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The submitted papers should be professional, and in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e., notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory, every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjointed two by two. But, since in many cases, the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and intuitionistic fuzzy logic). In neutrosophic logic, a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of *]-0, 1+[*.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of classical statistics.

What distinguishes neutrosophic from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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NCMPy: A Modelling Software for Neutrosophic Cognitive Maps based on Python Package

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Abstract: Cognitive maps are a vital tool that can be used for knowledge representation and reasoning. Fuzzy Cognitive Maps (FCMs) are popular soft computing techniques used to model large and complex systems, and they can aid in explainable artificial intelligence (AI). FCMs, however, cannot model the indeterminacy that arises in a system due to various uncertainties. Neutrosophic Cognitive Maps (NCMs), upgraded FCMs that could model indeterminacy, were introduced to address this issue. NCMs are a generalization of FCMs, a field of cognitive science firmly based on neural networks. NCMs have been used to solve a wide range of problems. NCMs were introduced in 2002, and even after 20 years, NCMs do not have any supportive software, package, toolbox, or visualization software like FCMs. The main reason for the absence of dedicated software is due to the indeterminacy concept 'I' and how it has to be handled. This paper presents the dedicated Python package created for handling the functioning of NCMs. The modelling software presented in this paper aids in visualizing the NCMs as a signed digraph with indeterminacy that is a directed signed neutrosophic graph. This package implements a sample case study using NCMs.

Keywords: Neutrosophy; Neutrosophic Cognitive Maps; Python Package; Visualization of NCMs.

1. Introduction

Fuzzy theory is a branch of mathematics that deals with vagueness and uncertainty in decision-making [1]. Fuzzy sets and logic model complex problems involving imprecise terms or partial truths. It has many applications in engineering fields, the healthcare sector, economics, and social science, which pertain to real-world problems. Fuzzy logic and its models have many applications in various fields, such as engineering, artificial intelligence, medicine, economics, and social problems. It can help model multifaceted problems that involve human knowledge, preferences, or emotions [2].

A fuzzy cognitive map (FCM) [3] represents a mental landscape within which the connections between the nodes (e.g., events, concepts, resources, or attributes) are used to compute the "strength of impact" of these elements. FCMs are signed fuzzy digraphs introduced by Bart Kosko [3].

FCMs have been used to analyze several socio-economic, healthcare, and decision-making problems. The applications and extensions of FCMs are vast and widely researched; a few are presented here.

In [4], the authors explore using FCMs as an agency for collective decision-making and how FCMs can capture the cognitive models and group beliefs of different stakeholders. FCMs were used as a learning assessment tool in [5] to understand the planning of children by stimulating cognitive function.

In [6], the Multi-Agent Genetic Algorithm (MAGA) is proposed to optimize convergence error for learning FCMs. A GIS-dependent crisis management tool to predict earthquakes in Tehran using FCMs was proposed in [7].

An evolutionary algorithm, known as IBMTEA-FCM, was proposed in [8] for learning large-scale FCMs. A qualitative analytical method using FCMs to specify causal-effect links between the interdependent SDGs by considering the long-term effect of COVID-19 was presented in [9].

In [10], the PRescriptiVe FCM (PRV-FCM) was introduced, based on FCMs and metaheuristic algorithms, to develop prescriptive models. In [11], three federated learning approaches were combined with FCMs for mortality prediction and treatment prescription in severe dengue cases.

Iran's population's health was analyzed [12] using FCMs. Since the concept of health is a complex and comprehensive system, other sector policies profoundly affect health. Borrero-Domínguez and Escobar-Rodríguez [13] proposed a decision support system for crowdfunding using FCMs. The various extensions of FCMs have been systemically reviewed by [14].

Several software packages are available for modelling FCMs. One such software, Mental Modeler, helps build FCMs intuitively and easily. After creating the models, decreasing or increasing the model's elements allows us to examine various change tactics. FCMexpert is a software tool for FCM-based scenario analysis and pattern classification presented in [15]. Over 10 FCM extensions were handled by supporting interoperability in the FCM extensions in [16].

Python packages are also available for modelling FCMs. FCMpy [17] is a recently introduced open-source package for building and analyzing FCMs. Notably, FCMPy allows simulating system behaviour using qualitative data to create fuzzy causal weights, applying ML algorithms to modify the FCMs matrix to aid in classification, and executing scenario examination by simulating theoretical interventions.

The package also helps apply ML algorithms (e.g., nonlinear and active Hebbian learning, deterministic learning, and genetic algorithms) to adjust the FCM weight matrix.

Neutrosophy is a branch of philosophy investigating neutralities' origin, nature, and scope and their interactions. Florentin Smarandache introduced neutrosophy in the 1990s [18]. Neutrosophy regards a proposition, hypothesis, concept, event, or entity depending on the modelling. Neutrosophy is the basis of the neutrosophic set, logic, probability, and statistics. Indeterminacy ("I") is a concept in neutrosophy that measures the degree of neutrality or uncertainty of a proposition, event, theory, entity, or concept.

Neutrosophic Cognitive Maps (NCMs) are an extension of FCMs that can handle indeterminate relationships between two concepts, obtaining more significant and sensitive results. It was introduced in [19] to analyze diverse social issues. NCMs have been modelled considerably on the AI focus to mimic the thinking-humanly approach. Here, it is unsupervised data and has a limited set of features.

Over the past two decades, many investigators have utilized NCMs to analyze diverse problems like situation analysis [20], pest analysis [21], transgressions against people experiencing homelessness [22], and imaginative play in children [23]. FCMs and NCMs on COVID variants were compared in [24]. SWOT analysis and NCMs were combined to analyze organic farming in India [25]. Al-Subhi et al. [26] proposed triangular NCMs and used them in multistage decision-making with a use case of evaluation. NCMs and cloud data were used in [27] to detect violence, and several datasets were used. NCMs and FCMs were compared in this analysis, and it clearly states that NCMs are better at handling indeterminacy than FCMs.

Dynamic NCMs [28], enhanced cuckoo search, and ensemble classifiers were presented for acquiring the profile of gene expression and differentiating between the individuals affected by rheumatoid arthritis and possible control subjects. Bhutani et al. [29] proposed a technique combining pest analysis based on fuzzy and neutrosophic logic to analyze the food industry.

In [30], the various factors impacting the paper-packaging industry are analyzed to provide a notional representation using NCMs since sustainable supply chains can be attained with repeated product use and recycling.

In [31], NCMs were used to analyze the various causes and effects that lead to violent behaviour. NCMs were used to determine the elements that enable proper decision-making to confirm a precise diagnosis of conversion disorder [32]. The various factors that affect homeless people were analyzed using NCMs in [33]. A neutrosophic sociogram-based NCM approach was introduced in [34]. FCMs and NCMs have been used in health care to analyze dengue fever [35].

NCMs have been applied in various domains such as health, agriculture, engineering, social problems, business, law, environment, and medicine. The substantial advantage of NCMs over other cognitive maps is their capacity to capture data realistically and consistently represent expert opinions, making them a valuable tool for decision-making strategies where there is an advanced degree of indeterminacy or uncertainty. NCMs can be constructed either by a data-driven approach or experts' opinions.

NCMs have not been integrated with machine-learning algorithms like FCMs. So, little research combines various machine learning algorithms that have been utilized in adjusting weights in an NCM, like in FCM. Despite the various applications of NCMs, there is no dedicated software or Python package for NCMs. This paper presents a dedicated software and Python package for constructing, analyzing, and visualizing NCMs.

The dedicated modules of our modelling software function in the following way:

i. Generating the neutrosophic graph and the related connection matrix using expert opinion
    - From linguistic terms.
    - From edge weights.
    - From a file as input from the user.
ii. Visualizing the NCMs as a neutrosophic digraph.
iii. Simulating the NCMs using various state vectors.
iv. Analysis of various case scenarios for given NCMs.

The paper is organized as follows: Section 2 recalls the workings of NCMs and their construction. Section 3 provides the software-based visualization and simulation of NCM for a case study. The conclusions and results are presented in the last section, together with suggestions for future research.

2. Working of Neutrosophic Cognitive Maps (NCMs)

When data is unsupervised and the association between two concepts is indeterminate, the indeterminacy can be captured by using neutrosophy. [19] introduced the concept of indeterminacy in FCMs, called NCMs. The basic properties of NCM are recalled to make this section self-contained.

NCM is a digraph with concepts as nodes and their causal relationships as edges. These concepts can be events, strategies, or policies as nodes of the graph and relationships as edges, where each concept is mathematically represented as a neutrosophic vector from the neutrosophic vector space. Every node, in its vector form, is represented by $(x_1, , x_n)$; $x_i \in \{0,1,I\}$, where 0 is off state, 1 is on state and I is the indeterminate state.

Consider two nodes N_a and N_b of the NCM; their relationship is given by the edge e_{ab}. Every weighted edge e_{ab} is from {–1, 0, 1, I}, where 0 means no impact, positive value means increase in N_a implies increase in N_b, similarly decrease in N_a implies decreases in N_b. If e_{ab} takes a negative value like -1, it implies that an increase in N_a implies a decrease in N_b, or similarly, a decrease in N_a implies an increase in N_b. The edge weight is assigned a value I if the effect from N_a to N_b can not be determined. The edge weights of simple NCMs are from {–1, 0, 1, I}. The NCM's adjacency matrix is denoted by $N(E) = (e_{ab})$, $e_{ab} \in \{-1,0,1,I\}$.

NCMs are expert opinions constructed based on data obtained from the expert, where they identify the factors or concepts relevant to the domain and associated causal relationships in terms of numbers and indeterminacy or linguistic terms.

Generally, the neutrosophic connection matrix is obtained directly from the digraph of the expert's opinion. Here, we have introduced the concept of dealing with neutrosophic linguistic terms using the following logic. Algorithm 1 provides neutrosophic edge weights from the linguistic terms.

Algorithm 1: Generate NCM Matrix

```
Data: matrix ← Empty Matrix; row ← Empty List; n ← No of nodes
Result: NCM Matrix;
i,j ← 0 while i ≠ n do
    string ← " "
    while j ≠ n do
        term ← input linguistic term
        if i = j then
            element ← 0
        else
            if term = "-VH" then
                element ← random(−5, −2.75)
            else if term = "-H" then
                element ← random(−2.75, −2)
            else if term = "-M" then
                element ← random(−2, −1.5)
            else if term = "-L" then
                element ← random(−1.5, −1)
            else if term = "-VL" then
                element ← random(−1, 0)
            else if term = "NC" then
                element ← 0
            else if term = "+VH" then
                element ← random(2.75, 5)
            else if term = "+H" then
                element ← random(2, 2.75)
            else if term = "+M" then
                element ← random(1.5, 2)
            else if term = "+L" then
                element ← random(1, 1.5)
            else if term = "+VL" then
                element ← random(0, 1)
            else if term = "NaN" then
                element ← I
        string ← string + element+','
        j ← j + 1
    for x ∈ temp do
        if x = "I" then
            continue
    row ← row.append(temp)
    i ← i + 1
i ← 0
while i ≠ n do
    matrix ← matrix.row_insert(i, row[i])
    i ← i + 1
return matrix
```

In the case of linguistic terms, these edges can be considered as negative very high (-VH), negative high (-H), negative medium (-M), negative low (-L), and negative very low (-VL). It implies a negative influence when the decrease of the influence of the node N_a results in the increase of the influence of the node N_b or the increase of N_a results in the decrease of N_b. Similarly, these edges can be considered as positive very high (+VH), positive high (+H), positive medium (+M), positive low (+L), and positive very low (+VL). It implies a positive influence when the decrease of N_a results in the decrease of N_b or the increase of N_a results in the increase of N_b. If the linguistic term is no causality, 0 is used; if it is indeterminate, "NaN" (not a number) is used. These linguistic terms are converted to numerical values using simple assignments as follows:

- If $term =$ "-VH" then the element gets a random value from [-5, -2.75).
- If $term =$ "-H", then the element gets a random value from [-2.75, -2).
- If $term =$ "-M", then the element gets a random value from [-2, -1.5).

- If $term =$ "-L", then the element gets a random value from [-1.5, -1).
- If $term =$ "-VL", then the element gets a random value from [-1, 0).
- If $term =$ "-NC" then the element gets 0.
- If $term =$ "+VH" then the element gets a random value from (2.75,5].
- If $term =$ "+H", then the element gets a random value from (2, 2.75].
- If $term =$ "+M", then the element gets a random value from (1.5,2].
- If $term =$ "+L", then the element gets a random value from (1, 1.5].
- If $term =$ "+VL", then the element gets a random value from (0, 1].
- If $term =$ "NaN" then the element gets I.

The values generated by the algorithm range from [-5, 5]; since the non-indeterminate edge weights of NCM are from [-1, 1], the edge weights are normalized before the NCM/neutrosophic digraph is constructed.

The expert opinion obtained can be used to generate the neutrosophic adjacency matrix of the NCMs. Generation can be done by using linguistic terms. According to the linguistic terms, the neutrosophic matrix can be obtained using random values between the ranges.

Example 1: Consider the graph of the NCM, using the seven concepts (or nodes or attributes) $L_1, L_2, \ldots, L_7$ and the expert opinion is obtained in terms of linguistic terms, using Algorithm 1, the weights for the edges are provided.

For illustration, assuming that the connection from L_2 to L_7 has a very high positive influence, and the connection between from L_1 to L_3 is indeterminate, and the connection from L_4 to L_7 is a very high negative influence. The edge weights generated by the algorithm using the linguistic terms are tabulated in the Table 1:

Table 1. The edge weights assigned by the algorithm.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7
L_1	0	-2.5	I	1.5	0	3.5	0
L_2	0	0	0	0	1.5	0	5
L_3	0	0	0	0	5	2	0
L_4	0	I	0	0	0	0	-4.5
L_5	0	0	0	0	0	0	-1
L_6	0.5	0	0	0	0	0	I
L_7	0	0	0	0	0	0	0

The NCM will be given in Figure 1. Each edge is weighted and directed. The dashed lines are used to represent indeterminate edges. The weight of each edge is given in the graph.

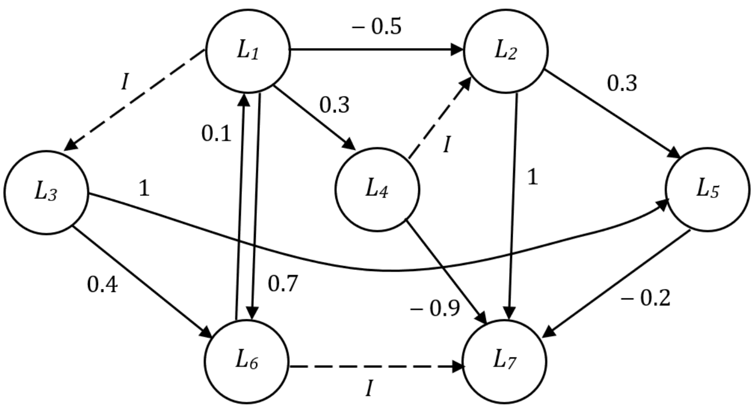

Figure 1. An illustration of the neutrosophic directed graph.

It is then normalized to weigh between $[-1, 1]$ and indeterminacy I. The connection matrix for the neutrosophic directed graph is given in Eq. (1).

$$E_{normalized} = \begin{pmatrix} 0 & -0.5 & I & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0.4 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & -0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (1)$$

The edge weights can also be obtained from the expert. The edges of simple NCMs are from {–1, 0, 1, *I*}.

Example 2: Consider a simple NCM given by an expert with seven concepts $L_1, L_2, \ldots, L_7$ as the nodes of the directed neutrosophic graph. The expert opinion is obtained in terms of edge weights.

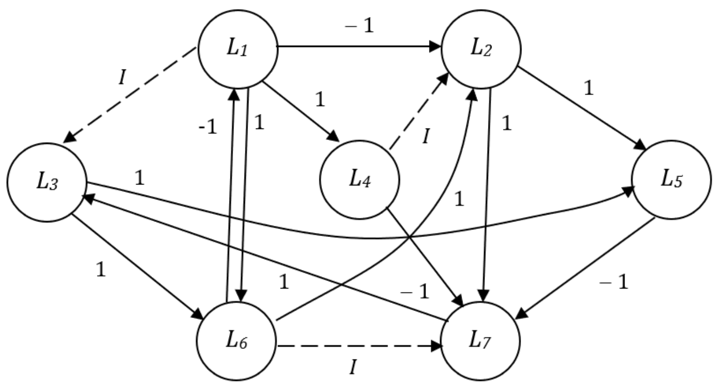

Figure 2. Neutrosophic directed graph for simple NCM.

The edge weights are from $\{-1, 0, 1, I\}$, and the indeterminate edges are represented by dotted lines. The NCM's connection matrix is denoted by $N(E) = (e_{ab})$, where $e_{ab} \in \{0, 1, -1, I\}$.

$$N(E) = \begin{pmatrix} 0 & -1 & I & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

The notion of state vector, dynamical system and its functioning are described. The neutrosophic state vector $S = (s_1, \ldots, s_n); s_i \in \{0,1,I\}$; where 0 indicates the off state, 1 is in the on state, and I indicates the indeterminate state. Let $\overleftarrow{L_1L_2}$, $\overleftarrow{L_2L_3}$, $\overleftarrow{L_3L_4}$, … , $\overleftarrow{L_aL_b}$ be the NCM's directed edges. Given the edges creating a cycle, the NCM becomes cyclic; otherwise, it is acyclic. Consider $\overleftarrow{L_1L_2}$, $\overleftarrow{L_2L_3}, \ldots, \overleftarrow{L_{n-1}L_n}$ to be cyclic, if node L_a is on the influence will flow via the existing cycle and cause L_a to be on again. This state of equilibrium of the dynamical system is called as a hidden pattern.

Various state vectors with different nodes in on state are considered to activate the system Consider the NCM with feedback given in Figure 2; its neutrosophic adjacency matrix is *N(E)* given in Eq. (2).

The state vector S_1 = (1, 0, 0, , 0) where L_1 is in on state is considered. The data needs to be transformed by *N(E)*, so we multiply S_1 by *N(E)*.

The state vector multiplied by the neutrosophic matrix $N(E)$ is given in Algorithm 3. In the given algorithm, S is a 1-dimensional matrix (row vector) and the neutrosophic adjacency matrix $N(E)$ is a 2-dimensional matrix denoted by B. The resultant vector Res of the multiplication of $S \times N(E)$ is returned.

An example illustrates this: Consider the graph in Figure 2 and its related connection matrix. Take the state vector $S_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0)$. The state vector S_1 is multiplied with the neutrosophic adjacency matrix $N(E)$.

$$S_1 \times N(E) = (1\ 0\ 0\ 0\ 0\ 0\ 0) \times \begin{pmatrix} 0 & -1 & I & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

$$= (0\ -1\ I\ 1\ 0\ 1\ 0) \quad (4)$$

Algorithm 2: `Multiply`

Data: 1D Matrix S and $B \leftarrow$ 2D Neutrosophic Adjacency Matrix $N(E)$
Result: Resultant Matrix
$Res \leftarrow$ 1D Matrix
$r \leftarrow S.size$
$R \leftarrow B.size$
$i, j \leftarrow 0$
while $i \neq r$ **do**
  while $j \neq R$ **do**
    $Res[i] \leftarrow Res[i] + (S[i] * B[j][i])$
    $j \leftarrow j + 1$
  $i \leftarrow i + 1$
return Res

After obtaining the resultant vector Res, it must undergo the threshold and update operations. By the definition of NCM, the threshold and update operation is denoted by the $\hookrightarrow$ symbol.

It is important to note here that working with I (indeterminate) needs to be done carefully. As by the definition of I

$$I \times I = I^2 = I \quad (5)$$

Any power of I gets mapped to I, as shown in Algorithm 3.

Algorithm 3: `UpdateIPower`

Data: $V \leftarrow$ state vector
Result: V state vector after updating I's
$i \leftarrow 0$
for $i \in V$ & $j \in b$ **do**
  if $i = I^2$ **then** $i \leftarrow$ I
return V

Similarly, during the updating and threshold operation, any constant into I is also mapped into I, as shown in Eq. (6)

$$n \times I = I \tag{6}$$

The resultant vector from the multiplication of the state vector with $N(E)$ is thresholded and updated.
Let $X = S_1 N(E) = (s_1, s_2, , s_n)$ is thresholded by replacing s_i accordingly to the Eq. (7).

$$s_i = \begin{cases} 1 & if \;\; s_i > t \\ 0 & if \;\; s_i < t \;\; (\; t \; is \; a \; suitable \; positive \; integer) \\ I & if \;\; s_i \text{ not } an \; integer \end{cases} \tag{7}$$

The resultant X is updated to ensure that the state considered on in the initial state vector S_1 is on in the resulting vector. Here, it is updated to make the concept N_1 as 1 in the resulting vector.
The algorithm for thresholding and updating is given in Algorithm 4.

Algorithm 4: `ThresholdandUpdate`

```
Data: X ← resultant vector and t threshold value, Initial On state
Result: Thresholded and updated resultant vector
X ← updateIPower(X)
len ← X.size
i ← 0
while i ≠ len do
    temp_expr ← X[i]
    if temp_expr = J then
        temp_expr ← 0
    if temp_expr ≥ t then
        X[i] ← t
    else if temp_expr = 0 then
        if X[i] ≠ 0 then
            X[i] ← J
    else
        X[i] ← 0
    i ← i + 1
X[state − 1] ← 1
return X
```

Here, the Algorithm 4 illustrates where only one node is taken in the on state in the initial state vector.

The thresholding and updating operation is mathematically denoted by $\hookrightarrow$. For example, consider the resultant vector of $S_1 \times N(E)$, the thresholding and updating result in S_2.

$$S_1 \times N(E) = (0 \;\; -1 \; I \; 1 \; 0 \; 1 \; 0) \hookrightarrow (1 \; 0 \; I \; 1 \; 0 \; 1 \; 0) = S_2 \tag{8}$$

Here, during the threshold operation, -1 is made 0, and during the update operation, the very first state is made on again.

Generally, in any NCM, more than one concept/ node can be considered in the on state. This will deal with the combined effect of both states being on. The proposed model can handle more than one node in the on state, and it can work to analyse the effect of a combination of various nodes. The multiplication of resultant vectors with the neutrosophic adjacency matrix $N(E)$ will continue until the resultant vector yields a fixed point or limit cycle.

If the NCM settles to a neutrosophic state vector repeating in the form

$$S_1 \hookrightarrow S_2 \hookrightarrow S_j \ldots \hookrightarrow S_i \hookrightarrow S_j,$$

Then the dynamic system's equilibrium is called NCM's limit cycle. Suppose, $S_1 N(E) \hookrightarrow S_2$ (where $\hookrightarrow$ denotes the resultant vector of $S_1 N(E)$ which is thresholded and updated) and for $S_2 N(E)$ we repeat the same procedure until we attain the fixed point / limit cycle.

Two vectors are compared in Algorithm 5.

```
Algorithm 5: Compare
  Data: a, b - 1D vectors
  Result: Boolean value
  i,j ← 0
  for i ∈ a & j ∈ b do
  |  if i ≠ j then
  |  |  return false
  return true
```

A fixed point or limit cycle is attained when both the vectors under comparison are the same. For example, if we compare S_1 with S_2, we can see that $S_1 \neq S_2$.

$$(1\ 0\ 0\ 0\ 0\ 0\ 0) \neq (1\ 0\ I\ 1\ 0\ 1\ 0) = S_2 \tag{9}$$

Since the fixed point nor limit cycle is reached, i.e., $S_1 \neq S_2$, the process is continued.

$$S_2 \times N(E) = (1\ I\ I\ 1\ I\ 1+I\ \ -1+I) \hookrightarrow (1\ I\ I\ 1\ I\ 1\ 0) = S_3 \tag{10}$$

Since the fixed point still needs to be reached, i.e., $S_3 \neq S_2$, the process is continued.

$$S_3 \times N(E) = (1+I\ \ 2I\ \ 2I\ \ 1\ \ 2I\ \ 1+I\ \ -1+2I) \hookrightarrow (1\ I\ I\ 1\ I\ 1\ 0) = S_4 \tag{11}$$

Here $S_3 = S_4$. The fixed point has been reached.

The Algorithm 6 is used to determine if a limit cycle is reached. It compares with the previous resultant vectors using the previously described in compare Algorithm 5 to check if the limit cycle is reached. In case it is reached, it returns a true or a false.

```
Algorithm 6: Check_cycle
  Data: b ← list of vectors
  Result: Boolean value
  n ← b.size
  i ← 0
  while i ≠ n − 1 do
  |  j ← i + 1
  |  while j ≠ n do
  |  |  v1 ← b[i]
  |  |  v2 ← b[j]
  |  |  if compare(v1, v2) = true then
  |  |  |  return true
  |  |  j ← j + 1
  |  i ← i + 1
  return false
```

To illustrate an example of limit cycle, consider the state vector $P_1 = (0\ 1\ 0\ 0\ 0\ 0\ 0)$, and the connection matrix $N(E)$ as given in Eq. (2), that is the related adjacency matrix for the NCM given in Figure 2.

$$P_1 \times N(E) = (0\ 0\ 0\ 0\ 1\ 0\ 1) \hookrightarrow (0\ 1\ 0\ 0\ 1\ 0\ 1) = P_2$$
$$P_2 \times N(E) = (0\ 0\ 1\ 0\ 1\ 0\ 0) \hookrightarrow (0\ 1\ 1\ 0\ 1\ 0\ 0) = P_3$$
$$P_3 \times N(E) = (0\ 0\ 0\ 0\ 2\ 1\ 0) \hookrightarrow (0\ 1\ 0\ 0\ 1\ 1\ 0) = P_4$$
$$P_4 \times N(E) = (1\ 1\ 0\ 0\ 1\ 0\ I) \hookrightarrow (1\ 1\ 0\ 0\ 1\ 0\ I) = P_5$$

$$P_5 \times N(E) = (0\ -1\ 2*I\ 1\ 1\ 1\ 0)$$
$$\hookrightarrow (0\ 1\ I\ 1\ 1\ 1\ 0) = P_6$$
$$P_6 \times N(E) = (1\ I+1\ 0\ 0\ I+1\ I\ I-1)$$
$$\hookrightarrow (1\ 1\ 0\ 0\ 1\ I\ 0) = P_7$$
$$P_7 \times N(E) = (I\ I-1\ I\ 1\ 1\ 1\ I^2)$$
$$\hookrightarrow (I\ 1\ I\ 1\ 1\ 1\ I) = P_8$$
$$P_8 \times N(E) = (1\ 1\ I^2+I\ \ I\ \ I+1\ 2*I\ I-1)$$
$$\hookrightarrow (1\ 1\ I\ I\ 1\ I\ 0) = P_9$$
$$P_9 \times N(E) = (I\ I^2+I-1\ \ I\ 1\ I+1\ I+1\ \ I^2-I)$$
$$\hookrightarrow (I\ 1\ I\ 1\ 1\ 1\ 0) = P_{10}$$
$$P_{10} \times N(E) = (1\ 1\ I^2\ I\ I+1\ 2*I\ I-1)$$
$$\hookrightarrow (1\ 1\ I\ I\ 1\ I\ 0) = P_{11} = P_9 \quad (12)$$

Since $P_{11} = P_9$, the iteration is stopped since a limit cycle has been achieved, enabling the determination of the hidden pattern. The limit cycle is as follows: P_8 gives P_9, P_9 gives P_{10}, P_{10} gives P_{11}; that is same as P_9.

The process of multiplication of the resultant vector with the matrix $N(E)$ is continued until a limit cycle / fixed point is reached.

The Algorithm 7 takes an adjacency/connection matrix and state vector (an integer as input denoting the state to be activated) and an integer denoting the threshold values as parameters.

```
Algorithm 7: Iteration
  Data: an Adjacency matrix E, state ← state to be activated, t threshold value
  Result: list of vectors
  len ← E.size
  c1 ← list[len] filled with 0
  c1[state − 1] ← 1
  start ← Matrix(c1)
  flag ← false
  vectors ← Empty List
  while flag = false do
  |  y ← multiply(start, E)
  |  y ← thresholdAndUpdate(y, t)
  |  vectors ← vectors.append(y)
  |  start ← y
  |  flag ← check_cycle(vectors)
  return vectors
```

The algorithm 7 continues multiplying the resultant vector with the *N(E)* until a fixed point or a limit cycle is reached. It is dependent on several previously described algorithms. Every concept must be made in the active state to capture the hidden pattern and understand the effect of the concept or node on others to analyse the NCM thoroughly. Only in NCMs can we signify that the clout of a node on different nodes can be indeterminate, and this vision needs to be revised in the case of FCMs.

For the given example, the results have been tabulated in Table 2 for various state vectors.

Table 2. Determining the Hidden Pattern.

Input state vector	limit cycle / fixed point	Resultant vector
(1 0 0 0 0 0 0)	fixed point	(1 I I 1 I I 0)
(0 1 0 0 0 0 0)	limit cycle	(I 1 I 1 1 1 0)
		(1 1 I I 1 I 0)
		(I 1 I 1 1 1 0)
		(1 1 I I 1 I 0)
(0 0 1 0 0 0 0)	fixed point	(1 I 1 1 1 1 0)
(0 0 0 1 0 0 0)	fixed point	(0 I 0 1 I 0 0)
(0 0 0 0 1 0 0)	fixed point	(0 0 0 0 1 0 0)
(0 0 0 0 0 1 0)	fixed point	(1 I I 1 I 1 0)
(0 0 0 0 0 0 1)	fixed point	(1 I 1 1 1 1 1)

It is seen that only the active state of concept L_2 results in a limit cycle—the rest results in a fixed point.

The conclusions that can be drawn from the equilibrium state of the dynamic system are as follows: When node L_1 is active, all nodes are either indeterminate or on state. When L_2 is in the on state, it results in a limited cycle, which affects all nodes other than L_7. Similarly, when nodes L_3, L_4, L_6 or L_7 alone are in the on state, other nodes are either indeterminate or on state. Whereas when node L_5 is in on state, no other node is affected; all of them remain in the off state.

3. Description of the modelling package

The overall flow of the modelling software is as given in Figure 3. The first module is for the input module, which can either be linguistic term-based or edge-weight-based.

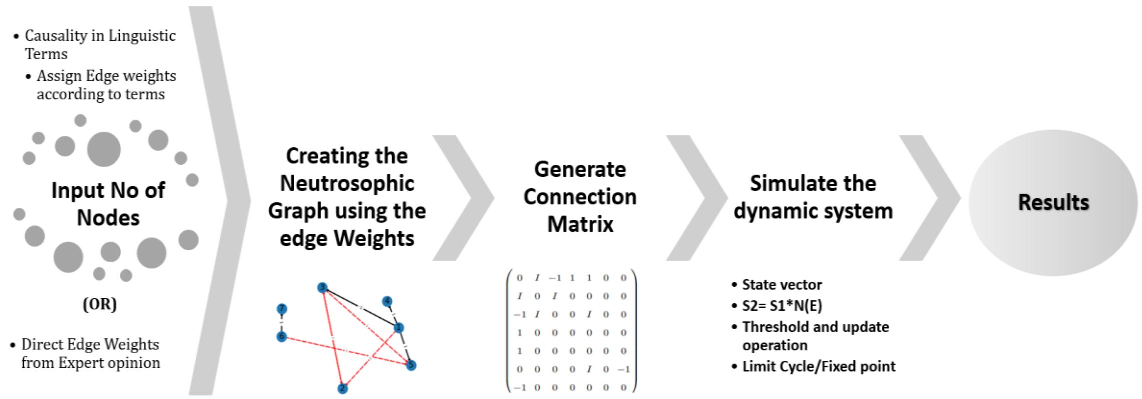

Figure 3. Various modules of NCM modelling software.

1. Input module: There are two methods in which the model can be created using expert opinion. They can enter by either method as described below:
   - *Linguistic terms:* The user can enter the linguistic terms to describe the relationship between two concepts in NCM as very negative or positive. According to the previously discussed algorithm, the edge weights are assigned based on the linguistic term.
   - *Edge weights:* The user can instead directly enter the edge weight to denote the causality between two nodes.
2. Neutrosophic digraph: The neutrosophic digraph is generated and visualised using the edge weights obtained from the user.

3. Connection matrix: The related connection matrix $N(E)$ is obtained from the neutrosophic digraph of the NCM.
4. Dynamical system: Using various state vectors, the dynamical system is simulated to analyse the effect of the on state of the various nodes.
5. Results: The effect of the on state of various nodes and combination of various states is consolidated.

The NCMpy python package performs a simulation of the dynamic system—a detailed description of the package is given in the next subsection.

3.1 Description of the NCMPy Python Package

The flowchart of the NCMpy package is given in the following Figure 4.

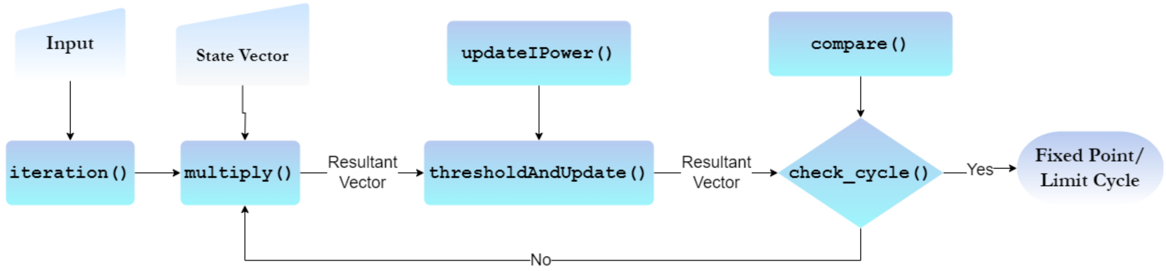

Figure 4. Flowchart of the NCM package.

The SymPy library has been used to handle indeterminate values in modelling the NCMs in this proposed package. In SymPy package, the names E, I, O, S, N, and Q collide with names already defined in the package. Hence, I can not be used to represent indeterminacy, so J is used instead of I. Throughout the coding snippets, J is used instead of I.

The various functions/modules are described here.

1. compare(x, y): This function takes two vectors as input and checks for their equality; it is dependent on Algorithm 6. The code snippet is as follows:

```
def compare(x, y):
    res = True
    for k in range(len(x)):
        if x[k] != y[k]:
            return False
    return True
```

Assume that compare function takes $S_7 = (I\ I\ I\ I\ I\ I\ 0)$ and $S_8 = (I\ I\ I\ I\ I\ I\ 0)$ as x and y. In that case, compare(x, y) will return True, in case it was $S_5 = (1\ I\ I\ 1\ I\ I\ I)$ and $S_6 = (I\ I\ I\ 1\ I\ I\ 0)$ under consideration, then it would return False.

2. check_cycle(b): This function is dependent on Algorithm 6; it is used to check if the resultant vector is a fixed point or limit cycle.

```
def check_cycle(b) :
  for i in range(len(b) - 1) :
    for j in range(i + 1 , len(b)) :
      v1 = b[i]
      v2 = b[j]
      if(compare(v1 , v2) == True) :
        return True
  return False
```

The fixed point is achieved when the resultant vector is the same as the previous resultant vector, that is $P_i N(E) \hookrightarrow P_i$. The limit cycle is achieved when the recently calculated state vector is the same as any one of the previously calculated resultant vector, which results in a cycle.

$$P_i N(E) \hookrightarrow P_{i+1}; P_{i+1} N(E) \hookrightarrow P_{i+2}; \dots P_x N(E) \hookrightarrow P_i;$$

In Example 2, considering the active state $P_1 = (0\ 1\ 0\ 0\ 0\ 0\ 0)$ results in a limit cycle.

3. updateIPower(x): This function takes a 1D vector as input and converts the indeterminate quadratic polynomial to a linear polynomial. This function is based on Algorithm 3. This function is used by the threshold and update function.

```
def updateIPower(x):
    for i in range((np.shape(x))[1]):
        if x[i].find(J**2):
            x[i] = x[i].xreplace({J**2: J})
    return x
```

Consider the resultant vector;

$$R_1 = (J ** 2\ J\ J + 1\ 1\ 0\ J ** 2\ 2J)$$

For a sample scenario. The updateIPower(R_1) will change this vector P_1 into

$$R_1 = (J\ J\ J + 1\ 1\ 0\ J\ 2J)$$

4. thresholdAndUpdate(X,threshold_value,state) : This function takes a 1D vector and a threshold value as a parameter and updates each vector value according to the defined thresholding operation. Also, the updation operation must see to it that the node which was on in the initial state is on in the next state and so on in the resultant state also; if not, it is set to 1 again.

```
def thresholdAndUpdate(X , threshold_value, state) :
    X=updateIPower(X)
    for i in range((np.shape(X)[1])):
        if(X[i].find(-1)):
            X[i]=0
        temp_expr=X[i].subs(J,0)
        if(temp_expr>=threshold_value):
            X[i]=X[i].subs(X[i],threshold_value)
        elif(temp_expr==0):
            if(X[i].find(J)):
                X[i]=J
        else:
            X[i]=X[i].subs(X[i],0)
    if isinstance(state, list):
        activated_states = [i for i, x in enumerate(state) if x == 1]
        for s in activated_states:
            X[s] = 1
    else:
        X[state - 1] = 1
    return X
```

$$X_{10} \times N(E) = (I + 1\ I + 1\ I\ I\ I + 1\ 2 * I\ I - 1) \hookrightarrow (1\ 1\ I\ I\ 1\ I\ 0) = X_{11}$$

5. iteration(E,state,threshold_value) : This function takes an adjacency/connection matrix, an integer denoting the state to be activated and an integer denoting the threshold values as parameters. This function is based on the Algorithm 7 .

```
def iteration(E , state, threshold_value = 1) :
    if isinstance(state, list):
        start = sym.Matrix(state)
    else:
        c1 = np.zeros((np.shape(E)[1]))
        c1[state - 1] = 1
        start = sym.Matrix(c1)

    flag = False
    start = start.T
    vectors = []
    while flag == False :
      y = multiply(start , E)
      # Performing the thresholding operation on output vector y
      y = thresholdAndUpdate(y , threshold_value, state)
      vectors.append(y)
      # Updating start vector to start with new state vector
      start = y
      # Checking for cycle among state vectors
      flag = check_cycle(vectors)
    return vectors
```

Consider Example 2, where the working out is done with active state P_1 = (0 1 0 0 0 0 0); it iterates until the limit cycle is achieved.

3.2 *Modelling NCMs for sample case study*

The opening page for the modelling software is given in Figure 5. Here, the user can select the option of working with linguistic terms or directly entering the neutrosophic edge weights given by the expert as shown in Figure 6.

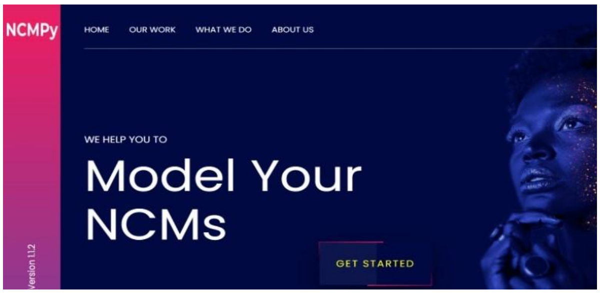

Figure 5. Homepage.

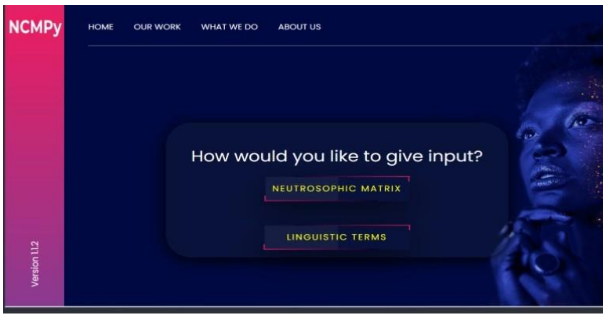

Figure 6. Option selection.

3.3 *NCMs using Linguistic Terms*

The neutrosophic linguistic terms are obtained from the user, as shown in Figure 7.

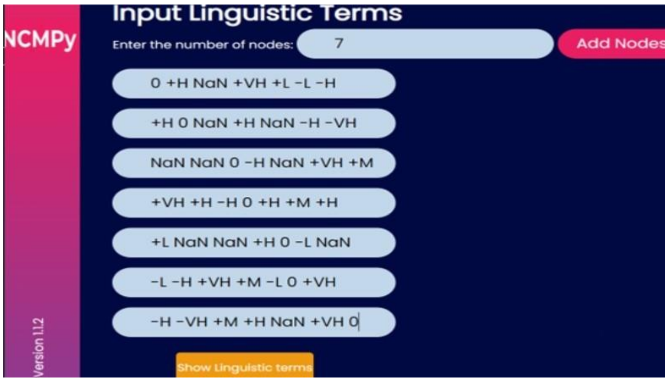

Figure 7. Sample input page for the linguistic terms related to the neutrosophic directed graph.

The NCMpy package runs through the Alogrithm GenerateNCM 2 and creates the necessary edge weights and normalises them as shown in Table 3, providing the neutrosophic bigraphs. The neutrosophic-directed graph of the NCM is given in Figure 8.

Table 3. The edge weights assigned.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7
L_1	0	4.88	I	3.58	1.91	-2.39	-2.81
L_2	4.22	0	I	2.13	I	-2.74	-4.68
L_3	I	I	0	-4.77	I	3.23	1.93
L_4	2.98	1.58	-2.41	0	2.41	1.81	3.11
L_5	1.22	I	I	4.2	0	-2.12	I
L_6	-1.20	-4.36	4.03	1.71	-2.40	0	2.85
L_7	-2.03	-4.77	1.34	3.89	I	4.44	0

The resultant neutrosophic connection matrix of the graph is

```
[
[0, 0.98, J, 0.72, 0.38, -0.48, -0.56],
[0.84, 0, J, 0.43, J, -0.55, -0.94],
[J, J, 0, -0.95, J, 0.65, 0.39],
[0.60, 0.32, -0.48, 0, 0.48, 0.36, 0.62],
[0.24, J, J, 0.84, 0, -0.42, J],
[-0.24, -0.87, 0.81, 0.34, -0.48, 0, 0.57],
[-0.41, -0.95, 0.27, 0.78, J, 0.89, 0]
]
```

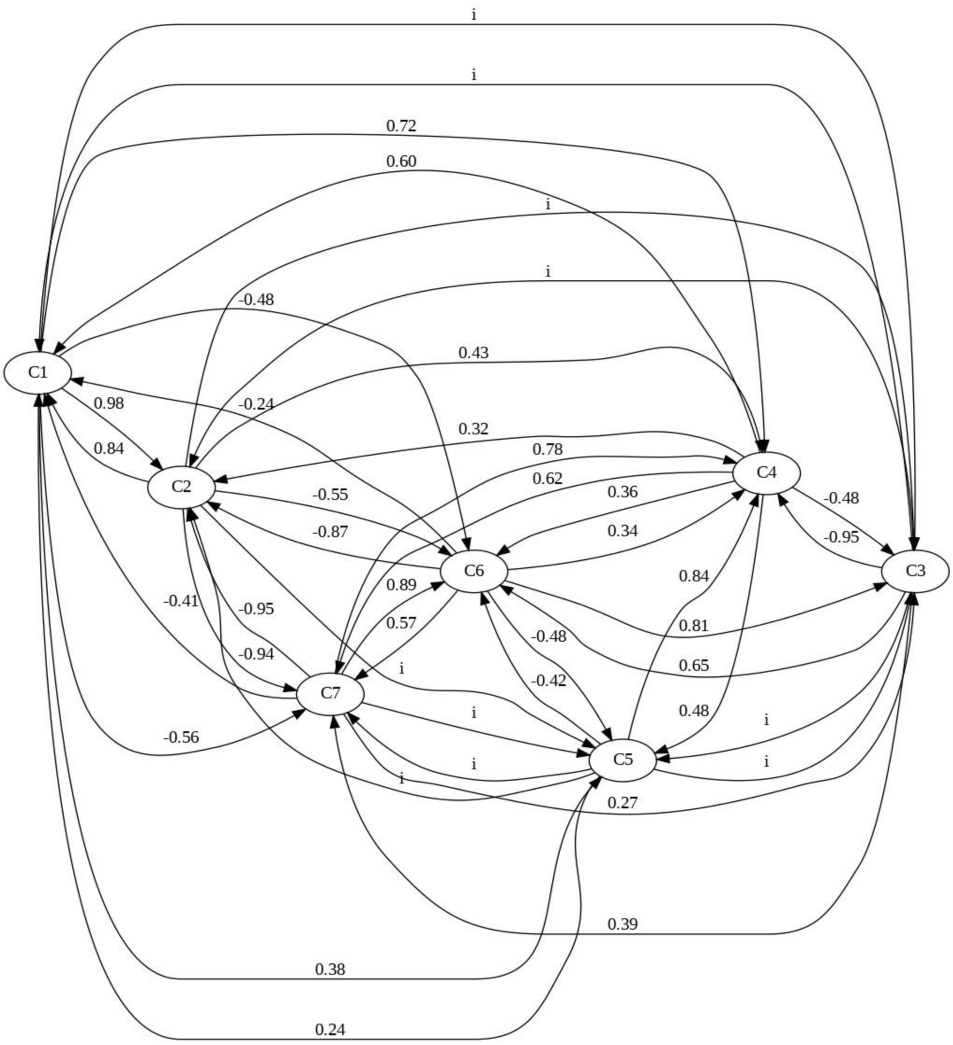

Figure 8. Neutrosophic directed graph.

To show the plotting capacity of the visualizing module, we have taken a matrix with all connections for the sample. The threshold value is obtained from the user. According to the threshold value set by the user, the thresholding and updating of state vectors are done.

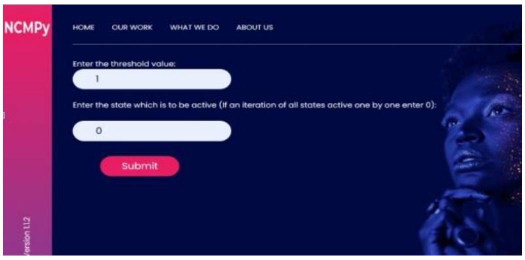

Figure 9. Insert the threshold value and state vector.

The threshold value of 1 was taken here, and the following results were obtained for various state vectors. The working out for the first state vector $S_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0)$, where the concept C_1 is on is shown here:

```
FOR ACTIVE STATE  1
Matrix([[0, 0.98, 1.0*J, 0.72, 0.38, -0.48, -0.56]])
Matrix([[0, 0.98, 1.0*J, 0.72, 0.38, -0.48, -0.56]])
Matrix([[0, 0.98, 1.0*J, 0.72, 0.38, -0.48, -0.56]])
Matrix([[0, 0, 1.0*J, 0.72, 0.38, -0.48, -0.56]])
Matrix([[0, 0, J, 0.72, 0.38, -0.48, -0.56]])
Matrix([[0, 0, J, 0, 0.38, -0.48, -0.56]])
Matrix([[0, 0, J, 0, 0, -0.48, -0.56]])
Matrix([[0, 0, J, 0, 0, 0, -0.56]])
After Thresholding and Updating
Matrix([[1, 0, J, 0, 0, 0, 0]])
Matrix([[J**2, J**2 + 0.98, J, 0.72 - 0.95*J, J**2 + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, J + 0.98, J, 0.72 - 0.95*J, J + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, J + 0.98, J, 0.72 - 0.95*J, J + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, 0, J, 0.72 - 0.95*J, J + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, 0, J, 0.72 - 0.95*J, J + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, 0, J, 0, J + 0.38, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, 0, J, 0, 0, 0.65*J - 0.48, 0.39*J - 0.56]])
Matrix([[J, 0, J, 0, 0, 0, 0.39*J - 0.56]])
After Thresholding and Updating
Matrix([[1, 0, J, 0, 0, 0, 0]])
[Matrix([[1, 0, J, 0, 0, 0, 0]]), Matrix([[1, 0, J, 0, 0, 0, 0]])]
```

Similarly, the working out for each and every state vector is carried out. For state vector $B_1 =$ (0 1 0 0 0 0 0), where the concept C_2 is on, the resultant vectors will be:

```
FOR ACTIVE STATE  2
Matrix([[0.84, 0, 1.0*J, 0.43, 1.0*J, -0.55, -0.94]])
After Thresholding and Updating
Matrix([[0, 1, J, 0, J, 0, 0]])
Matrix([[J**2 + 0.24*J + 0.84, 2*J**2, J**2 + J, 0.43 - 0.11*J, J**2 + J, 0.23*J - 0.55, J**2 + 0.39*J - 0.94]])
After Thresholding and Updating
Matrix([[0, 1, J, 0, J, 0, 0]])
Resultant vector(s) [Matrix([[0, 1, J, 0, J, 0, 0]]), Matrix([[0, 1, J, 0, J, 0, 0]])]
```

For state vector $G_1 =$ (0 0 1 0 0 0 0), where the concept C_3 is on, the resultant vectors will be:

```
FOR ACTIVE STATE  3
Matrix([[1.0*J, 1.0*J, 0, -0.95, 1.0*J, 0.65, 0.39]])
After Thresholding and Updating
Matrix([[J, J, 1, 0, J, 0, 0]])
Matrix([[2.1*J, J**2 + 2.0*J, 3*J**2, 2.0*J - 0.95, J**2 + 1.4*J, 0.65 - 1.4*J, J**2 - 1.5*J + 0.39]])
After Thresholding and Updating
Matrix([[J, J, 1, 0, J, 0, 0]])
Resultant vector(s) [Matrix([[J, J, 1, 0, J, 0, 0]]), Matrix([[J, J, 1, 0, J, 0, 0]])]
```

For state vector $X_1 =$ (0 0 0 1 0 0 0), where the concept C_4 is on, the resultant vectors will be:

```
FOR ACTIVE STATE  4
Matrix([[0.60, 0.32, -0.48, 0, 0.48, 0.36, 0.62]])
After Thresholding and Updating
Matrix([[0, 0, 0, 1, 0, 0, 0]])
Matrix([[0.60, 0.32, -0.48, 0, 0.48, 0.36, 0.62]])
After Thresholding and Updating
Matrix([[0, 0, 0, 1, 0, 0, 0]])
Resultant vector(s) [Matrix([[0, 0, 0, 1, 0, 0, 0]]), Matrix([[0, 0, 0, 1, 0, 0, 0]])]
```

For state vector $Y_1 =$ (0 0 0 0 1 0 0), where the concept C_5 is on, the resultant vectors will be:

```
FOR ACTIVE STATE  5
Matrix([[0.24, 1.0*J, 1.0*J, 0.84, 0, -0.42, 1.0*J]])
After Thresholding and Updating
Matrix([[0, J, J, 0, 1, 0, J]])
Matrix([[J**2 + 0.43*J + 0.24, J**2 + 0.05*J, J**2 + 1.3*J, 0.26*J + 0.84, 3*J**2, 0.99*J - 0.42, 0.45*J]])
After Thresholding and Updating
Matrix([[0, J, J, 0, 1, 0, J]])
Resultant vector(s) [Matrix([[0, J, J, 0, 1, 0, J]]), Matrix([[0, J, J, 0, 1, 0, J]])]
```

For state vector $Z_1 =$ (0 0 0 0 0 1 0), where the concept C_6 is on, the resultant vectors will be:

```
FOR ACTIVE STATE  6
Matrix([[-0.24, -0.87, 0.81, 0.34, -0.48, 0, 0.57]])
After Thresholding and Updating
Matrix([[0, 0, 0, 0, 0, 1, 0]])
Matrix([[-0.24, -0.87, 0.81, 0.34, -0.48, 0, 0.57]])
After Thresholding and Updating
Matrix([[0, 0, 0, 0, 0, 1, 0]])
Resultant vector(s) [Matrix([[0, 0, 0, 0, 0, 1, 0]]), Matrix([[0, 0, 0, 0, 0, 1, 0]])]
```

```
FOR ACTIVE STATE  7
Matrix([[-0.41, -0.95, 0.27, 0.78, 1.0*J, 0.89, 0]])
After Thresholding and Updating
Matrix([[0, 0, 0, 0, J, 0, 1]])
Matrix([[0.24*J - 0.41, J**2 - 0.95, J**2 + 0.27, 0.84*J + 0.78, J, 0.89 - 0.42*J, J**2]])
After Thresholding and Updating
Matrix([[0, 0, 0, 0, J, 0, 1]])
Resultant vector(s) [Matrix([[0, 0, 0, 0, J, 0, 1]]), Matrix([[0, 0, 0, 0, J, 0, 1]])]
```

For state vector $A_1 = (0\ 0\ 0\ 0\ 0\ 0\ 1)$, where the concept C_7 is on, the resultant vectors will be:

The result vector in each case has been shown. Results regarding the resultant vectors that can be discussed

1. Maximum Influence: The nodes C_2, C_3 and C_5 are the most influential since they do affect many other nodes and turn to an indeterminate state.
2. Least Influential nodes: The nodes C_4 and C_6 are the least influential since they do not affect any other node than itself.

3.4 NCMs using Edge Weights

The edge weights are obtained from the expert, as shown in Figure 10.

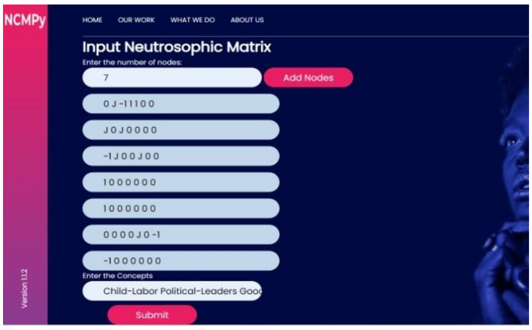

Figure 10. Input screen.

Using the edge weights obtained, the NCM is created. It is the visualization of the same represented as a NCMs digraph as shown in Figure 11.

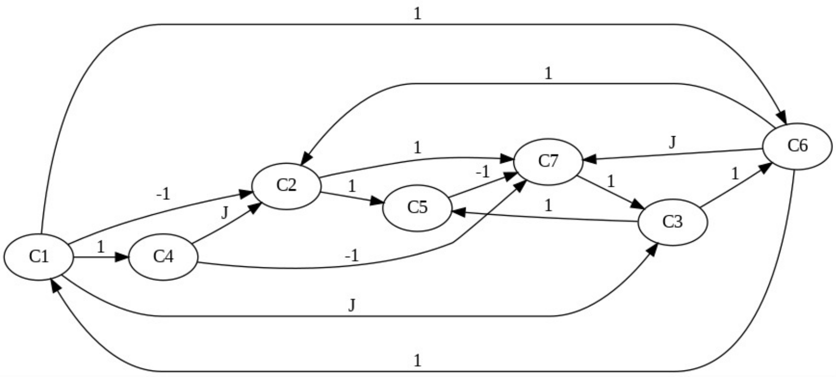

Figure 11. Visualization of the neutrosophic graph.

The related neutrosophic connection matrix $N(E)$ is given in Eq. (2). The obtained matrix is

```
[
  [0, -1, 'J', 1, 0, 1, 0],
  [0, 0, 0, 0, 1, 0, 1],
  [0, 0, 0, 0, 1, 1, 0],
  [0, 'J', 0, 0, 0, 0, -1],
  [0, 0, 0, 0, 0, 0, -1],
  [1, 1, 0, 0, 0, 0, 'J'],
  [0, 0, 1, 0, 0, 0, 0]
]
```

The threshold value of 1 was taken here, and the following results were obtained for various state vectors. The working out for the first state vector $S_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0)$, where the concept C_1 is on is shown here:

```
FOR ACTIVE STATE  1
Matrix([[0, -1.00000000000000, 1.0*J, 1.00000000000000, 0, 1.00000000000000, 0]])
After Thresholding and Updating Matrix([[1, 0, J, 1, 0, 1, 0]])
Matrix([[1, J, J, 1, J, J + 1, J - 1]])
After Thresholding and Updating Matrix([[1, J, J, 1, J, 1, 0]])
Matrix([[1, J, J, 1, 2*J, J + 1, J - 1]])
After Thresholding and Updating Matrix([[1, J, J, 1, J, 1, 0]])
Resultant vector(s) [Matrix([[1, 0, J, 1, 0, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]])]
```

The resultant vectors for all active states are given in the snippet below:

```
FOR ACTIVE STATE  2
Resultant vector(s) [Matrix([[0, 1, 0, 0, 1, 0, 1]]), Matrix([[0, 1, 1, 0, 1, 0, 0]]), Matrix([[0, 1, 0,
0, 1, 1, 0]]), Matrix([[1, 1, 0, 0, 1, 0, J]]), Matrix([[0, 1, J, 1, 1, 1, 0]]), Matrix([[1, 1, 0, 0, 1,
J, 0]]), Matrix([[J, 1, J, 1, 1, 1, J]]), Matrix([[1, 1, J, J, 1, J, 0]]), Matrix([[J, 1, J, 1, 1, 1,
0]]), Matrix([[1, 1, J, J, 1, J, 0]])]
FOR ACTIVE STATE  3
Resultant vector(s) [Matrix([[0, 0, 1, 0, 1, 1, 0]]), Matrix([[1, 1, 1, 0, 1, 1, 0]]), Matrix([[1, 0, 1,
1, 1, 1, J]]), Matrix([[1, J, 1, 1, 1, 1, 0]]), Matrix([[1, J, 1, 1, 1, 1, 0]])]
FOR ACTIVE STATE  4
Resultant vector(s) [Matrix([[0, J, 0, 1, 0, 0, 0]]), Matrix([[0, J, 0, 1, J, 0, 0]]), Matrix([[0, J, 0,
1, J, 0, 0]])]
FOR ACTIVE STATE  5
Resultant vector(s) [Matrix([[0, 0, 0, 0, 1, 0, 0]]), Matrix([[0, 0, 0, 0, 1, 0, 0]])]
FOR ACTIVE STATE  6
Resultant vector(s) [Matrix([[1, 1, 0, 0, 0, 1, J]]), Matrix([[1, 0, J, 1, 1, 1, 1]]), Matrix([[1, J, 1,
1, J, 1, 0]]), Matrix([[1, J, J, 1, 1, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]]), Matrix([[1, J, J, 1, J,
1, 0]])]
FOR ACTIVE STATE  7
Resultant vector(s) [Matrix([[0, 0, 1, 0, 0, 0, 1]]), Matrix([[0, 0, 1, 0, 1, 1, 1]]), Matrix([[1, 1, 1,
0, 1, 1, 1]]), Matrix([[1, 0, 1, 1, 1, 1, 1]]), Matrix([[1, J, 1, 1, 1, 1, 1]]), Matrix([[1, J, 1, 1, 1,
1, 1]])]
```

The combination of two concepts being in the on state is also handled by the package done there. Similarly, several other combinations can be done, according to the user's choice. A sample of the combinations of various concepts begins in on state, and the working is given in the snippet.

```
Activating state: [1, 1, 0, 0, 0, 0, 0]
[Matrix([[1, 1, J, 1, 1, 1, 1]]), Matrix([[1, 1, 1, 1, 1, 1, 0]]), Matrix([[1, 1, J, 1, 1, 1, 0]]),
Matrix([[1, 1, J, 1, 1, 1, 0]])]
Activating state: [1, 0, 1, 0, 0, 0, 0]
[Matrix([[1, 0, 1, 1, 1, 1, 0]]), Matrix([[1, J, 1, 1, 1, 1, 0]]), Matrix([[1, J, 1, 1, 1, 1, 0]])]
Activating state: [1, 0, 0, 1, 0, 0, 0]
[Matrix([[1, 0, J, 1, 0, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]])]
Activating state: [1, 0, 0, 0, 1, 0, 0]
[Matrix([[1, 0, J, 1, 1, 1, 0]]), Matrix([[1, J, J, 1, 1, 1, 0]]), Matrix([[1, J, J, 1, 1, 1, 0]])]
Activating state: [1, 0, 0, 0, 0, 1, 0]
[Matrix([[1, 0, J, 1, 0, 1, J]]), Matrix([[1, J, J, 1, J, 1, 0]]), Matrix([[1, J, J, 1, J, 1, 0]])]
Activating state: [1, 0, 0, 0, 0, 0, 1]
[Matrix([[1, 0, 1, 1, 0, 1, 1]]), Matrix([[1, J, 1, 1, 1, 1, 1]]), Matrix([[1, J, 1, 1, 1, 1, 1]])]
Activating state: [0, 1, 1, 0, 0, 0, 0]
[Matrix([[0, 1, 1, 0, 1, 1, 1]]), Matrix([[1, 1, 1, 0, 1, 1, J]]), Matrix([[1, 1, 1, 1, 1, 1, J]]),
Matrix([[1, 1, 1, 1, 1, 1, 0]]), Matrix([[1, 1, 1, 1, 1, 1, 0]])]
```

The complete NCMPy python package will be made available online.

4. Conclusions and discussions

As the world moves towards no code or less coding paradigms, a Python package for NCM will aid and help mathematicians, social scientists, economists, strategists, and other policymakers analyze real-world problems without worrying about the mathematical background or computational complexities of NCM. A visualization tool and a Python package to help in the working of NCM were presented in this paper. The NCMpy package and modelling software provide the functions essential to studying problems involving indeterminacy, which can be done using NCMs.

This package and modelling tool are open-source, written in Python, straightforward to implement, and provide the required functionality for handling models with indeterminacy.

This tool implementation is a collaboration with the founding and leading experts in the field of NCMs. This tool will facilitate research and enable new researchers and scientists to apply NCMs to their projects that involve indeterminacy. We plan to update our library and constantly welcome all scientific community contributions.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. Fuzzy sets and systems, 1(1), 3-28. https://doi.org/10.1016/0165-0114(78)90029-5.
2. Felix, G., Nápoles, G., Falcon, R., Froelich, W., Vanhoof, K., & Bello, R. (2019). A review on methods and software for fuzzy cognitive maps. Artificial intelligence review, 52, 1707-1737. https://doi.org/10.1007/s10462-017-9575-1.
3. Kosko, B. (1986). Fuzzy cognitive maps. International journal of man-machine studies, 24(1), 65-75. https://doi.org/10.1016/S0020-7373(86)80040-2.
4. Gray, S. A., Zanre, E., & Gray, S. R. (2013). Fuzzy cognitive maps as representations of mental models and group beliefs. In Fuzzy cognitive maps for applied sciences and engineering: From fundamentals to extensions and learning algorithms (pp. 29-48). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-39739-4_2.
5. Barón, H. B., Crespo, R. G., Pascual Espada, J., & Martínez, O. S. (2015). Assessment of learning in environments interactive through fuzzy cognitive maps. Soft Computing, 19, 1037-1050. https://doi.org/10.1007/s00500-014-1313-x.
6. Yang, Z., & Liu, J. (2020). Learning fuzzy cognitive maps with convergence using a multi-agent genetic algorithm. Soft Computing, 24, 4055-4066. https://doi.org/10.1007/s00500-019-04173-2.
7. Choukolaei, H. A., Mirani, S. E., Hejazi, M., & Ghasemi, P. (2023). A GIS-based crisis management using fuzzy cognitive mapping: PROMETHEE approach (a potential earthquake in Tehran). Soft Computing, 1-22. https://doi.org/10.1007/s00500-023-09260-z.
8. Wang, C., Liu, J., Wu, K., & Ying, C. (2021). Learning large-scale fuzzy cognitive maps using an evolutionary many-task algorithm. Applied Soft Computing, 108, 107441. https://doi.org/10.1016/j.asoc.2021.107441.
9. Ameli, M., Esfandabadi, Z. S., Sadeghi, S., Ranjbari, M., & Zanetti, M. C. (2023). COVID-19 and Sustainable Development Goals (SDGs): Scenario analysis through fuzzy cognitive map modeling. Gondwana Research, 114, 138-155. https://doi.org/10.1016/j.gr.2021.12.014.

10. Hoyos, W., Aguilar, J., & Toro, M. (2023). PRV-FCM: An extension of fuzzy cognitive maps for prescriptive modeling. Expert Systems with Applications, 120729. https://doi.org/10.1016/j.eswa.2023.120729.
11. Hoyos, W., Aguilar, J., & Toro, M. (2023). Federated learning approaches for fuzzy cognitive maps to support clinical decision-making in dengue. Engineering Applications of Artificial Intelligence, 123, 106371. https://doi.org/10.1016/j.engappai.2023.106371.
12. Salehnia, N., Arani, A. A., Olyaeemanesh, A., & Saghdel, H. S. (2022). Analysis of Casual Relationships between Social Determinants of Health in Iran: Using Fuzzy Cognitive Map. Journal of Research in Health Sciences, 22(4). https://doi.org/10.34172/jrhs.2022.101.
13. Borrero-Domínguez, C., & Escobar-Rodríguez, T. (2023). Decision support systems in crowdfunding: A fuzzy cognitive maps (FCM) approach. Decision Support Systems. https://doi.org/10.1016/j.dss.2023.114000
14. Schuerkamp, R., & Giabbanelli, P. J. (2023). Extensions of fuzzy cognitive maps: A systematic review. ACM Computing Surveys, 56(2), 1-36. https://doi.org/10.1145/3610771.
15. Nápoles, G., Espinosa, M. L., Grau, I., & Vanhoof, K. (2018). FCM expert: software tool for scenario analysis and pattern classification based on fuzzy cognitive maps. International Journal on Artificial Intelligence Tools, 27(07), 1860010. https://doi.org/10.1142/S0218213018600102.
16. Schuerkamp, R., Giabbanelli, P. J., & Daclin, N. (2023, May). Facilitating the Interoperability and Reuse of Extensions of Fuzzy Cognitive Maps. In 2023 Annual Modeling and Simulation Conference (ANNSIM) (pp. 708-719). IEEE.
17. Mkhitaryan, S., Giabbanelli, P., Wozniak, M. K., Nápoles, G., De Vries, N., & Crutzen, R. (2022). FCMpy: a python module for constructing and analyzing fuzzy cognitive maps. PeerJ Computer Science, 8, e1078. https://doi.org/10.7717/peerj-cs.1078.
18. Smarandache, F. (2000). Neutrosophy. arXiv preprint math/0010099.
19. Kandasamy, W. V., & Smarandache, F. (2003). Fuzzy cognitive maps and neutrosophic cognitive maps. Infinite Study.
20. Zafar, A., & Wajid, M. A. (2019). Neutrosophic cognitive maps for situation analysis. Neutrosophic Sets and Systems , 29-78.
21. Alava, R. P., Murillo, J. M., Zambrano, R. B., Vélez, M. I. Z., & Vazquez, M. L. (2018). PEST Analysis Based on Neutrosophic Cognitive Maps: A Case Study for Food Industry. Neutrosophic Sets and Systems, 21, 84-92.
22. Palacios, A. J. P., Ricardo, J. E., Piza, I. A. C., & Herrería, M. E. E. (2021). Phenomenological hermeneutical method and neutrosophic cognitive maps in the causal analysis of transgressions against the homeless. Neutrosophic sets and systems, 44, 147-156.
23. Vasantha, W. B., Kandasamy, I., Devvrat, V., & Ghildiyal, S. (2019). Study of imaginative play in children using neutrosophic cognitive maps model. Neutrosophic Sets and Systems, 30, 241–252.
24. Murugesan, R., Parthiban, Y., Devi, R., Mohan, K. R., & Kumarave, S. K. (2023). A Comparative Study of Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps on Covid Variants. Neutrosophic Sets and Systems, 55(1), 20.
25. Obbineni, J., Kandasamy, I., Vasantha, W. B., & Smarandache, F. (2023). Combining SWOT analysis and neutrosophic cognitive maps for multi-criteria decision making: a case study of organic agriculture in India. Soft Computing, 1-22. https://doi.org/10.1007/s00500-023-08097-w.
26. Al-subhi, S. H., Papageorgiou, E. I., Pérez, P. P., Mahdi, G. S. S., & Acuña, L. A. (2021). Triangular neutrosophic cognitive map for multistage sequential decision-making problems. International Journal of Fuzzy Systems, 23, 657-679. https://doi.org/10.1007/s40815-020-01014-5.
27. Wajid, M. S., Terashima-Marin, H., Paul Rad, P. N., & Wajid, M. A. (2022). Violence detection approach based on cloud data and Neutrosophic cognitive maps. Journal of Cloud Computing, 11(1), 1-18. https://doi.org/10.1186/s13677-022-00369-4.
28. Chithra, B., & Nedunchezhian, R. (2022). Dynamic neutrosophic cognitive map with improved cuckoo search algorithm (DNCM-ICSA) and ensemble classifier for rheumatoid arthritis (RA) disease. Journal of King Saud University-Computer and Information Sciences, 34(6), 3236-3246. https://doi.org/10.1016/j.jksuci.2020.06.011.
29. Bhutani, K., Gaur, S., Panwar, P., & Garg, S. (2023, February). A Neutrosophic Cognitive Maps Approach for Pestle Analysis in Food Industry. In International Conference On Emerging Trends In Expert

Applications & Security (pp. 349-360). Singapore: Springer Nature Singapore. https://doi.org/10.1007/978-981-99-1909-3_30.
30. Karadayi-Usta, S. (2022). The Role of the Paper Packaging Industry in the Circular Economy: The Causal Relationship Analysis via Neutrosophic Cognitive Maps. In Handbook of Research on Advances and Applications of Fuzzy Sets and Logic (pp. 605-618). IGI Global. https://doi.org/10.4018/978-1-7998-7979-4.ch027.
31. Rojas, H. E., Nelson, F. S., & Cabrita Marina, M. (2022). Neutrosophic Cognitive Maps for Violence Cause Analysis. International Journal of Neutrosophic Science (IJNS), 19(1).
32. Rodríguez, J. O. C., Veloz, Á. P. M., & Andi, S. T. T. (2022). Neutrosophic Cognitive Maps for the Analysis of the Factors in the proper Diagnosis of Conversion Disorder. Neutrosophic Sets and Systems, 52(1), 36.
33. Raúl, D. C., Fabricio, T. T., & Elizabeth, D. P. F. (2022). Analyzing the impact of social factors on homelessness with neutrosophic cognitive maps. International Journal of Neutrosophic Science (IJNS), 19(1).
34. Priya, R., & Martin, N. (2022). Neutrosophic Sociogram Approach to Neutrosophic Cognitive Maps in Swift Language. Neutrosophic Sets and Systems, 49, 111-144.
35. Kumaravel, S. K., Murugesan, R., Nagaram, N. B., Rasappan, S., & Yamini, G. (2023). Applications of fuzzy cognitive maps and neutrosophic cognitive maps on analysis of dengue fever. Fuzzy Logic Applications in Computer Science and Mathematics, 249-265. https://doi.org/10.1002/9781394175130.ch16.

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On b-anti-Open Sets: A Formal Definition, Proofs, and Examples

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Abstract: The concepts of open sets, closed sets, the interior of a set, and the exterior of a set are the most basic concepts in the study of topological spaces in any setting. When we turn our attention to the concept of anti-topological spaces, we encounter analogous fundamental concepts, such as the definition of anti-open sets, anti-closed sets, anti-interior, anti-exterior, etc. These concepts have already been introduced and studied by mathematicians worldwide. In this article, we introduce and study the concepts of b-anti-open set, b-anti-closed set, anti-b-interior, and anti-b-closure in the context of anti-topological spaces and investigate some of their basic properties.

Keywords: b-anti-Open Set; b-anti-Closed Set; b-anti-Interior; b-anti-Closure.

1. Introduction

In the age of artificial intelligence (AI), decision-making assumes a pivotal role within this technological landscape. AI technologies like cognitive computing and machine learning have the capacity to enhance the decision-making process by scrutinizing extensive data sets, identifying patterns, and suggesting the most advantageous solutions. These capabilities prove invaluable for decision-makers grappling with intricate situations, be it in the realm of medical diagnosis or strategic planning.

Many mathematicians from around the world are actively engaged in the development of decision-making theories utilizing the concept of neutrosophic logic. Haque et al. [13, 16] have adeptly employed neutrosophic logic in the formulation of decision-making theories. Furthermore, recent research by Banik et al. [14, 15, 17] has leveraged both fuzzy logic and neutrosophic logic in various modeling applications within the field of agriculture science. Neutrosophic logic also proves valuable in medical science, as exemplified by its application in [17] and several other studies.

Since the introduction of neutrosophic logic in 1995 by Florentin Smarandache [18], along with the subsequent development of neutrosophic topological spaces, various applications of neutrosophic theories have emerged in the literature. Similarly, with the theoretical advancement of anti-topological spaces and anti-algebra, we anticipate similar applications in the near future. Thus, we are also motivated to delve into the study of anti-topological spaces and their associated concepts with the aim of yielding future benefits.

In the year 2021, Şahin et al. [11] introduced the notion of anti-topological spaces. Subsequently, Witczak [12] conducted a comprehensive study on anti-topological spaces, providing valuable insights into the emerging field. In that work, the author introduced the concepts of anti-interior and anti-closure of a set, accompanied by a thorough examination of various properties associated with these notions. Furthermore, the author defined anti-dense sets and anti-nowhere-dense sets, shedding light on their essential properties. Additionally, the concept of anti-continuity was explored within this framework.

Over the years, researchers have introduced and investigated a multitude of open and closed sets [1, 2, 3, 4, 5, 7, 8, 9, 10] within various settings. Witczak [12] extended this line of research by introducing anti-semi-open sets, pseudo-anti-open sets, and anti-genuine sets. More recently, Khaklary and Ray [6] introduced and studied a diverse range of open sets, including anti-pre-open sets, anti-pre-closed sets, regular open sets, regular closed sets, α-open sets, α-closed sets, and more, in the context of anti-topological spaces.

In this article, we further advance the field by introducing the novel concepts of b-anti-open sets and b-anti-closed sets within the realm of anti-topological spaces. We delve into a comprehensive study of their properties, offering fresh insights into this intriguing domain. Figure 1 presents the flowchart of the proposed work.

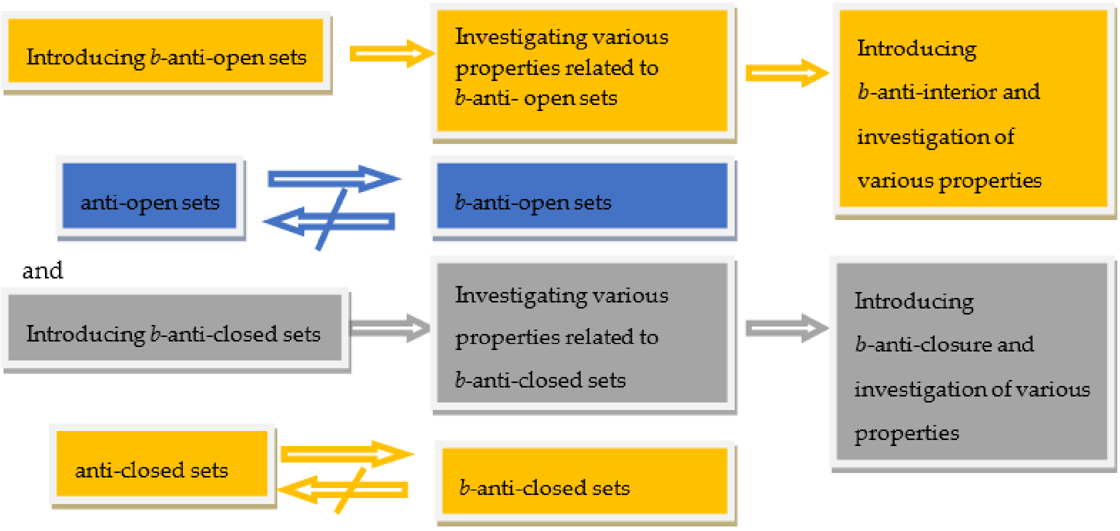

Figure 1. Flowchart of the proposed work.

2. Preliminaries

Definition 2.1: [11] Let X be a non-empty universe and τ be a collection of subsets of X. Then τ is called an anti-topology on X and (X, τ) is called an anti-topological space if the following three conditions are satisfied.

(i) $\varphi, X \notin \tau$
(ii) For all $q_1, q_2, \dots q_n \in \tau$, then $\bigcap_{i=1}^{n} q_i \notin \tau$ when any n is finite.
(iii) For all $q_1, q_2, \dots q_n \in \tau$, $\bigcup_{i \in I} q_i \notin \tau$.

Definition 2.2: [12] Let X be a non-empty universe and τ be a collection of subsets of X. We say (X, τ) is an anti-topological space if the following conditions are satisfied.

(i) $\varphi, X \notin \tau$
(ii) For any $n \in \mathbb{N}$, if $A_1, A_2, \dots A_n \in \tau$, then $\bigcap_{i=1}^{n} A_i \notin \tau$ (with the assumption that the sets in question are not all identical, i.e. the intersection is non-trivial).
(iii) For any collection $\{A_i\}_{i \in J \neq \varphi}$ such that $A_i \in \tau$ for each $i \in J$, $\bigcup_{i \in J} A_i \notin \tau$(with the assumption that the sets in question are not all identical, i.e. the union is non-trivial).

The elements of τ are called anti-open sets, while their complements are anti-closed sets. The set of all anti-closed sets will be denoted by τ_{Cl}. We say that every anti-topology is anti-closed under finite intersections and arbitrary unions (this refers respectively to condition (ii) and condition (iii) above). It is assumed that the property of being anti-closed refers only to non-trivial unions or intersections. The notion of non-trivial family is used to speak about those families of sets which contain at least two (different) sets.

Definition 2.3: [12] Let (X, τ) be an anti-topological space and $A \subseteq X$. Then anti-interior of A, denoted by $aInt(A)$, is defined as $aInt(A) = \cup\{U: U \subseteq A \text{ and } U \in \tau\}$.

Definition 2.4: [12] Let (X, τ) be an anti-topological space and $A \subseteq X$. Then anti-closure of A, denoted by $aCl(A)$, is defined as $aCl(A) = \cap\{F: A \subseteq F \text{ and } A \in \tau_{Cl}\}$.

Theorem 2.1: Let (X, τ) be an anti-topological space and $A, B \subseteq X$. Then the following hold:

(i) $aInt(A) \subseteq A$
(ii) If $A \in \tau$ then $aInt(A) = A$
(iii) $A \subseteq B$ then $aInt(A) \subseteq aInt(B)$
(iv) $aInt(aInt(A)) = aInt(A)$
(v) $A \subseteq aCl(A)$
(vi) If A is an anti-closed set then $aCl(A) = A$
(vii) $A \subseteq B$ then $aCl(A) \subseteq aCl(B)$
(viii) $aCl(aCl(A)) = aCl(A)$

Definition 2.5: [6] Let (X, τ) be an anti-topological space and $A \subseteq X$. Then A will be called an anti-pre-open set if $A \subseteq aInt(aCl(A))$.

Definition 2.6: [12] Let (X, τ) be an anti-topological space and $A \subseteq X$. Then A will be called an anti-semi-open set if $A \subseteq aCl(aInt(A))$.

3. b-anti-open sets

Definition 3.1: Let (X, τ) be an anti-topological space. A subset A of X will be called a b-anti-open set iff $A \subseteq aInt(aCl(A)) \cup aCl(aInt(A))$.

Example 3.2:

(i) Let $X = \{1,3,5,7,9\}$, $\tau = \{\{3\}, \{1,5,7\}, \{7,9\}\}$. Clearly (X, τ) is an anti-topological space and $\tau_{Cl} = \{\{1,5,7,9\}, \{3,9\}, \{1,3,5\}\}$. Let us take $A = \{1,5,7\} \subseteq X$. Now, $aInt(aCl(A)) \cup aCl(aInt(A)) = aInt(\{1,5,7,9\}) \cup aCl(\{1,5,7\}) = \{1,5,7,9\} \cup \{1,5,7,9\} = \{1,5,7,9\}$. Therefore, $A \subseteq aInt(aCl(A)) \cup aCl(aInt(A))$, i.e., A is a b-anti-open set.

(ii) Let $X = \{1,3,5,7,9\}$, $\tau = \{\{3\}, \{1,5,7\}, \{7,9\}\}$. Clearly, (X, τ) is an anti-topological space and $\tau_{Cl} = \{\{1,5,7,9\}, \{3,9\}, \{1,3,5\}\}$. Let $A = \{1,3,5\} \subseteq X$. Then $aInt(aCl(A)) \cup aCl(aInt(A)) = aInt(\{1,3,5\}) \cup aCl(\{3\}) = \{3\} \cup \{3\} = \{3\}$. Clearly, $A \nsubseteq aInt(aCl(A)) \cup aCl(aInt(A))$. So, A is not a b-anti-open set.

Proposition 3.1: In an anti-topological space, every anti-open set is a b-anti-open set.

Proof: Let (X, τ) be an anti-topological space and $A \subseteq X$ such that A is anti-open. Since A is anti-open, so we have, $A \in \tau \Rightarrow aInt(A) = A$. Now, we have $A \subseteq aCl(A) \Rightarrow aInt(A) \subseteq aInt(aCl(A)) \Rightarrow A \subseteq aInt(aCl(A)) \Rightarrow A \subseteq aInt(aCl(A)) \cup aCl(aInt(A)) \Rightarrow A$ is a b-anti-open set. Thus, every anti-open set is a b-anti-open set.

Remark 3.1: Converse of the prop. 3.1 is not true. We establish it by the following counterexample. Let $X = \{1,2,3,4,5\}$, $\tau = \{\{1\}, \{4\}, \{2,3\}, \{3,5\}\}$. Clearly (X, τ) is an anti-topological space and the anti-closed sets of X are $\{2,3,4,5\}, \{1,2,3,5\}, \{1,4,5\}, \{1,2,4\}$. Let us take $A = \{2,3,4\} \subseteq X$. Clearly, A is not an anti-open set. Now $aInt(aCl(A)) \cup aCl(aInt(A)) = aInt(\{2,3,4,5\}) \cup aCl(\{2,3,4\}) = \{2,3,4,5\} \cup \{2,3,4,5\} = \{2,3,4,5\}$. Clearly, $A \subseteq aInt(aCl(A)) \cup aCl(aInt(A))$ and so, A is a b-anti-open set. Thus A is not a b-anti-open set but not an anti-open set.

Proposition 3.2: In an anti-topological space, union of an arbitrary number of b-anti-open sets is a b-anti-open set.

Proof: Let (X,τ) be an anti-topological space and $\{A_i: i \in \Delta\}$ be an arbitrary collection of b-anti-open sets in X where Δ is an index set. Let $x \in \bigcup_{i\in\Delta} A_i \Rightarrow x \in A_k$, for some $k \in \Delta$. Since A_k is a b-anti-open set, so $A_k \subseteq aInt(aCl(A_k)) \cup aCl(aInt(A_k))$ and so, $x \in aInt(aCl(A_k)) \cup aCl(aInt(A_k))$. Now $A_k \subseteq \bigcup_{i\in\Delta} A_i \Rightarrow aCl(A_k) \subseteq aCl(\bigcup_{i\in\Delta} A_i) \Rightarrow aInt(aCl(A_k)) \subseteq aInt(aCl(\bigcup_{i\in\Delta} A_i))$. Similarly, $aCl(aInt(A_k)) \subseteq aCl(aInt(\bigcup_{i\in\Delta} A_i))$. Therefore, $A_k \subseteq aInt(aCl(A_k)) \cup aCl(aInt(A_k)) \subseteq aInt(aCl(\bigcup_{i\in\Delta} A_i)) \cup aCl(aInt(\bigcup_{i\in\Delta} A_i)) \Rightarrow x \in aInt(aCl(\bigcup_{i\in\Delta} A_i)) \cup aCl(aInt(\bigcup_{i\in\Delta} A_i))$. This gives $\bigcup_{i\in\Delta} A_i \subseteq aInt(aCl(\bigcup_{i\in\Delta} A_i)) \cup aCl(aInt(\bigcup_{i\in\Delta} A_i))$, i.e., $\bigcup_{i\in\Delta} A_i$ is a b-anti-open set. Hence proved.

Remark 3.2: In an anti-topological space, intersection of two b-anti-open sets may not be a b-anti-open set.
Let $X = \{1,2,3,4,5\}, \tau = \{\{1\},\{4\},\{2,3\},\{3,5\}\}$. Clearly (X, τ) is an anti-topological space and the anti-closed sets of X are $\{2,3,4,5\},\{1,2,3,5\},\{1,4,5\},\{1,2,4\}$. Let us consider the subsets $A = \{2,3,4\}$ and $B = \{2,4,5\}$ of X. Obviously A and B are b-anti-open sets. Now $A \cap B = \{2,4\}$ and $aInt(aCl(A \cap B)) \cup aCl(aInt(A \cap B)) = aInt(\{2,4\}) \cup aCl(\{4\}) = \{4\} \cup \{4\} = \{4\}$. Therefore, $A \cap B \nsubseteq aInt(aCl(A \cap B)) \cup aCl(aInt(A \cap B))$, i.e., $A \cap B$ is not a b-anti-open set.

Proposition 3.3: In an anti-topological space,
(i) Every anti-pre-open set is a b-anti-open set.
(ii) Every anti-semi-open set is a b-anti-open set.

Proof:
(i) Let (X,τ) be an anti-topological space and let A be an anti-pre-open subset of X. Then $A \subseteq aInt(aCl(A)) \Rightarrow A \subseteq aInt(aCl(A)) \cup aCl(aInt(A)) \Rightarrow A$ is a b-anti-openset.
(ii) Let (X,τ) be an anti-topological space and let A be an anti-semi-open subset of X. Then $A \subseteq aCl(aInt(A)) \Rightarrow A \subseteq aInt(aCl(A)) \cup aCl(aInt(A)) \Rightarrow A$ is a b-anti-open set.

Definition3.2: Let (X,τ) be an anti-topological space. A subset A of X will be called a b-anti-closed set if $aInt(aCl(A)) \cap aCl(aInt(A)) \subseteq A$.

Example 3.1:
(i) Let $X = \{a,b,c,d,e\}$. Clearly, $\tau = \{\{a\},\{b,c\},\{c,d,e\}\}$ is an anti-topology for X and $\tau_{Cl} = \{\{b,c,d,e\},\{a,d,e\},\{a,b\}\}$. Let us take $A = \{a,b\} \subseteq X$. Then $aInt(aCl(A)) \cap aCl(aInt(A)) = \{a\} \subseteq A$. Therefore, A is a b-anti-closed set.
(ii) Let $X = \{1,2,3,4,5\}$. Clearly, $\tau = \{\{1\},\{4\},\{2,3\},\{3,5\}\}$ is an anti-topology for X and $\tau_{Cl} = \{\{1,2,3,5\},\{1,4,5\},\{1,2,4\}\}$. Let us take $B = \{1,3,5\} \subseteq X$. Then $aInt(aCl(B)) \cap aCl(aInt(B)) = \{1,2,3,5\} \nsubseteq B$. Therefore, B is not a b-anti-closed set.

Proposition 3.4: Let (X,τ) be an anti-topological space and $A \subseteq X$. Then A is a b-anti-open set iff A^c is a b-anti-closed set.

Proof: A is a b-anti-open set

$$\Leftrightarrow A \subseteq aInt(aCl(A)) \cup aCl(aInt(A))$$
$$\Leftrightarrow A^c \supseteq [aInt(aCl(A)) \cup aCl(aInt(A))]^c$$
$$\Leftrightarrow A^c \supseteq [aInt(aCl(A))]^c \cap [aCl(aInt(A))]^c$$
$$\Leftrightarrow A^c \supseteq [aCl(aCl(A))^c] \cap [aInt(aInt(A))^c]$$
$$\Leftrightarrow A^c \supseteq aCl(aInt(A^c)) \cap aInt(aCl(A^c))$$
$$\Leftrightarrow A^c \text{ is a } b\text{-anti-closed set. Hence proved.}$$

Proposition 3.5: In an anti-topological space, every anti-closed set is a b-anti-closed set.

Proof: Let (X, τ) be an anti-topological space and $A \subseteq X$ such that A is anti-closed. Then A^c is anti-open set and from the proposition 3.1, it follows that A^c is a b-anti-open set. Therefore, by the proposition 3.4, A is a b-anti-closed set. Hence proved.

Remark 3.3: Converse of the prop. 3.5 is not true. We establish it by the following counterexample. Let $X = \{1, 2, 3, 4, 5\}$. Clearly, $\tau = \{\{1\}, \{4\}, \{2, 3\}, \{3, 5\}\}$ is an anti-topology for X and $\tau_{Cl} = \{\{1, 2, 3, 5\}, \{1, 4, 5\}, \{1, 2, 4\}\}$. Let us take $A = \{1,5\} \subseteq X$. Obviously A is not an anti-closed set. Now $aInt(aCl(A)) \cap aCl(aInt(A)) = \{1\} \subseteq A$. Therefore, A is a b-anti-closed set. Thus A is a b-anti-closed set but not an anti-closed set.

Proposition 3.6: In an anti-topological space,

(i) Every anti-pre-closed set is a b-anti-closed set.
(ii) Every anti-semi-closed set is b-anti-closed set.

Proof:

(i) Let A be an anti-pre-closed subset of X. Then $aCl(aInt(A)) \subseteq A \Rightarrow aCl(aInt(A)) \cap aInt(aCl(A)) \subseteq A \Rightarrow A$ is a b-anti-closed set.
(ii) Let A be an anti-semi-closed subset of X. Then $aInt(aCl(A)) \subseteq A \Rightarrow aInt(aCl(A)) \cap aInt(aCl(A)) \subseteq A \Rightarrow A$ is a b-anti-closed set.

Proposition 3.7: In an anti-topological space, intersection of arbitrary number of b-anti-closed sets is b-anti-closed.
Proof: Let (X, τ) be an anti-topological space and $\{A_i : i \in \Delta\}$ be an arbitrary collection of b-anti-closed sets in X where Δ is an index set. Then ${A_i}^c$ is a b-anti-open set for each $i \in \Delta \Rightarrow \bigcup_{i\in\Delta} {A_i}^c$ is a b-anti-open set [by prop.3.2] $\Rightarrow (\bigcap_{i\in\Delta} A_i)^c$ is a b-anti-open set $\Rightarrow \bigcap_{i\in\Delta} A_i$ is an$b$-anti-closed set. Hence proved.

Remark 3.4: In an anti-topological space, union of two b-anti-closed sets may not be a b-anti-closed set. We establish it by the following counterexample:
Let $X = \{1,2,3,4,5\}$. Clearly, $\tau = \{\{1\}, \{4\}, \{2, 3\}, \{3, 5\}\}$ is an anti-topology for X and $\tau_{Cl} = \{\{1, 2, 3, 5\}, \{1, 4, 5\}, \{1, 2, 4\}\}$. Let us take $A = \{1, 5\}$and $B = \{1, 3\} \subseteq X$. Clearly, A and B are two b-anti-closed sets in X. Now $A \cup B = \{1, 3, 5\}$ and $aInt(aCl(A \cup B)) \cap aCl(aInt(A \cup B)) = \{1, 2, 3, 5\} \nsubseteq A \cup B$. Therefore, $A \cup B$ is not a b-anti-closed set. Thus, the union of two b-anti-closed sets may not be a b-anti-closed set.

Definition 3.3: Let (X, τ) be an anti-topological space and $A \subseteq X$. Then the b-anti-interior of A, denoted by $b - aInt(A)$, is defined as $b - aInt(A) = \bigcup\{G: G \text{ is } b - \text{anti} - \text{open set in } X \text{ and } G \subseteq A\}$.

Proposition 3.8: Let (X, τ) be an anti-topological space and $A \subseteq X$. Then the following hold:

(i) $b - aInt(A)$ is a b-anti-open set.
(ii) $b - aInt(A) \subseteq A$.
(iii) A is b-anti-open set iff $b - aInt(A) = A$.
(iv) $b - aInt(b - aInt(A)) = b - aInt(A)$.

Proof:

(i) Since $b - aInt(A) = \bigcup\{G: G \text{ is } b - \text{anti} - \text{open set in } X \text{ and } G \subseteq A\}$ and union of arbitrary number of b-anti-open sets is a b-open set, so $b - aInt(A)$ is a b-anti-open set.
(ii) Since $b - aInt(A)$ is the union of all b-anti-open sets contained in A, so $b - aInt(A) \subseteq A$.
(iii) Let A be a b-anti-open set. Since $b - aInt(A)$ is the union of all b-anti-open sets which are contained in A and since, A is a b-anti-open set contained in A, so $A \subseteq b - aInt(A)$. Also from (ii), $b - aInt(A) \subseteq A$. Therefore, $b - aInt(A) = A$. Conversely let $b - aInt(A) = A$. Since $b - aInt(A)$ is a b-anti-open set, so A is also a b-anti-open set.

(iv) Since $b-aInt(A)$ is a b-anti-open set, so by (iii), $b-aInt\big(b-aInt(A)\big) = b-aInt(A)$.

Proposition 3.9: Let (X,τ) be an anti-topological space and A,B be subsets of X. Then the following hold:

(i) $A \subseteq B \Rightarrow b-aInt(A) \subseteq b-aInt(B)$.

(ii) $b-aInt(A\cup B) \supseteq b-aInt(A) \cup b-aInt(B)$.

(iii) $b-aInt(A\cap B) \subseteq b-aInt(A) \cap b-aInt(B)$.

Proof:

(i) We have $x \in b-aInt(A) \Rightarrow x \in \bigcup\{G: G \text{ is } b-\text{anti}-\text{open set in } X \text{ and } G \subseteq A\} \Rightarrow x \in \bigcup\{G: G \text{ is } b-\text{anti}-\text{open set in } X \text{ and } G \subseteq B\}$ $(\because A \subseteq B) \Rightarrow x \in b-aInt(B)$.

(ii) $A \subseteq A\cup B \Rightarrow b-aInt(A) \subseteq b-aInt(A\cup B)$. Similarly, $b-aInt(B) \subseteq b-aInt(A\cup B)$. Therefore, $b-aInt(A\cup B) \supseteq b-aInt(A) \cup b-aInt(B)$.

(iii) $A\cap B \subseteq A \Rightarrow b-aInt(A\cap B) \subseteq b-aInt(A)$. Similarly, $b-aInt(A\cap B) \subseteq b-aInt(B)$. Therefore, $b-aInt(A\cap B) \subseteq b-aInt(A) \cap b-aInt(B)$.

Definition 3.4: Let (X,τ) be an anti-topological space and $A \subseteq X$. Then the b-anti-closure of A, denoted by $b-aCl(A)$, is defined as $b-aCl(A) = \bigcap\{G: G \text{ isa } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\}$.

Proposition 3.10: Let (X,τ) be an anti-topological space and $A \subseteq X$. Then the following hold:

(i) $b-aCl(A)$is a $b-\text{anti}-\text{closed}$ set.

(ii) $A \subseteq b-aCl(A)$.

(iii) A is b-anti-closed set iff $b-aCl(A) = A$.

(iv) $b-aCl\big(b-aCl(A)\big)$= $b-aCl(A)$.

Proof:

(i) Since $b-aCl(A) = \bigcap\{G: G \text{ is a } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\}$ and intersection of arbitrary number of b-anti-closed sets is a b-anti-closed, so $b-aCl(A)$ is a b-anti-closed set.

(ii) Since $b-aCl(A)$ is the intersection of all b-anti-closed sets containing$A$, so $A \subseteq b-aCl(A)$.

(iii) Let A be a b -anti-closed set. Since $b-aCl(A) = \bigcap\{G: G \text{ is a } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\}$ and since A is also a b-anti-closed set so, $A \in \{G: G \text{ is a } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\}$ and therefore, $b-aCl(A) \subseteq A$. Also from (ii), $A \subseteq b-aCl(A)$. Hence, $b-aCl(A) = A$. Conversely, suppose that $b-aCl(A) = A$. Since $b-aCl(A)$ is a b-anti-closed set, so A is also a b-anti-closed set.

(iv) Since $b-aCl(A)$ is a b-anti-closed, so by (iii), $b-aCl\big(b-aCl(A)\big)$= $b-aCl(A)$.

Proposition 3.11: Let (X,τ) be an anti-topological space and A,B be subsets of X. Then the following hold.

(i) $A \subseteq B \Rightarrow b-aCl(B) \subseteq b-aCl(A)$.

(ii) $b-aCl(A\cup B) \subseteq b-aCl(A) \cup b-aCl(B)$.

(iii) $b-aCl(A\cap B) \supseteq b-aCl(A) \cap b-aCl(B)$

Proof:

(i) We have $x \in b-aCl(B) \Rightarrow x \in \bigcap\{G: G \text{ is } b-\text{anti}-\text{closed set in } X \text{ and } B \subseteq G\} \Rightarrow x \in \bigcap\{G: G \text{ is } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\}$ $(\because A \subseteq B) \Rightarrow x \in b-aCl(A)$. Hence $b-aCl(B) \subseteq b-aCl(A)$.

(ii) $A \subseteq A\cup B \Rightarrow b-aCl(A\cup B) \subseteq b-aCl(A)$. Similarly,$b-aCl(A\cup B) \subseteq b-aCl(B)$. Therefore, $b-aCl(A\cup B) \subseteq b-aCl(A) \cup b-aCl(B)$.

(iii) $A\cap B \subseteq A \Rightarrow b-aCl(A) \subseteq b-aCl(A\cap B)$. Similarly, $b-aCl(B) \subseteq b-aCl(A\cap B)$. Therefore, $b-aCl(A\cap B) \supseteq b-aCl(A) \cap b-aCl(B)$.

Proposition 3.12: Let (X,τ) be an anti-topological space and A,B be two subsets of X. Then the following hold:

(i) $\big(b-aCl(A)\big)^c = b-aInt(A^c)$.

(ii) $b-aCl(A^c) = \big(b-aInt(A)\big)^c$

Proof:

(i) We have $x \in \big(b-aCl(A)\big)^c$

$\Rightarrow x \in (\bigcap\{G{:}\, G \text{ is a } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G\})^c$

$\Rightarrow x \in \bigcup\{G^c : G^c \text{ is a } b-\text{anti}-\text{open set in } X \text{ and } G^c \subseteq A^c\}$

$\Rightarrow x \in b-aInt(A^c)$.

Hence, $\big(b-aCl(A)\big)^c \subseteq b-aInt(A^c)$.

Again, let $x \in b-aInt(A^c)$

$\Rightarrow x \in \bigcup\{G{:}\, G \text{ is } b-\text{anti}-\text{open set in } X \text{ and } G \subseteq A^c\}$

$\Rightarrow x \in (\bigcap\{G^c : G^c \text{ is } b-\text{anti}-\text{closed set in } X \text{ and } A \subseteq G^c\})^c$

$\Rightarrow x \in \big(b-aCl(A)\big)^c$.

Therefore, $b-aInt(A^c) \subseteq \big(b-aCl(A)\big)^c$.

and so, $b-aInt(A^c) = \big(b-aCl(A)\big)^c$.

(ii) Replacing A by A^c in (i), we get $\big(b-aCl(A^c)\big)^c$=$b-aInt((A^c)^c)$

$\Rightarrow\big(b-aCl(A^c)\big)^c$=$b-aInt(A)\Rightarrow b-aCl(A^c) = \big(b-aInt(A)\big)^c$.

4. Discussion

In this study, we explored the intricate relationships within anti-topological spaces, shedding light on the properties of *b*-anti-open sets and *b*-anti-closed sets. Notably, the observation that every anti-open (resp. anti-closed) set is a *b*-anti-open (resp. *b*-anti-closed) set suggests a broader characterization of *b*-anti-open sets (*b*-anti-closed sets). The exploration also uncovered nuanced aspects, such as the non-preservation of the *b*-anti-open property under the intersection of two *b*-anti-open sets, challenging conventional notions. Counterexamples, particularly the non-closure of unions of *b*-anti-closed sets, highlighted the counterintuitive nature of these spaces, prompting careful consideration in their analysis. The study further revealed intriguing properties regarding closure and interior operations. The observed reversal of conventional inclusions in the closure operation introduces a noteworthy departure from typical topological expectations. Unlike the standard relationship where the closure of a subset is contained within the closure of its superset, our findings reveal a reversal: if A is a subset of B then the *b*-anti-closure of B is a subset of the *b*-anti-closure of A. This counterintuitive result challenges the conventional understanding of closure operations and prompts a reevaluation of the underlying principles governing these relationships. Similarly, the outcomes concerning the closure of unions and intersections add another layer of complexity. These findings deepen our understanding of anti-topological spaces, revealing their complexities and inviting further exploration into their properties and applications.

5. Conclusion

In this article, we have introduced the concepts of b-anti-open sets and b-anti-closed sets in connection with anti-topological spaces and then explored their fundamental properties. Furthermore, we have defined the b-anti-interior and b-anti-closure of a set, delving into an in-depth analysis of their associated properties. From the above discussion, we have found that classes of b-anti-open sets and b-anti-closed sets in anti-topological spaces are finer than classes of anti-open sets and anti-closed sets, respectively. Also, the deviations from standard topological expectations signify the unique characteristics of the anti-topological space under consideration.

As we move forward, our future research endeavors will aim to investigate novel concepts and ideas related to anti-topological spaces. We anticipate that the insights presented in this article will contribute to the advancement of various facets within the field of anti-topological spaces, aiding researchers in their exploration and development of this intriguing domain.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Azad, K. K. (1981). On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity. Journal of mathematical analysis and applications, 82(1), 14-32. https://doi.org/10.1016/0022-247X(81)90222-5.
2. Alabdulsada, L. M. H. (2019). On B-Open Sets. European Journal of Pure and Applied Mathematics, 12(2), 358–369. https://doi.org/10.29020/nybg.ejpam.v12i2.3401.
3. Benchalli, S. S., & Karnel, J. J. (2010). On fuzzy b-open sets in fuzzy topological spaces. Journal of Computer and Mathematical Sciences, 1(2), 103-273.
4. Ebenanjar, P. E., Immaculate, H. J., & Wilfred, C. B. (2018, December). On neutrosophic b-open sets in neutrosophic topological space. In Journal of Physics: Conference Series (Vol. 1139, No. 1, p. 012062). IOP Publishing. https://doi.org/10.1088/1742-6596/1139/1/012062.
5. Iswarya, P., & Bageerathi, K. (2016). On neutrosophic semi-open sets in neutrosophic topological spaces. International Journal of Math. Trends and Tech., 37(3), 214-223. https://doi.org/10.14445/22315373/IJMTT-V37P529.
6. Khaklary, J. K., & Ray, G. C. (2023). A new class of NeutroOpen, NeutroClosed, AntiOpen and AntiClosed sets in NeutroTopological and AntiTopological spaces. International Journal of Neutrosophic Science (IJNS), 20(2). https://doi.org/10.54216/IJNS.200206.
7. Levine, N. (1963). Semi-open sets and semi-continuity in topological spaces. The American mathematical monthly, 70(1), 36-41. https://doi.org/10.1080/00029890.1963.11990039.
8. Mashhour, A. S. (1982). On precontinuous and weak precontinuous mappings. In Proc. Math. Phys. Soc. Egypt. (Vol. 53, pp. 47-53). https://doi.org/10.1080/00029890.1963.11990039.
9. Rao, V. V., & Rao, Y. S. (2017). Neutrosophic Pre-open sets and Pre-closed sets in Neutrosophic Topology. International Journal of Chem Tech Research, 10(10), 449-458.
10. Shahna, A. S. B. (1991). On fuzzy strong semicontinuity and fuzzy precontinuity. Fuzzy sets and systems, 44(2), 303-308. https://doi.org/10.1016/0165-0114(91)90013-G.
11. Şahin, M., Kargın, A., & Yücel, M. (2021). Neutro-topological space and anti-topological space. NeutroAlgebra Theory, 1, 16.
12. Witczak, T. (2022). Some remarks on anti-topological spaces. arXiv preprint arXiv:2202.10013, Asia Mathematika, 6(1), 23-34. https://doi.org/10.48550/arXiv.2202.10013.

13. Haque, T. S., Chakraborty, A., Alrabaiah, H., & Alam, S. (2022). Multiattribute decision-making by logarithmic operational laws in interval neutrosophic environments. Granular Computing, 7(4), 837-860. https://doi.org/10.1007/s41066-021-00299-7.
14. Banik, B., Alam, S., & Chakraborty, A. (2023). Comparative study between GRA and MEREC technique on an agricultural-based MCGDM problem in pentagonal neutrosophic environment. International Journal of Environmental Science and Technology, 1-16. https://doi.org/10.1007/s13762-023-04768-1.
15. Banik, B., Alam, S., & Chakraborty, A. (2023). A novel integrated neutrosophic cosine operator based linear programming ANP-EDAS MCGDM strategy to select anti-pegasus software. International Journal of Information Technology & Decision Making, 1-37. https://doi.org/10.1142/S0219622023500529.
16. Haque, T. S., Chakraborty, A., Mondal, S. P., & Alam, S. (2020). Approach to solve multi-criteria group decision-making problems by exponential operational law in generalised spherical fuzzy environment. CAAI Transactions on Intelligence Technology, 5(2), 106-114. https://doi.org/10.1049/trit.2019.0078.
17. Haque, T. S., Alam, S., & Chakraborty, A. (2022). Selection of most effective COVID-19 virus protector using a novel MCGDM technique under linguistic generalised spherical fuzzy environment. Computational and Applied Mathematics, 41(2), 84. https://doi.org/10.1007/s40314-022-01776-8.
18. Smarandache, F. (2010). Neutrosophic logic-a generalization of the intuitionistic fuzzy logic. Multispace & multistructure. Neutrosophic transdisciplinarity (100 collected papers of science), 4, 396.

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Some Special Refined Neutrosophic Ideals in Refined Neutrosophic Rings: A Proof-of-Concept Study

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Abstract: In this research, we created notions of a refined neutrosophic prime (completely prime, semiprime, and completely semiprime) ideal in a refined neutrosophic ring. If $R(I_1, I_2)$ is a refined neutrosophic ring, then each ideal of $R(I_1, I_2)$ has the form $J + KI_1 + LI_2$, where $J \subseteq L \subseteq K$ are ideals of the classical ring R. The objective of this work is to find the necessary and sufficient condition on classical ideals $J, L, and\ K$ that makes $J + KI_1 + LI_2$ a prime (completely prime, semiprime, and completely semiprime) ideal in $R(I_1, I_2)$. We studied some of the elementary properties of these concepts and the most important properties that link them.

We reached several results, the most important of which are as follows:

- If $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)} \Leftrightarrow J, K, and\ L \in S\wp_R$.
- $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, then $J, K, L \in \wp_R$.
- Assuming that $R(I_1, I_2)$ is a finite unity commutation, then $RN\mathcal{M}_{R(I_1,I_2)} = RN\wp_{R(I_1,I_2)}$.
- $R(I_1, I_2)$ is a refined neutrosophic field $\Leftrightarrow$ $\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)$ are only refined neutrosophic ideals in $R(I_1, I_2)$.
- We call $R(I_1, I_2)$ a refined neutrosophic prime ring if $RI_1 + RI_2 \in RN\wp_{R(I_1,I_2)}$ and a fully prime ring if $RN\mathfrak{T}_{R(I_1,I_2)} \backslash \{0\} = RN\wp_{R(I_1,I_2)}$.

Keywords: Refined Neutrosophic Ring; Refined Neutrosophic Ideal; Completely Semiprime; Fully Prime; Fully Semiprime.

1. Introduction

Neutrosophy is a broad view of intuitionistic fuzzy logic that represents a new development of fuzzy notions. This strategy has a fascinating impact on applied science [1, 2, 3, 4, 5]. Neutrosophy can be applied to algebraic structures as a new branch of philosophy, leading to a better understanding and evolution of these structures. Kandasamy and Smarandache presented the concept of neutrosophic groups, rings, and fields [6], which has been widely investigated [7, 8, 9, 10] and is still being studied. Numerous intriguing discoveries about neutrosophic rings have recently been discussed [11, 12, 13].

Adeleke et al. [14, 15] generalized neutrosophic sets by dividing the degree of indeterminacy I into two degrees of indeterminacy I_1, and I_2. This concept has been widely employed in algebra by analyzing refined neutrosophic rings [14, 15] and n-refined neutrosophic rings and modules [16, 17, 18], and many intriguing findings have been established [19]. Abobala [20] characterized the maximal and minimal ideals in a refined neutrosophic ring.

We present a characterization of refined neutrosophic prime (completely prime, semiprime, and completely semiprime) ideals by depending on the properties of classical ideals. This study aims to describe the structure and properties of prime, completely prime, semiprime, and completely semiprime ideals of refined neutrosophic rings.

Our motivation is to close an important research gap by determining all prime, completely prime, semiprime, and completely semiprime ideals and their properties in refined neutrosophic rings. This paper continues the work begun in "On Neutrosophic Prime, Completely Prime, Semiprime, and Completely Semiprime Ideals in Neutrosophic Ring."

2. Definitions and notations

Since most academics interested in the subject are already familiar with classical rings and their ideals, this section will focus on numerous definitions and major results relevant to refined neutrosophic rings and their ideals.

Definition 2.1: [14, 15] Let R be a ring, the collection $R(I_1, I_2) = \{a + bI\ ; a, b, c \in R$ and $I_1{}^2 = I_1, I_2{}^2 = I_2, I_1I_2 = I_2I_1 = I_1\}$ is called a refined neutrosophic ring. $R(I_1, I_2)$ is referred to as a refined neutrosophic field when R is a field.

Properties 2.2: [14, 15]

(i) R is a unity commutative ring iff $R(I_1, I_2)$ is a unity commutative refined neutrosophic ring.

(ii) $(I_1)^n = I_1$ *and* $(I_2)^n = I_2$ for each $n \in \mathbb{Z}^+$.

(iii) $aI_1 = I_1a$ *and* $aI_2 = I_2a\ \forall a \in R$.

(iv) $0I_1 = 0 = 0I_2$, $\underbrace{I_1 + I_1 + \cdots + I_1}_{n\ time} = nI_1$ *and* $\underbrace{I_2 + I_2 + \cdots + I_2}_{n\ time} = nI_2$.

Theorem 2.3: [20] If $R(I_1, I_2)$ is a refined neutrosophic ring, and $J + KI_1 + LI_2 \subseteq R(I_1, I_2)$, then $J + KI_1 + LI_2$ is a neutrosophic ideal iff $J, K, and\ L$ are ideals of R, where $J \subseteq L \subseteq K$.

Theorem 2.4: [20] If $R(I_1, I_2)$ is a refined neutrosophic ring, and $J + KI_1 + LI_2$ is an ideal of $R(I_1, I_2)$, then $J + KI_1 + LI_2$ is a neutrosophic maximal ideal iff J is a maximal of R, where $L = K = R\ or\ J + KI_1 + LI_2 = R(I_1, I_2)$.

3. Results

In a refined neutrosophic ring $R(I_1, I_2)$, we indicate by $RN\mathfrak{I}_{R(I_1,I_2)}$ is the set of refined neutrosophic ideals, $RN\wp_{R(I_1,I_2)}$ the set of refined neutrosophic prime ideals, $RNC\wp_{R(I_1,I_2)}$ the set of refined neutrosophic completely prime ideals, $RNS\wp_{R(I_1,I_2)}$ the set of refined neutrosophic semiprime ideals, $RNCS\wp_{R(I_1,I_2)}$ the collection of refined neutrosophic completely semiprime ideals, and $RN\mathcal{M}_{R(I_1,I_2)}$ the collection of refined neutrosophic maximal ideals. In addition, in classical ring R, we indicate by $\mathfrak{I}_R$ the collection of ideals, $\wp_R$ the collection of prime ideals, $C\wp_R$ the collection of completely prime ideals, $S\wp_R$ the collection of semiprime ideals, $SC\wp_R$ the collection of completely semiprime ideals, and $\mathcal{M}_R$ the collection of maximal ideals.

Definition 3.1: If $J + KI_1 + LI_2 \in RN\mathfrak{I}_{R(I_1,I_2)}$; $J \subseteq L \subseteq K$, then

(i) $J + KI_1 + LI_2$ is a refined neutrosophic semiprime ideal if the following condition is satisfied: $\forall J_1 + K_1I_1 + L_1I_2 \in RN\mathfrak{I}_{R(I_1,I_2)}; J_1 \subseteq L_1 \subseteq K_1;\ (J_1 + K_1I_1 + L_1I_2)^2 \subseteq J + KI_1 + LI_2 \Rightarrow J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$.

(ii) $J + KI_1 + LI_2$ is a refined neutrosophic completely semiprime ideal if the following condition is satisfied: $\forall a + bI_1 + cI_2 \in R(I_1, I_2);\ (a + bI_1 + cI_2)^2 \in J + KI_1 + LI_2 \Rightarrow a + bI_1 + cI_2 \in J + KI_1 + LI_2$.

(iii) $J + KI_1 + LI_2$ is a refined neutrosophic prime ideal if the following condition is satisfied:
$\forall J_1 + K_1I_1 + L_1I_2, J_2 + K_2I_1 + L_2I_2 \in RN\mathfrak{I}_{R(I_1,I_2)}; J_1 \subseteq L_1 \subseteq K_1\ and\ J_2 \subseteq L_2 \subseteq K_2;$

$$(J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$$
$$\Rightarrow\ J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2\ \ or\ J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$$

(iv) $J + KI_1 + LI_2$ is a refined neutrosophic completely prime ideal if the following condition is satisfied: $\forall a + bI_1 + cI_2\ and\ \ e + fI_1 + gI_2 \in R(I_1, I_2);\ (a + bI_1 + cI_2)(e + fI_1 + gI_2) \in J + KI$

$$\Rightarrow\ a + bI_1 + cI_2 \in J + KI_1 + LI_2\ \vee\ e + fI_1 + gI_2 \in J + KI_1 + LI_2$$

Theorem 3.2: If $J+KI_1+LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then $J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)} \Leftrightarrow J, K, and\ L \in S\wp_R$.

Proof.

Firstly, $\forall J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)}$. Now, suppose that $J_1, K_1, L_1 \in \mathfrak{T}_R$, where $J_1^2 \subseteq J, K_1^2 \subseteq K, and\ L_1^2 \subseteq L$. Subsequently, $J_1^2 \subseteq L \subseteq K\ and\ L_1^2 \subseteq K$.

We have $J_1+J_1I_1+J_1I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$, and we note

$$(J_1+J_1I_1+J_1I_2)^2 = J_1^2+(J_1^2+J_1^2+J_1^2+J_1^2+J_1^2)I_1+(J_1^2+J_1^2+J_1^2)I_2 \subseteq J+KI_1+LI_2$$

Since $J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)}$, so $J_1+J_1I_1+J_1I_2 \subseteq J+KI_1+LI_2$ and from which $J_1 \subseteq J$. Therefore, $J \in S\wp_R$.

On the other hand, $\{0\}+L_1 I_1+L_1 I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$, and we note $(\{0\}+L_1 I_1+L_1 I_2)^2 = \{0\}^2 + (\{0\}.L_1+L_1.\{0\}+L_1^2+L_1^2+L_1^2)I_2+(\{0\}.L_1+L_1.\{0\}+L_1^2)I_2 \subseteq J+KI_1+LI_2$

Since $J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)}$, so $\{0\}+L_1 I_1+L_1 I_2 \subseteq J+KI_1+LI_2$ and from which $L_1 \subseteq L$. Therefore, $L \in S\wp_R$.

And on the other hand, $\{0\}+K_1I_1+\{0\}I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$, and we note

$$(\{0\}+K_1I_1+\{0\}I_2)^2 = \{0\}^2+(\{0\}.K_1+K_1.\{0\}+K_1^2+K_1.\{0\}+\{0\}.K_1)I_2+(\{0\}^2+\{0\}^2+\{0\}^2)I_2 \subseteq J+KI_1+LI_2$$

Since $J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)}$, $\{0\}+K_1I_1+\{0\}I_2 \subseteq J+KI_1+LI_2$ and from which $K_1 \subseteq K$. Therefore, $K \in S\wp_R$.

Conversely, suppose that $J, K, L \in S\wp_R$.

Now, if $J_1+K_1I_1+L_1I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$; $J_1 \subseteq L_1 \subseteq K_1$, where

$$(J_1+K_1I_1+L_1I_2)^2 \subseteq J+KI_1+LI_2 \Rightarrow J_1^2+(J_1K_1+K_1J_1+K_1^2+K_1L_1+L_1K_1)I_1+(J_1L_1+L_1J_1+L_1^2)I_2 \subseteq J+KI_1+LI_2$$

Therefore, $J_1^2 \subseteq J\ and\ J_1K_1+K_1J_1+K_1^2+K_1L_1+L_1K_1 \subseteq K\ and\ J_1L_1+L_1J_1+L_1^2 \subseteq L$

Since $J_1^2 \subseteq J \subseteq L\ and\ J_1L_1+L_1J_1+L_1^2 \subseteq L \subseteq K$, so $J_1^2+J_1L_1+L_1J_1+L_1^2 = (J_1+L_1)^2 \subseteq L\ and\ J_1^2 + J_1K_1+K_1J_1+K_1^2+K_1L_1+L_1K_1+J_1L_1+L_1J_1+L_1^2 = (J_1+K_1+L_1)^2 \subseteq K$.

Since $J, K\ and\ L \in S\wp_R$, so $J_1 \subseteq J\ and\ J_1+L_1 \subseteq L \subseteq K\ and\ J_1+K_1+L_1 \subseteq K$.

Since $J_1 \subseteq J \subseteq L$, so $L_1 \subseteq L \subseteq K\ and\ K_1 \subseteq K$.

Therefore, $J_1+K_1I_1+L_1I_2 \subseteq J+KI_1+LI_2$. Thus $J+KI_1+LI_2 \in RNS\wp_{R(I_1,I_2)}$.

Theorem 3.3: If $J+KI_1+LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then $J+KI_1+LI_2 \in RNCS\wp_{R(I_1,I_2)} \Leftrightarrow J, K, L \in CS\wp_R$.

Proof.

Firstly, $\forall J+KI_1+LI_2 \in RNCS\wp_{R(I_1,I_2)}$.

Now, if $j, k, l \in R$, where $j^2 \in J \subseteq L\ and\ l^2 \in L \subseteq K\ and\ k^2 \in K$.

We have $j+jI_1+jI_2 \in R(I_1,I_2)$ and we note

$$(j+jI_1+jI_2)^2 = j^2+(j^2+j^2+j^2+j^2+j^2)I_1+(j^2+j^2+j^2)I_2 \subseteq J+KI_1+LI_2$$

Since $J+KI_1+LI_2 \in RNCS\wp_{R(I_1,I_2)}$, so $j+jI_1+jI_2 \in J+KI_1+LI_2$ and from which $j \in J$. Therefore, $J \in CS\wp_R$.

On the other hand, we have $0+l\,I_1+l\,I_2 \in R(I_1,I_2)$, and we note

$$(0+l\,I_1+l\,I_2)^2 = 0^2+(0.l+l.0+l^2+l^2+l^2)I_1+(0.l+l.0+l^2)I_2 \subseteq J+KI_1+LI_2$$

Since $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$, so $0 + l\,I_1 + l\,I_2 \in J + KI_1 + LI_2$ and from which $l \in L$. Therefore, $L \in CS\wp_R$.

And on the other hand, we have $0 + kI_1 + 0I_2 \in R(I_1, I_2)$, and we note

$$(0 + kI_1 + 0I_2)^2 = 0^2 + (0.k + k.0 + k^2 + k.0 + 0.k)I_1 + (0^2 + 0^2 + 0^2)I_2 \in J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$, so $0 + kI_1 + 0I_2 \in J + KI_1 + LI_2$ and from which $k \in K$. Therefore, $K \in CS\wp_R$.

Conversely, suppose that $J, K, and\ L \in CS\wp_R$.
Now, if $j_1 + k_1I_1 + l_1I_2 \in R(I_1, I_2)$, where $(j_1 + k_1I_1 + l_1I_2)^2 \in J + KI_1 + LI_2 \Rightarrow j_1^2 + (j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1)I_1 + (j_1l_1 + l_1j_1 + l_1^2)I_2 \in J + KI_1 + LI_2$
Therefore, $j_1^2 \in J\ and\ j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1 \in K\ and\ j_1l_1 + l_1j_1 + l_1^2 \in L$
Since $j_1^2 \in J \subseteq L\ and\ j_1l_1 + l_1j_1 + l_1^2 \in L \subseteq K$, so $j_1^2 + j_1l_1 + l_1j_1 + l_1^2 = (j_1 + l_1)^2 \in L\ and\ j_1^2 + j_1k_1 + k_1j_1 + k_1^2 + k_1l_1 + l_1k_1 + j_1l_1 + l_1j_1 + l_1^2 = (j_1 + k_1 + l_1)^2 \in K$.
Since $J, L, and\ K \in CS\wp_R$, so $J_1 \in J\ and\ J_1 + L_1 \in L \subseteq K\ and\ j_1 + k_1 + l_1 \in K$.
Since $j_1 \in J \subseteq L$, so $l_1 \in L \subseteq K\ and\ k_1 \in K$. Subsequently, $j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$.

Theorem 3.4: If $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, then $J, K, L \in \wp_R$.
Proof.
Suppose that $J_1, J_2, K_1, K_2, L_1, L_2 \in \mathfrak{T}_R$, where $J_1 J_2 \subseteq J, K_1K_2 \subseteq K, and\ L_1L_2 \subseteq L$.

Firstly, we have $J_1 + J_1I_1 + J_1I_2\ and\ J_2 + J_2I_1 + J_2I_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, and we note

$$(J_1 + J_1I_1 + J_1I_2)(J_2 + J_2I_1 + J_2I_2) = J_1J_2 + (J_1J_2 + J_1J_2 + J_1J_2 + J_1J_2 + J_1J_2)I_1 + (J_1J_2 + J_1J_2 + J_1J_2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, so $J_1 + J_1I_1 + J_1I_2 \subseteq J + KI_1 + LI_2\ or\ J_2 + J_2I_1 + J_2I_2 \subseteq J + KI_1 + LI_2$.
Subsequently, $J_1 \subseteq J\ or\ J_2 \subseteq J$. Thus $J \in \wp_R$.
On the other hand, we have $\{0\} + L_1I_1 + L_1I_2 and\ \{0\} + L_2I_1 + L_2I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$, and we note
$(\{0\} + L_1I_1 + L_1I_2)(\{0\} + L_2I_1 + L_2I_2) = \{0\}^2 + (\{0\}.L_2 + L_1.\{0\} + L_1L_2 + L_1L_2 + L_1L_2)I_1 + (\{0\}.L_2 + L_1.\{0\} + L_1L_2)I_2 \subseteq J + KI_1 + LI_2$
Since $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, so $\{0\} + L_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2\ or\ \{0\} + L_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$. Subsequently, $L_1 \subseteq L\ or\ L_2 \subseteq L$. Thus $L \in \wp_R$.

Also, we have $\{0\} + K_1I_1 + \{0\}I_2\ and\ \{0\} + K_2I_1 + \{0\}I_2 \in N\mathfrak{T}_{R(I_1,I_2)}$, and we note

$$(\{0\} + K_1I_1 + \{0\}I_2)(\{0\} + K_2I_1 + \{0\}I_2) = \{0\}^2 + (\{0\}.K_2 + K_1.\{0\} + K_1K_2 + K_1.\{0\} + \{0\}.K_2)I_1 + (\{0\}^2 + \{0\}^2 + \{0\}^2)I_2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, so $\{0\} + K_1I_1 + \{0\}I_2 \subseteq J + KI_1 + LI_2\ or\ \{0\} + K_2I_1 + \{0\}I_2 \subseteq J + KI_1 + LI_2$ and from which $K_1 \subseteq K\ or\ K_2 \subseteq K$. Thus $K \in \wp_R$.

Corollary 3.5: If $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, and $J, K, L \in \wp_R$, then not necessarily $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$.
Because.

Suppose that $J, K,$ and $L \in \wp_R$. Now, if $J_1 + K_1I_1 + L_1I_2,$ and $J_2 + K_2I_1 + L_2I_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, where,

$$(J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$$

$$\Rightarrow J_1J_2 + (J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2)I_1 + (J_1L_2 + L_1J_2 + L_1L_2)I_2 \subseteq J + KI_1 + LI_2, so$$

$$J_1J_2 \subseteq J, \quad J_1L_2 + L_1J_2 + L_1L_2 \subseteq L, \quad and\ J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2 \subseteq K$$

Since $J_1J_2 \subseteq J \subseteq L$, so $J_1J_2 + J_1L_2 + L_1J_2 + L_1L_2 = (J_1 + L_1)(J_2 + L_2) \subseteq L \subseteq K$

$and\ J_1J_2 + J_1K_2 + K_1J_2 + K_1K_2 + K_1L_2 + L_1K_2 + J_1L_2 + L_1J_2 + L_1L_2 = (J_1 + K_1 + L_1)(J_2 + K_2 + L_2) \subseteq K$

Since $J, K, and\ L \in \wp_R$, so

$(J_1 \subseteq J\ or\ J_2 \subseteq J), (J_1 + L_1 \subseteq L\ or\ J_2 + L_2 \subseteq L),\ and\ (J_1 + K_1 + L_1 \subseteq K\ or\ J_2 + K_2 + L_2 \subseteq K)$. Thus not necessarily $J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$ or $J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$

Subsequently, not necessarily $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$.

Theorem 3.6: If $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$, then $J, K, and\ L \in C\wp_R$.

Proof.

If $j_1, j_2, k_1, k_2, l_1, l_2 \in R$, where $j_1j_2 \in J,$ $k_1k_2 \in K,$ and $l_1l_2 \in L$.

Firstly, $j_1 + j_1I_1 + j_1I_2$ and $j_2 + j_2I_1 + j_2I_2 \in R(I_1, I_2)$ and we note

$$(j_1 + j_1I_1 + j_1I_2)(j_2 + j_2I_1 + j_2I_2) = j_1j_2 + (j_1j_2 + j_1j_2 + j_1j_2 + j_1j_2 + j_1j_2)I_1 + (j_1j_2 + j_1j_2 + j_1j_2)I_2 \in J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$, so

$j_1 + j_1I_1 + j_1I_2 \in J + KI_1 + LI_2\ or\ j_2 + j_2I_1 + j_2I_2 \in J + KI_1 + LI_2$.

Therefore, $j_1 \in J\ or\ j_2 \in J$. Thus $J \in C\wp_R$.

On the other hand, $0 + l_1I_1 + l_1I_2 and\ 0 + l_2I_1 + l_2I_2 \in R(I_1, I_2)$, and we note

$$(0 + l_1I_1 + l_1I_2)(0 + l_2I_1 + l_2I_2) = 0^2 + (0.l_2 + l_1.0 + l_1l_2 + l_1l_2 + l_1l_2)I_1 + (0.l_2 + l_1.0 + l_1l_2)I_2 \in J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$, so $0 + l_1I_1 + l_1I_2 \in J + KI_1 + LI_2\ or\ 0 + l_2I_1 + l_2I_2 \in J + KI_1 + LI_2$.

Therefore, $l_1 \in L\ or\ l_2 \in L$. Thus $L \in C\wp_R$.

Also, we have $0 + k_1I_1 + 0I_2 and\ 0 + k_2I_1 + 0I_2 \in R(I_1, I_2)$, and we note

$$(0 + k_1I_1 + 0I_2)(0 + k_2I_1 + 0I_2) = 0^2 + (0.k_2 + k_1.0 + k_1k_2 + k_1.0 + \{0\}.k_2)I_1 + (0^2 + 0^2 + 0^2)I_2 \in J + KI_1 + LI_2$$

since $J + KI_1 + LI_2 \in NC\wp_{R(I_1,I_2)}$, so $0 + k_1I_1 + 0I_2 \in J + KI_1 + LI_2\ or\ 0 + k_1I_1 + 0I_2 \in J + KI_1 + LI_2$ and from which $k_1 \in K\ or\ k_2 \in K$. Thus $K \in C\wp_R$.

Corollary 3.7: If $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$ and $J, K, and\ L \in C\wp_R$, then not necessarily $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$.

Because.

Suppose that $j_1 + k_1I_1 + l_1I_2,$ $j_2 + k_2I_1 + l_2I_2 \in R(I_1, I_2)$, where,

$$(j_1 + k_1I_1 + l_1I_2)(j_2 + k_2I_1 + l_2I_2) \in J + KI_1 + LI_2$$

$$\Rightarrow j_1j_2 + (j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2)I_1 + (j_1l_2 + l_1j_2 + l_1l_2)I_2 \in J + KI_1 + LI_2, so$$

$$j_1j_2 \in J\ and\ j_1l_2 + l_1j_2 + l_1l_2 \in L\ and\ j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2 \in K$$

Since $j_1j_2 \in J \subseteq L$, so $j_1j_2 + j_1l_2 + l_1j_2 + l_1l_2 = (j_1 + l_1)(j_2 + l_2) \in L \subseteq K$

$and\ j_1j_2 + j_1k_2 + k_1j_2 + k_1k_2 + k_1l_2 + l_1k_2 + j_1l_2 + l_1j_2 + l_1l_2 = (j_1 + k_1 + l_1)(j_2 + k_2 + l_2) \in K$

Since $J, K, L \in C\wp_R$, so $(j_1 \in J\ or\ j_2 \in J)$, $(j_1 + l_1 \in L\ or\ j_2 + l_2 \in L)$, $and\ (j_1 + k_1 + l_1 \in K\ or\ j_2 + k_2 + l_2 \in K\ )$. So not necessarily

$j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2\ or\ j_2 + k_2I_1 + l_2I_2 \in J + KI_1 + LI_2$. Therefore, not necessarily $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$.

Theorem 3.8: If $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I)}$, then

(i) $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)} \Rightarrow J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$.

(ii) $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)} \Rightarrow J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$.

Proof.

(i) Since $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$, so $J,\ K,\ and\ L \in CS\wp_R$ according to Theorem.3.3. Therefore, $J, K, and\ L \in S\wp_R$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$ according to Theorem.3.2.

(ii) Suppose that $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$. Now, if $J_1 + K_1I_1 + L_1I_2\ and\ J_2 + K_2I_1 + L_2I_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$ in which $(J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2) \subseteq J + KI_1 + LI_2$.

Firstly, suppose that

$J_1 + K_1I_1 + L_1I_2 \nsubseteq J + KI_1 + LI_2\ and\ J_2 + K_2I_1 + L_2I_2 \nsubseteq J + KI_1 + LI_2$. Therefore,

$\exists j_1 + k_1I_1 + l_1I_2 \in J_1 + K_1I_1 + L_1I_2\ and\ j_2 + k_2I_1 + l_2I_2 \in J_2 + K_2I_1 + L_2I_2,\ where$

$j_1 + k_1I_1 + l_1I_2 \notin J + KI_1 + LI_2\ and\ J_2 + K_2I_1 + L_2I_2 \notin J + KI_1 + LI_2$.

On the other hand, we have $(j_1 + k_1I_1 + l_1I_2)(j_2 + k_2I_1 + l_2I_2) \in (J_1 + K_1I_1 + L_1I_2)(J_2 + K_2I_1 + L_2I_2)$

$$\subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$, so

$j_1 + k_1I_1 + l_1I_2 \in J + KI_1 + LI_2\ or j_2 + k_2I_1 + l_2I_2 \in J + KI_1 + LI_2$. This is a contradiction. Therefore, $J + KI_1 + LI_2 \subseteq J + KI_1 + LI_2\ or\ J_2 + K_2I_1 + L_2I_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$.

Remark 3.9: Figure 1 shows the resulting relationship between the prime (completely prime, semiprime, and completely semiprime) ideals in any refined neutrosophic and classical ring, as follows:

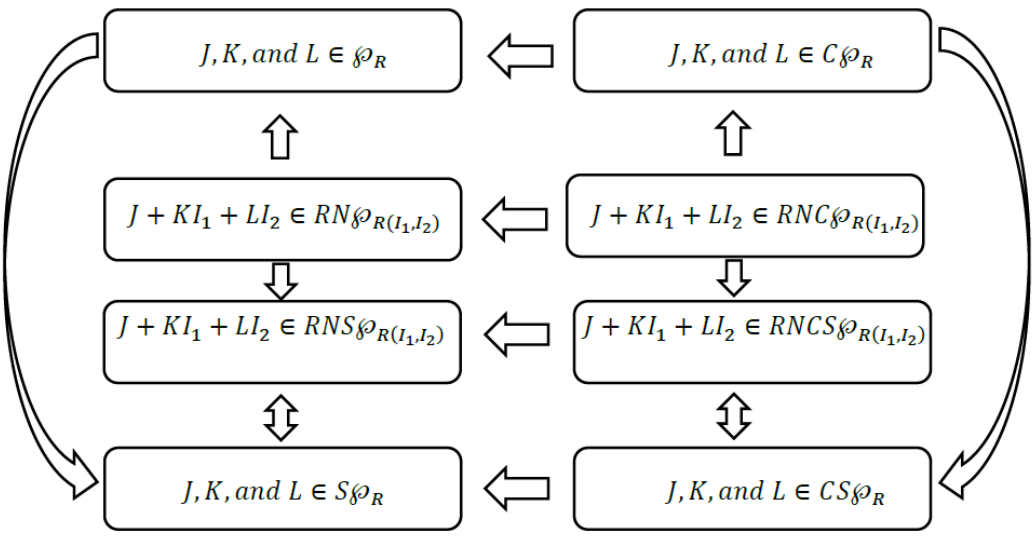

Figure 1. The relationship between the ideals of the refined neutrosophic and classical ring.

Theorem 3.10: If $R(I_1, I_2)$ is a unity, and $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)} \Leftrightarrow \forall r_1 + r_2I_1 + r_3I_2 \in R(I_1, I_2); (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) \subseteq J + KI_1 + LI_2 \Rightarrow r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$, and we will prove that the condition is satisfied.

$$\forall r_1 + r_2I_1 + r_3I_2 \in R(I_1, I_2); (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) \subseteq J + KI_1 + LI_2$$

$$\Rightarrow (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq (J + KI_1 + LI_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2$$

$$\Rightarrow [(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)]^2 \subseteq J + KI_1 + LI_2$$

Since $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$, so $(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2$.

On the other hand, we have

$$r_1 + r_2I_1 + r_3I_2 = (r_1 + r_2I_1 + r_3I_2).1 \in (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2 \Rightarrow r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$$

Conversely, suppose that the condition is true and we will prove that $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$.

Suppose that $J_1 + K_1I_1 + L_1I_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, where,

$[J_1 + K_1I_1 + L_1I_2]^2 \subseteq J + KI_1 + LI_2$.

If we assume the argument $J_1 + K_1I_1 + L_1I_2 \nsubseteq J + KI_1 + LI_2$. Therefore, there is an element $r_1 + r_2I_1 + r_3I_2 \in J_1 + K_1I_1 + L_1I_2$ *and* $r_1 + r_2I_1 + r_3I_2 \notin J + KI_1 + LI_2$

On the other hand, we have

$$r_1 + r_2I_1 + r_3I_2 \in J_1 + K_1I_1 + L_1I_2 \Rightarrow (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2) \subseteq J_1 + K_1I_1 + L_1I_2$$

$$\Rightarrow (r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1 + r_2I_1 + r_3I_2) \subseteq (J_1 + K_1I_1 + L_1I_2)(r_1 + r_2I_1 + r_3I_2)$$
$$\subseteq (J_1 + K_1I_1 + L_1I_2)(J_1 + K_1I_1 + L_1I_2) = [J_1 + K_1I_1 + L_1I_2]^2 \subseteq J + KI_1 + LI_2$$

Therefore, $r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2$, which is a contradiction.

So $J_1 + K_1I_1 + L_1I_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$.

Theorem.3.11 If $R(I_1, I_2)$ is a unity, and $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$ iff the condition is satisfied:

$$\forall r_1 + r_2I_1 + r_3I_2 \text{ and } r_1' + r_2'I_1 + r_3'I_2 \in R(I_1, I_2);$$

$$(r_1 + r_2I_1 + r_3I_2)R(I_1, I_2)(r_1' + r_2'I_1 + r_3'I_2) \subseteq J + KI_1 + LI_2$$

$$\Rightarrow r_1 + r_2I_1 + r_3I_2 \in J + KI_1 + LI_2 \text{ or } r_1' + r_2'I_1 + r_3'I_2 \in J + KI_1 + LI_2$$

Proof. In a similar way to proof of the theorem.3.10.

Corollary 3.12: Let $R(I_1, I_2)$ be a unity commutation.

(i) If $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}, then\, J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$.

(ii) If $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}, then\, J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$.

Proof.

1. Suppose that $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, and $r_1 + r_2I_1 + r_3I_2$, $r_1' + r_2'I_1 + r_3'I_2 \in R(I_1, I_2)$, where, $(r_1 + r_2I_1 + r_3I_2)(r_1' + r_2'I_1 + r_3'I_2) \in J + KI_1 + LI_2$.

$$\Rightarrow (r_1 + r_2I_1 + r_3I_2)(r_1' + r_2'I_1 + r_3'I_2)R(I_1, I_2) \subseteq (J + KI_1 + LI_2)R(I_1, I_2) \subseteq J + KI_1 + LI_2$$

Since $R(I_1, I_2)$ is a commutative, so

$$(r_1 + r_2 I_1 + r_3 I_2)R(I_1, I_2)(r_1' + r_2' I_1 + r_3' I_2) \subseteq J + KI_1 + LI_2$$

And since $J + KI_1 + LI_2 \in N\wp_{R(I_1,I_2)}$, and according to theorem.3.11, so
$r_1 + r_2 I_1 + r_3 I_2 \in J + KI_1 + LI_2 \; or \; r_1' + r_2' I_1 + r_3' I_2 \in J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNC\wp_{R(I_1,I_2)}$.
2. In a similar way to proof.1. Or in another way, since $R(I_1, I_2)$ is a unity commutative, so R is a unity commutative ring.
We have $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$, therefore, $J, K, and \; L \in S\wp_R$ according to Theorem.3.2.
Since R is a unity commutative ring, so J, K, $and \; L \in CS\wp_R$. Using the Theorem.3.3, $J + KI_1 + LI_2 \in RNCS\wp_{R(I_1,I_2)}$.

Theorem 3.13: Assuming that $R(I_1, I_2)$ is a unity. $If \; J + KI_1 + LI_2 \in RN\mathcal{M}_{R(I_1,I_2)}, then \; J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$.

Proof.

Since $J + KI_1 + LI_2 \in RN\mathcal{M}_{R(I_1,I_2)}$, so $(J \in \mathcal{M}_R \; and \; K = L = R) \; or \; J + KI_1 + LI_2 = R(I_1, I_2)$ according to theorem.2.4. If $J + KI_1 + LI_2 = R(I_1, I_2)$, then the desired is achieved. Now, suppose that $J + KI_1 + LI_2 \neq R(I_1, I_2)$.
We have $R(I_1, I_2)$ is a unity, therefore, we may apply the condition specified in the theorem.3.11.

$$\forall r_1 + r_2 I_1 + r_3 I_2 \; and \; r_1' + r_2' I_1 + r_3' I_2 \in R(I_1, I_2);$$

$$(r_1 + r_2 I_1 + r_3 I_2)R(I_1, I_2)(r_1' + r_2' I_1 + r_3' I_2) \subseteq J + KI_1 + LI_2$$

Now, we will prove that $r_1 + r_2 I_1 + r_3 I_2 \in J + RI_1 + RI_2 \; or \; r_1' + r_2' I_1 + r_3' I_2 \in J + RI_1 + RI_2$.

In fact, it suffices to demonstrate that $r_1 \in J \; or \; r_1' \in J$.

Firstly, $(r_1 + r_2 I_1 + r_3 I_2)(R + RI_1 + RI_2)(r_1' + r_2' I_1 + r_3' I_2) \subseteq J + RI_1 + RI_2$

$$\Rightarrow r_1 R r_1' + [r_1 R r_1' + r_2 R r_1' + r_2 R r_1' + r_2 R r_1' + r_3 R r_1' + r_1 R r_2' + r_1 R r_2' + r_2 R r_2' + r_2 R r_2' + r_2 R r_2' + r_3 R r_2' + r_1 R r_2' + r_3 R r_2' + r_3 R r_2' + r_1 R r_3' + r_2 R r_3' + r_2 R r_3' + r_2 R r_3' + r_3 R r_3']I_1 + [r_1 R r_1' + r_3 R r_1' + r_3 R r_1' + r_1 R r_3' + r_1 R r_3' + r_3 R r_3' + r_3 R r_3']I_2 \subseteq J + RI_1 + RI_2$$

Therefore, $r_1 R r_1' \subseteq J$.

Suppose that $r_1 \notin J$. Since $J \in \mathcal{M}_R$, so $J + r_1 R = R \Rightarrow J r_1' + r_1 R r_1' = R r_1'$

On the other hand, we have $J r_1' \subseteq J \; and \; r_1 R r_1' \subseteq J$. Therefore, $r_1' = 1.r_1' \in R r_1' \subseteq J$.

Subsequently, $r_1' + r_2' I_1 + r_3' I_2 \in J + RI_1 + RI_2$. Thus $J + KI_1 + LI_2 \in N\wp_{R(I_1,I_2)}$.

Remark 3.14: Figure 2 shows the resulting relationship between the prime (completely prime, semiprime, completely semiprime, and maximal) ideals in the unity refined neutrosophic and classical rings, as follows:

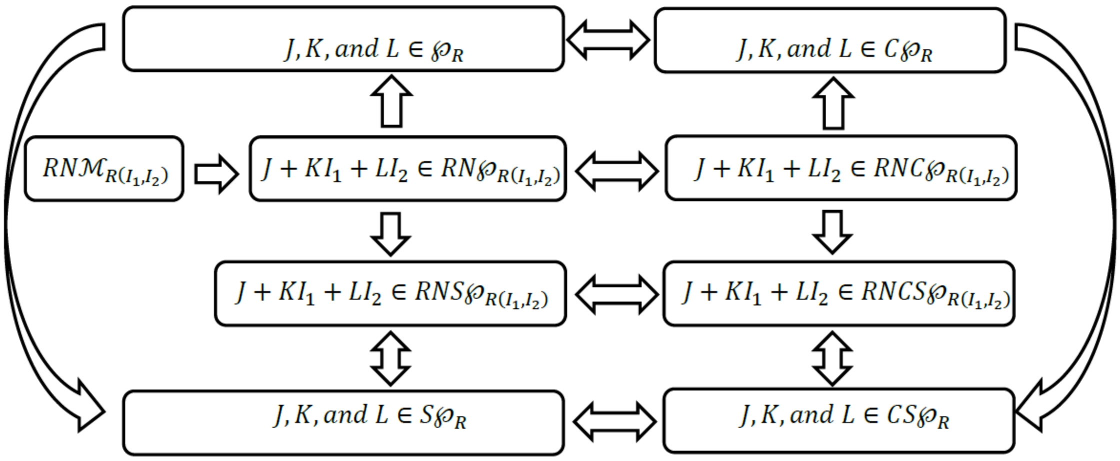

Figure 2. The relationship between the ideals of the unity refined neutrosophic and classical ring.

Theorem 3.15: Assuming that $R(I_1, I_2)$ is a finite unity commutation, then $RN\mathcal{M}_{R(I_1,I_2)} = RN\wp_{R(I_1,I_2)}$.

Proof.

Since $R(I_1, I_2)$ is a unity, so $RN\mathcal{M}_{R(I_1,I_2)} \subseteq RN\wp_{R(I_1,I_2)}$ according to theorem.3.13.

Now, if $J + KI_1 + LI_2 \in RN\wp_{R(I_1,I_2)}$, then $J, K, and\ L \in \wp_R$ according to theorem.3.4. Since $J \in \wp_R$, and R is a finite unity commutation, so $J \in \mathcal{M}_R$. Since $J \subseteq L \subseteq K$, so $K = L = R$. Thus $J + KI_1 + LI_2 \in RN\mathcal{M}_{R(I_1,I_2)}$.

Examples and Notes 3.16:

(1) In $Z(I)$, we have $10Z + 10ZI_1 + 10ZI_2 \in RNS\wp_{Z(I_1,I_2)}$, because $\forall\, r_1 + r_2I_1 + r_3I_2 \in Z(I_1, I_2)$; $(r_1 + r_2I_1 + r_3I_2)^2 \in 10Z + 10ZI_1 + 10ZI_2$, then $r_1{}^2 \in 10Z\ and\ (r_1 + r_3)^2 \in 10Z\ and\ (r_1 + r_2 + r_3)^2 \in 10Z$. Since $10Z \in S\wp_Z$, so $r_1 \in 10Z, r_1 + r_3 \in 10Z, and\ r_1 + r_2 + r_3 \in 10Z$. Therefore $r_1, r_2, r_3 \in 10Z$. Thus $r_1 + r_2I_1 + r_3I_2 \in 10Z + 10ZI_1 + 10ZI_2 \in RNS\wp_Z$.

By the same way we find that $< 0 > + < 0 > I_1 + < 0 > I_2 = \{0\} \in RNS\wp_{Z(I_1,I_2)}$.

(2) By the same way we find that $< 2 > + < 2 > I_1 + < 2 > I_2 \in RNS\wp_{Z_4(I_1,I_2)}$.

(3) In $Z_4(I_1, I_2)$, we have $< 2 >= \{0,2\} \in \wp_{Z_4}$, but $< 2 > + < 2 > I_1 + < 2 > I_2 = \{0, 2I_2, 2I_1, 2I_1 + 2I_2, 2, 2 + 2I_2, 2 + 2I_1, 2 + 2I_1 + 2I_2\} \notin RN\wp_{Z_4(I_1,I_2)}$, because we have $(I_1 + I_2)(2 + I_1) = 2I_2 \in < 2 > + < 2 > I_1 + < 2 > I_2$,

but $I_1 + I_2\ and\ 2 + I_1 \notin < 2 > + < 2 > I_1 + < 2 > I_2$.

(4) In $Z(I_1, I_2)$, we have $< 0 >\ and\ < 3 > \in \wp_Z$, but $< 0 > + < 3 > I_1 + < 0 > I_2 = < 3 > I_1 \notin RN\wp_{Z(I_1,I_2)}$, because we have $(0 + 2I_1)(3 + I_1) = 9I_1 \in < 3 > I_1$, but $3 + I_1\ and\ 2I_1 \notin < 3 > I_1$. By the same way, we find $< 3 > I_1 + < 3 > I_2 \notin RN\wp_{Z(I_1,I_2)}$.

(5) In $Z_6(I_1, I_2)$, we have $< 3 > + Z_6I_1 + < 3 > I_2 \notin RN\wp_{Z_6(I_1,I_2)}$, because we have $(0 + I_2)(2 + I_2) = 3I_2 \in < 3 > + Z_6I_1 + < 3 > I_2$, but $I_2\ and\ 2 + I_2 \notin < 3 > + Z_6I_1 + < 3 > I_2$.

(6) We note $< 2 > + Z_6I_1 + Z_6I_2 = \{0,2\} + Z_6I_1 + Z_6I_2 \in N\wp_{Z_6(I_1,I_2)}$, because $< 2 > \in \mathcal{M}_{Z_6}$, so $< 2 > + Z_6I_1 + Z_6I_2 \in RN\mathcal{M}_{Z_6(I_1,I_2)}$ according to theorem.2.7. Therefore, $< 2 > + Z_6I_1 + Z_6I_2 \in RN\wp_{Z_6(I_1,I_2)}$ according to Theorem.3.13.

(7) In $Z_7(I_1, I_2)$, we have $< 0 > + < 0 > I_1 + < 0 > I_2 = \{0\} \notin RN\wp_{Z_7(I_1,I_2)}$, because we have

$(6 + I_2)(I_1 + I_2) = 0 \in Z_7(I_1, I_2)$, but $6 + I_2\ and\ I_1 + I_2 \notin < 0 > + < 0 > I_1 + < 0 > I_2$.

(8) In any refined neutrosophic field $R(I_1, I_2)$, $RI_1 + RI_2 \in RN\wp_{R(I_1,I_2)}$, because $RI_1 + RI_2 =< 0 > +RI_1 + RI_2$, where $< 0 > \in N\mathcal{M}_R$. Using theorem.3.13, we find $RI_1 + RI_2 \in RN\wp_{R(I_1,I_2)}$.

It can be proven in another way:

If $a + bI_1 + cI_2$ *and* $d + eI_1 + fI_2 \in R(I_1, I_2)$ where $(a + bI_1 + cI_2)(d + eI_1 + fI_2) \in RI_1 + RI_2 \Rightarrow \exists r$, $r' \in R$ *inwhich* $(a + bI_1 + cI_2)(d + eI_1 + fI_2) = rI_1 + r'I_2$

So $ad + [ae + bd + be + bf + ce]I_1 + [af + cd + cf]I_2 = 0 + rI_1 + r'I_2$

Therefore, $ad = 0$. *So* $a = 0$ *or* $d = 0$

$$if\ a = 0\ then\ \ a + bI_1 + cI_2 \in RI_1 + RI_2$$
$$if\ d = 0\ then\ \ d + eI_1 + fI_2 \in RI_1 + RI_2$$

(9) Generally, in refined neutrosophic rings, $RI_1 + RI_2$ is not necessarily belongs to $RN\wp_{R(I_1,I_2)}$.

(10) $Z_9I_1 + Z_9I_2 \notin RN\wp_{Z_9(I_1,I_2)}$, because we have $(3 + I_1 + 2I_2)^2 = 2I_1 + 4I_2 \in Z_9I_1 + Z_9I_2$, but $3 + I_1 + 2I_2 \notin Z_9I_1 + Z_9I_2$.

(11) In $Z(I_1, I_2)$, we have $< 0 > +ZI_1 + ZI_2$ *and* $< p > +ZI_1 + ZI_2 \in RN\wp_{Z(I_1,I_2)}$, where p is prime.

Theorem 3.17: Assuming that $R(I_1, I_2)$ is a unity. Then $R(I_1, I_2)$ is a refined neutrosophic field $\Leftrightarrow$ $\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)$ are only refined neutrosophic ideals in $R(I_1, I_2)$.

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$. Since $R(I_1, I_2)$ is a refined neutrosophic field, so R is a field. Therefore, R contains only two ideals $\{0\}$ *and* R. Thus

$$J, K, and\ L = \{0\}\ or\ R$$

We have $J \subseteq L \subseteq K$and we note

$$if\ J = L = K = \{0\}, then\ \ J + KI_1 + LI_2 = \{0\}$$
$$if\ J = \{0\} \wedge K = L = R, then\ J + KI_1 + LI_2 = RI_1 + RI_2$$
$$if\ J = L = \{0\} \wedge K = R, then\ J + KI_1 + LI_2 = RI_1$$
$$if\ J = L = K = R, \qquad then\ J + KI_1 + LI_2 = R + RI_1 + RI_2$$

Subsequently, $RN\mathfrak{T}_{R(I_1,I_2)} = \{\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)\}$.

Conversely, suppose that $RN\mathfrak{T}_{R(I_1,I_2)} = \{\{0\}, RI_1 + RI_2, RI_1, R(I_1, I_2)\}$.

Now, If $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$, then

$$J + KI_1 + LI_2 = R + RI_1 + RI_2 \vee \{0\} + RI_1 + \{0\}I_2 \vee \{0\} + RI_1 + RI_2 \vee \{0\} + \{0\}I_1 + \{0\}I_2$$

In every case, we see that $J, K, and\ L = \{0\} \vee R$. Therefore, R contains only two ideals $\{0\}$ *and* R. Subsequently, R is a field. Thus $R(I_1, I_2)$ is a refined neutrosophic field.

Definition 3.18: Assuming that $R(I_1, I_2)$ is a refined neutrosophic ring.

(i) We call $R(I_1, I_2)$ a refined neutrosophic semiprime ring if $\{0\} \in RNS\wp_{R(I_1,I_2)}$ and a fully semiprime ring if $RN\mathfrak{T}_{R(I_1,I_2)} = RNS\wp_{R(I_1,I_2)}$.

(ii) We call $R(I_1, I_2)$ a refined neutrosophic prime ring if $RI_1 + RI_2 \in RN\wp_{R(I_1,I_2)}$ and a fully prime ring if $RN\mathfrak{T}_{R(I_1,I_2)} \setminus \{0\} = RN\wp_{R(I_1,I_2)}$.

(iii) We call $R(I_1, I_2)$ a refined neutrosophic fully idempotent if all its neutrosophic ideals are idempotent.

Examples 3.19:

(1) $Z(I_1, I_2)$ is a refined neutrosophic semiprime ring.

(2) $R(I_1, I_2)$ is a refined neutrosophic semiprime (fully semiprime) ring, where R is a field.

(3) $R(I_1, I_2)$ is a refined neutrosophic prime (fully prime) ring, where R is a field.

Theorem 3.20: Assuming that $R(I_1, I_2)$ is a refined neutrosophic ring,
$R(I_1, I_2)$ is a refined neutrosophic fully semiprime $\Leftrightarrow$ $R(I_1, I_2)$ is a refined neutrosophic fully idempotent.

Proof.

Firstly, suppose that $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$. Now, we have $(J + KI_1 + LI_2)^2 \in RN\mathfrak{T}_{R(I_1,I_2)}$.
Therefore, it belongs to $RNS\wp_{R(I_1,I_2)}$.

Also, we have $(J + KI_1 + LI_2)^2 \subseteq (J + KI_1 + LI_2)^2 \underbrace{\Rightarrow}_{(J+KI_1+LI_2)^2 \in RNS\wp_{R(I_1,I_2)}.}$

$$J + KI_1 + LI_2 \subseteq (J + KI_1 + LI_2)^2$$

On the other hand, $(J + KI_1 + LI_2)^2 \subseteq J + KI_1 + LI_2$. So $(J + KI_1 + LI_2)^2 = J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2$ is a refined neutrosophic idempotent ideal.

Conversely, suppose that $J + KI_1 + LI_2 \in RN\mathfrak{T}_{R(I_1,I_2)}$.

Now, let's prove that

$\forall P + QI_1 + SI_2 \in N\mathfrak{T}_{R(I_1,I_2)},\ where,\ (P + QI_1 + SI_2)^2 \subseteq J + KI_1 + LI_2,$

then $P + QI_1 + SI_2 \subseteq J + KI_1 + LI_2$.

Since $(P + QI_1 + SI_2)^2 = P + QI_1 + SI_2$ (because it is idempotent), then
$P + QI_1 + SI_2 \subseteq J + KI_1 + LI_2$. Thus $J + KI_1 + LI_2 \in RNS\wp_{R(I_1,I_2)}$.

Example.3.21 According to the theorem 3.17, in $\mathbb{Z}_3(I_1, I_2)$, we have $\{0\}$, $\mathbb{Z}_3 I_1 + \mathbb{Z}_3 I_2$, $\mathbb{Z}_3 I_1$, and $\mathbb{Z}_3(I_1, I_2)$ are the only neutrosophic ideals. Now we note $\{0\}$, $\mathbb{Z}_3 I_1 + \mathbb{Z}_3 I_2$, $\mathbb{Z}_3 I_1$, and $\mathbb{Z}_3(I_1, I_2)$ are refined neutrosophic idempotent ideals. According to definition.3.18, $\mathbb{Z}_3(I_1, I_2)$ is a refined neutrosophic semiprime ideals. Conversely, according to the theorem.3.20, $\mathbb{Z}_3(I_1, I_2)$ is a refined neutrosophic fully semiprime.

Finally, Table 1 depicts the key distinctions between the classical and refined neutrosophic rings.

Table 1. Key distinctions between the classical and refined neutrosophic rings.

R	$\mathbf{R(I_1, I_2)}$
R is a field $\Leftrightarrow$ $\{0\}$, R are only ideals in R.	$\mathbf{R(I_1, I_2)}$ is a refined neutrosophic field $\Leftrightarrow$ $\{\mathbf{0}\}, \mathbf{RI_1 + RI_2}, \mathbf{RI_1}, \mathbf{R(I_1, I_2)}$ are only refined neutrosophic ideals.
R is a prime ring if $\{0\} \in \wp_R$	$\mathbf{R(I_1, I_2)}$ is a refined neutrosophic prime ring if $\mathbf{RI_1 + RI_2} \in \mathbf{RN}\wp_{\mathbf{R(I_1,I_2)}}$
$R(I_1, I_2)$ is a fully prime ring if $\mathfrak{T}_R = \wp_R$.	$\mathbf{R(I_1, I_2)}$ is a fully prime ring if $\mathbf{RN}\mathfrak{T}_{\mathbf{R(I_1,I_2)}} \backslash \{\mathbf{0}\} = \mathbf{RN}\wp_{\mathbf{R(I_1,I_2)}}$.

4. Conclusion and future works

In this study, the structure and properties of all prime, completely prime, semiprime, and completely semiprime ideals in refined neutrosophic rings were determined. Herein, we present the concept of fully prime (fully prime) and fully semiprime (fully semiprime) refined neutrosophic rings. In addition, many examples were built to clarify the validity of this work. Certainly, these ideals will find applications in all places where they find their applications, with some indeterminacy. In the future, we plan to generalize the prime (completely prime, semiprime, and completely semiprime) ideals of the n-refined neutrosophic rings.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Banik, B., Alam, S., & Chakraborty, A. (2023). Comparative study between GRA and MEREC technique on an agricultural-based MCGDM problem in pentagonal neutrosophic environment. International Journal of Environmental Science and Technology, 1-16. https://doi.org/10.1007/s13762-023-04768-1.
2. Abdel-Basset, M., Manogaran, G., Gamal, A., & Chang, V. (2019). A novel intelligent medical decision support model based on soft computing and IoT. IEEE Internet of Things Journal, 7(5), 4160-4170. https://doi.org/10.1109/JIOT.2019.2931647.
3. Gamal, A., Abdel-Basset, M., Hezam, I. M., Sallam, K. M., & Hameed, I. A. (2023). An Interactive Multi-Criteria Decision-Making Approach for Autonomous Vehicles and Distributed Resources Based on Logistic Systems: Challenges for a Sustainable Future. Sustainability, 15(17), 12844. https://doi.org/10.3390/su151712844.
4. Haque, T. S., Chakraborty, A., Alrabaiah, H., & Alam, S. (2022). Multiattribute decision-making by logarithmic operational laws in interval neutrosophic environments. Granular Computing, 7(4), 837-860. https://doi.org/10.1007/s41066-021-00299-7.
5. Banik, B., Alam, S., & Chakraborty, A. (2023). A novel integrated neutrosophic cosine operator based linear programming ANP-EDAS MCGDM strategy to select anti-pegasus software. International Journal of Information Technology & Decision Making, 1-37. https://doi.org/10.1142/S0219622023500529.
6. Kandasamy, W. V., & Smarandache, F. (2006). Some neutrosophic algebraic structures and neutrosophic N-algebraic structures. Hexis, Phonex, Arizona 2006.
7. Agboola, A. A. A., Akinola, A. D., & Oyebola, O. Y. (2013). Neutrosophic rings I. Smarandache Multispace & Multistructure, 28, 114.
8. Agboola, A. A. A., Adeleke, E. O., & Akinleye, S. A. (2012). Neutrosophic rings II. Int. J. Math. Comb, 2, 1-8.
9. Chalapathi, T., & Kumar, R. V. K. (2017). Neutrosophic graphs of finite groups. Neutrosophic Sets and Systems, 15, 22-30.
10. Chalapathi, T., & Kumar, R. K. (2018). Neutrosophic units of neutrosophic rings and fields. Neutrosophic Sets and Systems, 21, 5-9.

11. Abobala, M. (2020). On Some Special Substructures of Neutrosophic Rings and Their Properties. International Journal of Neutrosophic Science, 4(2), 72-81.
12. Sankari, H., & Abobala, M. (2020). AH-homomorphisms in neutrosophic rings and refined neutrosophic rings (Vol. 38). Neutrosophic Sets and Systems, 38, 524-536.
13. Chalapathi, T., & Madhavi, L. (2020). Neutrosophic boolean rings. Neutrosophic Sets and Systems, 33, 57-66.
14. Adeleke, E.O.; Agboola, A.A.A. & Smarandache, F. (2020). Refined Neutrosophic Rings I. International Journal of Neutrosophic Science, 2, 77-81.
15. Adeleke, E.O.; Agboola, A.A.A. & Smarandache, F. (2020). Refined Neutrosophic Rings II. International Journal of Neutrosophic Science, 2, 89-94.
16. Smarandache, F., & Abobala, M. (2020). n-Refined neutrosophic Rings. International Journal of Neutrosophic Science, 5, 83-90.
17. Sankari, H., & Abobala, M. (2020). n-Refined neutrosophic modules (Vol. 36). Neutrosophic Sets and Systems, 36, 1-11.
18. Smarandache, F., & Abobala, M. (2020). n-Refined neutrosophic vector spaces. International Journal of Neutrosophic Science, 7, 47-54.
19. Abobala, M. (2020). On Some Special Substructures of Refined Neutrosophic Rings. International Journal of Neutrosophic Science, 5, 59-66.
20. Abobala, M. (2021). On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings. Neutrosophic sets and systems, 45, 62-73.

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Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions

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Abstract: We now found nine new topologies, such as: NonStandard Topology, Largest Extended NonStandard Real Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet Topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology, and recall and improve the seven previously founded topologies in the years (2019-2023), namely: NonStandard Neutrosophic Topology, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology, and Neutrosophic SuperHyperTopology. They are called avantgarde topologies because of their innovative forms.

Keywords: Classical Topology; Topological Space; NeutroSophication; AntiSophication; NeutroTopology; AntiTopology; Refined Neutrosophic Topology; Refined Neutrosophic Crisp Topology; SuperHyperTopology; Neutrosophic SuperHyperTopology; Extended NonStandard Real Set; NonStandard Topology; NonStandard Neutrosophic Topology; Largest Extended NonStandard Real Topology; left monad; Right Monad; Pierced Binad; Left Monad Closed to the Right; Right Monad Closed to the Left, Unpierced Binad; Neutrosophic OverTopology; Neutrosophic UnderTopology; Neutrosophic OffTopology; (Fuzzy & Fuzzy-Extensions) Over/Under/Off-Topologies; Neutrosophic MultiSet Topology.

1. Introduction

The foundation of new topologies raised from development of other fields such as NeutroAlgebra and AntiAlgebra (that gave birth to NeutroTopology and AntiTopology), SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra (that gave birth to SuperHyperTopology and Neutrosophic SuperHyperTopology), Refined Crisp Set (that gave birth to the Refined Crisp Topology), and Refined Neutrosophic Set (that gave birth to refined Neutrosophic Topology), and NonStandard Set (that gives birth to NonStandard Topology and NonStandard Neutrosophic Topology), Neutrosophic Triplet Set, Neutrosophic Extended Triplet Set, Neutrosophic Dual Set, Neutrosophic Extended Dual Set, and Neutrosophic MultiSet.

This is almost a virgin territory of research since little research has been done, mostly about the AntiTopology [8]. Nevertheless, it is a promising field to study in the future, since it better reflects our real world, where the laws (axioms) do not apply in the same degree to all people (powerful people are above the law, others immune to the law, and many feel the full hardship of the law); since the world as a dynamic system is formed by sub-systems, and each sub-system by sub-sub-systems and so on (whence the necessity to introduce the SuperHyperStructure based on the n-th PowerSet of a Set, whose particular cases are the SuperHyperAlgebra and SuperHyperTopology), etc.

We recall the classical definition of Topology, then the procedures of NeutroSophication and respectively AntiSophication of it, that result in adding in two new types of topologies: NeutroTopology and respectively AntiTopology.

Then we define topology on Refined Neutrosophic Set (2013), Refined Neutrosophic Crisp Set [3]. Afterwards, we extend the topology on the framework of SuperHyperAlgebra [6], then the NonStandard Neutrosophic Set to NonStandard Topology and NonStandard Neutrosophic Topology (never defined before).

The corresponding neutrosophic topological spaces are presented.

This research is an improvement of paper [7] and book [12, sections 4.8 and 4.9].

2. Classical Topology

Let $\mathcal{U}$ be a non-empty set, and $P(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $\tau \subseteq P(\mathcal{U})$ be a family of subsets of $\mathcal{U}$.

Then τ is called a Classical Topology on $\mathcal{U}$ if it satisfies the following axioms: (CT-1) ϕ and $\mathcal{U}$ belong to τ.

(CT-2) The intersection of any finite number of elements in τ is in τ.

(CT-3) The union of any finite or infinite number of elements in τ is in τ.

All three axioms are totally (100%) true (or T = 1, I = 0, F = 0). We simply call them (classical) *Axioms*.

Then $(\mathcal{U}, \tau)$ is called a *Classical Topological Space* on $\mathcal{U}$.

3. NeutroSophication of the Topological Axioms

NeutroSophication of the topological axioms means that the axioms become partially true, partially indeterminate, and partially false. They are called NeutroAxioms.

(NCT-1) Either $\{\phi \notin \tau$ and $\mathcal{U} \in \tau\}$, or $\{\phi \in \tau$ and $\mathcal{U} \notin \tau\}$.

(NCT-2) There exist a finite number of elements in τ whose intersection belong to τ (degree of truth *T*); and a finite number of elements in τ whose intersection is indeterminate (degree of indeterminacy *I*); and a finite number of elements in τ whose intersection does not belong to τ (degree of falsehood *F*); where $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$ since (1, 0, 0) represents the above Classical Topology, while (0, 0, 1) the below AntiTopology.

(NCT-3) There exist a finite or infinite number of elements in τ whose union belongs to τ (degree of truth *T*); and a finite or infinite number of elements in τ whose union is indeterminate (degree of indeterminacy *I*); and a finite or infinite number of elements in τ whose union does not belong to τ (degree of falsehood *F*); where of course $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$.

4. AntiSophication of the Classical Topological Axioms

AntiSophication of the topological axioms means to negate (anti) the axioms, the axioms become totally (100%) false (or T = 0, I = 0, F = 1). They are called *AntiAxioms*.

(ACT-1) $\phi \notin \tau$ and $\mathcal{U} \notin \tau$.

(ACT-2) The intersection of any finite number ($n \geq 2$) of elements in τ is not in τ.

(ACT-3) The union of any finite or infinite number ($n \geq 2$) of elements in τ is not in τ.

5. <Topology, NeutroTopology, AntiTopology>

As such, we have a neutrosophic triplet of the form:

<Axiom(1, 0, 0), NeutroAxiom(*T, I, F*), AntiAxiom(0, 0, 1)>,

where $(T, I, F) \neq (1, 0, 0)$ and $(T, I, F) \neq (0, 0, 1)$.

Correspondingly, one has:

<Topology, NeutroTopology, AntiTopology>.

Therefore, in general:

(Classical) Topology is a topology that has all axioms totally true. We simply call them *Axioms*.

NeutroTopology is a topology that has at least one *NeutroAxiom* and the others are all *classical Axioms* [therefore, no AntiAxiom].

AntiTopology is a topology that has one or more *AntiAxioms*, no matter what the others are (*classical Axioms*, or *NeutroAxioms*).

6. Theorem on the number of Structures/NeutroStructures/AntiStructures

If a Structure has m axioms, with $m \geq 1$, then after NeutroSophication and AntiSophication one obtains 3^m types of structures, categorized as follows:

1*Classical Structure* + $(2^m - 1)$*NeutroStructures* + $(3^m - 2^m)$*AntiStructures* = 3^m *Structures.*

7. Consequence on the number of Topologies/NeutroTopologies/AntiTopologies

As a particular case of the previous theorem, from a Topology which has $m = 3$ axioms, one makes, after NeutrosSophication and AntiSophication, $3^3 = 27$ types of structures, as follows: 1 classical Topology, $2^3 - 1 = 7$ NeutroTopologies, and $3^3 - 2^2 = 19$ AntiTopologies.

1*Classical Topology* + 7*NeutroTopologies* + 19*AntiTopologies* = 3^3 Topologies are presented below:

There is 1 (one) type of Classical Topology, whose axioms are listed below:

1 *Classical Topology*

$$\begin{pmatrix} CT-1 \\ CT-2 \\ CT-3 \end{pmatrix}$$

8. Definition of NeutroTopology [4, 5]

It is a topology that has at least one topological axiom which is partially true, partially indeterminate, and partially false, or *(T, I, F)*, where T = True, *I* = Indeterminacy, *F* = False, and no topological axiom is totally false, in other words: $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$, where *(1, 0, 0)* represents the classical Topology, while *(0, 0, 1)* represents the below AntiTopology.

Therefore, the NeutroTopology is a topology in between the classical Topology and the AntiTopology.

There are 7 types of different NeutroTopologies, whose axioms, for each type, are listed below:

7 *NeutroTopologies*

$$\begin{pmatrix} NCT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$
$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$
$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}.$$

9. Definition of AntiTopology [4, 5]

It is a topology that has at least one topological axiom that is 100% false (T, I, F) = (0, 0, 1). The NeutroTopology and AntiTopology are particular cases of NeutroAlgebra and AntiAlgebra [4] and, in general, they all are particular cases of the NeutroStructure and AntiStructure respectively, since we consider "Structure" in any field of knowledge [5].

There are 19 types of different AntiTopologies, whose axioms, for each type, are listed below:

$$19\ AntiTopologies$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ CT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}.$$

10. Refined Neutrosophic Set

Let U be a universe of discourse, and a non-empty subset R of it,

$$R = \left\{ \begin{matrix} x\left(T_1(x), T_2(x), \ldots, T_p(x)\right); \\ \left(I_1(x), I_2(x), \ldots, I_r(x)\right); \\ \left(F_1(x), F_2(x), \ldots, F_s(x)\right); \end{matrix} \right\}$$

with all $T_j, I_k, F_l \in [0,1], 1 \le j \le p, 1 \le k \le r, 1 \le l \le s,$ and no restriction on their sums $0 \le T_m + I_m + F_m \le 3$, with $1 \le m \le \max\{p, r, s\}$, where *p, r, s* $\geq$ 0 are fixed integers, and at least one of them is $\geq$ *2,* in order to ensure the refinement (sub-parts) or multiplicity (multi-parts) – depending on the application, of at least one neutrosophic component amongst *T* (truth), *I* (indeterminacy), *F* (falsehood); and of course $x \in \mathcal{U}$.

By notation we consider that index zero means the empty-set, i.e. $T_0 = I_0 = F_0 = \phi$ (or zero), and the same for the missing sub-parts (or multi-parts).

For example, the below (2,3,1)-Refined Neutrosophic Set is identical to a (3,3,3)-Refined Neutrosophic Set: $(T_1, T_2; I_1, I_2, I_3; F_1) \equiv (T_1, T_2, 0; I_1, I_2, I_3; F_1, 0, 0)$, where the missing components T_3, and F_2, F_3 were replaced each of them by 0 (zero) *R* is called a *(p, r, s)-refined neutrosophic set* { or *(p, r, s)-RNT* }.

The neutrosophic set has been extended to the Refined Neutrosophic Set (Logic, and Probability) by Smarandache [1] in 2013, where there are multiple parts of the neutrosophic components, as such T was split into subcomponents T_1, T_2, ..., T_p, and I into I_1, I_2, ..., I_r, and F into F_1, F_2, ...,F_s, with p + r + s

= n ≥ 2 and integers p, r, s ≥ 0 and at least one of them is ≥ 2 in order to ensure the refinement (or multiplicity) of at least one neutrosophic component amongst T, I, and F.

Even more: the subcomponents T_j, I_k, and/or F_l can be countable or uncountable infinite subsets of [0, 1].

This definition also includes the *Refined Fuzzy Set*, when $r = s = 0$ and $p \geq 2$;

and the definition of the *Refined Intuitionistic Fuzzy Set*, when $r = 0$, and either $p \geq 2$ and $s \geq 1$, or $p \geq 1$ and $s \geq 2$.

All other fuzzy extension sets (Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.) can be refined/multiplicated in a similar way.

11. Definition of Refined Neutrosophic Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.
Let $\tau_{RNT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.
Then τ_{RNT} is called a *Refined Neutrosophic Topology* (*RNT*) if it satisfies the axioms:
(RNT-1) ϕ and $\mathcal{U}$ belong to τ_{RNT};
(RNT-2) The intersection of any finite number of elements in τ_{RNT} is in τ_{RNT};
(RNT-3) The union of any finite or infinite number of elements in τ_{RNT} is in τ_{RNT};
Then $(\mathcal{U}, \tau_{RNT})$ is called a Refined Neutrosophic Topological Space on $\mathcal{U}$.

The *Refined Neutrosophic Topology* is a topology defined on a Refined Neutrosophic Set.
{Similarly, the Refined Fuzzy Topology is defined on a Refined Fuzzy Set, while the Refined Intuitionistic Fuzzy Topology is defined on a Refined Intuitionistic Fuzzy Set, etc.
And, as a generalization, on any type of fuzzy extension set [such as: Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.] one can define a corresponding fuzzy extension topology}.

12. Neutrosophic Crisp Set

The *Neutrosophic Crisp Set* was defined by Salama and Smarandache in 2014 and 2015.

Let X be a non-empty fixed space. And let *D* be a Neutrosophic Crisp Set [2],

where $D = \langle A, B, C\rangle$, with *A, B, C* as subsets of *X*.

Depending on the intersections and unions between these three sets *A, B, C* one gets several:

Types of Neutrosophic Crisp Sets [2, 3].

The object having the form $D = \langle A, B, C\rangle$ is called:

(a) A neutrosophic crisp set of Type 1 (NCS-Type1) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ (empty set).

(b) A neutrosophic crisp set of Type 2 (NCS-Type2) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ and $A \cup B \cup C = X$.

(c) A neutrosophic crisp set of Type 3 (NCS-Type3) if it satisfies:
$A \cap B \cap C = \phi$ and $A \cup B \cup C = X$.

Of course, more types of Neutrosophic Crisp Sets may be defined by modifying the intersections and unions of the subsets *A*, *B*, and *C*.

13. Refined Neutrosophic Crisp Set

The *Refined Neutrosophic Crisp Set* [3] was introduced by Smarandache in 2019, by refining/multiplication of *D* (and denoting it by *RD* = Refined D) by refining/multiplication of its sets *A, B, C* into sub-subsets/multi-sets as follows:
$RD = (A_1, ..., A_p; B_1, ..., B_r; C_1, ..., C_s)$, with $p, r, s \geq 1$ be positive integers and at least one of them be ≥ 2 in order to ensure the refinement/multiplication of at least one component amongs A, B, C, where

$$A = \bigcup_{i=1}^{p} A_i, B = \bigcup_{j=1}^{r} B_j, C = \bigcup_{k=1}^{s} C_k$$

and many types of Refined Neutrosophic Crisp Sets may be defined by modifying the intersections or unions of the subsets/multisets $A_i, B_j, C_k, 1 \leq i \leq p, 1 \leq j \leq r, 1 \leq k \leq s$, depending on each application.

14. Definition of Refined Neutrosophic Crisp Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Let $\tau_{RNCT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Then τ_{RNCT} is called a *Refined Neutrosophic Crisp Topology* (*RNCT*) if it satisfies the axioms:
(RNCT-1) ϕ and $\mathcal{U}$ belong to τ_{RNCT};
(RNCT-2) The intersection of any finite number of elements in τ_{RNCT} is in τ_{RNCT};
(RNCT-3) The union of any finite or infinite number of elements in τ_{RNCT} is in τ_{RNCT}.
Then $(\mathcal{U}, \tau_{RNCT})$ is called a Refined Neutrosophic Crisp Topological Space on $\mathcal{U}$.
Therefore, the *Refined Neutrosophic Crisp Topology* is a topology defined on the Refined Neutrosophic Crisp Set.

15. Definition of the n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$.

The n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$ of the set *H*, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system *H* (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each of them in sub-sub-systems, and so on.
The n[th]-PowerSet $P^n(H)$ is defined recursively:

$$P^0(H) \overset{def}{=} H$$

$$P^1(H) = P(H)$$

$$P^2(H) = P(P(H))$$

$$P^3(H) = P(P^2(H)) = P(P(P(H)))$$

....

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where *P* is repeated *n* times into the last formula, and the empty-set ϕ (that represents indeterminacy, uncertainty) is allowed in all sequence terms:

$$H, P(H), P^2(H), P^3(H), ..., P^n(H).$$

Similarly,
The n[th]-PowerSet $P_*^n(H)$ is defined recursively:

$$P_*^0(H) \stackrel{def}{=} H$$
$$P_*^1(H) = P_*(H)$$
$$P_*^2(H) = P_*(P_*(H))$$
$$P_*^3(H) = P_*(P_*^2(H)) = P_*(P_*(P_*(H)))$$
....
$$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$$

where P is repeated n times into the last formula, and the empty-set ϕ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms: $H, P_*(H), P_*^2(H), P_*^3(H), ..., P_*^n(H)$.

16. SuperHyperOperation

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [6].

Let $P_*^n(H)$ be the n[th]-powerset of the set H such that none of $P(H), P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

Also, let $P_n(H)$ be the n[th]-powerset of the set H such that at least one of the $P(H), P^2(H), \ldots, P^n(H)$ contain the empty set ϕ. For any subset A, we identify {A} with A.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has classical-type SuperHyperOperations, or $P^n(H)$ and in this case one has Neutrosophic SuperHyperOperations, for integer $n \geq 2$.

17. The n[th]-PowerSet better describe our real world

The n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system H (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each in sub-sub-systems, and so on.

18. SuperHyperAxiom

A classical-type SuperHyperAxiom or more accurately a *(m, n)*-SuperHyperAxiom is an axiom based on classical-type SuperHyperOperations.

Similarly, a Neutrosophic SuperHyperAxiom {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.
There are:

- Strong SuperHyperAxioms, when the left-hand side is equal to the right-hand side as in non-hyper axioms.
- And Week SuperHyperAxioms, when the intersection between the left-hand side and the right-hand side is non-empty.

19. SuperHyperAlgebra and SuperHyperStructure

A SuperHyperAlgebra or more accurately *(m-n)*-SuperHyperAlgebra is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a Neutrosophic SuperHyperAlgebra {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have SuperHyperStructures {or *(m-n)*-SuperHyperStructures}, and corresponding Neutrosophic SuperHyperStructures.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

20. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

- If none of the power sets $P^k(H)$, $1 \le k \le n$, do not include the empty set ϕ, then one has a classical-type SuperHyperAlgebra;
- If at least one power set, $P^k(H)$, $1 \le k \le n$, includes the empty set ϕ, then one has a Neutrosophic SuperHyperAlgebra.

21. Definition of SuperHyperTopology (SHT) [6]

It is a topology designed on the nth-PowerSet of a given non-empty set H, that excludes the empty-set, denoted as $P_*^n(H)$, built as follows:

$P_*(H)$ is the first powerset of the set *H*, and the index $*$ means without the empty-set (Ø);

$P_*^2(H) = P_*(P_*(H))$ is the second powerset of *H* (or the powerset of the powerset of *H*), without the empty-sets; and so on, the *n*-th powerset of *H*,

$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$, where P_* is repeated *n* time ($n \ge 2$), and without the empty-sets.

Let consider τ_{SHT} be a family of subsets of $P_*^n(H)$.

Then τ_{SHT} is called a Neutrosophic SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) ϕ and $P_*^n(H)$ belong to τ_{SHT}.

(SHT-2) The intersection of any finite number of elements in τ_{SHT} is in τ_{SHT}.

(SHT-3) The union of any finite or infinite number of elements in τ_{SHT} is in τ_{SHT}.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

22. Definition of Neutrosophic SuperHyperTopology (NSHT) [6]

It is, similarly, a topology designed on the *n*-th PowerSet of a given non-empty set *H*, but includes the empty-sets [that represent indeterminacies] too.

As such, in the above formulas, $P_*(H)$ that excludes the empty-set, is replaced by $P(H)$ that includes the empty-set.

$P(H)$ is the first powerset of the set *H*, including the empty-set (Ø);

$P^2(H) = P(P(H))$ is the second powerset of *H* (or the powerset of the powerset of *H*), that includes the empty-sets; and so on, the *n*-th powerset of *H*,

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where P is repeated n times ($n \geq 2$), and includes the empty-sets (Ø).

Let consider τ_{NSHT} be a family of subsets of $P^n(H)$.

Then τ_{NSHT} is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:

(NSHT-1) ϕ and $P^n(H)$ belong to τ_{NSHT}.

(NSHT-2) The intersection of any finite number of elements in τ_{NSHT} is in τ_{NSHT}.

(NSHT-3) The union of any finite or infinite number of elements in τ_{NSHT} is in τ_{NSHT}.

Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

23. Introduction to NonStandard Analysis [9-12]

An ***infinitesimal*** [or infinitesimal number] (ε) is a number ε such that $|\varepsilon| < 1/n$, for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero. Infinitesimals are used in calculus.

An ***infinite*** [or infinite number] (ω) is a number greater than anything:

$1 + 1 + 1 + \ldots + 1$ (for any finite number terms)

The infinites are reciprocals of infinitesimals.

The set of ***hyperreals*** (or *non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$1/\varepsilon = \omega/1$.

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

24. First Extension of NonStandard Analysis [13]

Let's denote by R_+^* the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:

Left Monad { that we denote, for simplicity, by (^-a) or only ^-a } is defined as:

$\mu(^-a) = (^-a) = {}^-a = \overset{-}{a} = \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

Right Monad { that we denote, for simplicity, by (a^+) or only by a^+ } is defined as:

$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

Pierced Binad { that we denote, for simplicity, by $(^-a^+)$ or only $^-a^+$ } is defined as:

$\mu(^-a^+) = (^-a^+) = {}^-a^+ = \overset{-+}{a} =$

$= \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$

$= \{a \pm x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

The left monad, right monad, and the pierced binad are subsets of R^*.

25. Second Extension of NonStandard Analysis

For necessity of doing calculations that will be used in NonStandard neutrosophic logic in order to calculate the NonStandard neutrosophic logic operators (conjunction, disjunction, negation, implication, equivalence) and in order to have the NonStandard Real MoBiNad Set closed under arithmetic operations, Smarandache extended in 2019: the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left; and the Pierced Binad to the Unpierced Binad, defined as follows:

Left Monad Closed to the Right

$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \overset{-0}{a} = \{a - x \mid x = 0, \text{ or } x \in R_+^*$ and x is infinitesimal$\} = \mu(^-a) \cup \{a\} = (^-a) \cup \{a\}$
$= {}^-a \cup \{a\}$.

Right Monad Closed to the Left

$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \overset{0+}{a} = \{a + x \mid x = 0, \text{ or } x \in R_+^*$ and x is infinitesimal$\} = \mu(a^+) \cup \{a\} = (a^+) \cup \{a\}$
$= a^+ \cup \{a\}$.

Unpierced Binad

$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \overset{-0+}{a} = \{a - x \mid x \in R_+^*$ and x is infinitesimal$\}$
$\cup \{a + x \mid x \in R_+^*$ and x is infinitesimal$\} \cup \{a\} =$
$= \{a \pm x \mid x = 0, \text{ or } x \in R_+^*$ and x is infinitesimal$\}$
$= \mu(^-a^+) \cup \{a\} = (^-a^+) \cup \{a\} = {}^-a^+ \cup \{a\}$

The element *{a}* has been included into the left monad, right monad, and pierced binad respectively.

26. NonStandard Neutrosophic Topology

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval

$I_{nontandard} =]^-0,1^+[$,

we have founded [13] since 1998, and we have previously [13-15] proposed it, where:

$I_{nontandard} =]^-0,1^+[= \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; 0 \le x \le, x \in R\}$, where *R* is the set of real numbers.

Let $P(]^-0,1^+[)$ be the powerset of $]^-0,1^+[$.

Let τ = $P(]^-0,1^+[)$, which means that τ is the family of all subsets of $P(]^-0,1^+[)$. Of course:

(i). ϕ and $]^-0,1^+[$ belong to τ.
(ii). The intersection of any finite number of elements in τ is in τ.
(iii). The union of any number of finite or infinite number of elements in τ is in τ.

Therefore, τ is a NonStandard Neutrosophic Topology.

Then ($]^-0,1^+[$, τ) is called a NonStandard Neutrosophic Topological Space.

27. NonStandard Topology

As a generalization of NonStandard Neutrosophic Topology one propose now the NonStandard Topology.

Let's consider the real numbers $a, b \in R$ and the real interval $[a, b]$. Let's extend it to a non-standard interval $]^{-}a, b^{+}[$ is the same way as for the NonStandard Neutrosophic Logic and Set.

Let's have by convention the same meaning of the following notations:

$x \overset{def}{=} \overset{0}{x}$, and $^{-}x = \overset{-}{x}$, also $x^{+} = \overset{+}{x}$ for any real number *x*.

Then:

$]^{-}a, b^{+}[= \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; a \le x \le b, x \in R\}$, where *R* is the set of real numbers.

Let $U_{NonStandard} = \,]^{-}a, b^{+}[$ be a NonStandard interval, for $a < b$, where *a* and *b* are real numbers, and $P(U_{NonStandard})$ be the power set of $U_{NonStandard}$.

Then $P(U_{NonStandard})$ is formed by the empty set (ϕ) and itself $U_{NonStandard}$, together with all standard and NonStandard subsets of $]^{-}a, b^{+}[$.

The finite intersections, and finite or infinite unions of any standard and NonStandard subsets are still (standard or NonStandard) subsets of $U_{NonStandard}$.

Let $\tau_{NonStandard} \subseteq P(U_{NonStandard})$ be a family of standard or NonStandard subsets of $P(U_{NonStandard})$.

Then $\tau_{NonStandard}$ is called a NonStandard Topology on $U_{NonStandard}$ if it satisfies the following axioms:

(i). The empty set (ϕ) and $U_{NonStandard}$ belong to $\tau_{NonStandard}$.

(ii). The intersection of finite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

(iii). The union of any finite or infinite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

Then ($U_{NonStandard}$, $\tau_{NonStandard}$) is called a NonStandard Topological Space.

28. Extended NonStandard Real Set ($\overset{-0+}{ER}$)

We introduce it now for the first time:

$\overset{-0+}{ER} = \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; x \in R\}$, actually:

$\overset{-0+}{ER} = \overset{0}{R} \cup \overset{-}{R} \cup \overset{+}{R} \cup \overset{-0}{R} \cup \overset{0+}{R} \cup \overset{-+}{R} \cup \overset{-0+}{R}$,

where one uses the notations:

$$\overset{0}{R} \overset{def}{\equiv} R$$

$$\overset{-}{R} = \{\overset{-}{x}, x \in R\}$$

$$\overset{+}{R} = \{\overset{+}{x}, x \in R\}$$

$$\overset{-0}{R} = \{\overset{-0}{x}, x \in R\}$$

$$\overset{0+}{R} = \{\overset{0+}{x}, x \in R\}$$

$$\overset{-+}{R} = \{\overset{-+}{x}, x \in R\}$$

$$\overset{-0+}{R} = \{\overset{-0+}{x}, x \in R\}$$

29. Largest Extended NonStandard Real Topology

$P\left(\overset{-0+}{ER}\right)$, which is the powerset of $\overset{-0+}{ER}$, generates the Largest Extended NonStandard Real Topology on the whole Extended NonStandard Real Set $\overset{-0+}{ER}$.

30. Over/Under/Off-Sets and Logics and Probabilities

The Neutrosophic Set was extended [Smarandache, 2007] to Neutrosophic Overset (when some Neutrosophic component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;

and to Neutrosophic Underset (when some Neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0;

and to and to Neutrosophic Offset (when some Neutrosophic components are off the interval [0, 1], i.e. some Neutrosophic component > 1 and some Neutrosophic component < 0).

Similarly for Over/Under/Off-Logic and respectively Over/Under/Off-Topology [16 - 19].

Since these ideas look counter-intuitive and totally different from the mainstream framework, we present below elementary examples from our real world of such degrees that are outside the box {we mean outside the interval [0, 1]}.

31. Real Example of OverMembership and UnderMembership

In a company a full-time employer works 40 hours per week. Let's consider the last week period.

Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 hours, so he had the membership degree 40/40 = 1, with respect to this company.

But George worked overtime 5 hours, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1.

Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was 0/40 = 0.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was - 20/40 = - 0.50 < 0.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Unde-r/Off-Logic, -Measure, -Probability, - Statistics* etc. (Smarandache, 2007).

32. Definition of the Single-Valued Neutrosophic OverSet

Let U_{over} be an OverUniverse of Discourse {i.e. there exist some elements in U_{over} whose degrees of membership are > 1 }, and the Neutrosophic OverSet $A_{over} \subseteq U_{over}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{over}$, with respect to the Neutrosophic OverSet A_{over}:

$T(x), I(x), F(x) : U_{over} \to [0,\Omega]$ where $0 < 1 < \Omega$, and Ω is called OverLimit,

$T(x), I(x), F(x) \in [0,\Omega]$, for all $x \in U_{over}$.

A Single-Valued Neutrosophic OverSet A_{over} is defined as:

$A_{over} = \{(x, <T(x), I(x), F(x)>), x \in U_{over}\}$, such that there exist some elements in A_{over} that have at least one neutrosophic component that is > 1.

33. Definition of the Single-Valued Neutrosophic OverTopology

Let U_{over} be an OverUniverse of Discourse, and $P(U_{over})$ the powerset of U_{over}.

Let $\tau_{over} \subseteq P(U_{over})$ be a family of Single-Valued Neutrosophic OverSets of U_{over}.

Then τ_{over} is called a Single-Valued Neutrosophic OverTopology on U_{over} if it satisfies the following axioms:

(i). ϕ and U_{over} belong to τ_{over}.

(ii). The intersection of any finite number of single-valued Neutrosophic OverSets in τ_{over} is in τ_{over}.

(iii). The union of any finite or infinite number of single-valued Neutrosophic OverSets in τ_{over} is in τ_{over}.

Then (U_{over}, τ_{over}) is called a Neutrosophic OverTopological Space.

34. Definition of the Single-Valued Neutrosophic UnderSet

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let U_{under} be an UnderUniverse of Discourse { i.e. there exist some elements in U_{under} whose degrees of membership are < 0 }, and the Neutrosophic UnderSet $A_{under} \subseteq U_{under}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{under}$, with respect to the Neutrosophic UnderSet A_{under}:

$T(x), I(x), F(x) : U_{under} \to [\Psi,1]$

where $\Psi < 0 < 1$, and Ψ is called UnderLimit,

$T(x), I(x), F(x) \in [\Psi,1]$, for all $x \in U_{under}$.

A Single-Valued Neutrosophic UnderSet A_{under} is defined as:

A_{under} = *{(x, <T(x), I(x), F(x)>), x* $\in U_{under}$*}*, such that there exist some elements in A_{under} that have at least one neutrosophic component that is < 0.

35. Definition of the Single-Valued Neutrosophic UnderTopology

Let U_{under} be an UnderUniverse of Discourse, and $P(U_{under})$ the powerset of U_{under}.

Let $\tau_{under} \subseteq P(U_{under})$ be a family of Single-Valued Neutrosophic UnderSets of U_{under}.

Then τ_{under} is called a Single-Valued Neutrosophic UnderTopology on U_{under} if it satisfies the following axioms:

(i). ϕ and U_{under} belong to τ_{under}.

(ii). The intersection of any finite number of single-valued neutrosophic undersets in τ_{under} is in τ_{under}.

(iii). The union of any finite or infinite number of single-valued neutrosophic undersets in τ_{under} is in τ_{under}.

Then (U_{under}, τ_{under}) is called a Neutrosophic UnderTopological Space.

36. Definition of the Single-Valued Neutrosophic OffSet

Let U_{off} be an OffUniverse of Discourse {i.e. there exist elements of U_{off} whose degrees of membership are outside the interval [0, 1], some < 0 and others > 1}, and the Neutrosophic OffSet $A_{off} \subseteq U$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{off}$, with respect to the neutrosophic offset A_{off}:

$$T(x), I(x), F(x) : U_{off} \rightarrow [\Psi, \Omega]$$

where $\Psi < 0 < 1 < \Omega$, and Ψ is called UnderLimit, while Ω is called OverLimit, $T(x), I(x), F(x) \in [\Psi, \Omega]$, for all $x \in U_{off}$.

A Single-Valued Neutrosophic Offset A_{off} is defined as:

A_{off} = {(x, <T(x), I(x), F(x)>), x ∈ U_{off}}, such that there exist some elements in A_{off} that have at least one neutrosophic component that is > 1, and at least one neutrosophic component that is < 0.

37. Definition of the Single-Valued Neutrosophic OffTopology

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let U_{off} be an OffUniverse of Discourse, and $P(U_{off})$ the powerset of U_{off}.

Let $\tau_{off} \subseteq P(U_{off})$ be a family of Single-Valued Neutrosophic OffSets of U_{off}.

Then τ_{off} is called a Single-Valued Neutrosophic OffTopology on U_{off} if it satisfies the following axioms:

(i). ϕ and U_{off} belong to τ_{off}.

(ii). The intersection of any finite number of single-valued neutrosophic offsets in τ_{off} is in τ_{off} .

(iii). The union of any finite or infinite number of single-valued neutrosophic offsets in τ_{off} is in τ_{off} .

Then (U_{off} , τ_{off}) is called a Neutrosophic OffTopological Space.

38. Neutrosophic Triplet Weak/Strong Set (N)

Let *(N, *)* be a groupoid, or non-empty set endowed with a well-defined binary operation *.
A **Neutrosophic Triplet** is an object of the form <*x, neut(x), anti(x)*>, for $x \in N$,
where $neut(x) \in N$ is the neutral of *x*, different from the classical algebraic unitary element if any, such that:

$$x * neut(x) = neut(x) * x = x$$

and $anti(x) \in N$ is the opposite of x such that:

$$x * anti(x) = anti(x) * x = neut(x).$$

In general, an element x may have more neutrals (*neut*'s) and more opposites (*anti*'s).
The neutrosophic triplets and their neutrosophic triplet algebraic structures were first introduced by Florentin Smarandache and Mumtaz Ali [20 - 23] in 2014 - 2016.

39. Definition of the Neutrosophic Triplet Weak Set (NTS, *) is a set such that each element $a \in NTS$ is part of a neutrosophic triplet <*b, neut(b), anti(b)*>, i.e. $a = b$, or $a = neut(b)$, or $a = anti(b)$.

40. Definition of the Single-Valued Neutrosophic Triplet Weak Topology

Let $U_{Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Weak Set, and $P(U_{Triplet-Weak})$ the powerset of $U_{Triplet-Weak}$.
Let $\tau_{Triplet-Weak} \subseteq P(U_{Triplet-Weak})$ be a family of Single-Valued Neutrosophic Triplet Weak Sets of $U_{Triplet-Weak}$.
Then $\tau_{Triplet-Weak}$ is called a Single-Valued Neutrosophic Triplet Weak Topology on $U_{Triplet-Weak}$ if it satisfies the following axioms:

(i). ϕ and $U_{Triplet-Weak}$ belong to $\tau_{Triplet-Weak}$.

(i). The intersection of any finite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

(ii). The union of any finite or infinite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

Then ($U_{Triplet-Weak}$, $\tau_{Triplet-Weak}$) is called a Neutrosophic Triplet Weak Topological Space.

41. Definition of Neutrosophic Triplet Strong Set (or Neutrosophic Triplet Set)

The groupoid *(N, *)* is called a neutrosophic triplet strong set if for any $a \in N$ there exist some neutral of *a*, denoted $neut(a) \in N$, different from the classical algebraic unitary element (if any), and some opposite of *a*, called $anti(a) \in N$.

Table 1. Example of Neutrosophic Triplet Strong Set.

*	1	2
1	2	1
2	1	1

The set ({1,2}, *) is a groupoid, without classical unit element.
Then <1, 2, 1> and <2, 1, 2> and are neutrosophic triplets.
The neutrosophic triplet strong set is N = {1, 2}.

42. Theorem on the Neutrosophic Triplet Strong and Weak Sets

Any neutrosophic triplet strong set is a neutrosophic triplet weak set, but not conversely.
Proof.

Let *(N, *)* be a neutrosophic triplet strong set. If $a \in N$, then is also included in N, therefore there exists a neutrosophic triplet in *N* that includes a, whence N is a neutrosophic triplet weak set.

Conversely, we prove by using a counterexample.

Let $Z_3 = \{0, 1, 2\}$, embedded with the multiplication × modulo *3*, which is a well-defined law. The classical unitary element in Z_3 is *1*.

$(Z_3, \times)$ is a neutrosophic triplet weak set, since the neutrosophic triplets formed in Z_3 with respect to the law × contain all elements *0, 1, 2,*

i.e. *<0, 0, 0>*, *<0, 0, 1>*, and *<0, 0, 2>.*

But $(Z_3, \times)$ is not a neutrosophic triplet strong set, since, for example, for $2 \in Z_3$ there is no $neut(2) \neq 1$ and no *anti(2).*

43. Definition of the Single-Valued Neutrosophic Triplet Strong Topology

Let $U_{Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Strong Set, and $P(U_{Triplet-Strong})$ the powerset of $U_{Triplet-Strong}$.

Let $\tau_{Triplet-Strong} \subseteq P(U_{Triplet-Strong})$ be a family of Single-Valued Neutrosophic Triplet Strong Sets of $U_{Triplet-Strong}$.

Then $\tau_{Triplet-Strong}$ is called a Single-Valued Neutrosophic Triplet Strong Topology on $U_{Triplet-Strong}$ if it satisfies the following axioms:

(i). ϕ and $U_{Triplet-Strong}$ belong to $\tau_{Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

Then ($U_{Triplet-Strong}$, $\tau_{Triplet-Strong}$) is called a Neutrosophic Triplet Strong Topological Space.

44. Neutrosophic Extended Triplet

A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the *neutral* of *x* {denoted by $_e neut(x)$ and called "extended neutral", where "e" in front stands for 'extended'} is allowed to be equal to the classical algebraic unitary element (if any) of the law * defined on the set. Therefore, the restriction "different from the classical algebraic unitary element if any" is released.

Thus, a neutrosophic extended triplet is an object of the form <*x*, *eneut(x)*, *eanti(x)*>, for $x \in N$, where $_e neut(x) \in N$ is the *extended neutral* of *x*, which can be equal or different from the classical algebraic unitary element if any, such that:

$X * {}_e neut(x) = {}_e neut(x) * x = x$

and $anti(x) \in N$ is the *extended opposite* of *x* such that:

$x * {}_e anti(x) = {}_e anti(x) * x = {}_e neut(x)$.

In general, for each $x \in$ N there are exist many *eneut*'s (extended neutrals) and *eanti*'s (extended opposites). The neutrosophic extended triplets were introduced by Smarandache in *2016*.

45. Definition of Neutrosophic Extended Triplet Weak Set

The set *N* is called a neutrosophic extended triplet weak set if for any $x \in N$ there exist a neutrosophic extended triplet <*y*, *eneut(y)*, *eanti(y)*> included in *N*, such that $x = y$ or $x = {}_e neut(y)$ or $x = {}_e anti(y)$.

46. Definition of the Single-Valued Neutrosophic Extended Triplet Weak Topology

Let $U_{Extended-Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Weak Set, and $P(U_{Extended-Triplet-Weak})$ the powerset of $U_{Extended-Triplet-Weak}$.

Let $\tau_{Extended-Triplet-Weak} \subseteq P(U_{Extended-Triplet-Weak})$ be a family of Single-Valued Neutrosophic Extended Triplet Weak Sets of $U_{Extended-Triplet-Weak}$.

Then $\tau_{Extended-Triplet-Weak}$ is called a Single-Valued Neutrosophic Extended Triplet Weak Topology on $U_{Extended-Triplet-Weak}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Triplet-Weak}$ belong to $\tau_{Extended-Triplet-Weak}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

Then ($U_{Extended-Triplet-Weak}$, $\tau_{Extended-Triplet-Weak}$) is called a Neutrosophic Extended Triplet Weak Topological Space.

47. Definition of Neutrosophic Extended Triplet Strong Set

The set *N* is called a neutrosophic extended triplet strong set if for any $x \in N$ there exist $_e neut(x) \in N$ and $_e anti(x) \in N$.

48. Definition of the Single-Valued Neutrosophic Extended Triplet Strong Topology

Let $U_{Extended-Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Strong Set, and $P(U_{Extended-Triplet-Strong})$ the powerset of $U_{Extended-Triplet-Strong}$.

Let $\tau_{Extended-Triplet-Strong} \subseteq P(U_{Extended-Triplet-Strong})$ be a family of Single-Valued Neutrosophic Extended Triplet Strong Sets of $U_{Extended-Triplet-Strong}$.

Then $\tau_{Extended-Triplet-Strong}$ is called a Single-Valued Neutrosophic Extended Triplet Strong Topology on $U_{Extended-Triplet-Strong}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Triplet-Strong}$ belong to $\tau_{Extended-Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

Then ($U_{Extended-Triplet-Strong}$, $\tau_{Extended-Triplet-Strong}$) is called a Neutrosophic Extended Triplet Strong Topological Space.

49. Neutrosophic Duplets

The Neutrosophic Duplets and the Neutrosophic Duplet Algebraic Structures were introduced by Florentin Smarandache in 2016.

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #.

50. Definition of the Neutrosophic Duplet

We say that <*a, neut(a)*>, where *a*, and its neutral *neut(a)* belong to D, is a neutrosophic duplet if:

(i). *neut(a)* is different from the unitary element of *D* with respect to the law # (if any);

(ii). *a # neut(a) = neut(a) # a = a;*

(iii). there is no opposite *anti(a)* belonging to *D* for which *a # anti(a) = anti(a) # a = neut(a)*.

51. Example of Neutrosophic Duplets

In *(Z₈, #)*, the set of integers with respect to the regular multiplication modulo 8, one has the following neutrosophic duplets:

<2, 5 >, <4, 3>, <4, 5>, <4, 7>, and <6, 5>.

Proof:

Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, having the unitary element *1* with respect to the multiplication *# modulo 8*.

2 # 5 = 5 # 2 = 10 = 2 (mod 8), so *neut(2) = 5 ≠ 1*.

There is no *anti(2)* ∈ Z_8, because:

2 # *anti(2) = 5 (mod 8)*, or *2y = 5 (mod 8)* by denoting *anti(2) = y*, is equivalent to:

2y - 5 = M_8 {multiple of *8*}, or *2y - 5 = 8k*, where *k* is an integer, or *2(y - 4k) = 5*, where both *y* and *k* are integers, or: e*ven number = odd number*, which is impossible.

Therefore, we proved that <*2, 5*> is a neutrosophic duplet.

Similarly for <*4, 5*>, <*4, 3*>, <*4, 7*>, and <*6, 5*>.

A counter-example: <*0, 0*> is not a neutrosophic duplet, because it is a neutrosophic triplet: <*0, 0, 0*>, where there exists an *anti(0) = 0*.

52. Definition of the Single-Valued Neutrosophic Duplet Topology

Let U_{Duplet} be a Universe of Discourse which has the structure of a Neutrosophic Duplet Set, and $P(U_{Duplet})$ the powerset of U_{Duplet}.

Let $\tau_{Duplet} \subseteq P(U_{Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of U_{Duplet}.

Then τ_{Duplet} is called a Single-Valued Neutrosophic Duplet Topology on U_{Duplet} if it satisfies the following axioms:

(i). ϕ and U_{Duplet} belong to τ_{Duplet}.

(ii). The intersection of any finite number of single-valued neutrosophic duplet sets in τ_{Duplet} is in τ_{Duplet}.

(iii). The union of any finite or infinite number of single-valued neutrosophic duplet sets in τ_{Duplet} is in τ_{Duplet}.

Then (U_{Duplet}, τ_{Duplet}) is called a Neutrosophic Duplet Topological Space.

53. Definition of the Neutrosophic Extended Duplet

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #. We say that <*a*, $_e$*neut(a)*>, where *a*, and its extended neutral $_e$*neut(a)* belong to *D*, such that:

(i). $_e$*neut(a)* may be equal or different from the unitary element of *D* with respect to the law # (if any);

(ii). *a* # $_e$*neut(a)* = $_e$*neut(a)* # *a* = *a;*

(iii). There is no extended opposite $_e$*anti(a)* belonging to *D* for which
a # $_e$*anti(a)* = $_e$*anti(a)* # *a* = $_e$*neut(a)*.

54. Definition of the Single-Valued Neutrosophic Extended Duplet Topology

Let $U_{Extended-Duplet}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Duplet Set, and $P(U_{Extended-Duplet})$ the powerset of $U_{Extended-Duplet}$.

Let $\tau_{Extended-Duplet} \subseteq P(U_{Extended-Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of $U_{Extended-Duplet}$.

Then $\tau_{Extended-Duplet}$ is called a Single-Valued Neutrosophic Duplet Topology on $U_{Extended-D}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Duplet}$ belong to $\tau_{Extended-Duplet}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

Then ($U_{Extended-Duplet}$, $\tau_{Extended-Duplet}$) is called a Neutrosophic Extended Duplet Topological Space.

55. Definition of Neutrosophic MultiSet

The Neutrosophic MultiSet and the Neutrosophic Multiset Algebraic Structures were introduced by Florentin Smarandache [23] in 2016.

Let $\mathcal{U}$ be a universe of discourse, and a set $M \subseteq U$.

A *Neutrosophic Multiset M* is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

It is an extension of the classical multiset, fuzzy multiset, intuitionistic fuzzy multiset, etc.

56. Examples of Neutrosophic MultiSets

A = {(0.6, 0.3, 0.1), (0.8, 0.4, 0.2), c(0.5, 0.1, 0.3)} is a neutrosophic set (not multiset).
But B = {(0.6, 0.3, 0.1),(0.6, 0.3, 0.1),b(0.8, 0.4, 0.2)} is a neutrosophic multiset, since the element a is repeated; we say that the element a has the *neutrosophic multiplicity* 2 with the same neutrosophic components.
While C = {(0.6, 0.3, 0.1), (0.7, 0.1, 0.2), a(0.5, 0.4, 0.3), c(0.5, 0.1, 0.3)} is also a neutrosophic multiset, because the element a is repeated (it has the *neutrosophic multiplicity* 3), but with different neutrosophic components, since, for example, during the time, the neutrosophic membership of an element may change.

If the element a is repeated k times, keeping the same neutrosophic components ($ta,,fa$), we say that a has *multiplicity* k.
But if there is some change in the neutrosophic components of a, we say that a has the *neutrosophic multiplicity* k.

Therefore, we define in general the *Neutrosophic Multiplicity Function (nm)*:

$nm: \mathcal{U} \to \mathbb{N} = \{1, 2, 3, \ldots, \infty\}$, and for any $a \in A$ one has

$(a) = \{(k_1, \langle t_1, i_1, f_1\rangle), (k_2, \langle t_2, i_2, f_2\rangle), \ldots, (k_j, \langle t_j, i_j, f_j\rangle), \ldots\}$ which means that

a is repeated k_1 times with the neutrosophic components $\langle t_1, i_1, f_1\rangle$;

a is repeated k_2 times with the neutrosophic components $\langle t_2, i_2, f_2\rangle$, ...,

a is repeated k_j times with the neutrosophic components $\langle t_j, i_j, f_j\rangle$, ..., and so on.

Then, a neutrosophic multiset A can be written as:

$$A = \{(a, (a)), \text{ for } a \in A)\}.$$

57. Examples of operations with neutrosophic multisets

Let's have:

$A = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.4, 0.1, 0.3\rangle, 6\langle 0.2, 0.7, 0.0\rangle\}$;

$B = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.8, 0.1, 0.1\rangle, 6\langle 0.9, 0.0, 0.0\rangle\}$;

$C = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle\}$.

Then:

Intersection of Neutrosophic Multisets.

$A \cap B = \{5\langle 0.6, 0.3, 0.2\rangle\}$.

Union of Neutrosophic Multisets

$A \cup B = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.4, 0.1, 0.3\rangle, 5\langle 0.8, 0.1, 0.1\rangle, 6\langle 0.2, 0.7, 0.0\rangle, 6\langle 0.9, 0.0, 0.0\rangle\}$.

Inclusion of Neutrosophic Multisets

$C \subset A$, but $C \not\subset B$.

58. Definition of the Single-Valued Neutrosophic MultiSet Topology

Let $U_{MultiSet}$ be a Universe of Discourse which has the structure of a Neutrosophic MultiSet, and $P(U_{MultiSet})$ the powerset of $U_{MultiSet}$.

Let $\tau_{MultiSet} \subseteq P(U_{MultiSet})$ be a family of Single-Valued Neutrosophic MultiSets of $U_{MultiSet}$.

Then $\tau_{MultiSet}$ is called a Single-Valued Neutrosophic MultiSet Topology on $U_{MultiSet}$ if it satisfies the following axioms:

(i). ϕ and $U_{MultiSet}$ belong to $\tau_{MultiSet}$.

(ii). The intersection of any finite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

Then ($U_{MultiSet}$, $\tau_{MultiSet}$) is called a Neutrosophic MultiSet Topological Space.

59. Conclusion

These eight new avantgarde topologies, together with the previous six new topologies and their corresponding topological space, were introduced by Smarandache in 2019-2023, but they have not yet been much studied and applied, except the NeutroTopologies and AntiTopologies [8] which got some attention from researchers. While NonStandard Neutrosophic Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology are proposed now for the first time. As future research would be to study their large applications in our real world.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Smarandache, F. n-valued Refined Neutrosophic Set and Logic and its Applications in Physics. Progress in Physics, 143-146.
2. Salama, A. A., & Smarandache, F. (2015). Neutrosophic crisp set theory. Infinite Study.
3. Smarandache, F. (2019). Refined Neutrosophic Crisp Set (RNCS), in the book Nidus Idearum, 2, 114-116.
4. Smarandache, F., & Al-Tahan, M. (2022). NeutroAlgebra and AntiAlgebra Are Generalizations of Classical Algebras. In Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras (pp. 1-10). IGI Global.
5. Smarandache, F. (2021). Structure, NeutroStructure, and AntiStructure in Science. International Journal of Neutrosophic Science, 13(1): 28-33. https://doi.org/10.54216/IJNS.130104.
6. Smarandache, F. (2022). The SuperHyperFunction and the Neutrosophic SuperHyperFunction. Neutrosophic Sets and Systems, 49, 594-600. http://fs.unm.edu/NSS/SuperHyperFunction37.pdf.
7. Smarandache, F. (2023). New Types of Topologies and Neutrosophic Topologies. Neutrosophic Systems With Applications, 1, 1–3. https://doi.org/10.61356/j.nswa.2023.1.
8. Witczak, T. (2023). Interior and closure in anti-minimal and anti-biminimal spaces in the frame of anti-topology. Neutrosophic Sets and Systems, 56(1), 42-440. https://fs.unm.edu/NSS/InteriorClosure29.pdf.
9. Robinson, A. (2016). Non-standard analysis. Princeton University Press.
10. Insall, M., & Weisstein, E. W. Nonstandard Analysis. From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/NonStandardAnalysis.html.

11. Insall, Matt. "Transfer Principle." From MathWorld--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/TransferPrinciple.html
12. Smarandache, F. (2019). Advances of standard and nonstandard neutrosophic theories, PONS Publishing House Brussels, Belgium. https://fs.unm.edu/AdvancesOfStandardAndNonStandard.pdf
13. Smarandache, F. (1998- 2007). A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics (sixth edition), USA, https://fs.unm.edu/eBook-Neutrosophics6.pdf (sixth edition online).
14. Smarandache, F. About NonStandard Neutrosophic Logic (Answers to Imamura 'Note on the Definition of Neutrosophic Logic'), pp. 1-16, Cornell University, New York City, USA, (Submitted on 24 Nov 2018 (v1), last revised 13 Feb 2019 (this version, v2)) Abstract: https://arxiv.org/abs/1812.02534v2, Full paper: https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf
15. Smarandache, F. (2011). A Geometric Interpretation of the Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set, 2011 IEEE International Conference on Granular Computing, edited by TzungPei Hong, Yasuo Kudo, Mineichi Kudo, Tsau-Young Lin, Been-Chian Chien, Shyue-Liang Wang, Masahiro Inuiguchi, GuiLong Liu, IEEE Computer Society, National University of Kaohsiung, Taiwan, 602-606, http://fs.unm.edu/IFS-generalized.pdf.
16. Smarandache, F. (2016). Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset / Similarly for Neutrosophic Over-/Under-/OffLogic, Probability, and Statistics, Pons Editions, Brussels, Belgium, 170 pages book. https://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf
17. Smarandache, F. (2016). Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets, Journal of Mathematics and Informatics, 5, 63-67. https://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf.
18. Smarandache, F. (2016). Interval-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets, International Journal of Science and Engineering Investigations, 5(54), 1-4. https://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf
19. Smarandache, F. (2016). Degrees of Membership > 1 and < 0 of the Elements with Respect to a Neutrosophic OffSet, Neutrosophic Sets and Systems, 12, 3-8. https://fs.unm.edu/NSS/DegreesOf-Over-Under-Off-Membership.pdf
20. Smarandache, F., & Ali, M. (2018). Neutrosophic triplet group. Neural Computing and Applications, 29(7), 595-601. https://doi.org/10.1007/s00521-016-2535-x.
21. Smarandache, F., & Ali, M. (2016). Neutrosophic triplet as extension of matter plasma, unmatter plasma, and antimatter plasma. In APS Annual Gaseous Electronics Meeting Abstracts, 6-110. http://meetings.aps.org/Meeting/GEC16/Session/HT6.111.
22. Smarandache, F. and Ali, M. (2014). The Neutrosophic Triplet Group and its Application to Physics, presented by F. S. to Universidad Nacional de Quilmes, Department of Science and Technology, Bernal, Buenos Aires, Argentina.
23. Smarandache, F. (2017). Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras and Applications. http://fs.unm.edu/NeutrosophicPerspectives-ed2.pdf.

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The Energy of Interval-Valued Complex Neutrosophic Graph Structures: Framework, Application and Future Research Directions

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Abstract: Graph structure is a developing field with many real-world applications and advancements, particularly effective frameworks for integrative problem-solving in computer networks and artificial intelligence systems. To define the idea of an Interval-Valued Complex Neutrosophic Graph Structure (IVCNGS), the concept of an Interval-Valued Complex Neutrosophic Set (IVCNS) is applied to the graph structure. Using the adjacency matrix to calculate the degree of vertex, we have defined some findings about the IVCNGS. Further, we compute the energy and Laplacian energy of IVCNGS. Moreover, we derive the lower and upper bounds for the energy and Laplacian energy of IVCNGS, and we have discussed their application in IVCNGS. Finally, we develop an algorithm that clarifies the fundamental processes of the application.

Keywords: Graph Structure; Interval-Valued Complex Neutrosophic Graph Structure; Energy and Laplacian Energy; Applications.

1. Introduction

Real-world problems with uncertainty and ambiguity are not always amenable to the standard techniques of classical mathematics. The concept of a fuzzy set (FS) was first proposed by Zadeh [1] in 1965 as an extension of the conventional notion of sets. A gradual determination of an element's membership in a set is allowed by the fuzzy set theory, as represented by a membership function with a value in the real unit interval [0, 1]. Since then, numerous scholars have investigated the concept of fuzzy logic and fuzzy sets to resolve a range of ambiguous and uncertain real-world problems. Interval-valued fuzzy sets are the development that the author initiated in Turksen [2] in 1986. As a result of using numbers as the membership function, it also takes into account the values of number intervals to account for uncertainty. Usually, it is indicated by the symbol $[\mu_{AL}^{-}(x), \mu_{AU}^{+}(x)]$. Use the equation $0 \leq \mu_{AL}^{-}(x) + \mu_{AU}^{+}(x) \leq 1$ to represent the degree of membership of the fuzzy set A.

Likewise, the membership function is single-valued and it is not always possible to use it to capture both support and objection evidence. The intuitionistic fuzzy set (IFS) was developed by Atanassov [3] as a generalization of Zadeh's fuzzy set. IFS, which has both a membership and a non-membership function, can be created by deriving a new component, the degree of membership and non-membership, from the fuzzy set's properties. When defining intuitionistic fuzzy sets, he also included interval-valued intuitionistic fuzzy sets [4] for representing uncertainty, interval-valued intuitionistic fuzzy sets instead of traditional fuzzy sets are preferred. Defuzzification, a technique employed in fuzzy control in many ways, is the phase of the process that needs the most processing.

To interpret the degree of true and false membership functions, it is defined as a pair of intervals $[\mu^-, \mu^+], 0 \leq \mu^- + \mu^+ \leq 1$ and $[\lambda^-, \lambda^+], 0 \leq \lambda^- + \lambda^+ \leq 1$ with $0 \leq \mu^+ + \lambda^+ \leq 1$.

On the other hand, erroneous, inconsistent, and incomplete periodic information cannot be handled by FSs, IFSs, or IVIFSs. Although these theories have applications in many different scientific domains, they are all hampered by the inability to accurately describe two-dimensional events. Ramot [5] proposed the concept of a complex fuzzy set (CFS) in 2012 to address this problem. A helpful generalization of FS is the membership grade of this concept, which is expressed as $re^{i\theta}$, where r stands for the amplitude term and θ for the phase term. Values are restricted to only derived from the complex plane's unit circle. The phase term of CFS is significant since it is better equipped to control cyclical difficulties or recurrent troublesome phenomena. There will undoubtedly be circumstances where the second dimension is required because the phase term is present in CFS. This phrase distinguishes CFS from every other kind of information that is currently available. This use best exemplifies the original notion with a CF representation of solar activity. The concepts of complex intuitionistic fuzzy sets (CIFSs), which they translated to complex intuitionistic fuzzy sets using the degree of complex-valued non-membership functions, were initially described by Alkouri and Salleh [6] in 2012. Complex Interval-Valued Intuitionistic Fuzzy Sets (CIVIFSs) and its associated Aggregation Operator are novel concepts introduced by Harish Garg and Dimple Rani [9]. It is defined as a pair of intervals $[\mu^- e^{i\alpha^-}, \mu^+ e^{i\alpha^+}], 0 \leq \mu^- + \mu^+ \leq 1, 0 \leq \alpha^- + \alpha^+ \leq 2\pi$ and $[\lambda^- e^{i\beta^-}, \lambda^+ e^{i\beta^-}], 0 \leq \lambda^- + \lambda^+ \leq 1, 0 \leq \beta^- + \beta^+ \leq 2\pi$ with $0 \leq \mu^+ + \lambda^+ \leq 1$ and $0 \leq \alpha^+ + \beta^+ \leq 1$ to interpret the complex degree of true and false membership functions.

Unfortunately, it is limited to processing incomplete and ambiguous data; it is unable to process inconsistent and ambiguous data, which is common in situations in the real world. It cannot handle the kind of ambiguous and indeterminate information that frequently arises in real-life situations; it can only handle partial and ambiguous information. Thus, Florentin Smarandache introduces the terms neutrosophic set, a unifying field in logics, and A Generalization of the intuitionistic fuzzy sets [7-11] and they are used in many domains to handle contradictory and ambiguous data. Truth membership, indeterminacy membership, and false membership are defined completely independently if the sum of these values in the neutrosophic set lies between 0 and 3. This is known as the indeterminacy value. Neutrosophy: Neutral Logic, Neutral Set, and Neutral Probability Give a more thorough explanation of the ideas of neutrosophy, set, logic, and neutrosophic probability. The neutrosophic set has quickly attracted the attention of many scholars because of the wide range of descriptive situations it covers. Additionally, this new set aids in controlling the ambiguity resulting from the neutrosophic scope. A comprehensive bibliometric examination of the neutrosophic collection is showcased, encompassing the years from 1998 to 2017. Mumtaz Ali and Florentin Smarandache developed the idea of a Complex neutrosophic set in 2016 [12]. When a set of real-valued amplitude terms for truth, indeterminacy, and falsehood are combined with their corresponding phase terms, we have a complex neutrosophic set. This set has a complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsehood membership function. The complex neutrosophic set extends the neutrosophic set. Moreover, Atiqe U. R., Muhammad.S, Florentin Smarandache, and Muhammad R. A. [13] present the development of hybrids of hypersoft sets with complex fuzzy sets, complex intuitionistic fuzzy sets, and complex neutrosophic sets in 2020.

Figure 1 presents the development of IVCNS, including the CS Crisp Set, FS Fuzzy Set, IFS Intuitionistic Fuzzy Set, IVFS Interval-Valued Fuzzy Set, CFS Complex Fuzzy Set, NS Neutrosophic Set, CIFS Complex Intuitionistic Fuzzy Set, CIVFS Complex Interval-Valued Fuzzy Set, CNS Complex Neutrosophic Set, CIVIFS Complex Interval-Valued Intuitionistic Fuzzy Set, and IVCNS Interval-Valued Complex Neutrosophic Set.

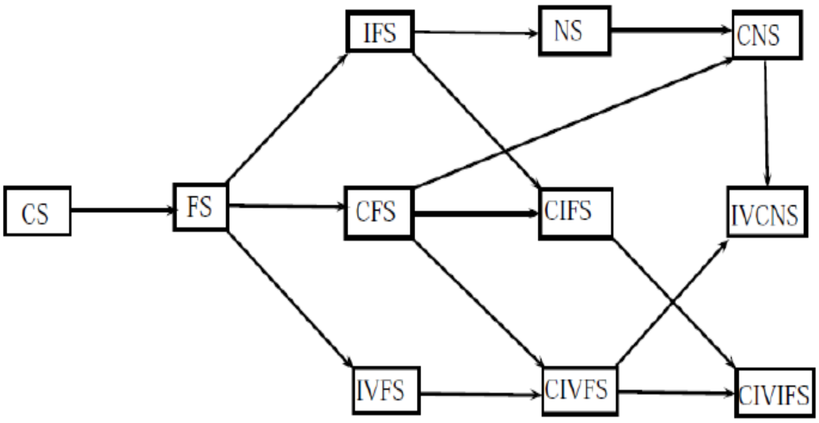

Figure 1. The development of IVNCNS.

Ivan Gutman and Bo Zhou [14] introduced the idea of a graph's Laplacian energy in 2006. Its definition is the sum of the absolute values of the adjacency matrix's eigenvalues for the graph. The energy of a graph is used in quantum theory and many other applications in the context of energy, and it is defined as the sum of the absolute values of the differences of the average vertex degree of the graph to the Laplacian eigenvalues of the graph. This is done by connecting the edge of a graph to the electron energy of a particular type of molecule. Rosenfeld [15] created fuzzy graph theory in 1975 and studied the fuzzy graphs that Kauffmann used to develop the basic idea in 1973. He explored some basic concepts in graph theory and established some of their characteristics. Bhattacharya [16] showed that the inferences from (crisp) graph theory are not always relevant to FGs in his remarks on FGs. Intuitionistic fuzzy relations and intuitionistic fuzzy graphs were introduced by Shannon and Atanassov in 1994. Fuzzy graphs with irregular interval values were examined by Rashmanlou [17]. Additionally, they defined fuzzy graphs [18] and various features of very irregular interval-valued fuzzy graphs. M.G. Karunambigai and K. Palanivel [19] first proposed the Edge Regular Intuitionistic Fuzzy Graph in 2015.

Thirunavukarasu et al. [20] created complex fuzzy graphs (CFGs) to handle uncertain and ambiguous relationships that have a periodic nature. According to Yaqoob et al. [21], complex intuitionistic fuzzy graphs (CIFGs) were defined. They looked into the homomorphisms of CIFG and demonstrated a CIFG application in cellular network provider companies to test their proposed approach. To broaden the concept of neutrosophic graphs and CIFGs, Yaqoob and Akram introduced complex neutrosophic graphs (CNGs) [22]. They covered several basic CNG functions and provided examples to illustrate them. They also presented the energy of CNGs. The concept of Complex Neutrosophic Hypergraphs: New Social Network Models was expounded upon in 2019 by Anam Luqman, Muhammad Akram, and Florentin Smarandache [23]. The best examples and motivation for CNS derive from two voting procedures, and they use this example to support the applicability of their proposed model in their introduction. Laplacian energy of fuzzy graphs is a concept introduced by Sharbaf and Fayazi [24], and some results on Laplacian energy bounds extend to fuzzy graphs. For more details, see the research papers by Soumitra Poulik and Ganesh Ghorai [25–28] on detour g-interior nodes and Detour g-boundary nodes in bipolar fuzzy graphs with applications, pragmatic results in Taiwan education system-based IVFG & IVNG, and empirical results on operations of Bipolar fuzzy graphs with their degree. Further, a note on "Bipolar fuzzy graphs with applications" was proposed in 2020. A graph structure can be produced by enlarging an undirected graph; this structure can then be used to investigate other sorts of structures, such as graphs and signed graphs. The concept of graph structures was first proposed by Sampath Kumar in his essay from 2006 [29]. The concept of a fuzzy graph structure was first proposed by T. Dinesh and T. V. Ramakrishnan in 2011 [30]. To use this model in IVCNGS, it can be rewritten in an abstract form. Muhammad Akram recently proposed the idea of Operations on Intuitionistic Fuzzy Graph Structures [31].

1.1 The framework of this research

This idea can be applied in IVCNGS after being restated abstractly. This work is structured as shown in Figure 2 and as follows:

- The concept of Interval-Valued Complex Neutrosophic Graph Structures (IVCNGS) is introduced in this work. Some results that we can share are that the IVCNGS adjacency matrix and the degree of vertex presence are being further examined.
- Further, the energy and Laplacian energy of IVCNGS are calculated. Also, we determine IVCNGS's energy and Laplacian energy upper and lower bounds.
- Moreover, IVCNGS applications and algorithm explanations were provided. Finally, an explanation of all these studies is provided in conclusion and future works.

In order for researchers to further investigate this theory using analysis of the energy and Laplacian energies of IVCNGS, we recommended readers to read this article.

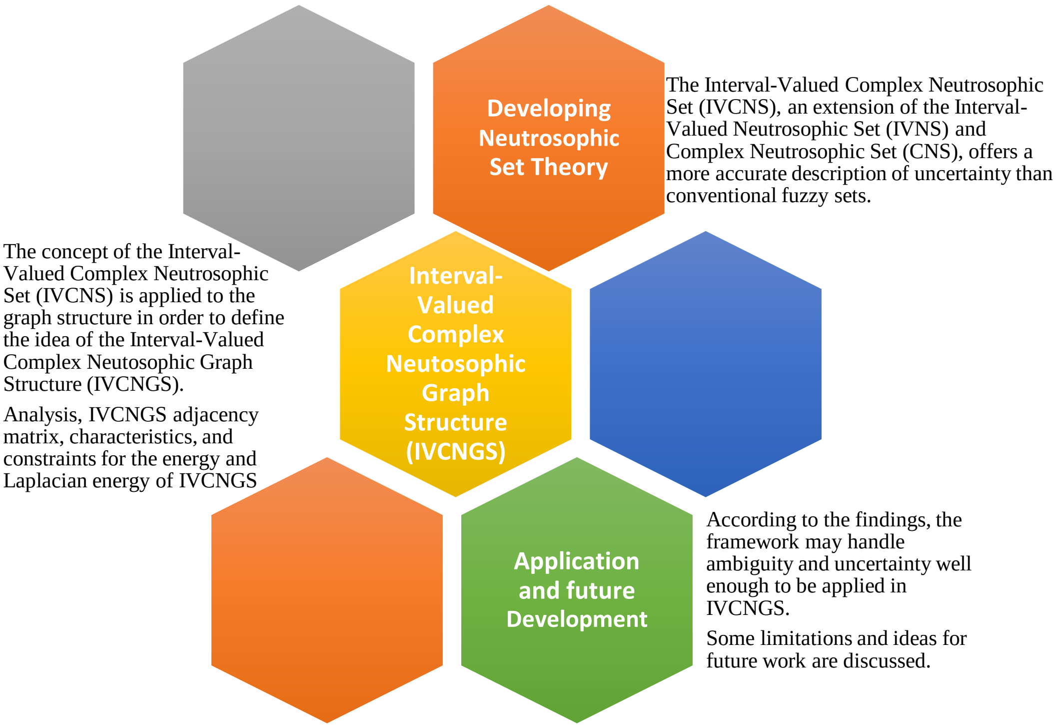

Figure 2. The development of IVNCNS.

2. Preliminaries

The development of the research work will be helped by the deliberation of some fundamental concepts and attributes in this field.

Definition 1. Let's say that conversation is the universe Y. Interval-Valued Complex Neutrosophic Set (IVCNS) A defined on Y is the object of the form.

$$A = \{(a, [\mu^-_{A_1}(a)e^{i\alpha^-_{A_1}(a)}, \mu^+_{A_1}(a)e^{i\alpha^+_{A_1}(a)}], [\mu^-_{A_2}(a)e^{i\alpha^-_{A_2}(a)}, \mu^+_{A_2}(a)e^{i\alpha^+_{A_2}(a)}], [\mu^-_{A_3}(a)e^{i\alpha^-_{A_3}(a)}, \mu^+_{A_3}(a)e^{i\alpha^+_{A_3}(a)}] : a \in Y\}, \text{ where}$$

$i=\sqrt{-1}$, $\mu^-_{A_1}(a), \mu^+_{A_1}(a), \mu^-_{A_2}(a), \mu^+_{A_2}(a), \mu^-_{A_3}(a), \mu^+_{A_3}(a) \in [0,1]$,
$\alpha^-_{A_1}(a), \alpha^+_{A_1}(a), \alpha^-_{A_2}(a), \alpha^+_{A_2}(a), \alpha^-_{A_3}(a), \alpha^+_{A_3}(a) \in [0,2\pi], 0 \le \left(\left(\mu^+_{A_1}(a)\right) + \left(\mu^+_{A_2}(a)\right) + \left(\mu^+_{A_3}(a)\right)\right) \le 3.$

Definition 2. Let $A = \{(a, \left[\mu^-_{A_1}(a)e^{i\alpha^-_{A_1}(a)}, \mu^+_{A_1}(a)e^{i\alpha^+_{A_1}(a)}\right], [\mu^-_{A_2}(a)e^{i\alpha^-_{A_2}(a)}, \mu^+_{A_2}(a)e^{i\alpha^+_{A_2}(a)}],$ $\left[\mu^-_{A_3}(a)e^{i\alpha^-_{A_3}(a)}, \mu^+_{A_3}(a)e^{i\alpha^+_{A_3}(a)}\right]: a \in Y\}$ and $B = \{(a, \left[\mu^-_{B_1}(a)e^{i\alpha^-_{B_1}(a)}, \mu^+_{B_1}(a)e^{i\alpha^+_{B_1}(a)}\right]$ $, \left[\mu^-_{B_2}(a)e^{i\alpha^-_{B_2}(a)}, \mu^+_{B_2}(a)e^{i\alpha^+_{B_2}(a)}\right], \left[\mu^-_{B_3}(a)e^{i\alpha^-_{B_3}(a)}, \mu^+_{B_3}(a)e^{i\alpha^+_{B_3}(a)}\right]: a \in Y\}$ be the two IVCNSs in Y, then

- $A \subseteq B$ if and only if $\mu^-_{A_1}(a) \le \mu^-_{B_1}(a)$, $\mu^+_{A_1}(a) \le \mu^+_{B_1}(a)$, $\mu^-_{A_2}(a) \le \mu^-_{B_2}(a)$, $\mu^+_{A_2}(a) \le \mu^+_{B_2}(a)$ and $\mu^-_{A_3}(a) \le \mu^-_{B_3}(a)$, $\mu^+_{A_3}(a) \le \mu^+_{B_3}(a)$ for amplitude terms and $\alpha^-_{A_1}(a) \le \alpha^-_{B_1}(a)$, $\alpha^+_{A_1}(a) \le \alpha^+_{A_1}(a)$, $\alpha^-_{A_2}(a) \le \alpha^-_{B_2}(a)$, $\alpha^+_{A_2}(a) \le \alpha^+_{A_2}(a)$ and $\alpha^-_{A_3}(a) \le \alpha^-_{B_3}(a)$, $\alpha^+_{A_3}(a) \le \alpha^+_{A_3}(a)$ for phase terms, for all $a \in Y$;
- $A = B$ if and only if $\mu^-_{A_1}(a) = \mu^-_{B_1}(a)$, $\mu^+_{A_1}(a) = \mu^+_{B_1}(a)$, $\mu^-_{A_2}(a) = \mu^-_{B_2}(a)$, $\mu^+_{A_2}(a) = \mu^+_{B_2}(a)$ and $\mu^-_{A_3}(a) = \mu^-_{B_3}(a)$, $\mu^+_{A_3}(a) = \mu^+_{B_3}(a)$ for amplitude terms and $\alpha^-_{A_1}(a) = \alpha^-_{B_1}(a)$, $\alpha^+_{A_1}(a) = \alpha^+_{A_1}(a)$, $\alpha^-_{A_2}(a) = \alpha^-_{B_2}(a)$, $\alpha^+_{A_2}(a) = \alpha^+_{A_2}(a)$ and $\alpha^-_{A_3}(a) = \alpha^-_{B_3}(a)$, $\alpha^+_{A_3}(a) = \alpha^+_{A_3}(a)$ for phase terms, for all $a \in Y$;

For simplicity, the $(\left[\mu^-_{A_1}(a)e^{i\alpha^-_{A_1}(a)}, \mu^+_{A_1}(a)e^{i\alpha^+_{A_1}(a)}\right], \left[\mu^-_{A_2}(a)e^{i\alpha^-_{A_2}(a)}, \mu^+_{A_2}(a)e^{i\alpha^+_{A_2}(a)}\right], \left[\mu^-_{A_3}(a)e^{i\alpha^-_{A_3}(a)}, \mu^+_{A_3}(a)e^{i\alpha^+_{A_3}(a)}\right])$ *is called the IVCNS, where,* $\mu^+_{A_1}, \mu^+_{A_2}, \mu^+_{A_3} \in [0,1]$ *such that* $\mu^+_{A_1} + \mu^+_{A_2} + \mu^+_{A_3} \le 3$.

Definition 3. A Interval-valued complex Neutrosophic relation in Y is described as a IVCNS X in $Y \times Y$ and is characterised by:
$X = \{(ab, \left[\mu^-_{X_1}(ab)e^{i\alpha^-_{X_1}(ab)}, \mu^+_{X_1}(ab)e^{i\alpha^+_{X_1}(ab)}\right], \left[\mu^-_{X_2}(ab)e^{i\alpha^-_{X_2}(ab)}, \mu^+_{X_2}(ab)e^{i\alpha^+_{X_2}(ab)}\right],$ $\left[\mu^-_{X_3}(ab)e^{i\alpha^-_{X_3}(ab)}, \mu^+_{X_3}(ab)e^{i\alpha^+_{X_3}(ab)}\right])/ab \in Y \times Y\}$ *where the Inter-valued complex Neutrosophic truth-membership, complex indeterminate-membership and complex false-membership functions of X are mapping to [0,1], such that* $0 \le \mu^+_{X_1}(rs) + \mu^+_{X_2}(rs) + \mu^+_{X_3}(rs) \le 3$ *for all* $rs \in Y \times Y$.

Definition 4. On a non-empty set X, a Interval-valued complex Neutrosophic graph is a pair $G = (A, B)$, where A and B are complex Neutrosophic sets on X and a Interval-valued complex Neutrosophic relation on X, respectively, such that:

$$\begin{aligned}
(i)\mu^-_{B_1}(rs)e^{i\alpha^-_{B_1}(rs)} &\le min\{\mu^-_{A_1}(r), \mu^-_{A_1}(s)\}e^{imin\{\alpha^-_{A_1}(r), \alpha^-_{A_1}(s)\}}\\
(ii)\mu^+_{B_1}(rs)e^{i\alpha^+_{B_1}(rs)} &\le min\{\mu^+_{A_1}(r), \mu^+_{A_1}(s)\}e^{imin\{\alpha^+_{A_1}(r), \alpha^+_{A_1}(s)\}}\\
(iii)\mu^-_{B_2}(rs)e^{i\alpha^-_{B_2}(rs)} &\le max\{\mu^-_{A_2}(r), \mu^-_{A_2}(s)\}e^{imax\{\alpha^-_{A_2}(r), \alpha^-_{A_2}(s)\}}\\
(iv)\mu^+_{B_2}(rs)e^{i\alpha^+_{B_2}(rs)} &\le max\{\mu^+_{A_2}(r), \mu^+_{A_2}(s)\}e^{imax\{\alpha^+_{A_2}(r), \alpha^+_{A_2}(s)\}}\\
(v)\mu^-_{B_3}(rs)e^{i\alpha^-_{B_3}(rs)} &\le max\{\mu^-_{A_3}(r), \mu^-_{A_3}(s)\}e^{imax\{\alpha^-_{A_3}(r), \alpha^-_{A_3}(s)\}}\\
(vi)\mu^+_{B_3}(rs)e^{i\alpha^+_{B_3}(rs)} &\le max\{\mu^+_{A_3}(r), \mu^+_{A_3}(s)\}e^{imax\{\alpha^+_{A_3}(r), \alpha^+_{A_3}(s)\}}
\end{aligned}$$

$0 \le \mu^+_{B_1}(rs) + \mu^+_{B_2}(rs) + \mu^+_{B_3}(rs) \le 3$ *for all* $rs \in Y \times Y$.

3. Energy of IVCNGS

In this part, the concept of routine IVCNGS is introduced. To further explain some of the fundamental IVCNGS features, examples are also provided.

Definition 5. Let $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is referred to as an IVCNGS of graph structure (GS) $\zeta^* = \{Q, R_1, R_2, \dots, R_k\}$ if $\eta = (\eta_1, \eta_2, \eta_3) = ([\eta_1^- e^{i\alpha_1^-}, \eta_1^+ e^{i\alpha_1^+}], [\eta_2^- e^{i\alpha_2^-}, \eta_2^+ e^{i\alpha_2^+}]\ [\eta_3^- e^{i\alpha_3^-}, \eta_3^+ e^{i\alpha_3^+}])$ is an IVCNS on Q and $\delta_J = (\delta_{1J}, \delta_{2J}, \delta_{3J}) = ([\delta_{1J}^- e^{i\beta_{1J}^-}, \delta_{1J}^+ e^{i\beta_{1J}^+}], [\delta_{2J}^- e^{i\beta_{2J}^-}, \delta_{2J}^+ e^{i\beta_{2J}^+}], [\delta_{3J}^- e^{i\beta_{3J}^-}, \delta_{3J}^+ e^{i\beta_{3J}^+}])$ are IVCNSs on Q and R_J such that

$$(i)\delta_{1J}^-(a,b)e^{i\beta_{1J}^-(a,b)} \le \min\{\eta_1^-(a), \eta_1^-(b)\}e^{i\min\{\alpha_1^-(a),\alpha_1^-(b)\}},$$

$$(ii)\delta_{1J}^+(a,b)e^{i\beta_{1J}^+(a,b)} \le \min\{\eta_1^+(a), \eta_1^+(b)\}e^{i\min\{\alpha_1^+(a),\alpha_1^+(b)\}},$$

$$(iii)\delta_{2J}^-(a,b)e^{i\beta_{2J}^-(a,b)} \le \max\{\eta_2^-(a), \eta_2^-(b)\}e^{i\max\{\alpha_2^-(a),\alpha_2^-(b)\}},$$

$$(iv)\delta_{2J}^+(a,b)e^{i\beta_{2J}^+(a,b)} \le \max\{\eta_2^+(a), \eta_2^+(b)\}e^{i\max\{\alpha_2^+(a),\alpha_2^+(b)\}},$$

$$(iv)\delta_{3J}^-(a,b)e^{i\beta_{3J}^-(a,b)} \le \max\{\eta_3^-(a), \eta_3^-(b)\}e^{i\max\{\alpha_3^-(a),\alpha_3^-(b)\}},$$

$$(iv)\delta_{3J}^+(a,b)e^{i\beta_{3J}^+(a,b)} \le \max\{\eta_3^+(a), \eta_3^+(b)\}e^{i\max\{\alpha_3^+(a),\alpha_3^+(b)\}},$$

$0 \le (\delta_{1J}^+(a,b)) + (\delta_{2J}^+(a,b)) + (\delta_{3J}^+(a,b)) \le 3$ and $(\beta_{1J}^+(ab)), (\beta_{2J}^+(ab)), (\beta_{3J}^+(ab)) \in [0,2\pi]\ \forall\ ab \in R_J, J = 1,2,\dots,k$.

Note : δ_{1J}^-, δ_{1J}^+, δ_{2J}^- , δ_{2J}^+ and δ_{3J}^- , δ_{3J}^+ are function from R_J to [0,1] such that $\delta_{1J}^-(a,b) \le \delta_{1J}^+(a,b)$, $\delta_{2J}^-(a,b) \le \delta_{2J}^+(a,b)$, $\delta_{3J}^-(a,b) \le \delta_{3J}^+(a,b)$, $\beta_{1J}^-(a,b) \le \beta_{1J}^+(a,b)$, $\beta_{2J}^-(a,b) \le \beta_{2J}^+(a,b)$ and $\beta_{3J}^-(a,b) \le \beta_{3J}^+(a,b)$ for all $(a,b) \in R_J, J = 1,2,\dots,k$.

Definition 6. The adjacency matrix $A\zeta = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ of a IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$, where $A\delta_J, (J = 1,2,\dots,k)$ is a square matrix as $[u_{jk}]$ in which $u_{jk} = ([\delta_{1J}^-(u_j u_k)e^{i\beta_{1J}^-(u_j u_k)}, \delta_{1J}^+(u_j u_k)e^{i\beta_{1J}^+(u_j u_k)}], [\delta_{2J}^-(u_j u_k)e^{i\beta_{2J}^-(u_j u_k)}, \delta_{2J}^+(u_j u_k)e^{i\beta_{2J}^+(u_j u_k)}],$

$[\delta_{3J}^-(u_j u_k)e^{i\beta_{3J}^-(u_j u_k)}, \delta_{3J}^+(u_j u_k)e^{i\beta_{3J}^+(u_j u_k)}])$, where $\delta_{1J}^-(u_j u_k), \delta_{1J}^+(u_j u_k)$ is represent the strength of interval-valued truth membership amplitude term and $\delta_{2J}^-(u_j u_k), \delta_{2J}^+(u_j u_k)$ is represent the strength of interval-valued indeterminate membership amplitude term between u_j and u_k and $\delta_{3J}^-(u_j u_k), \delta_{3J}^+(u_j u_k)$ is represent the strength of interval-valued false membership amplitude term between u_j and u_k and $\beta_{1J}^-(u_j u_k), \beta_{1J}^+(u_j u_k)$ is represent the strength of interval-valued truth membership phase term and $\beta_{2J}^-(u_j u_k), \beta_{2J}^+(u_j u_k)$ is represent the strength of interval-valued indeterminate membership phase term between u_j and u_k and $\beta_{3J}^-(u_j u_k), \beta_{3J}^+(u_j u_k)$ is represent the strength of interval-valued false membership phase term between u_j and u_k.

Definition 7. The adjacency matrix $A\zeta = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ of a IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. Then the δ_J - degree of vertex u in $A(\zeta)$ is defined as $Ad_{\delta_J}(u)e^{iAd_{\beta_J}(u)} =$

$$([Ad_{\delta_{1J}^-}(u)e^{iAd_{\beta_{1J}^-}(u)}, Ad_{\delta_{1J}^+}(u)e^{iAd_{\beta_{1J}^+}(u)}],$$

$$[Ad_{\delta_{2J}^-}(u)e^{iAd_{\beta_{2J}^-}(u)}, Ad_{\delta_{2J}^+}(u)e^{iAd_{\beta_{2J}^+}(u)}], [Ad_{\delta_{3J}^-}(u)e^{iAd_{\beta_{3J}^-}(u)}, Ad_{\delta_{3J}^+}(u)e^{iAd_{\beta_{3J}^+}(u)}]),$$

$$Ad_{\delta_{1J}^{-}}(u)e^{iAd_{\beta_{1J}^{-}}(u)} = \left(\sum_{z=1}^{k} \delta_{1J}^{-}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{1J}^{-}(u_{jz})}, Ad_{\delta_{1J}^{+}}(u)e^{iAd_{\beta_{1J}^{+}}(u)} = \left(\sum_{z=1}^{k} \delta_{1J}^{+}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{1J}^{+}(u_{jz})},$$

$$Ad_{\delta_{2J}^{-}}(u)e^{iAd_{\beta_{2J}^{-}}(u)} = \left(\sum_{z=1}^{k} \delta_{2J}^{-}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{2J}^{-}(u_{jz})}, Ad_{\delta_{2J}^{+}}(u)e^{iAd_{\beta_{2J}^{+}}(u)} = \left(\sum_{z=1}^{k} \delta_{2J}^{\mp}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{2J}^{+}(u_{jz})},$$

$$Ad_{\delta_{3J}^{-}}(u)e^{iAd_{\beta_{3J}^{-}}(u)} = \left(\sum_{z=1}^{k} \delta_{3J}^{-}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{3J}^{-}(u_{jz})}, Ad_{\delta_{3J}^{+}}(u)e^{iAd_{\beta_{3J}^{+}}(u)} = \left(\sum_{z=1}^{k} \delta_{3J}^{+}\left(u_{jz}\right)\right) e^{\sum_{z=1}^{k} \beta_{3J}^{+}(u_{jz})},$$

$\forall J = 1,2,\ldots,k.$

Example 1. An IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ of a GS $\zeta^* = (Q, R_1, R_2)$ given Figure 3 is a IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ such that $\eta = \{u_1([.4e^{i.3\pi}, .7e^{i.4\pi}], [.3e^{i.1\pi}, .6e^{i.3\pi}], [.2e^{i.1\pi}, .4e^{i.3\pi}]),$

$u_2([.4e^{i.2\pi}, .6e^{i.4\pi}], [.3e^{i..5\pi}, .5e^{i.5\pi}], [.4e^{i.3\pi}, .6e^{i.4\pi}]), u_3([.5e^{i.3\pi}, .6e^{i.4\pi}], [.5e^{i.1\pi}, .7e^{i.2\pi}], [.3e^{i.4\pi}, .4e^{i.5\pi}]),$

$u_4([.3e^{i.6\pi}, .6e^{i.7\pi}], [.4e^{i.4\pi}, .5e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]).$

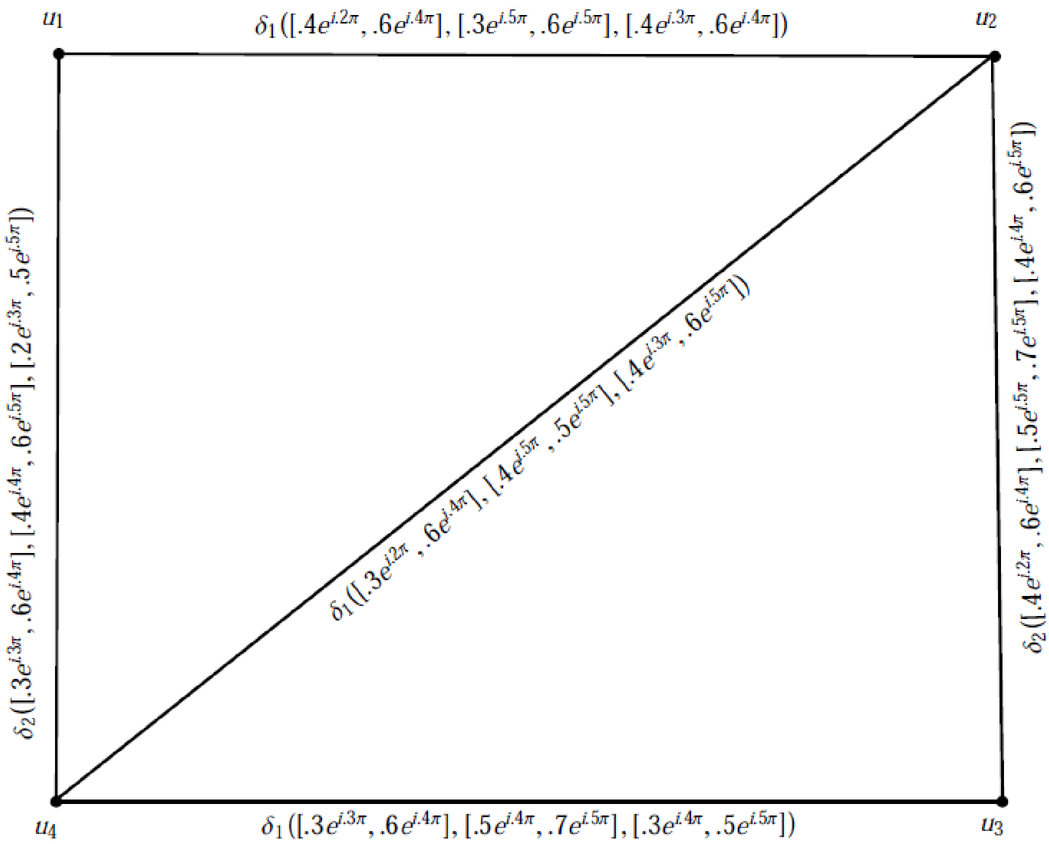

Figure 3. The adjacency matrix of the amplitude term of an IVCNGS.

The adjacency matrix of the amplitude term of an IVCNGS given in Figure 3 is:

$$A\delta_1 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .5 \\ .4 & .6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .5 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The adjacency matrix of the amplitude term of an IVCNGS given in Figure 3 is
The $\delta_1 -$ degree of vertex u_i in $A(\zeta)$ is (i=1, 2, 3, 4).

$$Ad_{\delta_1}(u_1) = \left(\left[Ad_{\delta_{11}^-}(u_1), Ad_{\delta_{11}^+}(u_1)\right], \left[Ad_{\delta_{21}^-}(u_1), Ad_{\delta_{21}^+}(u_1)\right], \left[Ad_{\delta_{31}^-}(u_1), Ad_{\delta_{31}^+}(u_1)\right]\right)$$

$$Ad_{\delta_1}(u_1)=([.4e^{i.2\pi}, .6e^{i.4\pi}], [.3e^{i.5\pi}, .6e^{i.5\pi}], [.4e^{i.3\pi}, .6e^{i.4\pi}]),$$

$$Ad_{\delta_1}(u_2) = ([.7e^{i.4\pi}, 1.2e^{i.8\pi}], [.7e^{i1.0\pi}, 1.1e^{i1.0\pi}], [.8e^{i.6\pi}, 1.2e^{i1.0\pi}]),$$

$$Ad_{\delta_1}(u_3) = ([.3e^{i.3\pi}, .6e^{i.4\pi}], [.5e^{i.4\pi}, .7e^{i.5\pi}], [.3e^{i.4\pi}, .5e^{i.5\pi}]),$$

The adjacency matrix of the phase term of an IVCNGS given in Figure 3 is:

$$A\beta_1 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .6 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

$$Ad_{\delta_1}(u_4) = ([.6e^{i.5\pi}, 1.2e^{i.8\pi}], [.9e^{i.9\pi}, 1.2e^{i1.0\pi}], [.7e^{i.7\pi}, 1.1e^{i1.1\pi}]).$$

Similarly, we calculate, the adjacency matrix of adjacency matrix of amplitude term of an IVCNGS given in Figure 3 is:

$$A\delta_2 = \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .4 & .6 \\ .2 & .5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .5 & .7 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .6 \\ .5 & .7 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} .3 & .6 \\ .4 & .6 \\ .2 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The adjacency matrix of the phase term of an IVCNGS given in Figure 3 is:

$$A\beta_2 = \begin{bmatrix} \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}.3&.4\\.4&.5\\.3&.5\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}.2&.4\\.5&.5\\.4&.5\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}.2&.4\\.5&.5\\.4&.5\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \\ \begin{pmatrix}.3&.4\\.4&.5\\.3&.5\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \end{bmatrix}$$

The $\delta_2 -$ degree of vertex u_i in $A(\zeta)$ is (i=1, 2, 3, 4).

$$Ad_{\delta_2}(u_1) = \left(\left[Ad_{\delta_{12}^-}(u_1), Ad_{\delta_{12}^+}(u_1)\right], \left[Ad_{\delta_{22}^-}(u_1), Ad_{\delta_{22}^+}(u_1)\right], \left[Ad_{\delta_{32}^-}(u_1), Ad_{\delta_{32}^+}(u_1)\right]\right)$$

$$Ad_{\delta_2}(u_1)=([.3e^{i.3\pi}, .6e^{i.4\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_2) = ([.4e^{i.2\pi}, .6e^{i.4\pi}], [.5e^{i.5\pi}, .7e^{i.5\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_3) = ([.4e^{i.2\pi}, .6e^{i.4\pi}], [.5e^{i.5\pi}, .7e^{i.5\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}]),$$

$$Ad_{\delta_2}(u_4) = ([.3e^{i.3\pi}, .6e^{i.4\pi}], [.4e^{i.4\pi}, .6e^{i.5\pi}], [.2e^{i.3\pi}, .5e^{i.5\pi}]).$$

Definition 8. The spectrum of an adjacency matrix of an IVCNGS is defined as $\langle P_1, Q_1, P_2, Q_2, P_3, Q_3\rangle$, where $P_1, Q_1, P_2, Q_2, P_3, Q_3$ is the amplitude term of the set eigenvalues of $A(\zeta)$ and $\langle P_1', Q_1', P_2', Q_2', P_3', Q_3'\rangle$, where $P_1', Q_1', P_2', Q_2', P_3', Q_3'$ is the phase term of the set eigenvalues of $A(\zeta)$ respectively.

Example 2. The spectrum of IVCPFGS, given in Figure 3 follows.

$$Spec\left(A\delta_{11}^-(u_j,u_k)\right) = \{-0.5389, -0.2227, 0.2227, 0.5389\},$$

$$Spec\left(A\delta_{11}^+(u_j,u_k)\right) = \{-0.9708, -0.3708, 0.3708, 0.9708\},$$

$$Spec\left(A\delta_{21}^-(u_j,u_k)\right) = \{-0.6708, -0.2236, 0.2236, 0.6708\},$$

$$Spec\left(A\delta_{21}^+(u_j,u_k)\right) = \{-0.9514, -0.4415, 0.4415, 0.9514\},$$

$$Spec\left(A\delta_{31}^-(u_j,u_k)\right) = \{-0.6093, -0.1970, 0.1970, 0.6093\},$$

$$Spec\left(A\delta_{31}^+(u_j,u_k)\right) = \{-0.9306, -0.3224, 0.3224, 0.9306\},$$

$$Spec\left(A\beta_{11}^-(u_j,u_k)\right) = \{-0.3811, -0.1575, 0.1575, 0.3811\},$$

$$Spec\left(A\beta_{11}^+(u_j,u_k)\right) = \{-0.6472, -0.2472, 0.2472, 0.6472\},$$

$$Spec\left(A\beta_{21}^-(u_j,u_k)\right) = \{-0.7697, -0.2598, 0.2598, 0.7697\},$$

$$Spec\left(A\beta_{21}^+(u_j,u_k)\right) = \{-0.8090, -0.3090, 0.3090, 0.8090\},$$

$$Spec\left(A\beta_{31}^-(u_j,u_k)\right) = \{-0.5389, -0.2227, 0.2227, 0.5389\},$$

$$Spec\left(A\beta_{31}^+(u_j,u_k)\right) = \{-0.8450, -0.2367, 0.2367, 0.8450\}.$$

Therefore, the spectrum of amplitude term is

$$Spec\big(A(\delta_1)\big) = \{\langle -0.5389, -0.9708, -0.6708, -0.9514, -0.6093, -0.9306 \rangle,$$

$$\langle -0.2227, -0.3708, -0.2236, 0.4415, -0.1970, -0.3224 \rangle,$$
$$\langle 0.2227, 0.3708, 0.2236, 0.4415, 0.1970, 0.3224 \rangle,$$
$$\langle 0.5389, 0.9708, 0.6708, 0.9514, 0.6093, 0.9306 \rangle\}$$

The spectrum of phase terms is

$$Spec\big(A(\beta_1)\big) = \{\langle -0.3811, -0.6472, -0.7697, -0.8090, -0.5389, -0.8450 \rangle,$$
$$\langle -0.1575, -0.2472, -0.2598, 0.3090, -0.2227, -0.2367 \rangle,$$
$$\langle 0.1575, 0.2472, 0.2598, 0.3090, 0.2227, 0.2367 \rangle,$$
$$\langle 0.3811, 0.6472, 0.7697, 0.8090, 0.5389, 0.8450 \rangle\}$$

Similarly, we calculate

The spectrum of amplitude term is

$$Spec\big(A(\delta_2)\big) = \{\langle -0.4000, -0.6000, -0.5000, -0.7000, -0.4000, -0.6000 \rangle,$$
$$\langle -0.3000, -0.6000, -0.4000, -0.6000, -0.2000, -0.5000 \rangle,$$
$$\langle 0.3000, 0.6000, 0.4000, 0.6000, 0.2000, 0.5000 \rangle,$$
$$\langle 0.4000, 0.6000, 0.5000, 0.6000, 0.4000, 0.6000 \rangle\}$$

The spectrum of phase terms is

$$Spec\big(A(\beta_2)\big) = \{\langle -0.3000, -0.4000, -0.5000, -0.5000, -0.5000, -0.5000 \rangle,$$
$$\langle -0.2000, -0.4000, -0.4000, -0.5000, -0.3000, -0.5000 \rangle,$$
$$\langle 0.2000, 0.4000, 0.4000, 0.5000, 0.3000, 0.5000 \rangle,$$
$$\langle 0.3000, 0.4000, 0.5000, 0.5000, 0.5000, 0.5000 \rangle\}$$

Definition 9. The energy of amplitude term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as the following;

$$\epsilon(\zeta) =< \epsilon(A\delta_1), \epsilon(A\delta_2), \ldots, \epsilon(A\delta_k) >$$
$$\epsilon\big(A\delta_J\big) = \left(\sum_{i=1}^{n}(\mu_i^-)_{\delta_J}, \sum_{i=1}^{n}(\mu_i^+)_{\delta_J}, \sum_{i=1}^{n}(\lambda_i^-)_{\delta_J}, \sum_{i=1}^{n}(\lambda_i^+)_{\delta_J}, \sum_{i=1}^{n}(\chi_i^-)_{\delta_J}, \sum_{i=1}^{n}(\chi_i^+)_{\delta_J}\right), \forall J = 1,2, \ldots, k,$$

and the energy of phase term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is defined as the following;

$$\epsilon(\zeta) =< \epsilon(A\beta_1), \epsilon(A\beta_2), \ldots, \epsilon(A\beta_k) >$$
$$\epsilon\big(A\beta_J\big) = \left(\sum_{i=1}^{n}(\vartheta_i^-)_{\beta_J}, \sum_{i=1}^{n}(\vartheta_i^+)_{\beta_J}, \sum_{i=1}^{n}(\rho_i^-)_{\beta_J}, \sum_{i=1}^{n}(\rho_i^+)_{\beta_J}, \sum_{i=1}^{n}(\gamma_i^-)_{\beta_J}, \sum_{i=1}^{n}(\gamma_i^+)_{\beta_J}\right), \forall J = 1,2, \ldots, k.$$

Example 3. The energy of amplitude term of an IVCNGS ζ given in Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(A\delta_1), \epsilon(A\delta_2) >$$
$$\epsilon(A\delta_1) =< 1.5232, 2.6833, 1.7889, 2.7857, 1.6125, 2.5060 >$$
$$\epsilon(A\delta_2) =< 1.4000, 2.4000, 1.8000, 2.6000, 1.2000, 2.2000 >$$

The energy of phase term of an IVCNGS ζ given in Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(A\beta_1), \epsilon(A\beta_2) >$$
$$\epsilon(A\beta_1) =< 1.0770, 1.7889, 2.0591, 2.2361, 1.5232, 2.1633 >$$
$$\epsilon(A\beta_2) =< 1.0000, 1.6000, 1.8000, 2.0000, 1.6000, 2.0000 >$$

Theorem 10. Let $A(\zeta) = \{A\delta_1, A\delta_2, \ldots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \geq (\mu_2^-)_{\delta_J} \geq \ldots \geq (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \geq (\mu_2^+)_{\delta_J} \geq \ldots \geq (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \geq (\lambda_2^-)_{\delta_J} \geq \ldots \geq (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \geq (\lambda_2^+)_{\delta_J} \geq \ldots \geq (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \geq (\chi_2^-)_{\delta_J} \geq \ldots \geq (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \geq (\chi_2^+)_{\delta_J} \geq \ldots \geq (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms, $(\vartheta_1^-)_{\beta_J} \geq (\vartheta_2^-)_{\beta_J} \geq \ldots \geq (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \geq (\vartheta_2^+)_{\beta_J} \geq \ldots \geq (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \geq (\rho_2^-)_{\beta_J} \geq \ldots \geq (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \geq (\rho_2^+)_{\beta_J} \geq \ldots \geq (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \geq (\gamma_2^-)_{\beta_J} \geq \ldots \geq (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \geq (\gamma_2^+)_{\beta_J} \geq \ldots \geq (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms. Then (i). $\sum_{i=1}^{n}(\mu_i^-)_{\delta_J} = \sum_{i=1}^{n}(\mu_i^+)_{\delta_J} = \sum_{i=1}^{n}(\lambda_i^-)_{\delta_J} = \sum_{i=1}^{n}(\lambda_i^+)_{\delta_J} = \sum_{i=1}^{n}(\chi_i^-)_{\delta_J} = \sum_{i=1}^{n}(\chi_i^+)_{\delta_J} = 0$ and $\sum_{i=1}^{n}(\vartheta_i^-)_{\beta_J} = \sum_{i=1}^{n}(\vartheta_i^+)_{\beta_J} = \sum_{i=1}^{n}(\rho_i^-)_{\beta_J} = \sum_{i=1}^{n}(\rho_i^+)_{\beta_J} =$

$$\sum_{i=1}^{n}(\gamma_i^-)_{\beta_J} = \sum_{i=1}^{n}(\gamma_i^+)_{\beta_J} = 0$$

(ii). $\sum_{i=1}^{n}(\mu_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{1J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\mu_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{1J}^+(u_j,u_k)\right)^2,$

$$\sum_{i=1}^{n}(\lambda_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\lambda_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\chi_i^-)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\chi_i^+)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^+(u_j,u_k)\right)^2, and$$

$$\sum_{i=1}^{n}(\vartheta_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\vartheta_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\rho_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\rho_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^+(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\gamma_i^-)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^-(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\gamma_i^+)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^+(u_j,u_k)\right)^2,$$

$$\forall J = 1,2,\dots,k.$$

Proof (i) since $A(\zeta)$ is a symmetric matrix with zero trace, its eigenvalues are real and have a total value of zero. (ii) By the trace properties of the matrix, we have:

$$tr\left(\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)^2\right) = \sum_{i=1}^{n}(\mu_i^S)^2_{\delta_J}, \text{ where}$$

$$tr\left(\left(A\left(\delta_{1J}^S(u_ju_k)\right)\right)^2\right) = \Big(0 + \left(\delta_{1J}^S(u_1u_2)\right)^2 + \dots + \left(\delta_{1J}^S(u_1u_n)\right)^2,$$

$$+\left(\delta_{1J}^S(u_2u_1)\right)^2 + \dots + \left(\delta_{1J}^S(u_1u_n)\right)^2,$$

$$\vdots$$

$$+\left(\delta_{1J}^S(u_nu_1)\right)^2 + \left(\delta_{1J}^S(u_nu_2)\right)^2 + \dots + 0\Big)$$

$$= 2\sum_{1\le j<k\le n}\left(\delta_{1J}^S(u_j,u_k)\right)^2$$

Similarly, we prove that

$$\sum_{i=1}^{n}(\lambda_i^S)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{2J}^S(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\chi_i^S)^2_{\delta_J} = 2\sum_{1\le j<k\le n}\left(\delta_{3J}^S(u_j,u_k)\right)^2$$

$$\text{and} \sum_{i=1}^{n}(\vartheta_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{1J}^S(u_j,u_k)\right)^2, \sum_{i=1}^{n}(\rho_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{2J}^S(u_j,u_k)\right)^2,$$

$$\sum_{i=1}^{n}(\gamma_i^S)^2_{\beta_J} = 2\sum_{1\le j<k\le n}\left(\beta_{3J}^S(u_j,u_k)\right)^2, \forall S = -,+ \text{ and } J = 1,2,\dots,k.$$

Example 4. Next, we show the example of the above Theorem 10. Let us consider $A(\zeta) = \{A\delta_1, A\delta_2\}$ be an adjacency matrix of an IVCNGS $\zeta = (\eta, \delta_1, \delta_2)$ as shown in Figure 3 in Example 1. Then:

(i). $\sum_{i=1}^{n}(\mu_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(\lambda_i^S)_{\delta_J} = 0, \sum_{i=1}^{n}(\chi_i^S)_{\delta_J} = 0$ and

$$\sum_{i=1}^{n}(\vartheta_i^S)_{\beta_J} = 0, \sum_{i=1}^{n}(\rho_i^S)_{\beta_J} = 0, \sum_{i=1}^{n}(\gamma_i^S)_{\beta_J} = 0, \forall\, S = -,+ \text{ and } J = 1,2.$$

(ii). $\sum_{u_j u_k \in R_1} (\mu_i^-)^2_{\delta_1} = 0.6800 = 2(0.34) = 2\sum_{u_j u_k \in R_1} \left(\delta_{11}^-(u_j,u_k)\right)^2,$

$$\sum_{u_j u_k \in R_1} (\mu_i^+)^2_{\delta_1} = 2.1600 = 2(1.08) = 2\sum_{u_j u_k \in R_1} \left(\delta_{11}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\lambda_i^-)^2_{\delta_1} = 1.0000 = 2(0.5) = 2\sum_{u_j u_k \in R_1} \left(\delta_{21}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\lambda_i^+)^2_{\delta_1} = 2.2000 = 2(1.1) = 2\sum_{u_j u_k \in R_1} \left(\delta_{21}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\chi_i^-)^2_{\delta_1} = 0.8200 = 2(0.41) = 2\sum_{u_j u_k \in R_1} \left(\delta_{31}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\chi_i^+)^2_{\delta_1} = 1.9400 = 2(0.97) = 2\sum_{u_j u_k \in R_1} \left(\delta_{31}^+(u_j,u_k)\right)^2, \text{and}$$

$$\sum_{u_j u_k \in R_1} (\vartheta_i^-)^2_{\beta_1} = 3.4000 = 2(0.17) = 2\sum_{u_j u_k \in R_1} \left(\beta_{11}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\vartheta_i^+)^2_{\beta_1} = 0.9600 = 2(0.48) = 2\sum_{u_j u_k \in R_1} \left(\beta_{11}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\rho_i^-)^2_{\beta_1} = 1.3200 = 2(0.66) = 2\sum_{u_j u_k \in R_1} \left(\beta_{21}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\rho_i^+)^2_{\beta_1} = 1.5000 = 2(0.75) = 2\sum_{u_j u_k \in R_1} \left(\beta_{21}^+(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\gamma_i^-)^2_{\beta_1} = 0.6800 = 2(0.34) = 2\sum_{u_j u_k \in R_1} \left(\beta_{31}^-(u_j,u_k)\right)^2,$$

$$\sum_{u_j u_k \in R_1} (\gamma_i^+)^2_{\beta_1} = 1.3200 = 2(0.66) = 2\sum_{u_j u_k \in R_1} \left(\beta_{31}^+(u_j,u_k)\right)^2$$

Similarly, we calculate J = 2.

Theorem 11. Let $A(\zeta) = \{A\delta_1, A\delta_2, \ldots, A\delta_k\}$ be an adjacency matrix of an IVCPFGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. Then:

(i). $\sqrt{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j,u_k)\right)^2 + n(n-1) \bmod \left(det\left(A\left(\delta_{1J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{1J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j,u_k)\right)^2},$

(ii). $\sqrt{2\sum_{u_j u_k \in R_J} \left(\delta_{2J}^S(u_j,u_k)\right)^2 + n(n-1) \bmod \left(det\left(A\left(\delta_{2J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{2J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_j u_k \in R_J} \left(\delta_{2J}^S(u_j,u_k)\right)^2},$

(iii). $\sqrt{2\sum_{u_j u_k \in R_J} \left(\delta_{3J}^S(u_j,u_k)\right)^2 + n(n-1) \bmod \left(det\left(A\left(\delta_{3J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \le \epsilon\left(\delta_{3J}^S(u_j,u_k)\right) \le \sqrt{2n\sum_{u_j u_k \in R_J} \left(\delta_{3J}^S(u_j,u_k)\right)^2},$

(iv). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2+n(n-1)\text{ mod }\left(det\left(A\left(\beta_{1J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}}\leq \epsilon\left(\beta_{1J}^S(u_j,u_k)\right)\leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2}$,

(v). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2+n(n-1)\text{ mod }\left(det\left(A\left(\beta_{2J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}}\leq \epsilon\left(\beta_{2J}^S(u_j,u_k)\right)\leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2}$,

(vi). $\sqrt{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2+n(n-1)\text{ mod }\left(det\left(A\left(\beta_{3J}^S(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}}\leq \epsilon\left(\beta_{3J}^S(u_j,u_k)\right)\leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2}$, $\forall S=-,+$ and $J=1,2,\ldots,k$.

Proof. (i) Upper bound:
The following results are obtained by applying the Cauchy-Schwarz inequality to the vectors $(1,1,\ldots,1)$ and $\left(\text{ mod }(\mu_1^S),\text{ mod }(\mu_2^S),\ldots,\text{ mod }(\mu_n^S)\right)$ with n entries, we get:

$$\sum_{i=1}^{n}\text{ mod }(\mu_i^S)\leq\sqrt{n}\sqrt{\sum_{i=1}^{n}\text{ mod }(\mu_i^S)^2} \qquad (1)$$

$$\left(\sum_{i=1}^{n}\mu_i^S\right)^2=\sum_{i=1}^{n}\text{ mod }(\mu_i^S)^2+2\sum_{1\leq i<j\leq n}\mu_i^S\mu_j^S \qquad (2)$$

By comparing the coefficients of $(\mu^S)^{n-2}$ in the characteristic polynomial:

$$\prod_{i=1}^{n}(\mu^S-\mu_i^S)=\text{ mod }(A(\zeta)-\mu^S I),\text{ we have:}$$

$$\sum_{1\leq i\leq j\leq n}\mu_i^S\mu_j^S=-\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2 \qquad (3)$$

Substituting 3 in 2, we obtain:

$$\sum_{i=1}^{n}\text{ mod }(\mu_i^S)^2=2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2 \qquad (4)$$

Substituting 4 in 1, we obtain:

$$\sum_{i=1}^{n}\text{ mod }(\mu_i^S)=\sqrt{n}\sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2}=\sqrt{2n\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2}$$

$$\text{Therefore, }\epsilon\left(\delta_{1J}^S(u_j,u_k)\right)\leq\sqrt{2n\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2}$$

Lower bound:

$$\left(\epsilon\left(\delta_{1J}^S(u_j,u_k)\right)\right)^2=\left(\sum_{i=1}^{n}\mu_i^S\right)^2=\sum_{i=1}^{n}\text{ mod }(\mu_i^S)^2+2\sum_{1\leq i<j\leq n}\text{ mod }\left(\mu_i^S\mu_j^S\right)$$

$$2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{2n(n-1)}{2}AM\{\text{ mod }\left(\mu_i^S\mu_j^S\right)\}$$

$$\text{Since, }AM\{\text{ mod }\left(\mu_i^S\mu_j^S\right)\}\geq GM\{\text{ mod }\left(\mu_i^S\mu_j^S\right)\},\ 1\leq i\leq j\leq n,$$

$$\text{So, } \epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \geq \sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 + n(n-1)GM\{\bmod\left(\mu_i^S\mu_j^S\right)\}}$$

Also since:

$$GM\{\bmod\left(\mu_i^S\mu_j^S\right)\} = \left(\prod_{1\leq i<j\leq n}\bmod\left(\mu_i^S\mu_j^S\right)\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n}\bmod\left(\mu_i^S\right)^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$\left(\prod_{i=1}^{n}\bmod\left(\mu_i^S\right)\right)^{\frac{2}{n}} = \bmod\left(det\left(A\left(\delta_{1J}^{S}(u_ju_k)\right)\right)\right)^{\frac{2}{n}}$$

$$\text{Therefore } \epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \geq \sqrt{2\sum_{1\leq j<k\leq n}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{1J}^{S}(u_ju_k)\right)\right)\right)^{\frac{2}{n}}}$$

$$\text{Thus, } \sqrt{2\sum_{u_j,u_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 + n(n-1)\bmod\left(det\left(A\left(\delta_{1J}^{S}(u_j,u_k)\right)\right)\right)^{\frac{2}{n}}} \leq$$

$$\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \leq \sqrt{2n\sum_{u_j,u_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}, \forall\, S = -,+ \text{ and } J = 1,2,\dots,k.$$

Likewise, we can demonstrate that (ii), (iii), (iv), (v), and (vi).

Theorem 12. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. If $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^{S}(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^{S}(u_j,u_k)\right)^2$, and $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2$, $n \leq 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2$, Then:

(i). $$\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(ii). $$\epsilon\left(\delta_{2J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(iii). $$\epsilon\left(\delta_{3J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(iv). $$\epsilon\left(\beta_{1J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$$

(v). $\epsilon\left(\beta_{2J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$

(vi). $\epsilon\left(\beta_{3J}^{S}(u_j,u_k)\right) \leq \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2}{n} + \sqrt{(n-1)\left\{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2 - \left(\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^{S}(u_j,u_k)\right)^2}{n}\right)^2\right\}}$

$\forall\, S = -,+$ and $J = 1,2,\ldots,k$.

***Proof*.** If $A = [a_{jk}]_{n\times n}$ is a symmetric matrix with zero trace, then $\mu_{max}^{S} \geq \frac{2\sum_{u_ju_k\in R_J} a_{jk}}{n}$, where μ_{max}^{S} is the maximum eigenvalue of A. If $A(\zeta)$ is the adjacency matrix of an IVCNGS ζ, then $\mu_1^S \geq \frac{2\sum_{u_ju_k\in R_J}\delta_{1J}^{S}(u_j,u_k)}{n}$, where $\mu_1^S \geq \mu_2^S \geq \ldots \geq \mu_n^S$.

$$\text{Moreover, since } \sum_{i=1}^{n}(\mu_i^S)^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2$$

$$\sum_{i=2}^{n}(\mu_i^S)^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 - (\mu_1^S)^2 \qquad (5)$$

With the vectors $(1,1,\ldots,1)$ and $\left(\text{mod }(\eta_1^S), \text{mod }(\eta_2^S),\ldots, \text{mod }(\eta_n^S)\right)$ with $n-1$ entries, the Cauchy-Schwarz inequality is applied, and the following result is obtained:

$$\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) - \mu_1^S = \sum_{i=2}^{n}\text{mod }(\mu_i^S) \leq \sqrt{(n-1)\sum_{i=2}^{n}\text{mod }(\mu_1^S)^2} \qquad (6)$$

Substituting 5 in 6, we must have:

$$\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) - \mu_1^S \leq \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 - (\mu_1^S)^2\right)}$$

$$\epsilon\left(\delta_{1J}^{S}(u_j,u_k)\right) \leq \mu_1^S + \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 - (\mu_1^S)^2\right)} \qquad (7)$$

Now, since the function:

$$F(u) = u + \sqrt{(n-1)\left(2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2 - u^2\right)}$$

decreases on the interval:

$$\left(\sqrt{\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}{n}}\ \sqrt{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^{S}(u_j,u_k)\right)^2}\right),$$

$$Also, n \leq 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2, 1 \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}. Therefore,$$

$$\sqrt{\frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}} \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n} \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)}{n}$$

$$\leq \mu_1^S \leq \sqrt{2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}.$$

Therefore, Eq. (7) implies:

$$\epsilon\left(\delta_{1J}^S(u_j, u_k)\right) \leq \frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n} +$$

$$\sqrt{(n-1)\left\{2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 - \left(\frac{2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2}{n}\right)^2\right\}}, \forall\, S = -, + \text{ and } J = 1,2, \dots, k.$$

Likewise, we can demonstrate that (ii), (iii), (iv), (v), and (vi).

Theorem 13. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. Then, $\epsilon(\zeta) \leq \frac{n}{2}(1 + \sqrt{n})$.

Proof. Let $A(\zeta) = \{A\delta_1, A\delta_2, \dots, A\delta_k\}$ be an adjacency matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$. If $n \leq 2\sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 = 2z$, it is simple to demonstrate using standard calculus that $f(z) = \frac{2y}{n} + \sqrt{(n-1)(2z - \left(\frac{2z}{n}\right)^2}$ is maximized when $z = \frac{n^2 + n\sqrt{n}}{4}$. We must have $\epsilon\left(\delta_{1J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$ if we replace this value of z with $z = \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2$ in Theorem 12. Similarly, to that, it is simple to demonstrate that $\epsilon\left(\delta_{2J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\delta_{3J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{1J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{2J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\epsilon\left(\beta_{3J}^S(u_j, u_k)\right) \leq \frac{n}{2}(1 + \sqrt{n})$, $\forall\, S = -, +$ and $J = 1,2, \dots, k$. Hence, $\epsilon(\zeta) \leq \frac{n}{2}(1 + \sqrt{n})$.

4. Laplacian Energy of IVCNGS

The Laplacian energy of an IVCNGS is defined and examined, and its specific properties are given in this section.

Definition 14. Let $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ be an IVCNGS on n vertices. The degree matrix in amplitude term $D\delta_J(\zeta) = \left(\left[D\delta_{1J}^-(u_i u_j), D\delta_{1J}^+(u_i u_j)\right], \left[D\delta_{2J}^-(u_i u_j), D\delta_{2J}^+(u_i u_j)\right], \left[D\delta_{3J}^-(u_i u_j), D\delta_{3J}^+(u_i u_j)\right]\right) = D\delta_J(ij)$

The degree matrix in amplitude term

$$D\beta_J(\zeta) = \left(\left[D\beta_{1J}^-(u_i u_j), D\beta_{1J}^+(u_i u_j)\right], \left[D\beta_{2J}^-(u_i u_j), D\beta_{2J}^+(u_i u_j)\right], \left[D\beta_{3J}^-(u_i u_j), D\beta_{3J}^+(u_i u_j)\right]\right) = D\beta_J(ij)$$

ζ is an $n \times n$ diagonal matrix of amplitude term, which is defined as $D\delta_J(ij) = \begin{cases} d_{\delta_J}(u_i), & i = j \\ 0, & i \neq j \end{cases}$

ζ is an $n \times n$ diagonal matrix of phase term, which is defined as $D\beta_J(ij) = \begin{cases} d_{\beta_J}(u_i), & i = j \\ 0, & i \neq j \end{cases}$

Definition 15. The Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is defined as $L(\zeta) = (L\delta_1, L\delta_2, \dots, L\delta_k)$, where $L\delta_J = D\delta_J - A\delta_J$, and $D\delta_J$ is a degree matrix of an IVCNGS ζ and $A\delta_J$ is an adjacency matrix for all $J = 1,2, \dots, k$.

Example 5. The Laplacian matrix of IVCNGS is shown in Figure 3 in Example 1. The $D\lambda_1$ degree matrix of amplitude term of an IVCNGS

$$D\delta_1 = \begin{bmatrix} \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .7 & 1.2 \\ .7 & 1.1 \\ .8 & 1.2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .6 & 1.2 \\ .9 & 1.2 \\ .7 & 1.1 \end{pmatrix} \end{bmatrix}$$

The $D\lambda_1$ degree matrix of phase term of an IVCNGS

$$D\beta_1 = \begin{bmatrix} \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .4 & .8 \\ 1 & 1 \\ .6 & .9 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .5 & .8 \\ .9 & 1 \\ .7 & 1 \end{pmatrix} \end{bmatrix}$$

The Laplacian matrix of amplitude term of an IVCNGS is

$$L\delta_1 = \begin{bmatrix} \begin{pmatrix} .4 & .6 \\ .3 & .6 \\ .4 & .6 \end{pmatrix} & \begin{pmatrix} -.4 & -.6 \\ -.3 & -.6 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -.4 & -.6 \\ -.3 & -.6 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} .7 & 1.2 \\ .7 & 1.1 \\ .8 & 1.2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.4 & -.5 \\ -.4 & -.6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .6 \\ .5 & .7 \\ .3 & .5 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.5 & -.7 \\ -.3 & -.5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.4 & -.5 \\ -.4 & -.6 \end{pmatrix} & \begin{pmatrix} -.3 & -.6 \\ -.5 & -.7 \\ -.3 & -.5 \end{pmatrix} & \begin{pmatrix} .6 & 1.2 \\ .9 & 1.2 \\ .7 & 1.1 \end{pmatrix} \end{bmatrix}$$

Laplacian matrix of phase term of an IVCNGS is

$$L\beta_1 = \begin{bmatrix} \begin{pmatrix} .2 & .4 \\ .5 & .5 \\ .3 & .4 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.4 \end{pmatrix} & \begin{pmatrix} .4 & .8 \\ 1 & 1 \\ .6 & .9 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} .3 & .4 \\ .4 & .5 \\ .4 & .5 \end{pmatrix} & \begin{pmatrix} -.3 & -.4 \\ -.4 & -.5 \\ -.4 & -.5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -.2 & -.4 \\ -.5 & -.5 \\ -.3 & -.6 \end{pmatrix} & \begin{pmatrix} -.3 & -.4 \\ -.4 & -.5 \\ -.4 & -.5 \end{pmatrix} & \begin{pmatrix} .5 & .8 \\ .9 & 1 \\ .7 & 1 \end{pmatrix} \end{bmatrix}$$

Similarly, we can calculate $L\delta_2$ and $L\beta_2$ Laplacian matrix

Definition 16. The spectrum of the Laplacian matrix of an IVCNGS is defined as $\langle P_{1L}, Q_{1L}, P_{2L}, Q_{2L}, P_{3L}, Q_{3L}, \rangle$, where $P_{1L}, Q_{1L}, P_{2L}, Q_{2L}, P_{3L}, Q_{3L}$ is the amplitude term of the set eigenvalues of $L(\zeta)$ and $\langle P'_{1L}, Q'_{1L}, P'_{2L}, Q'_{2L}, P'_{3L}, Q'_{3L}\rangle$, where $P'_{1L}, Q'_{1L}, P'_{2L}, Q'_{2L}, P'_{3L}, Q'_{3L}$ is the phase term of the set eigenvalues of $L(\zeta)$ respectively.

Example 6. The Laplacian spectrum of an IVCNGS shown in Figure 3 in Example 1 are as follows:

$$\text{Laplacian Spectrum } \left(L(\delta_{11}^-)\right) = (0.0000, 0.1866, 0.6819, 1.1314),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{11}^+)\right) = (-0.0000, 0.3515, 1.2000, 2.0485),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{21}^-)\right) = (-0.0000, 0.2236, 0.7553, 1.4211),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{21}^+)\right) = (-0.0000, 0.3289, 1.2879, 1.9832),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{31}^-)\right) = (-0.0000, 0.2149, 0.6896, 1.2955),$$
$$\text{Laplacian Spectrum } \left(L(\delta_{31}^+)\right) = (-0.0000, 0.3339, 1.0925, 1.9735),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{11}^-)\right) = (0.0000, 0.1268, 0.4732, 0.8000),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{11}^+)\right) = (0.0000, 0.2343, 0.8000, 1.3657),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{21}^-)\right) = (0.0000, 0.2746, 0.8913, 1.6341),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{21}^+)\right) = (0.0000, 0.2929, 1.0000, 1.7071)$$
$$\text{Laplacian Spectrum } \left(L(\beta_{31}^-)\right) = (0.0000, 0.1866, 0.6819, 1.1314),$$
$$\text{Laplacian Spectrum } \left(L(\beta_{31}^+)\right) = (-0.0528, 0.2861, 0.8466, 1.7200)$$

Therefore, the Laplacian spectrum of amplitude term is Laplacian

$$\text{spec}(L\lambda_1) = \{(0, -0, -0, -0, -0, -0), (0.1866, 0.3515, 0.2236, 0.3289, 0.2149, 0.3339), (0.6819, 1.2000, 0.7553, 1.2879, 0.6896, 1.0925), (1.1314, 2.0485, 1.4211, 1.9832, 1.2955, 1.9735)\}$$

And the Laplacian spectrum of phase term is

$$\text{spec}(L\beta_1) = \{(0,0,0,0,0,-0.0528), (0.1268, 0.2343, 0.2746, 0.2929, 0.1866, 0.2861), (0.4732, 0.8000, 0.8913, 1.0000, 0.6819, 0.8466), (0.8000, 1.3657, 1.6341, 1.7071, 1.1314, 1.7200)\}$$

Similarly, we can calculate Laplacian $\text{spec}(L\delta_2)$ and $\text{spec}(L\beta_2)$

Example 7. The Laplacian energy of amplitude term of an IVCNGS ζ given Figure 3 are as follows:

$$\epsilon(\zeta) = < \epsilon(L\delta_1), \epsilon(L\delta_2) >$$
$$\epsilon(L\delta_1) = < 1.6267, 2.8971, 1.9528, 2.9423, 1.7702, 2.7321 >$$

The Laplacian energy of phase term of an IVCNGS ζ given Figure 3 are as follows:

$$\epsilon(\zeta) =< \epsilon(L\delta_1), \epsilon(L\delta_2) >$$
$$\epsilon(L\delta_1) = < 1.1464, 1.9314, 2.2507, 2.4142, 1.6267, 2.3334 >$$

Similarly, we can calculate Laplacian $\epsilon(L\delta_2)$ *and* $\epsilon(L\beta_2)$

Theorem 17. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \geq (\mu_2^-)_{\delta_J} \geq \ldots \geq (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \geq (\mu_2^+)_{\delta_J} \geq \ldots \geq (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \geq (\lambda_2^-)_{\delta_J} \geq \ldots \geq (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \geq (\lambda_2^+)_{\delta_J} \geq \ldots \geq (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \geq (\chi_2^-)_{\delta_J} \geq \ldots \geq (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \geq (\chi_2^+)_{\delta_J} \geq \ldots \geq (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms, $(\vartheta_1^-)_{\beta_J} \geq (\vartheta_2^-)_{\beta_J} \geq \ldots \geq (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \geq (\vartheta_2^+)_{\beta_J} \geq \ldots \geq (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \geq (\rho_2^-)_{\beta_J} \geq \ldots \geq (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \geq (\rho_2^+)_{\beta_J} \geq \ldots \geq (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \geq (\gamma_2^-)_{\beta_J} \geq \ldots \geq (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \geq (\gamma_2^+)_{\beta_J} \geq \ldots \geq (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms. Then

(i) $$\sum_{i=1,(\mu_i^-)_{\delta_J} \in P_{1L}}^{n} (\mu_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\mu_i^+)_{\delta_J} \in Q_{1L}}^{n} (\mu_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\lambda_i^-)_{\delta_J} \in R_L}^{n} (\lambda_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\lambda_i^+)_{\delta_J} \in S_L}^{n} (\lambda_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\chi_i^-)_{\delta_J} \in R_L}^{n} (\chi_i^-)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{3J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\chi_i^+)_{\delta_J} \in S_L}^{n} (\chi_i^+)_{\delta_J} = 2 \sum_{u_j u_k \in R_J} \left(\delta_{3J}^+(u_j, u_k)\right), and$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J} \in P'_{1L}}^{n} (\vartheta_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{1J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\vartheta_i^+)_{\beta_J} \in Q'_{1L}}^{n} (\vartheta_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{1J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J} \in P'_{2L}}^{n} (\rho_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{2J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\rho_i^+)_{\beta_J} \in Q'_{2L}}^{n} (\rho_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{2J}^+(u_j, u_k)\right),$$

$$\sum_{i=1,(\gamma_i^-)_{\beta_J} \in P'_{3L}}^{n} (\gamma_i^-)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{3J}^-(u_j, u_k)\right), \quad \sum_{i=1,(\gamma_i^+)_{\beta_J} \in Q'_{3L}}^{n} (\gamma_i^+)_{\beta_J} = 2 \sum_{u_j u_k \in R_J} \left(\beta_{3J}^+(u_j, u_k)\right)$$

(ii) $$\sum_{i=1,(\mu_i^-)_{\delta_J} \in P_{1L}}^{n} (\mu_i^-)_{\delta_J}^2 = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^-(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\mu_i^+)_{\delta_J} \in Q_{1L}}^{n} (\mu_i^+)_{\delta_J}^2 = 2 \sum_{u_j u_k \in R_J} \left(\delta_{1J}^+(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\lambda_i^-)_{\delta_J} \in P_{2L}}^{n} (\lambda_i^-)_{\delta_J}^2 = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^-(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^-}(u_j),$$

$$\sum_{i=1,(\lambda_i^+)_{\delta_J} \in Q_{2L}}^{n} (\lambda_i^+)_{\delta_J}^2 = 2 \sum_{u_j u_k \in R_J} \left(\delta_{2J}^+(u_j, u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^+}(u_j),$$

$$\sum_{i=1,(\chi_i^-)_{\delta_J}\in P_{3L}}^{n}(\chi_i^-)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\delta_{3J}^-}(u_j),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in Q_{3L}}^{n}(\chi_i^+)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\delta_{3J}^+}(u_j)\text{ , and}$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P'_{1L}}^{n}(\vartheta_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q'_{1L}}^{n}(\vartheta_i^+)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J}\in P'_{2L}}^{n}(\rho_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)^2+\sum_{j=1}^{n}d_{\beta_{2J}^-}(u_j),$$

Proof. (i) Given that L(ζ) is a symmetric matrix with positive Laplacian eigenvalues, the following is true:

$$(i)\quad \sum_{i=1,(\mu_i^-)_{\delta_J}\in P_L}^{n}(\mu_i^-)_{\delta_J} = tr\left(L\delta_{1J}^-\right)=\sum_{j=1}^{n}d_{\delta_{1J}^-}(u_j)=2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)$$

Likewise, we can demonstrate that

$$\sum_{i=1,(\mu_i^+)_{\delta_J}\in Q_{1L}}^{n}(\mu_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right),\quad \sum_{i=1,(\lambda_i^-)_{\delta_J}\in R_L}^{n}(\lambda_i^-)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\lambda_i^+)_{\delta_J}\in S_L}^{n}(\lambda_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\chi_i^-)_{\delta_J}\in R_L}^{n}(\chi_i^-)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in S_L}^{n}(\chi_i^+)_{\delta_J} = 2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P'_{1L}}^{n}(\vartheta_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q'_{1L}}^{n}(\vartheta_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\rho_i^-)_{\beta_J}\in P'_{2L}}^{n}(\rho_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\rho_i^+)_{\beta_J}\in Q'_{2L}}^{n}(\rho_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)\quad \sum_{i=1,(\gamma_i^-)_{\beta_J}\in P'_{3L}}^{n}(\gamma_i^-)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right),$$

$$\sum_{i=1,(\gamma_i^+)_{\beta_J}\in Q'_{3L}}^{n}(\gamma_i^+)_{\beta_J} = 2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)$$

(ii) By the Definition 15 of Laplacian matrix, we have:

$$L\delta_{1J}^{-} = \begin{bmatrix} d_{\delta_{1J}^{-}}(u_1) & -\delta_{1J}^{-}(u_1u_2) & \cdots & -\delta_{1J}^{-}(z_1z_n) \\ -\delta_{1J}^{-}(u_2u_1) & d_{\delta_{1J}^{-}}(u_2) & \cdots & -\delta_{1J}^{-}(z_2z_n) \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_{1J}^{-}(u_nu_1) & -\delta_{1J}^{-}(u_nu_2) & \cdots & d_{\delta_{1J}^{-}}(u_n) \end{bmatrix}$$

By the trace properties of a matrix, we have:

$$tr\left(\left(L\left(\delta_{1J}^{-}\right)\right)^2\right) = \sum_{i=1,(\mu_i^{-})_{\delta_J}\in P_{1L}}^{n} (\mu_i^{-})_{\delta_J}^2$$

$$tr\left(\left(L\left(\delta_{1J}^{-}\right)\right)^2\right) = (d_{\delta_{1J}^{-}}^2(u_1) + \left(\delta_{1J}^{-}(u_1u_2)\right)^2 + \cdots + \left(\delta_{1J}^{-}(z_1z_n)\right)^2 +$$

$$\left(\delta_{1J}^{-}(u_2u_1)\right)^2 + d_{\delta_{1J}^{-}}^2(u_2) + \cdots + \left(\delta_{1J}^{-}(z_2z_n)\right)^2 + \cdots +$$

$$\left(\delta_{1J}^{-}(u_nu_1)\right)^2 + \left(\delta_{1J}^{-}(u_nu_2)\right)^2 + \cdots + d_{\delta_{1J}^{-}}^2(u_n)$$

$$= 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{-}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{-}}\left(u_j\right)$$

Therefore, $\sum_{i=1,(\mu_i^{-})_{\delta_J}\in P_{1L}}^{n} (\mu_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{-}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{-}}\left(u_j\right)$

Likewise, we can demonstrate that

$$\sum_{i=1,(\mu_i^{+})_{\delta_J}\in Q_{1L}}^{n} (\mu_i^{+})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{1J}^{+}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{1J}^{+}}\left(u_j\right),$$

$$\sum_{i=1,(\lambda_i^{-})_{\delta_J}\in P_{2L}}^{n} (\lambda_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{2J}^{-}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^{-}}\left(u_j\right),$$

$$\sum_{i=1,(\lambda_i^{+})_{\delta_J}\in Q_{2L}}^{n} (\lambda_i^{+})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{2J}^{+}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{2J}^{+}}\left(u_j\right),$$

$$\sum_{i=1,(\chi_i^{-})_{\delta_J}\in P_{3L}}^{n} (\chi_i^{-})_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{3J}^{-}\left(u_j,u_k\right)\right)^2 + \sum_{j=1}^{n} d_{\delta_{3J}^{-}}\left(u_j\right),$$

$$\sum_{i=1,(\chi_i^+)_{\delta_J}\in Q_{3L}}^{n} (\chi_i^+)_{\delta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\delta_{3J}^+(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\delta_{3J}^+}(u_j)\text{, and}$$

$$\sum_{i=1,(\vartheta_i^-)_{\beta_J}\in P_{1L}'}^{n} (\vartheta_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{1J}^-(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{1J}^-}(u_j),$$

$$\sum_{i=1,(\vartheta_i^+)_{\beta_J}\in Q_{1L}'}^{n} (\vartheta_i^+)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{1J}^+(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{1J}^+}(u_j),$$

$$\sum_{i=1,(\rho_i^-)_{\beta_J}\in P_{2L}'}^{n} (\rho_i^-)_{\beta_J}^2 = 2\sum_{u_ju_k\in R_J} \left(\beta_{2J}^-(u_j,u_k)\right)^2 + \sum_{j=1}^{n} d_{\beta_{2J}^-}(u_j), \forall J = 1,2,\dots,k.$$

Definition 18. The Laplacian energy of amplitude term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is defined as: $L\epsilon(\zeta) =< L\epsilon(\delta_1), L\epsilon(\delta_2), \dots, L\epsilon(\delta_k) >$

$$L\epsilon(\delta_J) = (\sum_{i=1}^{n} \text{mod}\left((L\mu_i^-)_{\delta_J}\right), \sum_{i=1}^{n} \text{mod}\left((L\mu_i^+)_{\delta_J}\right),$$

$$\sum_{i=1}^{n} \text{mod}\left((L\lambda_i^-)_{\delta_J}\right), \sum_{i=1}^{n} \text{mod}\left((L\lambda_i^+)_{\delta_J}\right)), \sum_{i=1}^{n} \text{mod}\left((L\chi_i^-)_{\delta_J}\right) \sum_{i=1}^{n} \text{mod}\left((L\chi_i^+)_{\delta_J}\right)), \text{ where}$$

$$(L\mu_i^-)_{\delta_J} = (\mu_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)}{n}, (L\mu_i^+)_{\delta_J} = (\mu_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right)}{n},$$
$$(L\lambda_i^-)_{\delta_J} = (\lambda_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right)}{n}, (L\lambda_i^+)_{\delta_J} = (\lambda_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)}{n},$$
$$(L\chi_i^-)_{\delta_J} = (\chi_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)}{n}, (L\chi_i^+)_{\delta_J} = (\chi_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)}{n},$$

For all $J = 1,2,\dots,k$. And the Laplacian energy of phase term of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ is defined as:

$$L\epsilon(\zeta) =< L\epsilon(\beta_1), L\epsilon(\beta_2), \dots, L\epsilon(\beta_k) >$$

$$L\epsilon(\beta_J) = (\sum_{i=1}^{n} \text{mod}\left((L\vartheta_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{mod}\ (L\vartheta_i^+)_{\beta_J}),$$

$$\sum_{i=1}^{n} \text{mod}\left((L\rho_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{mod}\left((L\rho_i^+)_{\beta_J}\right)),$$

$$\sum_{i=1}^{n} \text{mod}\left((L\gamma_i^-)_{\beta_J}\right), \sum_{i=1}^{n} \text{mod}\left((L\gamma_i^+)_{\beta_J}\right)), \text{ where}$$

$$(L\vartheta_i^-)_{\beta_J} = (\vartheta_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)}{n}, (L\vartheta_i^+)_{\beta_J} = (\vartheta_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)}{n},$$
$$(L\rho_i^-)_{\beta_J} = (\rho_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)}{n}, (L\rho_i^+)_{\beta_J} = (\rho_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)}{n},$$
$$(L\gamma_i^-)_{\beta_J} = (\gamma_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right)}{n}, (L\gamma_i^+)_{\beta_J} = (\gamma_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)}{n},$$
$\forall J = 1,2,\dots,k.$

Example 8. In Example 6, the Laplacian spectrum is found. An IVCNGS is Laplacian energy is shown in Figure 3 as follows:

$$(L\mu_i^-)_{\delta_1} = \text{mod}\left(0-\frac{2(1.0)}{4}\right)+\text{mod}\left(0.1866-\frac{2(1.0)}{4}\right)+\text{mod}\left(0.6819-\frac{2(1.0)}{4}\right)$$
$$+\text{mod}\left(1.1314-\frac{2(1.0)}{4}\right) = 1.6267$$

$$(L\mu_i^+)_{\delta_1} = \text{mod}\left(0-\frac{2(1.8)}{4}\right)+\text{mod}\left(0.3515-\frac{2(1.8)}{4}\right)+\text{mod}\left(1.2000-\frac{2(1.8)}{4}\right)$$
$$+\text{mod}\left(2.0485-\frac{2(1.8)}{4}\right) = 2.897$$

$$(L\lambda_i^-)_{\delta_1} = \text{mod}\left(0-\frac{2(1.2)}{4}\right)+\text{mod}\left(0.2236-\frac{2(1.2)}{4}\right)+\text{mod}\left(0.7553-\frac{2(1.2)}{4}\right)$$
$$+\text{mod}\left(1.4211-\frac{2(1.2)}{4}\right) = 1.9528$$

$$(L\lambda_i^+)_{\delta_1} = \text{mod}\left(0-\frac{2(1.8)}{4}\right)+\text{mod}\left(0.3289-\frac{2(1.8)}{4}\right)+\text{mod}\left(1.2879-\frac{2(1.8)}{4}\right)$$
$$+\text{mod}\left(1.9832-\frac{2(1.8)}{4}\right) = 2.9422$$

$$(L\chi_i^-)_{\delta_1} = \text{mod}\left(0-\frac{2(1.1)}{4}\right)+\text{mod}\left(0.2149-\frac{2(1.1)}{4}\right)+\text{mod}\left(0.6896-\frac{2(1.1)}{4}\right)$$
$$+\text{mod}\left(1.2955-\frac{2(1.1)}{4}\right) = 1.7702$$

$$(L\chi_i^+)_{\delta_1} = \text{mod}\left(0-\frac{2(1.7)}{4}\right)+\text{mod}\left(0.3339-\frac{2(1.7)}{4}\right)+\text{mod}\left(1.0925-\frac{2(1.7)}{4}\right)$$
$$+\text{mod}\left(1.9735-\frac{2(1.7)}{4}\right) = 2.7321$$

$$(L\vartheta_i^-)_{\beta_1} = \text{mod}\left(0-\frac{2(.7)}{4}\right)+\text{mod}\left(0.1268-\frac{2(.7)}{4}\right)+\text{mod}\left(0.4732-\frac{2(.7)}{4}\right)$$
$$+\text{mod}\left(0.8000-\frac{2(.7)}{4}\right) = 1.1464$$

$$(L\vartheta_i^+)_{\beta_1} = \text{ mod }\left(0-\frac{2(1.2)}{4}\right)+\text{ mod }\left(0.2343-\frac{2(1.2)}{4}\right)+\text{ mod }\left(0.8000-\frac{2(1.2)}{4}\right)$$
$$+\text{ mod }\left(1.3657-\frac{2(1.2)}{4}\right)=1.9314$$

$$(L\rho_i^-)_{\beta_1} = \text{ mod }\left(0-\frac{2(1.4)}{4}\right)+\text{ mod }\left(0.2746-\frac{2(1.4)}{4}\right)+\text{ mod }\left(0.8913-\frac{2(1.4)}{4}\right)$$
$$+\text{ mod }\left(1.6341-\frac{2(1.4)}{4}\right)=2.2508$$

$$(L\rho_i^+)_{\beta_1} = \text{ mod }\left(0-\frac{2(1.5)}{4}\right)+\text{ mod }\left(0.2929-\frac{2(1.5)}{4}\right)+\text{ mod }\left(1.0000-\frac{2(1.5)}{4}\right)$$
$$+\text{ mod }\left(1.7071-\frac{2(1.5)}{4}\right)=2.4142$$

$$(L\gamma_i^-)_{\beta_1} = \text{ mod }\left(0-\frac{2(1)}{4}\right)+\text{ mod }\left(0.1866-\frac{2(1)}{4}\right)+\text{ mod }\left(0.6819-\frac{2(1)}{4}\right)$$
$$+\text{ mod }\left(1.1314-\frac{2(1)}{4}\right)=1.6267$$

$$(L\gamma_i^+)_{\beta_1} = \text{ mod }\left(-0.0528-\frac{2(1.4)}{4}\right)+\text{ mod }\left(0.2861-\frac{2(1.4)}{4}\right)+\text{ mod }\left(0.8466-\frac{2(1.4)}{4}\right)$$
$$+\text{ mod }\left(1.7200-\frac{2(1.4)}{4}\right)=2.2805$$

Theorem 19. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$. If $(\mu_1^-)_{\delta_J} \geq (\mu_2^-)_{\delta_J} \geq \ldots \geq (\mu_n^-)_{\delta_J}$, $(\mu_1^+)_{\delta_J} \geq (\mu_2^+)_{\delta_J} \geq \ldots \geq (\mu_n^+)_{\delta_J}$ and $(\lambda_1^-)_{\delta_J} \geq (\lambda_2^-)_{\delta_J} \geq \ldots \geq (\lambda_n^-)_{\delta_J}$, $(\lambda_1^+)_{\delta_J} \geq (\lambda_2^+)_{\delta_J} \geq \ldots \geq (\lambda_n^+)_{\delta_J}$ and $(\chi_1^-)_{\delta_J} \geq (\chi_2^-)_{\delta_J} \geq \ldots \geq (\chi_n^-)_{\delta_J}$, $(\chi_1^+)_{\delta_J} \geq (\chi_2^+)_{\delta_J} \geq \ldots \geq (\chi_n^+)_{\delta_J}$ are the eigenvalues of the amplitude terms $L\delta_{1J}^-(u_ju_k), L\delta_{1J}^+(u_ju_k), L\delta_{2J}^-(u_ju_k)$, $L\delta_{2J}^+(u_ju_k)$ and $L\delta_{3J}^-(u_ju_k)$, $L\delta_{3J}^+(u_ju_k)$ respectively, and $(\vartheta_1^-)_{\beta_J} \geq (\vartheta_2^-)_{\beta_J} \geq \ldots \geq (\vartheta_n^-)_{\beta_J}$, $(\vartheta_1^+)_{\beta_J} \geq (\vartheta_2^+)_{\beta_J} \geq \ldots \geq (\vartheta_n^+)_{\beta_J}$ and $(\rho_1^-)_{\beta_J} \geq (\rho_2^-)_{\beta_J} \geq \ldots \geq (\rho_n^-)_{\beta_J}$, $(\rho_1^+)_{\beta_J} \geq (\rho_2^+)_{\beta_J} \geq \ldots \geq (\rho_n^+)_{\beta_J}$ and $(\gamma_1^-)_{\beta_J} \geq (\gamma_2^-)_{\beta_J} \geq \ldots \geq (\gamma_n^-)_{\beta_J}$, $(\gamma_1^+)_{\beta_J} \geq (\gamma_2^+)_{\beta_J} \geq \ldots \geq (\gamma_n^+)_{\beta_J}$ are the eigenvalues of the phase terms $L\beta_{1J}^-(u_ju_k), L\beta_{1J}^+(u_ju_k), L\beta_{2J}^-(u_ju_k)$, $L\beta_{2J}^+(u_ju_k)$ and $L\beta_{3J}^-(u_ju_k)$, $L\beta_{3J}^+(u_ju_k)$ respectively,

$$(L\mu_i^-)_{\delta_J} = (\mu_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^-(u_j,u_k)\right)}{n},\ (L\mu_i^+)_{\delta_J} = (\mu_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^+(u_j,u_k)\right)}{n},$$

$$(L\lambda_i^-)_{\delta_J} = (\lambda_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^-(u_j,u_k)\right)}{n},\ (L\lambda_i^+)_{\delta_J} = (\lambda_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^+(u_j,u_k)\right)}{n},$$

$$(L\chi_i^-)_{\delta_J} = (\chi_i^-)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^-(u_j,u_k)\right)}{n},\ (L\chi_i^+)_{\delta_J} = (\chi_i^+)_{\delta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^+(u_j,u_k)\right)}{n}$$

$$\textit{and } (L\vartheta_i^-)_{\beta_J} = \mathrm{s}(\vartheta_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^-(u_j,u_k)\right)}{n},\ (L\vartheta_i^+)_{\beta_J}(\vartheta_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^+(u_j,u_k)\right)}{n},$$

$$(L\rho_i^-)_{\beta_J} = (\rho_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^-(u_j,u_k)\right)}{n},\ (L\rho_i^+)_{\beta_J} = (\rho_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^+(u_j,u_k)\right)}{n}$$

$$(L\gamma_i^-)_{\beta_J} = (\gamma_i^-)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^-(u_j,u_k)\right)}{n},\ (L\gamma_i^+)_{\beta_J} = (\gamma_i^+)_{\beta_J} - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^+(u_j,u_k)\right)}{n},$$

then: $\sum_{i=1}^{n}(L\mu_i^S)_{\delta_J}=0, \sum_{i=1}^{n}(L\lambda_i^S)_{\delta_J}=0,\ \sum_{i=1}^{n}(L\chi_i^S)_{\delta_J}=0, \sum_{i=1}^{n}(L\vartheta_i^S)_{\beta_J}=0,$

$$\sum_{i=1}^{n}(L\rho_i^S)_{\beta_J}=0, \sum_{i=1}^{n}(L\gamma_i^S)_{\beta_J}=0,$$

$$\sum_{i=1}^{n}(L\mu_i^S)_{\delta_J}^2=2M_{\delta_{1J}^S}, \sum_{i=1}^{n}(L\lambda_i^S)_{\delta_J}^2=2M_{\delta_{2J}^S}, \sum_{i=1}^{n}(L\chi_i^S)_{\delta_J}^2=2M_{\delta_{3J}^S}$$

$$\sum_{i=1}^{n}(L\vartheta_i^S)_{\beta_J}^2=2M_{\beta_{1J}^S}, \sum_{i=1}^{n}(L\rho_i^S)_{\beta_J}^2=2M_{\beta_{2J}^S}, \sum_{i=1}^{n}(L\gamma_i^S)_{\beta_J}^2=2M_{\beta_{3J}^S}$$

where:

$$M_{\delta_{1J}^S}=\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\delta_{2J}^S}=\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\delta_{3J}^S}=\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{3J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{1J}^S}=\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{1J}^S}=\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{2J}^S}=\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$$M_{\beta_{3J}^S}=\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)}{n}\right)^2,$$

$\forall\ S=-,+$ and $J=1,2,\ldots,k$.

Theorem 20. Let $L(\zeta)=\{L\delta_1, L\delta_2,\ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta=\{\eta,\delta_1,\delta_2,\ldots,\delta_k\}$ on n vertices. Then,

(i) $(L\mu_i^S)_{\delta_J}\le\sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+n\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$

(ii) $(L\lambda_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)}{n}\right)}$

(iii) $(L\chi_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{3J}^S(u_j,u_k)\right)}{n}\right)}$

(iv) $(L\vartheta_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)}$

(v) $(L\rho_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)}{n}\right)}$

(vi) $(L\gamma_i^S)_{\beta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)}{n}\right)}$

$\forall\, S = -, +$ and $J = 1,2,\ldots,k$.

Proof. (i) By applying Cauchy-Schwarz inequality to the n numbers 1,1,…,1 and $\mathrm{mod}\left((L\mu_1^S)_{\delta_J}\right)$, $\mathrm{mod}\left((L\mu_2^S)_{\delta_J}\right)$,…, $\mathrm{mod}\left((L\mu_n^S)_{\delta_J}\right)$, we have:

$$\sum_{i=1}^{n}\mathrm{mod}\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}\mathrm{mod}\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$(L\mu_i^S)_{\delta_J} \leq \sqrt{n}\sqrt{2M_{\delta_{1J}^S}} = \sqrt{2nM_{\delta_{1J}^S}}$$

Since, $M_{\delta_{1J}^S} = \sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2$,

Therefore,

$(L\mu_i^S)_{\delta_J} \leq \sqrt{2n\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2 + n\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$ for all $S = -, +$ and $J = 1,2,\ldots,k$.

We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

Theorem 21. Let $L(\zeta) = \{L\delta_1, L\delta_2, \ldots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ on n vertices. Then,

(i) $$(L\mu_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$$

(ii) $$(L\lambda_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{2J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{2J}^S(u_j,u_k)\right)}{n}\right)}$$

(iii) $$(L\chi_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$$

(iv) $$(L\vartheta_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{1J}^S(u_j,u_k)\right)}{n}\right)}$$

(v) $$(L\rho_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{2J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{2J}^S(u_j,u_k)\right)}{n}\right)}$$

(vi) $$(L\gamma_i^S)_{\beta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{3J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\beta_{3J}^S(u_j,u_k)\right)}{n}\right)}$$

$\forall\ S=-,+$ and $J=1,2,\ldots,k$.

Proof. (i)

$$\left(\sum_{i=1}^{n}\ \mathrm{mod}\left((L\mu_i^S)_{\delta_J}\right)\right)^2=\sum_{i=1}^{n}\ \mathrm{mod}\left((L\mu_i^S)_{\delta_J}\right)^2+2\sum_{u_ju_k\in R_J}\ \mathrm{mod}\ \left((L\mu_i^S)_{\delta_J}(L\mu_j^S)_{\delta_J}\right)\geq 4M_{\delta_{1J}^S}$$

$$(L\mu_i^S)_{\delta_J}\geq 2\sqrt{M_{\delta_{1J}^S}}$$

Since, $M_{\delta_{1J}^S}=\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2$

Therefore, $(L\mu_i^S)_{\delta_J} \geq 2\sqrt{\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j)-\frac{2\sum_{u_ju_k\in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)}$

for all $S=-,+$ and $J=1,2,\ldots,k$. We can verify the other section (ii),(iii),(iv),(v), and (vi) in a similar manner.

Theorem 22. Let $L(\zeta) = \{L\delta_1, L\delta_2, \dots, L\delta_k\}$ be the Laplacian matrix of an IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \dots, \delta_k\}$ on n vertices. Then,

$$\text{(i)}\quad (L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{1J}^S} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right)^2\right)}$$

$$\text{(ii)}\quad (L\lambda_i^S)_{\delta_J} \leq \text{ mod }\left((L\delta_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{2J}^S} - \text{ mod }\left((L\lambda_1^S)_{\delta_J}\right)^2\right)}$$

$$\text{(iii)}\quad (L\chi_i^S)_{\delta_J} \leq \text{ mod }\left((L\delta_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{3J}^S} - \text{ mod }\left((L\chi_1^S)_{\delta_J}\right)^2\right)}$$

$$\text{(iv)}\quad (L\vartheta_i^S)_{\beta_J} \leq \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{1J}^S} - \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right)^2\right)}$$

$$\text{(v)}\quad (L\rho_i^S)_{\beta_J} \leq \text{ mod }\left((L\rho_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{2J}^S} - \text{ mod }\left((L\rho_1^S)_{\beta_J}\right)^2\right)}$$

$$\text{(vi)}\quad (L\gamma_i^S)_{\beta_J} \leq \text{ mod }\left((L\gamma_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2M_{\beta_{3J}^S} - \text{ mod }\left((L\rho_1^S)_{\beta_J}\right)^2\right)}$$

$\forall\, S = -, +$ and $J = 1,2, \dots, k$.

Proof. (i)

$$\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{n\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right) \leq \sqrt{(n-1)\sum_{i=1}^{n} \text{ mod }\left((L\mu_i^S)_{\delta_J}\right)^2}$$

$$(L\mu_i^S)_{\delta_J} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) \leq \sqrt{(n-1)\left(2M_{\delta_{1J}^S} - \text{ mod }\left((L\mu_1^S)_{\delta_J}\right)^2\right)}$$

That is $M_{\delta_{1J}^S} = \sum_{u_j u_k \in R_J} \left(\delta_{1J}^S(u_j, u_k)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\delta_{1J}^S}(u_j) - \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n}\right)^2,$

Therefore, $(L\mu_i^S)_{\delta_J} -$ mod $\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2M_{\delta_{1J}^S} \leq \text{ mod } \left((L\mu_1^S)_{\delta_J}\right)^2\right)}$ for all $S = -, +$ and $J = 1,2,\ldots,k$. We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

Theorem 23. If the IVCNGS $\zeta = \{\eta, \delta_1, \delta_2, \ldots, \delta_k\}$ is regular, then:

(i) $(L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right) - (L\mu_1^S)_{\delta_J}^2\right)}$;

(ii) $(L\lambda_i^S)_{\delta_J} \leq \text{ mod }\left((L\lambda_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{2J}^S(u_j,u_k)\right) - (L\lambda_1^S)_{\delta_J}^2\right)}$;

(ii) $(L\chi_i^S)_{\delta_J} \leq \text{ mod }\left((L\chi_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{3J}^S(u_j,u_k)\right) - (L\chi_1^S)_{\delta_J}^2\right)}$;

(iv) $(L\vartheta_i^S)_{\beta_J} \leq \text{ mod }\left((L\vartheta_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{1J}^S(u_j,u_k)\right) - (L\vartheta_1^S)_{\beta_J}^2\right)}$;

(v) $(L\rho_i^S)_{\beta_J} \leq \text{mod }\left((L\rho_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{2J}^S(u_j,u_k)\right) - (L\rho_1^S)_{\beta_J}^2\right)}$;

(vi) $(L\gamma_i^S)_{\beta_J} \leq \text{mod }\left((L\gamma_1^S)_{\beta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\beta_{3J}^S(u_j,u_k)\right) - (L\gamma_1^S)_{\beta_J}^2\right)}$; $\forall\ S = -, +$ and $J = 1,2,\ldots,k$.

Proof.

$$d_{\delta_{1J}^S}(u_j) = \frac{2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right)}{n} \qquad (9)$$

Substituting 9 in 8, we get

$$(L\mu_i^S)_{\delta_J} \leq \text{ mod }\left((L\mu_1^S)_{\delta_J}\right) + \sqrt{(n-1)\left(2\sum_{u_j u_k \in R_J}\left(\delta_{1J}^S(u_j,u_k)\right) - (L\mu_1^S)_{\delta_J}^2\right)};$$

We can verify the other sections (ii), (iii), (iv), (v), and (vi) in a similar manner.

5. Application

We evaluate the effectiveness of the proposed IVCNGS policies with real-world examples of medicine resource analyses based on the clinical field. The modern human life is heavily reliant on medicine. In the present context, it is the most important essential in the world. In our daily lives, we use a variety of medications, including herbal, homeopathic, and generic medications. Satisfying human demand and supplying a sufficient number of medicines at a reasonable cost to the market is extremely significant for the pharmaceutical industry.

Let's investigate how our IVCNGS concepts are applied to the pharmaceutical industry, which encompasses generic, homeopathic, herbal, and allopathic products, to explain its exceptional performance. The vertices in this example represent generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4). We examine the network analysis of best best-edition drugs in the pharmaceutical industry. The two intended relationships between the pharmaceutical effect of the introduced unit's impact on global demand $R_1$and the damage to medications R_2. According to the provided definition 5, impact on global demand R_1 and the damage to medications R_2. The offered definition 5 can be applied in any situation because it helps to take into account everything that has an uncertain value. In this case, a set of relations R_J and a vertex set Q are considered. Examine Q ={ generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic(u_4)}, as well as the effects on global demand R_1 and the relationships between components in the pharmaceutical industry that affect medication damage R_2. We assume in Example 1 and Figure 3 $\zeta = (\eta, \delta_1, \delta_2)$ is IVCNGS of a GS $\zeta^* = (Q, R_1, R_2)$. The greatest value of an IVCNGS amplitude term's energy ζ is $max(\epsilon(\zeta))$ =2.6833 and the greatest value of an IVCNGS phase term's energy ζ is $max(\epsilon(\zeta)) = 2.2361$. In this illustration, it is obvious that the components have a greater impact on each other when there is a greater quantity of energy present in their relationships. It is obvious that more energy exists in R_1. As a result, generic, homeopathic, herbal, and allopathic all have a greater impact on one another.

To assess each of these, we instructed two relation $e_k (k = 1,2)$ Interval-valued complex Neutrosophic Preference Relations (IVCNPRs) [32] to increase the degree of components in the pharmaceutical industry. Following is a formula to determine each expert's weight:

$$w_J = \left(\frac{\epsilon(A\delta_{1J}^S)}{\sum_{J=1}^2 \epsilon\left(A\delta_{1J}^S\right)} e^{i\frac{\epsilon(A\beta_{1J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{1J}^S)}}, \frac{\epsilon(A\delta_{2J}^S)}{\sum_{J=1}^2 \epsilon\left(A\delta_{2J}^S\right)} e^{i\frac{\epsilon(A\beta_{2J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{2J}^S)}}, \frac{\epsilon(A\delta_{3J}^S)}{\sum_{J=1}^2 \epsilon\left(A\delta_{3J}^S\right)} e^{i\frac{\epsilon(A\beta_{3J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{3J}^S)}} \right),$$
$$\forall\, S = -, + \; and\; J = 1,2.$$

Amplitude term of IVCNGS:

$$W_1 = ((0.5210, 0.5278), (0.4984, 0.5172), (0.5733, 0.5325)),$$
$$W_2 = ((0.4789, 0.4721), (0.5015, 0.4827), (0.4266, 0.4674))$$

Phase term of IVCNGS:

$$W_1 = ((0.5185, 0.5278), (0.5335, 0.5278), (0.4877, 0.5196)),$$
$$W_2 = ((0.4814, 0.4721), (0.4664, 0.4721), (0.5122, 0.4803))$$

By using the Interval-Valued Complex Neutrosophic Averaging (IVCNA) \label{0.1} operator, compute the averaged Interval-Valued Complex Neutrosophic element (IVCNE) u_i^k of the pharmaceutical industry u_i= {generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4)} over all other testing venues for the experts e_k *(k=1,2):*

$$u_i^k = IVCNA(u_{i1}^k, u_{i2}^k, \dots, u_{in}^k) =$$

$$\left(\sqrt{1 - \left(\prod_{i=1}^n \left(1 - (\delta_{1J}^-)^2{}_{ij}\right) \right)^{\frac{1}{n}}}, \left(\prod_{i=1}^n (\delta_{1J}^+)_{ij} \right)^{\frac{1}{n}} e^{i\sqrt{1-\left(\prod_{i=1}^n\left(1-(\beta_{1J}^-)^2{}_{ij}\right)\right)^{\frac{1}{n}}}, \left(\prod_{i=1}^n (\beta_{1J}^+)^2_{ij}\right)^{\frac{1}{n}}}, \right.$$

$$\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\delta_{2J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\delta_{2J}^{+}\right)_{ij}\right)^{\frac{1}{n}} e^{i\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\beta_{2J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\beta_{2J}^{+}\right)_{ij}\right)^{\frac{1}{n}}},$$

$$\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\delta_{3J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\delta_{3J}^{+}\right)_{ij}\right)^{\frac{1}{n}} e^{i\sqrt{1-\left(\prod_{i=1}^{n}\left(1-\left(\beta_{3J}^{-}\right)^{2}{}_{ij}\right)\right)^{\frac{1}{n}}},\left(\prod_{i=1}^{n}\left(\beta_{3J}^{+}\right)_{ij}\right)^{\frac{1}{n}}}) \text{for all } J=1,2,\dots,k.$$

Displays the findings as an aggregate Table 1 and 2. Calculate a collective IVCNE u_i $(i = 1,2,3,4)$ of the generic (u_1), homeopathic (u_2), herbal (u_3), and allopathic (u_4) using the Interval-Valued Complex Neutrosophic Weighted Averaging (IVCNWA) Operator.

$u_i^k = IVCNA(u_i^1, u_i^2, \dots, u_i^s) =$

$$(\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{1J}^{-}\right)_{k}\right)^{W_{1J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{1J}^{+}\right)_{k}^{w_{1J}^{+}}\right) e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{1J}^{-}\right)_{k}\right)^{W_{1J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{1J}^{+}\right)_{k}^{w_{1J}^{+}}\right)},$$

$$\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{2J}^{-}\right)_{k}\right)^{W_{2J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{2J}^{+}\right)_{k}^{w_{2J}^{+}}\right) e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{2J}^{-}\right)_{k}\right)^{W_{2J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{2J}^{+}\right)_{k}^{w_{2J}^{+}}\right)},$$

$$\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\delta_{3J}^{-}\right)_{k}\right)^{W_{3J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\delta_{3J}^{+}\right)_{k}^{w_{3J}^{+}}\right) e^{i\sqrt{1-\left(\prod_{k=1}^{2}\left(1-\left(\beta_{3J}^{-}\right)_{k}\right)^{W_{3J}^{-}}\right)},\left(\prod_{k=1}^{2}\left(\beta_{2J}^{+}\right)_{k}^{w_{3J}^{+}}\right)}, \forall\ J=1,2,\dots,k.$$

Table 1 The expert aggregation results in amplitude term.

Experts	The Overall Results of the Experts
e_1	$u_1^1 = \langle 0.2065, 0.8801, 0.1526, 0.8801, 0.2065, 0.8801\rangle$
	$u_2^1 = \langle 0.2548, 0.7745, 0.2548, 0.7406, 0.2889, 0.7745\rangle$
	$u_3^1 = \langle 0.1526, 0.8801, 0.2634, 0.9146, 0.1526, 0.8408\rangle$
	$u_4^1 = \langle 0.2146, 0.7745, 0.3302, 0.7691, 0.2548, 0.7406\rangle$
e_2	$u_1^2 = \langle 0.1526, 0.8801, 0.2065, 0.8801, 0.1007, 0.8408\rangle$
	$u_2^2 = \langle 0.2065, 0.8801, 0.2634, 0.9146, 0.2065, 0.8801\rangle$
	$u_3^2 = \langle 0.2065, 0.8801, 0.2634, 0.9146, 0.2065, 0.8801\rangle$
	$u_4^2 = \langle 0.1526, 0.8801, 0.2065, 0.8801, 0.1007, 0.8408\rangle$

Table 2 The expert aggregation results in phase term.

Experts	The Overall Results of the Experts
e_1	$u_1^1 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.1526, 0.7952\rangle$
	$u_2^1 = \langle 0.1426, 0.6324, 0.3660, 0.7071, 0.2146, 0.6999\rangle$
	$u_3^1 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.2065, 0.8408\rangle$
	$u_4^1 = \langle 0.1822, 0.6324, 0.3302, 0.7071, 0.2548, 0.7400\rangle$
e_2	$u_1^2 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.1526, 0.8408\rangle$
	$u_2^2 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.2065, 0.7952\rangle$
	$u_3^2 = \langle 0.1007, 0.7952, 0.2634, 0.8408, 0.2065, 0.8408\rangle$
	$u_4^2 = \langle 0.1526, 0.7952, 0.2065, 0.8408, 0.1526, 0.8408\rangle$

Therefore, generic (u_1), = (0.1828, 0.8801, 0.1817, 0.8801, 0.1700, 0.8615), homeopathic (u_2), = (0.2330, 0.8226, 0.2591, 0.8200, 0.2574, 0.8222), herbal (u_3) = (0.1805, 0.8801, 0.2633, 0.9146, 0.1777, 0.8589), and allopathic (u_4) = (0.1876, 0.8226, 0.2762, 0.8208, 0.2047, 0.7858). Evaluate the score function $S(u_k) = ((Ad^-_{\delta_{1J}})^2 - (Ad^+_{\delta_{1J}})^2) + ((Ad^-_{\delta_{2J}})^2 - (Ad^+_{\delta_{2J}})^2) + ((Ad^-_{\delta_{3J}})^2 - (Ad^+_{\delta_{3J}})^2)$ \cite{A1} of u_k $(k = 1,2,3,4)$ and rated all the testing venues $(u_i), i = 1,2,3,4$.

$$S(u_1) = -2.1960, \quad S(u_2) = -1.8374, \quad S(u_3) = -2.2152, \quad S(u\_4) = -1.8144.$$

Then $S(u_4) > S(u_2) > S(u_1) > S(u_3)$. Therefore, $S(u_4)$ is the best test venue.

Phase terms: Similarly, We can verify the phase terms.

5.1 Algorithm

We now explain our method's step-by-step computation process, which is used in the algorithm that follows.

(i). Input the set $Q = \{a_1, a_2, \ldots, a_n\}$ use a variety of medications (vertices) and put the membership values $\eta = (\eta_1, \eta_2, \eta_3) = ([\eta_1^- e^{i\alpha_1^-}, \eta_1^+ e^{i\alpha_1^+}], [\eta_2^- e^{i\alpha_2^-}, \eta_2^+ e^{i\alpha_2^+}], [\eta_3^- e^{i\alpha_3^-}, \eta_3^+ e^{i\alpha_3^+}])$ of the nodes a_i's, $\eta_1^S, \eta_2^S, \eta_3^S \in [0,1]$ and $\alpha_1^S, \alpha_2^S, \alpha_3^S \in [0,2\pi]$ for all $S = -,+$.

(ii). Input the membership values $\delta_J = (\delta_{1J}, \delta_{2J}, \delta_{3J}) = ([\delta_{1J}^- e^{i\beta_1^-}, \delta_{1J}^+ e^{i\beta_1^+}], [\delta_{2J}^- e^{i\beta_2^-}, \delta_{2J}^+ e^{i\beta_2^+}])$ of the edges $a_i a_j \in R_J$ such that

$$\delta_{1J}^S(a_i a_j) e^{i\beta_{1J}^S(a_i a_j)} \leq min\{\eta_1^S(a_i), \eta_1^S(a_j)\} e^{imin\{\alpha_1^S(a_i), \alpha_1^S(a_j)\}}$$
$$\delta_{2J}^S(a_i a_j) e^{i\beta_{2J}^S(a_i a_j)} \leq max\{\eta_2^S(a_i), \eta_2^S(a_j)\} e^{imax\{\alpha_2^S(a_i), \alpha_2^S(a_j)\}}$$
$$\delta_{3J}^S(a_i a_j) e^{i\beta_{3J}^S(a_i a_j)} \leq max\{\eta_3^S(a_i), \eta_3^S(a_j)\} e^{imax\{\alpha_3^S(a_i), \alpha_3^S(a_j)\}}$$

$0 \leq (\delta_{1J}^S(a_i a_j)) + (\delta_{2J}^S(a_i a_j)) + (\delta_{3J}^S(a_i a_j)) \leq 3$ and $(\beta_{1J}^S(a_i a_j)), (\beta_{2J}^S(a_i a_j)), (\beta_{3J}^S(a_i a_j)) \in [0,2\pi]$ $\forall$ $S = -,+$ and $a_i a_j \in R_J$, $J = 1,2,\ldots,k$.

(iii). On the set used variety of medications Q, develop mutually disjoint, irreflexive, symmetric relations $R_1, R_2, \ldots, R_k$. Give each relation an identity that reflects a particular stage of development between the two types of medications it represents.

(iv). Construct a graph structure on a set of medications with relation, then calculate the energy of each $A\eta_1, A\eta_2, \ldots, A\eta_k$.

(v). Input a calculation like IVCNPRs

$$w_J = \left(\frac{\epsilon(A\delta_{1J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{1J}^S)} e^{i\frac{\epsilon(A\beta_{1J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{1J}^S)}}, \frac{\epsilon(A\delta_{2J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{2J}^S)} e^{i\frac{\epsilon(A\beta_{2J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{2J}^S)}}, \frac{\epsilon(A\delta_{3J}^S)}{\sum_{J=1}^2 \epsilon(A\delta_{3J}^S)} e^{i\frac{\epsilon(A\beta_{3J}^S)}{\sum_{J=1}^2 \epsilon(A\beta_{3J}^S)}} \right),$$

$\forall$ $S = -,+$ *and* $J = 1,2$.

(vi). Calculate IVCNA and IVCNWA

(vii). Evaluate the score function $S(u_k) = (\left(Ad^{-}_{\delta_{1J}}\right)^2 - \left(Ad^{+}_{\delta_{1J}}\right)^2) + \left(\left(Ad^{-}_{\delta_{2J}}\right)^2 - \left(Ad^{+}_{\delta_{2J}}\right)^2\right) + (\left(Ad^{-}_{\delta_{3J}}\right)^2 - \left(Ad^{+}_{\delta_{3J}}\right)^2)$

(viii). Provide an optimal testing venue output.

6. Conclusions and Future Works

The idea of IVCNGS has been developed in this research article by the authors. A more realistic description of uncertainty is offered by the Set IVCNS, an extension of the CNS and IVNS, compared to conventional fuzzy sets. It can be applied in many different contexts through fuzzy control. Many of the mathematical properties of the energy graph have been studied. The integration of the adjacency matrix IVCNGS, the energy of IVCNGS, and Laplacian energy IVCNGS with their intriguing properties has been proposed in this paper. Using the adjacency matrix's eigenvalues, we computed the IVCNGS's spectrum and determined its energy. Moreover, we presented the application of the energy IVCNGS in decision-making, specifically in determining the optimal level of pharmaceutical sources. If the adjacency matrix IVCNGS is used, there are several possible directions for this field's further investigation. Extension of the graph Structures energy to Complex Bipolar Picture Fuzzy Graph Structures, Interval-Valued Spherical Fuzzy Graph Structures, and dominating Complex bipolar neutrosophic graph structures are recommended areas of future research. Some of the limitations of this work are as follows:

- IVCNGS was the main focus of the study and related network systems.
- This approach is only applicable when there are symmetric, irreflexive, and mutually disjoint relations on the IVCNGS.
- The IVCNGS idea is not relevant if the membership values of the characters are provided in distinct environments.
- Obtaining accurate data could sometimes not be possible.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L. A. (1965). Fuzzy sets. Inf. Control, 8: 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X.
2. Turksen. (1986). Interval-valued fuzzy sets based on normal forms. Fuzzy Sets Syst, 20: 19110; https://doi.org/1016/0165-0114(86)90077-1.
3. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets & Syst, 20(1), 87-96: https://doi.org/10.1016/S0165-0114(86)80034-3.

4. Atanassove, K.T. and Gargov, G. (2004). Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application fuzzy sets. Fuzzy Sets and Systems, Pages 55-95. https://doi.org/10.1016/S0888-613X(03)00072-0.
5. Ramot, D., Friedman, M., Langholz G.A., & Kandel. (2003). Complex fuzzy logic. IEEE Trans. Fuzzy Syst, 11: 450?61. DOI: 10.1109/TFUZZ.2003.814832.
6. Alkouri, A., Salleh, A. (2012). Complex intuitionistic fuzzy sets. AIP Conf. Proc, 14: 464–470. https://doi.org/10.1063/1.4757515.
7. F. Smarandache. (2010). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. International Journal of Pure and Applied Mathematics, 24: 289-297.
8. F. Smarandache, Neutrosophic Graphs, in his book Symbolic Neutrosophic Theory, Europa, Nova.
9. F. Smarandache. (2005). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter. J.Pure Appl. Math, 24: 287-297.
10. F. Smarandache. (1999). A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability. Set and Logic. Rehoboth: American Research Press.
11. F. Smarandache. (1998). Neutrosophy, Neutrosophic Probability, Set, and Logic. Amer. Res. Press, Rehoboth, USA, 105 pages. http://fs.gallup.unm.edu/eBookneutrosophics4.pdf(4th edition).
12. F. Smarandache., M. Ali. (2017). Complex neutrosophic set. Neural Comput. APPl., 28(7): 1817-1834. https://doi.org/10.1007/s00521-015-2154-y.
13. Atiqe U. R., Muhammad S., F. Smarandache., & Muhammad. R. A. (2020). Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set. Neutrosophic Sets and Systems, Vol. 38.
14. Ivan Gutman., Bo Zhou. (2006). Laplacian energy of a graph. Linear Algebra and its Applications, Volume 414, Issue 1, 1April, Pages 29-37. https://doi.org/10.1016/j.laa.2005.09.008.
15. Rosenfeld, A. (1975). Fuzzy graphs. Zadeh, L.A., K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and their Applications, Academic Press, New York, pp.77-95.
16. Bhattacharya, P. (1987) Some remarks on fuzzy graphs. Pattern Recognit Lett, 6: 297–302. https://doi.org/10.1016/0167-8655(87)90012-2.
17. Rashmanlou, H. & M. Pal. (2013). Some properties of highly irregular interval-valued fuzzy graphs. World Applied Sciences Journal, 27(12): 1756–1773. https://doi.org/10.5829/idosi.wasj.2013.27.12.1316.
18. Rashmanlou, H., & Pal, M. (2014). Antipodal interval-valued fuzzy graphs. https://doi.org/10.48550/arXiv.1407.6190
19. Karunambigai, M. G., Palanivel, K., & Sivasankar, S. (2015). Edge regular intuitionistic fuzzy graph. Advances in Fuzzy Sets and Systems, 20(1), 25-46. http://dx.doi.org/10.17654/AFSSSep2015_025_046.
20. Thirunavukarasu, P., Suresh, R., & Viswanathan, K. K. (2016). Energy of a complex fuzzy graph. Int. J. Math. Sci. Eng. Appl, 10(1), 243-248.
21. Yaqoob. N., Gulistan M., Kadry S., & Wahab H. (2019). Complex intuitionistic fuzzy graphs with application incellular network provider companies. Mathematics, 7: 35. https://doi.org/10.3390/MATH7010035.
22. Yaqoob. N., Akram .M. (2018). Complex neutrosophic graphs. Bull. Comput. Appl. Math, 6: 85–109.
23. Anam Luqman., Akram, M., & F. Smarandache. (2019). Complex Neutrosophic Hypergraphs: New Social Network Models. Algorithms, 12: 234. https://doi.org/10.3390/a12110234.
24. Shabaf, S. R., Fayazi, F. (2014). Laplacian Energy of a Fuzzy Graph, Iran. J. Math. Chem, 5, 1–10.
25. Soumitra Poulik., Ganesh Ghorai. (2020). Detour g-interior nodes and detour g-boundary nodes in bipolar fuzzy graph with applications. Hacettepe Journal of Mathematics \& Statistics, Volume 49 (1): 106 – 119. https://doi.org/10.15672/HJMS.2019.666.
26. Soumitra Poulik., Ganesh Ghorai., & Qin Xin. (2021). Pragmatic results in Taiwan education system based IVFG \& IVNG, 25, 711–724. DOI: https://doi.org/10.1007/s00500-020-05180-4.
27. Poulik, S., & Ghorai, G. (2020). Empirical results on operations of bipolar fuzzy graphs with their degree. Missouri Journal of Mathematical Sciences, 32(2), 211-226. https://doi.org/10.35834/2020/3202211.
28. Poulik, S., & Ghorai, G. (2020). Note on "Bipolar fuzzy graphs with applications". Knowledge-Based Systems, 192, 105315. https://doi.org/10.1016/j.knosys.2019.105315.

29. Sampathkumar, E. (2006). Generalized graph structures. Bull. Kerala Math. Assoc, 3(2), 65-123.
30. Dinesh, T., Ramakrishnan, T.V. (2011). Generalised Fuzzy Graph Structures. Applied Mathematical Sciences, 5(4), 173 -180.
31. Akram, M., & R. Akmal. (2016). operations on Intuitionistic Fuzzy Graph Structures. Fuzzy Inf.Eng, 8: 389-410. https://doi.org/10.1016/J.FIAE.2017.01.001.
32. Mohamad, S. N. F., Hasni, R., Smarandache, F., & Yusoff, B. (2021). Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. Axioms, 10(3), 172. https://doi.org/10.3390/axioms10030172.

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