak{Q}}\right)$.

Proof: Proof is similar to Prop 1.

Definition 3.3:

Let $\ddot{\mathcal{W}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \mathcal{T}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{I}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{F}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right) >: \breve{\varkappa} \in \tilde{\Delta}\right)$;

$\ddot{\mathcal{V}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \mathcal{T}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{I}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{F}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right) >: \breve{\varkappa} \in \tilde{\Delta}\right)$ be PNHSSs. The Cosine Similarity Measures (CSMs) between $\ddot{\mathcal{W}}, \ddot{\mathcal{V}}$ by using A.M is given by, $\grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) =$

$$\left(\ddot{\aleph}, < \breve{\varkappa}, \frac{1}{\acute{n}}, \sum_{i=1}^{n} \frac{\left(\mathcal{T}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)\left(\mathcal{T}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)+\left(\mathcal{I}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)\left(\mathcal{I}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)+\left(\mathcal{F}^{\ddot{\mathcal{W}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)\left(\mathcal{F}^{\ddot{\mathcal{V}}^2}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})\right)}{\sqrt{\mathcal{T}^{\ddot{\mathcal{W}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})+\mathcal{I}^{\ddot{\mathcal{W}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})+\mathcal{F}^{\ddot{\mathcal{W}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})}\sqrt{\mathcal{T}^{\ddot{\mathcal{V}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})+\mathcal{I}^{\ddot{\mathcal{V}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})+\mathcal{F}^{\ddot{\mathcal{V}}^4}_{(\eta(\ddot{\aleph}))_i}(\breve{\varkappa})}} >\right) \text{------} \quad (4)$$

Proposition 3:

The CSMs $\grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right)$ satisfies,

(1) $0 \leq \grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) \leq 1$

(2) $\grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) = 1$ iff $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$

(3) $\grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) = \grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{V}},\ddot{\mathcal{W}}\right)$.

Proof:

(1) Value of the Cosine function lies between [0, 1]. Hence $0 \leq \grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) \leq 1$.

(2) If $\ddot{\mathcal{W}} = \ddot{\mathcal{V}}$, then $\mathcal{T}^{\ddot{\mathcal{W}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right) = \mathcal{T}^{\ddot{\mathcal{V}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right)$, $\mathcal{I}^{\ddot{\mathcal{W}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right) = \mathcal{I}^{\ddot{\mathcal{V}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right)$, $\mathcal{F}^{\ddot{\mathcal{W}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right) = \mathcal{F}^{\ddot{\mathcal{V}}}_{(\eta(\ddot{\aleph}))_i}\left(\breve{\varkappa}\right)$

for $i = 1,2,\ldots n$. Hence, $\grave{C}^1{}_{PNHSS}\left(\ddot{\mathcal{W}},\ddot{\mathcal{V}}\right) = 1$.

(3) Proof is Straightforward.

Definition 3.4:

Let $\ddot{\mathcal{W}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \mathcal{T}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{I}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{F}^{\ddot{\mathcal{W}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right) >: \breve{\varkappa} \in \tilde{\Delta}\right)$;

$\ddot{\mathcal{V}} = \left(\ddot{\aleph}, < \breve{\varkappa}, \mathcal{T}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{I}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right), \mathcal{F}^{\ddot{\mathcal{V}}}_{\eta(\ddot{\aleph})}\left(\breve{\varkappa}\right) >: \breve{\varkappa} \in \tilde{\Delta}\right)$ be PNHSSs.

The CSMs between $\underbrace{\ddot{\mathcal{W}}}$, $\underbrace{\ddot{\mathcal{V}}}$ based on the cosine function is

$$\grave{C}^2{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right) = \left(\ddot{\aleph}, < \underbrace{\breve{\varkappa}}, \frac{1}{\acute{n}} \sum_{i=1}^{n} cos\left[\frac{\pi}{2}\left(\left|\acute{\mathcal{T}}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\mathcal{T}}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right| \vee \left|\acute{\jmath}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\jmath}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right| \vee \left|\acute{\mathcal{F}}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\mathcal{F}}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right|\right)\right] \quad (5)$$

$$\grave{C}^3{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right) = \left(\ddot{\aleph}, < \underbrace{\breve{\varkappa}}, \frac{1}{\acute{n}} \sum_{i=1}^{n} cos\left[\frac{\pi}{6}\left(\left|\acute{\mathcal{T}}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\mathcal{T}}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right| \vee \left|\acute{\jmath}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\jmath}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right| \vee \left|\acute{\mathcal{F}}^{\underbrace{\ddot{\mathcal{W}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right) - \acute{\mathcal{F}}^{\underbrace{\ddot{\mathcal{V}}}^2}_{\left(\eta(\ddot{\aleph})\right)_i}\left(\underbrace{\breve{\varkappa}}\right)\right|\right)\right] \quad (6)$$

Proposition 4:
The CSMs $\grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$ satisfies the following properties:

(1) $0 \leq \grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right) \leq 1$

(2) $\grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right) = \grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{V}}},\underbrace{\ddot{\mathcal{W}}}\right)$.

(3) $\grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right) = 1$ iff $\underbrace{\ddot{\mathcal{W}}} = \underbrace{\ddot{\mathcal{V}}}$

(4) If $\underbrace{\ddot{\mathfrak{Q}}}$ is a PNHSS set and $\underbrace{\ddot{\mathcal{W}}} \subset \underbrace{\ddot{\mathcal{V}}} \subset \underbrace{\ddot{\mathfrak{Q}}}$ then $\grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathfrak{Q}}}\right) \leq \grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$ and

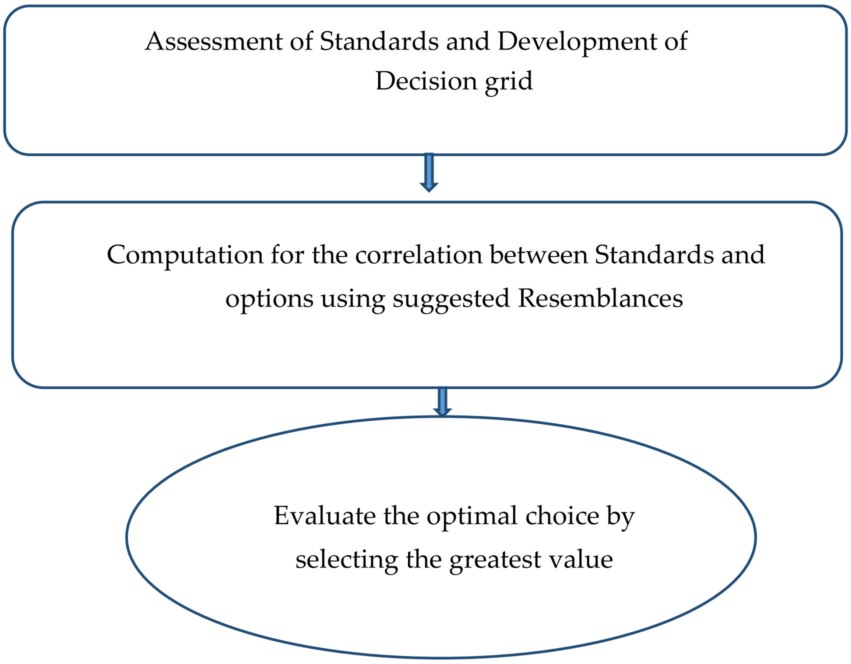
$\grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathfrak{Q}}}\right) \leq \grave{C}^{2,3}_{PNHSS}\left(\underbrace{\ddot{\mathcal{V}}},\underbrace{\ddot{\mathfrak{Q}}}\right)$.

Proof: The proof is similar to Prop.1.

4. Application of TSMs for PNHSS

All countries have been using different types of plastic like PETE, HDPE, PVC, LDPE, PP, PS and Mix plastic. Few countries are converting plastic waste into energy in the form of solid, liquid and gaseous fuels. Also, it's possible to convert waste plastics into Hydrogen, Methane and Ethylene. Both Hydrogen and Methane can be used for clean fuels. Few states are currently sending their collected plastic waste to cement plants for Co-Processing. The world is affected a lot due to usage of plastic. Plastic things are prohibited by many countries. But still we could not minimize as expected. Several techniques are used for converting plastic waste into energy. Pyrolysis is a common technique used to convert plastic waste into energy. We try to develop a mathematical model to overcome this world problem.

4.1 Methodology



Let $\acute{\mathcal{C}} = \{\acute{\mathcal{C}}^1, \acute{\mathcal{C}}^2, \acute{\mathcal{C}}^3, \acute{\mathcal{C}}^4, \acute{\mathcal{C}}^5, \acute{\mathcal{C}}^6, \acute{\mathcal{C}}^7, \acute{\mathcal{C}}^8, \acute{\mathcal{C}}^9, \acute{\mathcal{C}}^{10}, \acute{\mathcal{C}}^{11}, \acute{\mathcal{C}}^{12}\}$ be a set of Countries and $\acute{\mathrm{P}} = \{\mathit{Slow}\ \text{Pyrolysis}(SP), \mathit{Intermediate}\ \text{Pyrolysis}(IP), \mathit{Ultra\ Fast}\ \text{Pyrolysis(UFP)}\}$ be a types of M Pyrolysis process.

The collection of attributes to $\acute{\mathcal{C}}$ & $\acute{\mathrm{P}}$ be,

$$\ddot{\mathfrak{C}} = \left\{\begin{matrix} \ddot{\mathfrak{C}}^1\ (\text{Mean Amount of plastic usage[PU] (MT/Day)}) \\ \ddot{\mathfrak{C}}^2\ (\text{Typical Amount of PU (MT/ Year)}) \\ \ddot{\mathfrak{C}}^3\ (\text{Average Amount of PU (Grams/ Week)}) \\ \ddot{\mathfrak{C}}^4\ (\text{Typical Amount of PU}(kg/person)) \\ \ddot{\mathfrak{C}}^5\ (\text{Mean Amount of PU }(g/person)) \end{matrix}\right\}$$

Sub-Attributes are $\ddot{\mathfrak{C}}^1 = \{< 1.5\ \mathrm{M.T}, 1.5 - 2.5\ \mathrm{M.T}, 2.5 - 3.5\ \mathrm{M.T}, > 3.5\ \mathrm{M.T}\}$; $\ddot{\mathfrak{C}}^2 = \{< 10\ \mathrm{M.T}, 10 - 15\ \mathrm{M.T}, 15 - 16.5\ \mathrm{M.T}\}$; $\ddot{\mathfrak{C}}^3 = \{0.1 - 2\ \mathrm{G}, 2 - 5\ \mathrm{G}, > 5\ \mathrm{G}\}$;$\ddot{\mathfrak{C}}^4 = \{< 5\ \mathrm{kgs}, 5 - 10\ \mathrm{kgs}, 10 - 15\ \mathrm{kgs}\}$;$\ddot{\mathfrak{C}}^5 = \{0.1 - 2\ \mathrm{g}, 2 - 5\ \mathrm{g}, > 5\ \mathrm{g}\}$. The PNHSS be $\eta: (\ddot{\mathfrak{C}}^1 \times \ddot{\mathfrak{C}}^2 \times \ddot{\mathfrak{C}}^3 \times \ddot{\mathfrak{C}}^4 \times \ddot{\mathfrak{C}}^5) \to \mathcal{P}(\acute{\mathcal{C}})$ and $\ddot{\gamma}: (\ddot{\mathfrak{C}}^1 \times \ddot{\mathfrak{C}}^2 \times \ddot{\mathfrak{C}}^3 \times \ddot{\mathfrak{C}}^4 \times \ddot{\mathfrak{C}}^5) \to \mathcal{P}(\acute{\mathrm{P}})$.

Let $(\eta, \ddot{\zeta}) = \{2.5\ \mathrm{M} - 3.5\ \mathrm{M}, 10 - 15\ \mathrm{M.T}, 2 - 5\ \mathrm{G}, 10 - 15\ \mathrm{kgs}, > 5\ \mathrm{g}\}$.

Now using the proposed several SMs for PNHSSs, we will decide which country is widely using mentioned energy techniques.

For this purpose, we should first provide the relationship between $\{\acute{\mathcal{C}}^2, \acute{\mathcal{C}}^3, \acute{\mathcal{C}}^5, \acute{\mathcal{C}}^{11}\}$ and $\{2.5\ \mathrm{M} - 3.5\ \mathrm{M}, 10 - 15\ \mathrm{M.T}, 2 - 5\ \mathrm{G}, 10 - 15\ \mathrm{kgs}, > 5\ \mathrm{g}\}$ in terms of PNHSSs.

In the 2nd step, we should provide the relationship between $\{2.5\ \mathrm{M} - 3.5\ \mathrm{M}, 10 - 15\ \mathrm{M.T}, 2 - 5\ \mathrm{G}, 10 - 15\ \mathrm{kgs}, > 5\ \mathrm{g}\}$ and $\{(SP), (IP), (\mathrm{UFP})\}$.

In Step 3, we should find the correlation between $\{\acute{\mathcal{C}}^2, \acute{\mathcal{C}}^3, \acute{\mathcal{C}}^5, \acute{\mathcal{C}}^{11}\}$ and $\{(SP), (IP), (\mathrm{FP}), (\mathrm{UFP})\}$.

In step 4, The association is determined with the proposed TSMs for PNHSS by Equations (1-6).

In step 5, Finding the best selection.

Table 1. Relation between Regions and criteria

Regions	$\mathbf{2.5\ M - 3.5\ M}$	$\mathbf{10 - 15\ M.T}$	$\mathbf{2 - 5\ G}$	$\mathbf{10 - 15\ kgs}$	$\mathbf{> 5\ g}$
$\underbrace{\acute{\mathcal{C}}^{2}}$	(.5, .3, .4)	(.5, .4, .6)	(.9, .4, .3)	(.7, .3, .4)	(.6, .3, .7)
$\underbrace{\acute{\mathcal{C}}^{3}}$	(.6, .4, .5)	(.7, .4, .5)	(.8, .4, .1)	(.8, .3, .5)	(.8, .2, .6)
$\underbrace{\acute{\mathcal{C}}^{5}}$	(.8, .3, .2)	(.9, .3, .1)	(.6, .7, .8)	(.7, .5, .6)	(.5, .4, .6)
$\underbrace{\acute{\mathcal{C}}^{11}}$	(.7, .6, .1)	(.5, .2, .6)	(.4, .6, .7)	(.4, .5, .7)	(.7, .3, .6)

Table 2. Relation between sources and criteria.

Sources	$\mathbf{2.5\ M - 3.5\ M}$	$\mathbf{10 - 15\ M.T}$	$\mathbf{2 - 5\ G}$	$\mathbf{10 - 15\ kgs}$	$\mathbf{> 5\ g}$
SP	(.6, .3, .5)	(.8, .6, .4)	(.7, .2, .3)	(.8, .6, .5)	(.7, .6, .2)
IP	(.7, .5, .3)	(.7, .6, .4)	(.8, .6, .1)	(.6, .3, .7)	(.9, .7, .2)
UFP	(.7, .2, .6)	(.5, .6, .7)	(.9, .1, .2)	(.8, .7, .5)	(.4, .2, .9)

Table 3. SMs using $\breve{\mathbb{T}}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$.

SMs	**Regions**	**SP**	**IP**	**UFP**
$\breve{\mathbb{T}}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$	$\underbrace{\acute{\mathcal{C}}^{2}}$	.85229	.82273	**.88249**
	$\underbrace{\acute{\mathcal{C}}^{3}}$	**.88170**	.87670	.70570
	$\underbrace{\acute{\mathcal{C}}^{5}}$	**.82234**	.79836	.75526
	$\underbrace{\acute{\mathcal{C}}^{11}}$	.78589	**.82361**	.76558

Table 4. SMs using $CT^1{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$.

SMs	Regions	SP	IP	UFP
$\boldsymbol{CT^1{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)}$	$\underbrace{\acute{C}^2}$	.61163	.62552	**.71484**
	$\underbrace{\acute{C}^3}$	**.72953**	.66488	.63983
	$\underbrace{\acute{C}^5}$	**.60579**	.56037	.50495
	$\underbrace{\acute{C}^{11}}$	.53976	**.62016**	.51110

Table 5: SM using $CT^2{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$.

SMs	Regions	SP	IP	UFP
$\boldsymbol{CT^2{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)}$	$\underbrace{\acute{C}^2}$	.85186	.85851	**.87225**
	$\underbrace{\acute{C}^3}$	**.90033**	.87437	.86319
	$\underbrace{\acute{C}^5}$	**.84881**	.82829	.80667
	$\underbrace{\acute{C}^{11}}$	.82248	**.85499**	.78950

Table 6. SM using $\grave{C}^1{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$.

SMs	Regions	SP	IP	UFP
$\boldsymbol{\grave{C}^1{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)}$	$\underbrace{\acute{C}^2}$	.86637	.82112	**.94452**
	$\underbrace{\acute{C}^3}$	**.92401**	.89219	.85379
	$\underbrace{\acute{C}^5}$	**.79727**	.80323	.71249
	$\underbrace{\acute{C}^{11}}$	.67766	**.77675**	.66510

Table 7: SM using $\grave{C}^2{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$

SMs	Regions	SP	IP	UFP
$\grave{C}^2{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$	$\underbrace{\acute{C}^2}$	.87026	.88792	**.94036**
	$\underbrace{\acute{C}^3}$	**.94101**	.90718	.88230
	$\underbrace{\acute{C}^5}$	**.86286**	.79998	.82065
	$\underbrace{\acute{C}^{11}}$	.82876	**.87008**	.74815

Table 8: SM using $\grave{C}^3{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$.

SMs	Regions	SP	IP	UFP
$\grave{C}^3{}_{PNHSS}\left(\underbrace{\ddot{\mathcal{W}}},\underbrace{\ddot{\mathcal{V}}}\right)$	$\underbrace{\acute{C}^2}$	.98518	.98724	**.99328**
	$\underbrace{\acute{C}^3}$	**.99335**	.98943	.98651
	$\underbrace{\acute{C}^5}$	**.9842**	.97647	.97527
	$\underbrace{\acute{C}^{11}}$	.98039	**.98511**	.97056

The Highest Measure (Table values 3,4,5,6,7,8) reflects Region $\underbrace{\acute{C}^2}$ should be selected for UFP, Region $\underbrace{\acute{C}^3}$ should be selected for SP, Region $\underbrace{\acute{C}^5}$ should be selected for SP, Region $\underbrace{\acute{C}^{11}}$ should be selected for IP.

5. Conclusions

The aim of this paper is to establish Tangent, Cotangent and Cosine SMs of PNHSSs. The extension is very applicable to decision-making problems. We introduced six TSMs for PNHSSs with properties. Also, applied them to Energy source selection problem.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L.A. Fuzzy sets. Information and control **1965**, 8, 338-353.
2. Uma G, & S, N. (2023). An Investigative Study on Quick Switching System using Fuzzy and Neutrosophic Poisson Distribution. Neutrosophic Systems with Applications, 7, 61–70. https://doi.org/10.61356/j.nswa.2023.28.
3. Atanassov, K.T. Intuitionistic fuzzy sets. Fuzzy Sets Systems 1986, 20, 87-96.
4. Florentin Smarandache. A generalization of Intuitionistic fuzzy sets. Journal of Defence Resources Management, **2010**, 107-116.
5. Yager, RR. Pythagorean Fuzzy Subsets. In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada **2013**, 57-61.
6. Molodtsov, D. Soft set theory- first results. Computers and Mathematics with Applications, **1999**, 37, 19-31.
7. Smarandache, F. Extension of soft set to Hypersoft set, and then to Plithogenic Hypersoft set. Neutrosophic Sets and Systems **2018**, 22, 168-170.
8. Ramya, G.; Francina Shalini, A. An Introduction to Pythagorean Neutrosophic Hypersoft Topological Spaces. Mukt Shabd Journal, **2023**, XII (VIII), 807-822.

Received: Mar 02, 2023. Accepted: Sep 03, 2023

Neutrosophic Systems with Applications, Vol. 9, 2023
https://doi.org/10.61356/j.nswa.2023.68

Pura Vida Neutrosophic Algebra

Ranulfo Paiva Barbosa (Sobrinho) [1,*] and Florentin Smarandache [2]

[1] Web3 Blockchain Entrepreneur; 37 Dent Flats, Monte de Oca, 11501, San José, Costa Rica; ranulfo17@gmail.com.
[2] Math and Science Department, University of New Mexico, Gallup, NM, 87301, USA; smarand@unm.edu.

* Correspondence: ranulfo17@gmail.com.

Abstract: We introduce Pura Vida Neutrosophic Algebra, an algebraic structure consisting of neutrosophic numbers equipped with two binary operations namely addition and multiplication. The addition can be calculated sometimes with the function min and other times with the max function. The multiplication operation is the usual sum between numbers. Pura Vida Neutrosophic Algebra is an extension of both Tropical Algebra (also known as Min-Plus, or Min-Algebra) and Max-Plus Algebra (also known as Max-algebra). Tropical and Max-Plus algebras are algebraic structures included in semirings and their operations can be used in matrices and vectors. Pura Vida Neutrosophic Algebra is included in Neutrosophic semirings and can be used in Neutrosophic matrices and vectors.

Keywords: Tropical Algebra; Max-Plus Algebra; Pura Vida Neutrosophic Logic; Neutrosophic Number.

1. Introduction

Uncertain, indeterminacy, imprecise, and vague are common characteristics of data in real-life problems like decision-making, engineering, computer science, finance, etc. Several theories have been proposed to deal with these data characteristics, fuzzy set theory [1], intuitionistic fuzzy sets [2], rough set theory [3], Soft set [4], and Neutrosophy theory [5]. Since Smarandache introduced Neutrosophy to study the basis, nature, and range of neutralities as well as their contact with ideational spectra in the 1990s, we have seen the emergence of neutrosophic algebraic structures [6], neutrosophic probability and statistics [7, 8] neutrosophic numbers [8], single-valued neutrosophic sets (SVNSs) [9, 21], and several algebraic structures such as neutrosophic semirings [10], among others theoretical advances [11] and also applications [12].

Through neutrosophic semirings, we introduce Pura Vida (PV) Neutrosophic Algebra, an algebraic structure consisting of neutrosophic numbers equipped with two binary operations namely addition and multiplication. Pura Vida Neutrosophic Algebra is an extension of both Tropical Algebra (also known as Min-Plus) [13] and Max-Plus Algebra [14]. Both Tropical and Max-Plus algebra are algebraic structures included in semirings and were discovered independently by several researchers [13, 14]. They were defined on the real number domain and for the first time, we extended them to the neutrosophic domain.

2. Preliminaries

2.1. Semiring

A semiring [15] denoted (V, $\oplus$, $\otimes$, 0, 1) is a set V equipped with two binary operations, addition:

$$\oplus : V \times V \rightarrow V$$

And multiplication:

$\otimes$: V x V → V

Which satisfies the following axioms for any u, v, w ∈ V:

1. (V, $\oplus$, 0) is a commutative monoid and (V, $\otimes$, 1) is a monoid.

2. u $\otimes$ (v $\oplus$ w) = (u $\otimes$ v) $\oplus$ (u $\otimes$ w) and (v $\oplus$ w) $\otimes$ u = (v $\otimes$ u) $\oplus$ (w $\otimes$ u) (distributivity).

3. 0 annihilates V: v $\otimes$ 0 = 0 $\otimes$ v = 0.

When (V, $\otimes$, 1) is a commutative monoid, the semiring (V, $\oplus$, $\otimes$, 0, 1) is said to be a commutative semiring.

2.2 Tropical algebra

The tropical algebra is also referred to as tropical semiring T, which consists of the set of real numbers, R, extended with infinity, equipped with the operations of taking minimums (as semiring addition) and addition (as semiring multiplication) [14, 16]. Tropical algebra is also known as min-plus algebra. With minimum replaced by maximum, we get the isomorphic max-plus algebra [17]. According to [17], the adjective "tropical" was coined by French mathematicians to honor their Brazilian colleague Imre Simon [16], who pioneered the use of min-plus algebra in optimization theory.

$$T = (R \cup \{ \infty \}, \oplus, \otimes)$$

Addition operation:

$$a \oplus b = \min (a, b)$$

Multiplication operation:

$$a \otimes b = a + b$$

the operations of R, are extended to T in the usual way and the identities of $\oplus$ and $\otimes$ are, respectively, ∞ and 0. The element ∞ represents plus-infinity [13]. Given a real number, x ∈ T, its addition and multiplication identity are given, respectively:

$$x \oplus \infty = x$$

$$x \otimes 0 = x$$

Michaleck points out the following equations involving the two identity elements:

$$x \otimes \infty = \infty \qquad \text{and} \qquad x \oplus 0 = \begin{cases} 0, & if\ x \geq 0 \\ x, & if\ x < 0 \end{cases}$$

Michaleck said there is no subtraction in tropical arithmetic. Tropical division $\oslash$ is defined to be classical subtraction.

Tropical division, x $\oslash$ y = x, exists if and only if y $\otimes$ z = x [20].

In Tropical algebra the pairs of operations ($\oplus$,$\otimes$) is extended to matrices and vectors similarly as in linear algebra. That is if A = (a_{ij}), B = (b_{ij}) and C = (c_{ij}) are matrices with elements from R of compatible sizes, we write:

C = A $\oplus$ B if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j

C = A $\otimes$ B if $c_{ij} = \sum_k^{\oplus} a_{ik} \oplus b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in$ R.

2.3 Max-Plus algebra

The Max-Plus algebra is an algebraic structure semiring MP, which consists of the set of real numbers, R, extended with infinity, equipped with the operations of taking maximums (as semiring addition) and addition (as semiring multiplication) [14].

$$MP = (R \cup \{ -\infty \}, \oplus', \otimes)$$

Addition operation:

$$a \oplus' b = \max (a, b)$$

Multiplication operation:

$$a \otimes b = a + b$$

the operations of R, are extended to MP in the usual way and the identities of $\oplus'$ and $\otimes$ are, respectively, $-\infty$ and 0.

In max-plus algebra the pairs of operations ($\oplus'$,$\otimes$) is extended to matrices and vectors similarly as in linear algebra. That is if A = (a_{ij}), B = (b_{ij}) and C = (c_{ij}) are matrices with elements from R of compatible sizes, we write:

$C = A \oplus' B$ if $c_{ij} = a_{ij} \oplus' b_{ij}$ for all i, j

$C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus'} a_{ik} \oplus b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in R$.

2.4 Neutrosophic Set

Smarandache [5] defined Neutrosophic set as a set of elements composed of tripart structure: a Truth membership (T), an Indeterminacy membership (I) and a False membership (F). These parts are independent each other and can be represented by different functions. Together, <T, I, F>, these parts compose an element of Neutrosophic set.

2.5 Neutrosophic Number

According to [18] the neutrosophic number (NN) is a number which structure is given by "X = a + bI", where I represents the indeterminacy component of X, and 'a' and 'b' are real or complex numbers [19].

2.6 Neutrosophic Semiring

An algebraic structure (S∪I, $\oplus$, $\otimes$) is called neutrosophic semiring [10] if $\oplus$ and $\otimes$ are the closed and associative binary operations and $\otimes$ is distributive over $\oplus$, where S is semiring with respect to $\oplus$ and $\otimes$ and I is the neutrosophic element ($I = I^2$) and < S∪I> = { a + bI; a, b $\in$ S}.

2.7 Neutrosophic field [6]

Let K be the field of reals. We call the field generated by K ∪ I to be the neutrosophic field for it involves the indeterminacy (I) factor in it. We define I2 = I, I + I = 2I, i.e., I + … + I = nI, and if k $\in$ K then kI = Ik, 0I = 0. We denote the neutrosophic field by K(I).

2.8 Neutrosophic matrix [6]

Let M_{nxm} = {(a_{ij}) / $a_{ij} \in$ K(I) }, where K(I), is a neutrosophic field. We call Mnxm to be the neutrosophic matrix.

3. Pura Vida Neutrosophic Algebra

The Pura Vida Neutrosophic Algebra, PV, is an extension of the Tropical algebra and Max-Plus Algebra.

Pura Vida Neutrosophic Algebra is included in a Neutrosophic semiring, i.e., it has both associative binary operations, addition $\oplus$ and multiplication $\otimes$ where $\otimes$ is distributive over $\oplus$, and S is semiring with respect to $\oplus$ and $\otimes$ and I is the neutrosophic element ($I = I^2$) and < S∪I> = { a = bI; a, b $\in$ S}. The addition operation can use either the min function, $\oplus$, or the max function, $\oplus'$, depending on the situation.

$$PV = (S \cup I \{ -\infty, +\infty \}, \oplus, \oplus', \otimes)$$

Pura Vida Neutrosophic Algebra operations addition ($\oplus$, or, $\oplus'$) and multiplication ($\otimes$) are given:

3.1 Addition operation$\oplus$, or, $\oplus'$

Depending on the real-life applications, the addition operation can use the min or max function. Given two neutrosophic numbers x = a + bI, and z = c + dI $\in$ S, the addition of x and z:

3.1.1. $x \oplus z = (a \oplus c) + (b \oplus d)I = \min(a, c) + \min(b, d)I$

or,

3.1.2. $x \oplus' z = (a \oplus' c) + (b \oplus' d)I = \max(a, c) + \max(b, d)I$

3.2 Multiplication operation ⊗

Given two neutrosophic numbers x = a + bI, and z = c + dI ∈ S, the multiplication of x and z:

$x \otimes z = a \otimes c + (b \otimes d)I = (a + c) + (b + d)I$

3.3 Identities

In Pura Vida Neutrosophic Algebra, PV, the identities of the operators ⊕, ⊕′ and ⊗ are, respectively, ∞, -∞ and 0.

3.4 Properties

Next, we show that the PV attends the closure property and distributive and associative laws. We use min for the addition operation, but, one could use the max function to show that PV verifies the mentioned properties.

3.4.1 Closure property:

Let (a + bI) and (c + dI) ∈ S∪I then,

$(a + bI) \oplus (c + dI) = (a \oplus c) + (b \oplus d)I = \min(a, b) + \min(c, d)I$, ∈ S∪I. The addition operation verifies the closure property.

$(a + bI) \otimes (c + dI) = a \otimes c + (b \otimes d)I = (a + c) + (b + d)I$ ∈ S∪I. Which shows that the closure property is satisfied for the multiplication operation.

3.4.2 Distributive law:

Let (a + bI), (c + dI) and (e + fI) ∈ S∪I, then:

$(a + bI) \otimes [(c + dI) \oplus (e + fI)] = (a + bI) \otimes [\min(c, e) + \min(d, f)I] =$
$= [a + \min(c, e)] + [b + \min(d, f)]I.$
And $[(a + bI) \otimes (c + dI)] \oplus [(a + bI) \otimes (e + f)I] =$
$= [(a + c) + (b + d)I] \oplus [(a + e) + (b + f)I] =$
$= \min\{(a + c), (a + e)\} + \min\{(b + d) + (b + f)\}I =$
$= [a + \min(c, e)] + [b + \min(d, f)]I.$

3.4.3 Associative law:

Let (a + bI), (c + dI) and (e + fI) ∈ S∪I, then:

$[(a + bI) \oplus (c + dI)] \oplus (e + fI) =$
$[\min(a, c) + \min(b, d)I] \oplus (e + fI) = \min[\min(a, c), e] + \min[\min(b, d), f]I =$
$= (a \oplus c \oplus e) + (b \oplus d \oplus f)I.$
$(a + bI) \oplus [(c + dI) \oplus (e + fI)] =$
$(a + bI) \oplus [\min(c, e) + \min(d, f)I] = \min[a, \min(c, e)] + \min[b, \min(d, f)]I =$
$= (a \oplus c \oplus e) + (b \oplus d \oplus f)I.$
Again:
$[(a + bI) \otimes (c + dI)] \otimes (e + fI) = [a \otimes c + (b \otimes d)I] \otimes (e + fI) =$
$[(a + c) + (b + d)I] \otimes (e + fI) = (a + c) \otimes e + [(b + d) \otimes f]I =$
$(a + c + e) + (b + d + f)I.$
$(a + bI) \otimes [(c + dI)] \otimes (e + fI)] = (a + bI) \otimes [(c + e) + (d + f)I] =$
$= a \otimes (c + e) + b \otimes (d + f)I = (a + c + e) + (b + d + f)I.$

3.5 Pura Vida Neutrosophic Algebra on Matrices

In Pura Vida Neutrosophic Algebra the pairs of operations (⊕, ⊕′, ⊗) is extended to matrices and vectors similarly as in linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices with elements from R of compatible sizes, we write:

$C = A \oplus' B$ if $c_{ij} = a_{ij} \oplus' b_{ij}$ for all i, j
$C = A \otimes B$ if $c_{ij} = \sum_{k}^{\oplus'} a_{ik} \oplus b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in R$.

3.5.1 Matrices Addition using $\oplus$ operator

Given P and Q, both square neutrosophic matrices 2x2, their sum is D = P⊕Q.

$$P = \begin{vmatrix} -8+I & 5-I \\ 3+8I & 23-2I \end{vmatrix} \quad \text{and} \quad Q = \begin{vmatrix} 3+2I & 13+3I \\ 7+9I & 3+5I \end{vmatrix}$$

$$D = \begin{vmatrix} Min(-8,3)+Min(1,2)I = -8+I & Min(5,13)+Min(-1,3)I = 5-I \\ Min(3,7)+Min(8,9)I = 3+8I & Min(23,3)+Min(-2,5)I = -2+3I \end{vmatrix}$$

3.5.2 Matrices Addition using $\oplus'$ operator

Given X and Z, both square neutrosophic matrices 2x2, their sum is W = X⊕′Z.

$$X = \begin{vmatrix} -8+I & 5-I \\ 3+8I & 23-2I \end{vmatrix} \quad \text{and} \quad Z = \begin{vmatrix} 3+2I & 13+3I \\ 7+9I & 3+5I \end{vmatrix}$$

$$W = \begin{vmatrix} Max(-8,3)+Max(1,2)I = 3+2I & Max(5,13)+Max(-1,3)I = 13+3I \\ Max(3,7)+Max(8,9)I = 7+9I & Max(23,3)+Max(-2,5)I = 23+5I \end{vmatrix}$$

3.5.3 Matrices Multiplication using $\otimes$ operator

Given A and B, both rectangular neutrosophic matrices, their multiplication is C = A⊗B.

$$A = \begin{vmatrix} -1 & 2 & -I \\ 3 & I & 0 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} I & 1 & 2 & 4 \\ 1 & I & 0 & 2 \\ 5 & -2 & 3I & -I \end{vmatrix}$$

$$C = A \otimes B = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \end{vmatrix}$$

Where,

$C_{11} = (-1 \quad 2 \quad -I) \otimes (I \quad 1 \quad 5) = (-1 \otimes I + 2 \otimes 1 + -I \otimes 5) = (-1+I + 2+1 + -I+5) = 7$

$C_{21} = (3\ I\ 0) \otimes (I\ 1\ 5) = (3 \otimes I + I \otimes 1 + 0 \otimes 5) = (3 + I + 1 + I + 5) = 9 + 2I.$

$C_{12} = (-1 \quad 2 \quad -I) \otimes (1\ I \quad -2) = (-1 \otimes 1 + 2 \otimes I + -I \otimes -2) = (0 + 2 + I - I - 2) = 0.$

$C_{22} = (3\ I\ 0) \otimes (1\ I \quad -2) = 3 + 1 + I + I - 2 = 2 + 2I.$

$C_{13} = (-1 \quad 2 \quad -I) \otimes (2 \quad 0 \quad 3I) = 1 + 2 + 2I = 3 + 2I.$

$C_{23} = (3\ I\ 0) \otimes (2 \quad 0 \quad 3I) = 5 + I + 3I = 5 + 4I.$

$C_{14} = (-1 \quad 2 \quad -I) \otimes (4\ 2\ -I) = 3 + 4 - 2I = 7 - 2I.$

$C_{24} = (3\ I\ 0) \otimes (4\ 2\ -I) = 7 + I + 2 - I = 9.$

$$C = A \otimes B = \begin{vmatrix} 7 & 0 & 3+2I & 7-2I \\ 9+2I & 2+2I & 5+4I & 9 \end{vmatrix}$$

4. Conclusion

We introduced Pura Vida Neutrosophic Algebra through neutrosophic numbers and explored some its properties and applied to neutrosophic matrices.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L. Fuzzy sets. Information and Control 1965, 338-353.
2. Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1) (1986) 87 − 96.
3. Pawlak, Z.: Rough sets. Int. J. Comput. Inform. Sci. 11, 341–356, 1982.
4. Molodtsov, D.: Soft set theory-first results. Comput. Math. Appl. 37, 19–31, 1999.
5. Smarandache, F. A unifying field in Logics: Neutrosophic Logic. In Philosophy. American Research Press, 1-141, 1992.
6. Kandasamy, W.B.V, and Smarandache, F. Basic Neutrosophic algebraic structures and their applications to fuzzy and Neutrosophic models. Hexis, 2004.
7. Smarandache, F. Neutrosophy neutrosophic probability, set, and logic, American Research Press, Rehoboth, USA, 1998.
8. Smarandache, F. Introduction to neutrosophic statistics. 2014.
9. Wang, H.; Smarandache, F.; Zhang, Y.; and Sunderraman, R. 2010. Single valued SVNS sets, Multisp Multistruct 4: 410–413.
10. Gulistan, M., Rehman, I., Shazad, M., Nawaz, S., Khan, S. Generalized Neutrosophic semirings. Neutrosophic Sets and Systems, 47: 533-540, 2021.
11. https://fs.unm.edu/NSS.
12. El-Hefenaway, N., Metwally, M.A., Ahmed, Z.M., El-Henawy, I.M. A review on the applications of neutrosophic sets. J.Comp.Theor. Nanoscience, 13, 936-944, 2016
13. Michalek, M., Sturmfels, B. Tropical Algebra in Invitation to nonlinear algebra. 2021.
14. Butkovic, P. Max-linear systems: theory and algorithms. Springer Monographs in Mathematics. 2010.
15. Golan, J. S. Semirings and their applications. Springer Science & Business Media, 2013
16. Simon, I. 1988. On semigroups of matrices over the Tropical semiring. Theoretical Informatics and Applications, 28(3-4): 277-294.
17. Maclagan, D., Sturmfels, B. Introduction to tropical geometry. AMS. 2015.
18. Kandasamy WBV, Smarandache F. Fuzzy cognitive maps and neutrosophic cognitive maps. ProQuest Information and Learning, Ann Arbor, 2003.
19. Kandasamy WBV, Smarandache F. Neutrosophic rings. 2006 https://arxiv.org/ftp/math/papers/0607/0607765.pdf.
20. Grigoriev, D., Shpilrain, V. Tropical cryptography. Comm. Algebra, 6: 2624-2632, 2014.
21. Jdid, M., Smarandache, F., & Broumi, S. (2023). Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic. Neutrosophic Systems With Applications, 1, 4–13. https://doi.org/10.61356/j.nswa.2023.2

Received: May 15, 2023. Accepted: Aug 27, 2023

NEUTROSOPHIC
SYSTEMS WITH APPLICATIONS