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The submitted papers should be professional, and in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e., notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory, every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjointed two by two. But, since in many cases, the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and intuitionistic fuzzy logic). In neutrosophic logic, a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of classical statistics.

What distinguishes neutrosophic from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Unveiling Efficiency: Investigating Distance Measures in Wastewater Treatment Using Interval-Valued Neutrosophic Fuzzy Soft Set

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Abstract: To reduce the threats that wastewater poses to human health and the environment, water treatment techniques must be improved. The use of a procedure that includes preparation, testing, primary and secondary treatments, filtration, disinfection, and continuous monitoring is therefore required. The objective of this research is to create a hybrid notion that extends the idea of an interval-valued neutrosophic fuzzy soft set (IVNFSS) to an interval-valued neutrosophic fuzzy set. Operations like complement, union, and integration are included in the idea. To improve decision-making accuracy, a quality-assessment distance measure is incorporated, offering a numerical representation of the disparity between various factors. Furthermore, the IVNFSS defines distance measures, which are used in the wastewater treatment process. To monitor water quality, IVNFSS, in conjunction with a distance measure, is a potent instrument whose application extends to water waste treatment procedures. This can completely change the way that water quality management is now done by providing a methodical way to guarantee the security and quality of drinking water.

Keywords: Interval-Valued Neutrosophic Fuzzy Soft Set; Decision Making; Distance Measure; Unveiling Efficiency.

1. Introduction

Wastewater is a severe issue as it is composed of polluted water from several activities and processes. The release of this untreated water into rivers or seas has a negative impact on the ecosystem, endangering humans, animals, and plants. To avoid these problems and protect the environment and public health, it is imperative to upgrade water treatment techniques. Complete purification may be difficult to achieve since traces of impurities may remain after the first cleaning procedure. The difficult job is getting rid of these leftovers and handling any byproducts that are produced while treating water. There are several procedures involved in preparing water for human consumption. The first steps in eliminating impurities are screening and pre-treatment, which uses chemicals to help with particle clumping. Organic matter is broken down and settled by further primary and secondary treatments, and any leftover particles are removed by filtering Remaining dangerous germs are eliminated by disinfection using chemicals or UV radiation, and safety is guaranteed by pH correction. Water quality requirements are upheld by rigorous testing, and residences are supplied with purified water via pipelines. The water supply is kept secure via ongoing observation and strict conformity to the law.

Purifying water to fulfill stringent quality requirements is the goal of the painstaking process known as water treatment. It is more important to achieve predetermined standards to guarantee its safety for human use and environmental effects than to make a direct comparison to naturally clean water. The constant and trustworthy evaluation of water quality across several sources is ensured by this methodical technique. It might be challenging to implement conventional wastewater treatment in many situations due to the absence of clear, confusing, or sufficient data. Fuzzy logic and fuzzy sets are effective options for handling confusing data in wastewater treatment.

Fuzzy set theory, introduced by Zadeh [1] in 1965, is a mathematical framework that extends traditional set theory to handle uncertainty and vagueness. Fuzzy sets allow for partial membership, assigning degrees of membership between [0, 1] to elements. This enables the representation of gradual or fuzzy boundaries, capturing the inherent ambiguity and imprecision in real-world situations. Zadeh then expanded the idea of FS to interval-valued fuzzy sets in 1975 [2]. Because of this improvement, the membership function may now be divided into intervals rather than just one value, which is more appropriate for use in practical applications. Sometimes, it can be difficult to assign appropriate membership values to fuzzy sets. The use of IVFS was suggested by Turksen [3] as a solution for such scenarios. When faced with situations of anxiety and uncertainty, it is important to carefully consider the appropriate representation of an object. In such cases, the object's suitable representation values, i.e., unbiased membership and non-membership values, cannot be accurately calculated by either IVFS or fuzzy sets. In addition to making a substantial contribution to the subject of FS theory, IVFS sets have also created new opportunities for addressing and modeling uncertainty in a more thorough way [4]. IVFS addresses situations where the exact membership value is ambiguous or difficult to determine precisely, accommodating a wider range of uncertainty. These extended representations of IVFS have attracted significant research attention, particularly in the field of multi-criteria decision-making (MCDM) [5-7]. These developments offer valuable tools for handling uncertainty in decision-making processes.

Intuitionistic fuzzy set theory, introduced by Atanassov [8] in 1983, further extends fuzzy set theory by incorporating an additional function known as the non-membership degree along with the membership degree. In intuitionistic fuzzy sets, each element is characterized by its membership degree, non-membership degree, and hesitation degree. The sum of the membership degree, non-membership degree, and hesitation degree is always less than or equal to one. This property ensures that the degrees assigned to an element accurately represent the entire uncertainty spectrum without exceeding the bounds. The hesitation degree represents the lack of confidence in assigning a definite membership value and allows for a more comprehensive representation of uncertainty and imprecision in decision-making processes. By considering both membership and non-membership degrees, intuitionistic fuzzy sets provide a richer framework for modeling and reasoning with uncertain and vague information. Atanassov laid the groundwork for the voyage by first defining SM for IFS components [9], constructing on top of this. Quantifying similarity degrees for ambiguous sets was first proposed by Chen [10]. Cher measurements, according to Hong and Kim, displayed errors and indistinguishable results in severe circumstances, which called for the creation of modified SM [11]. Following this, Dengfen and Chuntian [12] concentrated on finding SMS for IFSS, particularly in the context of discrete or continuous universal sets for pattern recognition issues.

Smarandache expanded fuzzy set theory in [13] by incorporating a third element known as indeterminacy. This is known as the neutrosophic set theory. NS can more successfully enable approximation reasoning and is more prepared to maintain the fuzziness of information's contents. Traditional NS has a worse descriptive capacity than NS due to the addition of non-membership and indeterminacy-graded functions. The unit closed interval was given for each of the three uncertain NS components in Wang et al. [14] formulation of SVNS. Many researchers have contributed to NS for application in topological spaces, statistics, and the development of different hybridized frameworks to help with decision-making. In 1995, Smarandache introduced the pioneering concepts of Neutrosophic Over-/Under-/Off-Set and Logic, which were later published in 2007 [15]. These unique notions diverge from traditional sets, logic, and probabilities and have been showcased at global conferences from 1995 to 2016. Neutrosophic sets have been extended to include Neutrosophic Overset (where a component > 1) and Neutrosophic Underset (where a component < 0), allowing for the capture of intricate real-world scenarios. The framework's innovative approach, which includes neutrosophic over/under/off logic and probability, has unlocked valuable applications in technology, economics, and social sciences, making it ripe for further exploration. Pramanik et al. [16] thoroughly analyzed the Neutrosophic cubic set, which blends INS with SVNS to enable the simultaneous recording of hybrid information. Saqlain et al. [17] proposed the single and multi-valued neutrosophic hypersoft sets and the tangent similarity measure of single-valued neutrosophic hypersoft sets. In this work, we expand Said and Smarandache's [18] examination of Bhowmik and Pal's INS sets and incorporate it into the world of soft sets. The notation is because of INSS, along with the introduction of definitions, operations, and the construction of characteristics relevant to this merger. We define operations on INSS and validate assertions by bridging the gap between intuitionistic neutrosophic sets and soft sets, both of which naturally deal with imprecision. Additionally, the use of INSS in resolving a dilemma in decision-making serves to highlight how practically useful it is. Broumi [19] integrates A.A., generalizes NSS, and combines Molodtsov's soft set and Salama's neutrosophic set, introducing a new framework for handling imprecision and exploring interdisciplinary applications in engineering, mathematics, and computer science. Broumi et al. [20] propose an enhanced extension, Soft Relations IVNSS, encompassing various types of relations—soft, fuzzy, intuitionistic fuzzy, interval-valued intuitionistic fuzzy, and neutrosophic soft relations—with an exploration of reflexivity, symmetry, and transitivity, offering potential applications and prompting further research in the field. Deli [21] integrates interval-valued neutrosophic sets with soft sets, introducing the innovative concept of interval-valued neutrosophic soft, which generalizes various set types and demonstrates its efficacy in decision-making techniques [22-26].

Soft sets (SS) are a broad mathematical technique that Molodtsov [27] suggested to work with ambiguous, indeterminate, and uncertain substances. SS allow for the representation and manipulation of data without requiring crisp boundaries or precise definitions. Maji et al. [28] further on SS's work and specified a few operations and their characteristics. They also make judgments in [29-30] based on SS theory. Soft matrices with operations were introduced and their characteristics were examined by Cagman and Enginoglu [31]. They also proposed a technique for making decisions to deal with difficulties of uncertainty [32-33]. They altered the Molodtsov's SS-proposed activities in [34]. The author of [35] introduces novel methods for soft matrices, including soft difference products, to combine the characteristics of soft sets (SSs), fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), and neutrosophic sets (NSs), resulting in FS, IFSS, and NSS. Babitha and Sunil [36] further explore SSs, soft set relations, Cartesian soft set products, and related concepts, providing a foundation for addressing uncertainty in complex systems. They also investigate set theory using soft set relations, emphasizing the development of these ideas as a theoretical basis for future research and advancement in the field. By incorporating real-world applications [37-38], it expands the soft set theory. To ensure that soft set theory is correctly applied in a variety of disciplines, Broumi et al. developed the hybridized structure of NSS with SS for interval settings [39] and Das et al. [40]. In addition to talking about the basics of these models, they employed techniques to make these models applicable in varied settings. Attribute values are split up into sub-attributive values in several real-world situations.

To achieve this, an Interval-values neutrosophic fuzzy soft set environment was developed. These environments can more accurately handle uncertainty. As previously mentioned, there are some limitations to the existing studies, such as:

- Fuzzy soft sets deal with truthiness exclusively.
- An expansion of this idea is intuitionistic fuzzy soft, yet not a complete version. On the other hand, Neutrosophic Soft Sets handle three values, including truthiness, indeterminacy, and falsity, but lack sub-attributions.

- Neutrosophic fuzzy soft sets incorporate values: of truth, indeterminacy, and falsity, while also utilizing the expert value of fuzziness.
- Interval-values Neutrosophic fuzzy soft set incorporate values: of truth, indeterminacy, and falsity, while also utilizing the expert value of fuzziness.
- Some situations cannot be explained by current theories. In this thesis, we extended the theory and developed numerous techniques.

Working with neutrosophic fuzzy sets can be challenging due to their complex framework. To address this, we have developed the Interval-values neutrosophic fuzzy soft set. In this study, we aim to examine and address any issues. By doing so, we can apply all the definitions, operators, and properties of IVNFSS. These sets have been useful in various decision-making strategies. When faced with genuine logical and numerical problems, uncertainty can be helpful. To address this uncertainty, often turn to Multi-Criteria Decision-Making (MCDM). These types of problems involve various attributes, and we strive to find the best possible match. However, when faced with more complex selections like Multi-Criteria Multi Attributive Decision Problems (MCDM), then utilize tools such as Interval-valued Neutrosophic fuzzy soft sets. Additionally, researchers have dedicated significant effort towards developing distance measures (DM) to aid in these types of problems.

The proposed study has investigated and answered potential questions.

- Theoretical approach for IVNFSS.
- Development of Distance measure with the help of IVNFSS.
- Development of Algorithms (IVNFSS) to solve MCDM.

The following is how the study is set up: A few chosen preliminary definitions that are necessary for introducing IVNFSS are provided in Section 2. The idea of IVNFSS is defined in Section 3 along with the attributes and operators previously discussed. The applications of fuzzy distance measures are first discussed in Section 4, after which the development of Hamming and Normalized Hamming distance measures and the proofs necessary to support their usage are covered. An algorithm for computing similarity from an ideal solution is built in Section 5 to demonstrate the use of the distance measurements that have been constructed in the treatment of water waste. Next, using a comparison of the facilities with a reference set, this approach is employed to choose the best alternative among several. In the Conclusion section, the paper's main conclusions are summarized.

2. Preliminaries

In this section, some basic concepts are defined that are required for the development of the proposed structure.

2.1 Neutrosophic Fuzzy Set [40]

Neutrosophic fuzzy sets, which incorporate the indeterminacy dimension and combine fuzzy sets with neutrosophic set theory, are an important tool for efficiently managing uncertainty and imprecision in a variety of domains. Let °F is universal set and $\curlyvee \in$ °F and function $\ddot{\Upsilon}$ is defined as:

$$\ddot{\Upsilon} = \{\mathrm{e}, \langle \mathcal{T}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q})\rangle, \mathfrak{Q}_{\ddot{\Upsilon}}(\mathrm{e}) \ \forall \ \mathrm{e} \in {}^{\circ}\mathrm{F}\},$$

Where $\mathcal{T}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q})$ are truth, indeterminacy, and falsity membership functions of every fuzzy-membership value and $\mathcal{T}_{\ddot{\Upsilon}}, \mathcal{I}_{\ddot{\Upsilon}}$ *and* $\mathcal{F}_{\ddot{\Upsilon}}$ are the real standard or non-standard subsets of $\mathrm{I}_{0_-}^{1^+}$. Such that $\mathcal{T}_{\ddot{\Upsilon}}, \mathcal{I}_{\ddot{\Upsilon}}, \mathcal{F}_{\ddot{\Upsilon}} : {}^{\circ}\mathrm{F} \longrightarrow \mathrm{I}_{0_-}^{1^+}$ and no restrictions are there on the sum of $\mathcal{T}_{\ddot{\Upsilon}}, \mathcal{I}_{\ddot{\Upsilon}}$ and $\mathcal{F}_{\ddot{\Upsilon}}$
Hence,

$$0^- \leq \mathcal{T}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}) + \mathcal{I}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}) + \mathcal{F}_{\ddot{\Upsilon}}(\mathrm{e}, \mathfrak{Q}) \leq 3^+.$$

2.2 Single-Valued Neutrosophic Fuzzy Set [41]

A SVNFS $\ddot{\Upsilon}$ in universal set °F is defined as.

$$\ddot{\Upsilon} = \{e, \langle \mathcal{T}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(e, \mathfrak{Q})\rangle, \mathfrak{Q}_{\ddot{\Upsilon}}(e)\ \forall\ e \in {}^{\circ}\mathrm{F}\},$$

where,

$$\mathcal{T}_{\ddot{\Upsilon}}(e,\mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(e, \mathfrak{Q}) \in [0,1],$$

and

$$0 \leq \mathcal{T}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) + \mathcal{I}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) + \mathcal{F}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) \leq 3.$$

3. Proposed Structure based on IVNFSS

This section introduces the Interval-valued neutrosophic fuzzy soft set concept and defines its relevant operators.

3.1 Interval-Valued Neutrosophic Fuzzy Set

Interval-valued neutrosophic fuzzy sets, an extension of Zadeh's fuzzy set, effectively manage uncertainty and imprecision in a variety of disciplines by pro viding a thorough framework that combines intervals and the indeterminacy dimension. Let $^{\circ}$F is the Universal of discourse and $e \in {}^{\circ}$F an I-VNFS Υ in $^{\circ}$F is define as;

$$\ddot{\Upsilon} = \{e, \langle \mathcal{T}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(e, \mathfrak{Q})\rangle, \mathfrak{Q}_{\ddot{\Upsilon}}(e)\ \forall\ e \in {}^{\circ}\mathrm{F}\},$$

where,

$$\mathcal{T}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{I}_{\ddot{\Upsilon}}(e, \mathfrak{Q}), \mathcal{F}_{\ddot{\Upsilon}}(e, \mathfrak{Q}) \in [0,1],$$

and

$$0 \leq \mathcal{S}up\mathcal{T}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) + \mathcal{S}up\mathcal{I}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) + \mathcal{S}up\mathcal{F}_{\ddot{\Upsilon}}(e,\mathfrak{Q}) \leq 3.$$

3.2 Neutrosophic Fuzzy Soft Set

Let $^{\circ}$F be an initial universal set and let ℵ be the set of parameters. Consider $\mathcal{A}$ to be a subset of ℵ then a function Ï its mapping is given by: Ï = $\mathcal{A} \rightarrow N_{fss}(^{\circ}F)$. Then a function NFSS is characterized as:

$$\c{K} = \{e, \langle \mathcal{T}_{\c{K}}(e, \mathfrak{Q}), \mathcal{I}_{\c{K}}(e, \mathfrak{Q}), \mathcal{F}_{\c{K}}(e, \mathfrak{Q})\rangle, \mathfrak{Q}_{\c{K}}(e)\ \forall\ e \in {}^{\circ}\mathrm{F}\},$$

$$0^{-} \leq \mathcal{T}_{\c{K}}(e,\mathfrak{Q}) + \mathcal{I}_{\c{K}}(e,\mathfrak{Q}) + \mathcal{F}_{\c{K}}(e,\mathfrak{Q}) \leq 3^{+}.$$

3.3 Single-Valued Neutrosophic Fuzzy Soft Set

Let $^{\circ}$F be an initial universal set and let ℵ be the set of parameters. Consider $\mathcal{A}$ to be a subset of ℵ then a function Ï its mapping is given by: Ï = $\mathcal{A} \rightarrow S_{VNFSS}(^{\circ}F)$. Then a function IVNFSS is characterized as:

$$\c{K} = \{e, \langle \mathcal{T}_{\c{K}}(e, \mathfrak{Q}), \mathcal{I}_{\c{K}}(e, \mathfrak{Q}), \mathcal{F}_{\c{K}}(e, \mathfrak{Q})\rangle, \mathfrak{Q}_{\c{K}}(e)\ \forall\ e \in {}^{\circ}\mathrm{F}\},$$

where

$$\mathcal{T}_{\c{K}}(e, \mathfrak{Q}), \mathcal{I}_{\c{K}}(e, \mathfrak{Q}), \mathcal{F}_{\c{K}}(e, \mathfrak{Q}) \subseteq \mathrm{I}_0^1$$

and

$$0 \leq \mathcal{T}_{\c{K}}(e,\mathfrak{Q}) + \mathcal{I}_{\c{K}}(e,\mathfrak{Q}) + \mathcal{F}_{\c{K}}(e,\mathfrak{Q}) \leq 3.$$

3.4 Interval-Valued Neutrosophic Fuzzy Soft Set

Let $^{\circ}$F is the universal of discourse and ℵ be the set of attributes of elements in $^{\circ}$F. Take $\mathcal{A}$ to be a subset of ℵ then a function F its mapping is given by: F: $\mathcal{A} \rightarrow$P ($^{\circ}$F), Then the Interval-Valued Neutrosophic Fuzzy soft set can be generated as follow:

$$\mathcal{T\!D} = \{\langle \hbar, \mathfrak{f}_{\lambda}(\hbar)\rangle \mid \hbar \in \mathcal{A}, \mathfrak{f}_{\lambda}(\hbar) \in \mathrm{P}\ (^{\circ}\mathrm{F})\},$$

where P ($^{\circ}$F) is the IVNFSS,

$$\mathfrak{f}_{\lambda}(\hbar) = \{e, \langle \mathcal{T}_{\lambda}(e, \mathfrak{Q}), \mathcal{I}_{\lambda}(e, \mathfrak{Q}), \mathcal{F}_{\lambda}(e, \mathfrak{Q})\rangle, \mathfrak{Q}_{\lambda}(e)\ \forall\ e \in {}^{\circ}\mathrm{F}\},$$

and

$$0 \leq \mathcal{S}up\mathcal{T}_{\lambda}(\mathrm{e}, \mathfrak{Q}) + \mathcal{S}up\mathcal{I}_{\lambda}(\mathrm{e}, \mathfrak{Q}) + \mathcal{S}up\mathcal{F}_{\lambda}(\mathrm{e}, \mathfrak{Q}) \leq 3.$$

3.5 Complement of IVNFSS

The complement of an IVNFSS Ụ denoted by $Ụ^{c}$ defined as;

$$Ụ^{c} = \begin{matrix} [\boldsymbol{i_m}\boldsymbol{\mathcal{F}}_{Ụ}(\mathrm{e}, \mathfrak{Q}), \boldsymbol{\mathcal{S}_m}\boldsymbol{\mathcal{F}}_{Ụ}(\mathrm{e}, \mathfrak{Q})], \\ [\,\mathbf{1} - \boldsymbol{\mathcal{S}_m}\boldsymbol{\mathcal{I}}_{Ụ}(\mathrm{e}, \mathfrak{Q}), \mathbf{1} - \boldsymbol{i_m}\boldsymbol{\mathcal{I}}_{Ụ}(\mathrm{e}, \mathfrak{Q})], \\ [\boldsymbol{i_m}\boldsymbol{\mathcal{T}}_{Ụ}(\mathrm{e}, \mathfrak{Q}), \boldsymbol{\mathcal{S}_m}\boldsymbol{\mathcal{T}}_{Ụ}(\mathrm{e}, \mathfrak{Q})], \\ \mathbf{1} - \mathfrak{Q}_{Ụ}(\mathrm{e}), \end{matrix}$$

$\forall\ \mathrm{e} \in$ °F.

3.6 Example

$Ụ^{c}$= {⟨$\hbar_1$, (e_1 ,[0.3,0.6], [0.3,0.5] ,[0.4,0.6], 0.3), (e_2, [0.3, 0.6], [0.4, 0.6], [0.4, 0.8], 0.4), (e_3, [0.3, 0.5], [0.4, 0.6], [0.4, 0.6], 0.4)⟩, ⟨$\hbar_2$, (e_1, ([0.3, 0.6], [0.5, 0.6], [0.5, 0.8], 0.4), (e_2, ([0.5, 0.6], [0.5, 0.7], [0.5, 0.7], 0.3), (e_3, ([0.2, 0.5], [0.4, 0.7], [0.4, 0.6], 0.6)⟩, ⟨$\hbar_3$, (e_1, [0.3, 0.7], [0.4, 0.6], [0.7, 0.8], 0.2), (e_2, ([0.4, 0.6], [0.4, 0.7], [0.4, 0.8], 0.3), (e_3, ([0.2, 0.4], [0.5, 0.7], [0.6, 0.7], 0.3)⟩}.

3.7 Union of IVNFSS

The union of IVNFSS of Ọ ∪ Ụ in °F is given:

$$Ọ \cup Ụ = \begin{matrix} \left[\mathcal{M}_{\chi}\left(i_m\mathcal{T}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{T}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{\chi}\left(\mathcal{S}_m\mathcal{T}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{T}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \left[\mathcal{M}_{n}\left(i_m\mathcal{I}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{I}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{n}\left(\mathcal{S}_m\mathcal{I}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{I}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \left[\mathcal{M}_{n}\left(i_m\mathcal{F}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{F}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{n}\left(\mathcal{S}_m\mathcal{F}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{F}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \mathcal{M}_{\chi}\left(\mathfrak{Q}_{Ọ}(\mathrm{e}), \mathfrak{Q}_{Ụ}(\mathrm{e})\right), \end{matrix}$$

$\forall\ \mathrm{e} \in$ °F.

3.8 Example

The union of Ọ ∪ Ụ is

Ọ ∪ Ụ= {⟨$\hbar_1$,(e_1, [0.6,0.7], [0.3,0.6], [0.3,0.6], 0.6), (e_2, [0.4, 0.8], [0.2, 0.6], [0.2, 0.6], 0.5), (e_3, [0.5, 0.6], [0.4, 0.7], [0.3, 0.5], 0.6)⟩, ⟨$\hbar_2$, (e_1, [0.7, 0.8], [0.4, 0.5], [0.3, 0.6], 0.6), (e_2, [0.5, 0.7], [0.3, 0.5], [0.5, 0.7], 0.7), (e_3, [0.4, 0.6], [0.3, 0.6], [0.2, 0.5], 0.7)⟩, ⟨$\hbar_3$, (e_1,[0.7, 0.8], [0.4, 0.6], [0.3, 0.6], 0.8), (e_2, [0.5, 0.8], [0.3, 0.9], [0.5, 0.7], 0.6), (e_3, [0.6, 0.7], [0.3, 0.5], [0.2, 0.4], 0.7)⟩}.

3.9 Intersection of IVNFSS

The Intersection of IVNFSS of Ọ ∩ Ụ in °F is given:

$$Ọ \cap Ụ = \begin{matrix} \left[\mathcal{M}_{n}\left(i_m\mathcal{T}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{T}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{n}\left(\mathcal{S}_m\mathcal{T}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{T}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \left[\mathcal{M}_{\chi}\left(i_m\mathcal{I}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{I}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{\chi}\left(\mathcal{S}_m\mathcal{I}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{I}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \left[\mathcal{M}_{\chi}\left(i_m\mathcal{F}_{Ọ}(\mathrm{e}, \mathfrak{Q}), i_m\mathcal{F}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right), \mathcal{M}_{\chi}\left(\mathcal{S}_m\mathcal{F}_{Ọ}(\mathrm{e}, \mathfrak{Q}), \mathcal{S}_m\mathcal{F}_{Ụ}(\mathrm{e}, \mathfrak{Q})\right)\right], \\ \mathcal{M}_{n}\left(\mathfrak{Q}_{Ọ}(\mathrm{e}), \mathfrak{Q}_{Ụ}(\mathrm{e})\right), \end{matrix}$$

$\forall\ \mathrm{e} \in$ °F.

4. Decision Support System based on IVNFSS

The evaluation of the connections between the components of IVNFSS depends heavily on the distance measure. This metric is essential for applications like clustering and decision-making when using shaky data. It measures the degree of concordance or variance among neutrosophic memberships, fuzzy degrees of uncertainty, and memberships with interval values, providing crucial information for reliable data interpretation. To deal with the complexity of actual data sets, a variety of specialized distance metrics designed for IVNFSS are available. The Hamming and Normalized Hamming distances for interval-valued neutrosophic fuzzy soft sets IVNFSS are introduced. These measures of distance are useful in the scientific and engineering sectors and may be used in a variety of real-world situations to make analysis and comparisons easier inside the IVNFSS framework.

Let's take two IVNFSS $\check{K}$ *and* F a universe of discourse °F= $\{e_1, e_2, e_3, \ldots, e_n\}$ which are denoted by;

$$\check{K}= \{e\ [\mathcal{T}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{T}_{\check{K}}^{U}(e, \mathfrak{Q})],\ [\mathcal{I}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{I}_{\check{K}}^{U}(e, \mathfrak{Q})],\ [\mathcal{F}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{F}_{\check{K}}^{U}(e, \mathfrak{Q})]\rangle, \mathfrak{Q}_{\check{K}}(e)\ \forall:\ e \in {}^{\circ}F\}$$

$$F= \{e\ [\mathcal{T}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{T}_{F}^{U}(e, \mathfrak{Q})],\ [\mathcal{I}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{I}_{F}^{U}(e, \mathfrak{Q})],\ [\mathcal{F}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{F}_{F}^{U}(e, \mathfrak{Q})]\rangle, \mathfrak{Q}_{F}(e)\ \forall:\ e \in {}^{\circ}F\}$$

4.1 Hamming Distance

$$d^{\mathcal{H}}(\check{K}, F) = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7m}\begin{cases} \hbar_J\left|\mathcal{T}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{T}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{I}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{I}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{F}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{F}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathfrak{Q}_{\check{K}}(e) - \mathfrak{Q}_{F}(e)\right|. \end{cases}$$

4.2 Normalized Hamming Distance

$$d^{\mathcal{NH}}(\check{K}, F) = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7nm}\begin{cases} \hbar_J\left|\mathcal{T}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{T}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{I}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{I}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{F}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{F}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathfrak{Q}_{\check{K}}(e) - \mathfrak{Q}_{F}(e)\right|. \end{cases}$$

4.3 Theorem

Distance $d^{\mathcal{H}}(\check{K}, F)$ is said to be distance measure if it satisfies the following properties:

1. $0 \le d^{\mathcal{H}}(\check{K}, F) \le 1$;
2. $d^{\mathcal{H}}(\check{K}, F) = 0$ *iff* $\check{K} = F$;
3. $d^{\mathcal{H}}(\check{K}, F) = d^{\mathcal{H}}(F, \check{K})$;
4. If $\check{K} \subseteq F \subseteq H$, than $d^{\mathcal{H}}(\check{K}, F) \leqslant d^{\mathcal{H}}(\check{K}, H)$ and $d^{\mathcal{H}}(F, H) \leqslant d^{\mathcal{H}}(\check{K}, H)$.

Proof:

1. By the definition of distance, it is $d^{\mathcal{H}}(\check{K}, F) \geqslant 0$. For it to be valid $d^{\mathcal{H}}(\check{K}, F) \leqslant 1$. By using the definition of Interval-valued neutrosophic fuzzy soft set. Such That;
   $0 \leqslant \mathcal{T}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{T}_{\check{K}}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathcal{I}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{I}_{\check{K}}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathcal{F}_{\check{K}}^{L}(e, \mathfrak{Q}),\ \mathcal{F}_{\check{K}}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathfrak{Q}_{\check{K}}(e) \leqslant 1$.
   $0 \leqslant \mathcal{T}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{T}_{F}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathcal{I}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{I}_{F}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathcal{F}_{F}^{L}(e, \mathfrak{Q}),\ \mathcal{F}_{F}^{U}(e, \mathfrak{Q}) \leqslant 1$, $0 \leqslant \mathfrak{Q}_{F}(e) \leqslant 1$.

This implies that

$$0 \leqslant \left|\mathcal{T}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{L}(e,\mathfrak{Q})\right| \leqslant 1, 0 \leqslant \left|\mathcal{T}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{U}(e,\mathfrak{Q})\right| \leqslant 1$$
$$0 \leqslant \left|\mathcal{I}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{L}(e,\mathfrak{Q})\right| \leqslant 1, 0 \leqslant \left|\mathcal{I}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{U}(e,\mathfrak{Q})\right| \leqslant 1$$
$$0 \leqslant \left|\mathcal{F}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{L}(e,\mathfrak{Q})\right| \leqslant 1, 0 \leqslant \left|\mathcal{F}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{U}(e,\mathfrak{Q})\right| \leqslant 1$$
$$0 \leqslant \left|\mathfrak{Q}_{\check{K}}(e) - \mathfrak{Q}_{F}(e)\right| \leqslant 1$$

$$0 \leqslant \begin{pmatrix} \hbar_J\left|\mathcal{T}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{T}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{T}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{I}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{I}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{I}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{F}_{\check{K}}^{L}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{L}(e,\mathfrak{Q})\right| + \hbar_J\left|\mathcal{F}_{\check{K}}^{U}(e,\mathfrak{Q}) - \mathcal{F}_{F}^{U}(e,\mathfrak{Q})\right| + \\ \hbar_J\left|\mathfrak{Q}_{\check{K}}(e) - \mathfrak{Q}_{F}(e)\right|. \end{pmatrix} \leqslant 7$$

$\Rightarrow\ 0 \le d^{\mathcal{H}}(\check{K}, F) \le 1$

2. Let $d^{\mathcal{H}}(\check{K}, Ғ) = 0$ for two IVNFSS $\check{K}$ *and* Ғ

$$\Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7nm}\begin{pmatrix} \hbar_j\left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{Ғ}(\mathfrak{e})\right|. \end{pmatrix} = 0$$

iff for all:

$$\begin{gathered} \left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| = 0, \left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| = 0 \\ \left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| = 0, \left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| = 0 \\ \left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| = 0, \left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| = 0 \\ \left|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{Ғ}(\mathfrak{e})\right| = 0 \end{gathered}$$

which is equivalent to

$$\begin{gathered} \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) = \mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) = \mathcal{T}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q}) \\ \mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) = \mathcal{I}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) = \mathcal{I}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q}) \\ \mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) = \mathcal{F}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) = \mathcal{F}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q}) \\ \mathfrak{Q}_{\check{K}}(\mathfrak{e}) = \mathfrak{Q}_{Ғ}(\mathfrak{e}). \end{gathered}$$

Thus $d^{\mathcal{H}}(\check{K}, Ғ) = 0$

$\Rightarrow \check{K} = Ғ$

3. For two IVNFSS $\check{K}$ and Ғ are;

$$d^{\mathcal{H}}(\check{K}, Ғ) = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7m}\left\{\begin{matrix} \hbar_j\left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{Ғ}(\mathfrak{e})\right|. \end{matrix}\right.$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7m}\left\{\begin{matrix} \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_j\left|\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_j\left|\mathfrak{Q}_{Ғ}(\mathfrak{e}) - \mathfrak{Q}_{\check{K}}(\mathfrak{e})\right|. \end{matrix}\right.$$

$$= d^{\mathcal{H}}(Ғ, \check{K})$$

Hence, $d^{\mathcal{H}}(\check{K}, Ғ) = d^{\mathcal{H}}(Ғ, \check{K})$

4. If $\check{K} \subseteq Ғ \subseteq Ħ$, than

$$\begin{gathered} [\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q})] \supseteq [\mathcal{T}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{T}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})] \subseteq [\mathcal{T}_{Ħ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{T}_{Ħ}^{U}(\mathfrak{e},\mathfrak{Q})] \\ [\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q})] \supseteq [\mathcal{I}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{I}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})] \supseteq [\mathcal{I}_{Ħ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{I}_{Ħ}^{U}(\mathfrak{e},\mathfrak{Q})] \\ [\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q})] \supseteq [\mathcal{F}_{Ғ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{F}_{Ғ}^{U}(\mathfrak{e},\mathfrak{Q})] \supseteq [\mathcal{F}_{Ħ}^{L}(\mathfrak{e},\mathfrak{Q}), \mathcal{F}_{Ħ}^{U}(\mathfrak{e},\mathfrak{Q})] \end{gathered}$$

Also $\mathfrak{Q}_{\check{K}}(\mathfrak{e}) \geq \mathfrak{Q}_{Ғ}(\mathfrak{e}) \geq \mathfrak{Q}_{Ħ}(\mathfrak{e})$

Therefore,

$$\left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{H}^{L}(\mathfrak{e},\mathfrak{Q})\right|, \left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{H}^{U}(\mathfrak{e},\mathfrak{Q})\right|,$$

$$\left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{H}^{L}(\mathfrak{e},\mathfrak{Q})\right|, \left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{H}^{U}(\mathfrak{e},\mathfrak{Q})\right|$$

$$\left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{H}^{L}(\mathfrak{e},\mathfrak{Q})\right|, \left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| \leqslant \left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{H}^{U}(\mathfrak{e},\mathfrak{Q})\right|$$

$$|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{F}(\mathfrak{e})| \leqslant |\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{H}(\mathfrak{e})|$$

$$d^{\mathcal{H}}(\check{K},F) = \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7m}\begin{cases} \hbar_J\left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{F}(\mathfrak{e})|. \end{cases}$$

$$\geqslant \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{1}{7m}\begin{cases} \hbar_J\left|\mathcal{T}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{T}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{T}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{I}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{I}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{I}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J\left|\mathcal{F}_{\check{K}}^{L}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{L}(\mathfrak{e},\mathfrak{Q})\right| + \hbar_J\left|\mathcal{F}_{\check{K}}^{U}(\mathfrak{e},\mathfrak{Q}) - \mathcal{F}_{F}^{U}(\mathfrak{e},\mathfrak{Q})\right| + \\ \hbar_J|\mathfrak{Q}_{\check{K}}(\mathfrak{e}) - \mathfrak{Q}_{F}(\mathfrak{e})|. \end{cases}$$

$$= d^{\mathcal{H}}(\check{K},F)$$

Similarly $d^{\mathcal{H}}(F,H) \leqslant d^{\mathcal{H}}(\check{K},H)$

Hence, it is valid distance.

4.4 *Example*

Here we define example related hamming distance and normalized hamming distance:

Ǩ= {<$\hbar_1$, ($\mathfrak{e}_1$, [0.5,0.7], [0.6,0.8], [0.1,0.5], 0.5), ($\mathfrak{e}_2$, [0.7,0.8], [0.3,0.8], [0.7,0.8],0.6)>, <$\hbar_2$, ($\mathfrak{e}_1$, [0.6,0.9], [0.7,0.9], [0.6,0.7], 0.5), ($\mathfrak{e}_2$, [0.7,0.9], [0.6,0.9], [0.6,0.8], 0.6)>,< $\hbar_3$, ($\mathfrak{e}_1$, [0.6,0.8], [0.5,0.7], [0.4,0.6], 0.5), ($\mathfrak{e}_2$, [0.1,0.8], [0.5,0.9], [0.6,0.7], 0.6) >}.

Ŧ ={<$\hbar_1$,($\mathfrak{e}_1$, [0.4,0.5], [0.3,0.5], [0.5,0.7], 0.6),($\mathfrak{e}_2$, [0.6,0.7], [0.2,0.7], [0.6,0.8], 0.5)>, <$\hbar_2$, ($\mathfrak{e}_1$, [0.4, 0.5], [0.3, 0.5], [0.6, 0.7], 0.7), ($\mathfrak{e}_2$, [0.3, 0.4], [0.4, 0.8], [0.3, 0.6], 0.4)> , <$\hbar_3$, ($\mathfrak{e}_1$, [0.2, 0.5], [0.3, 0.6], [0.3, 0.7], 0.8), ($\mathfrak{e}_2$, [0.6, 0.8], [0.6, 0.8], [0.5, 0.7], 0.3)>}.

4.4.1 Example: Hamming Distance for IVNFSS

$$d^{\mathcal{H}}(\check{K},F) = \frac{1}{14}\begin{pmatrix} 0.1 + 0.2 + 0.3 + 0.3 + 0.4 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.2 + 0.1 + 0.1 \\ 0.2 + 0.4 + 0.4 + 0.4 + 0.1 + 0.2 + 0.4 + 0.5 + 0.2 + 0.1 + 0.3 + 0.2 + 0.2 \\ 0.4 + 0.3 + 0.2 + 0.1 + 0.1 + 0.1 + 0.3 + 0.5 + 0.1 + 0.1 + 0.1 + 0.3 \end{pmatrix}$$

= 0.5857

4.4.2 Example: Normalized Hamming Distance for IVNFSS

$$d^{\mathcal{NH}}(\check{K},F) = \frac{1}{42}\begin{pmatrix} 0.1 + 0.2 + 0.3 + 0.3 + 0.4 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.2 + 0.1 + 0.1 \\ 0.2 + 0.4 + 0.4 + 0.4 + 0.1 + 0.2 + 0.4 + 0.5 + 0.2 + 0.1 + 0.3 + 0.2 + 0.2 \\ 0.4 + 0.3 + 0.2 + 0.1 + 0.1 + 0.1 + 0.3 + 0.5 + 0.1 + 0.1 + 0.1 + 0.3 \end{pmatrix}$$

= 0.1976

5. Algorithm Design

In continuation with the previous section, this section offers a decision-making method that uses the earlier-defined distance measurements. The method is made to take full use of the pragmatic properties of the Interval-valued neutrosophic fuzzy soft set. The flow chart of the algorithm is presented in Figure 1, and methodically as follows:

Step 1. To analyze the data across universe $\mathcal{A}$, an IVNFSS element is built. It is founded on decision-makers.

Step 2. Build an IVNFSS set $(\check{K}, Ғ)$ based on the criteria decided upon by the decision-making committee for a corporation.

Step 3. Create m IVNFSS $(C_i, Ғ)$ sets using the decision-making team's ap praisal of the various possibilities, where i = 1, 2, ..., m.

Step 4. Calculate the distance Hamming distance and Normalized hamming distance between $(\check{K}, Ғ)$ and $(C_i, Ғ)$.

Step 5. Evaluate the ranking by using distance measure.

Figure 1. Flowchart of the proposed method.

5.1 *Application of IVNFSS for Decision Making*

To make sure that the water we use to drink is safe and free of toxins, water treatment is an essential procedure. Water treatment facilities strive to clean water from a variety of sources, including rivers, lakes, and groundwater, to make it appropriate for human use. Although water treatment considerably improves water quality, it's crucial to realize that the drinkability of the treated water relies on a number of factors, including the exact treatment methods employed, the quality of the source water, and the upkeep of the distribution system. In this conversation, we'll look at the key steps in water treatment as well as the variables that affect whether the water is safe to drink. For raw water from natural sources to be changed into safe and clean drinking water, the water treatment process normally entails multiple steps. The main elements of water treatment are summarized generally as follows:

- **Coagulation and Flocculation**: To destabilize particles and pollutants, chemicals like alum or ferric sulfate are added to the raw water. Due to the coagulation and flocculation that results, it is simpler to remove minute particles after they have clumped together.
- **Sedimentation:** The floc particles naturally sink to the bottom of the water while it sits in a sedimentation tank or basin. Solids and pollutants are removed from the water using this technique.
- **Filtration:** The water is filtered using several materials, including sand, gravel, and activated carbon, after sedimentation. These filters further eliminate any leftover pollutants, germs, and tiny suspended particles.
- **Disinfection**: Disinfection is an essential phase in the process of eradiating or rendering harmless microorganisms (bacteria, viruses, and parasites) inert. This objective is frequently served by the use of chlorine, chloramines, ozone, or ultraviolet (UV) radiation.
- **pH Adjustment:** An essential phase in the treatment process is adjusting the water's pH level to conform to legal requirements and make sure it is not excessively acidic or alkaline.

Now that the water has been filtered, we must compare it to pure water and determine the distance measure. The distance between the pure water and the filtered water is then calculated using the pure water as the ideal benchmark. To compute the quality control matrix for treatment, consider

{E_1, E_2, E_3, E_4} set of alternatives. Where E_1 = Ultrapure water, E_2 = Distilled water, E_3 = Deionized water, E_4 = Tap water. Let {$\hbar_1$ = Chemical Composition, $\hbar_2$ = Microbiological Quality, $\hbar_3$ = Regulatory Compliance, $\hbar_4$ = Continuous Monitoring, $\hbar_5$ = Health and Environ mental Impact Assessment} be the set of attributes.

The water treatment process encompasses crucial steps such as coagulation, sedimentation, filtration, disinfection, and pH adjustment to guarantee the production of safe drinking water. The quality control results for pure water in Table 1, comparing the treated water to different types of pure water, are detailed in Tables 2, 3, and 4. We excluded the Hamming distance and normalized Hamming for water treatment, basing our estimates on pure water as the ideal reference point. These tables' present similarities measured through Hamming Distance and Normalized Hamming Distance for alternative sets C1, C2, and C3, respectively. In Table 5, outcomes reveal that C1 exhibits the lowest similarity, followed by C3, whereas C2 demonstrates the highest similarity. We decided not to include Hamming distance or normalized Hamming values in our assessment of water treatment procedures, preferring to make pure water the gold standard. The decision to exclude Hamming distance was made due to its poor applicability in capturing fundamental aspects of water quality, whilst the decision to exclude normalized Hamming values was made to get a more contextually meaningful assessment. We attempted to determine whether consuming treated water is feasible by designating pure water as the optimal standard and assigning a value of 1 to the degree of similarity to its properties. The practical removal of some metrics supported a more accurate and pertinent assessment of water quality in the context of treatment procedures, especially as our estimations got closer to this benchmark, which is a sign of a successful treatment outcome.

Table 1. Tabular representation of pure water.

(Ǩ, Ғ)	E_1	E_2	E_3	E_4
$\hbar_1$	⟨[0.5, 0.6], [0.2, 0.3], [0.4,0.7],0.6⟩	⟨[0.2,0.7],[0.7, 0.9],[0.4,0.7], 0.4⟩	⟨[0.2,0.6], [0.3,0.7], [0.1,0.9],0.5⟩	⟨[0.4,0.6], [0.5,0.8], [0.2,0.8],0.7⟩
$\hbar_2$	⟨[0.2, 0.5], [0.4, 0.8], [0.5, 0.6],0.7⟩	⟨[0.4,0.7], [0.3, 0.8], [0.5,0.6],0.7⟩	⟨[0.3,0.7], [0.4,0.6][0.2,0.8],0.3⟩	⟨[0.4,0.7], [0.5,0.7],[0.3,0.8],0.7⟩
$\hbar_3$	⟨[0.1, 0.7], [0.3, 0.6], [0.4, 0.7], 0.8⟩	⟨[0.6,0.8], [0.1, 0.9], [0.2, 0.5], 0.5⟩	⟨[0.4,0.8], [0.5,0.8], [0.3,0.7],0.5⟩	⟨[0.4,0.8], [0.5,0.7], [0.4,0.7],0.6⟩
$\hbar_4$	⟨[0.2, 0.6], [0.7, 0.8], [0.4, 0.6], 0.7⟩	⟨[0.5,0.7], [0.7, 0.8], [0.6, 0.9], 0.2⟩	⟨[0.5,0.9], [0.6,0.7], [0.4,0.8],0.8⟩	⟨[0.2,0.7], [0.7,0.9], [0.4,0.7],0.4⟩
$\hbar_5$	⟨[0.2, 0.6], [0.7, 0.8], [0.4, 0.6], 0.7⟩	⟨[0.5,0.7], [0.7, 0.8], [0.6, 0.9], 0.2⟩	⟨[0.5,0.9], [0.6,0.7], [0.4,0.8],0.8⟩	⟨[0.2,0.7], [0.7,0.9], [0.4,0.7],0.4⟩

Table 2. Tabular representation of water treatment.

(C_1,Ғ)	**E_1**	**E_2**	**E_3**	**E_4**
$\hbar_1$	⟨[0.4,0.7],[0.3,0.8],[0.5, 0.7],0.4⟩	⟨[0.5,0.8],[0.6,0.9],[0. 1,0.7], 0.7⟩	⟨[0.4,0.7],[0.5,0.8],[0. 4,0.6],0.6⟩	⟨[0.5,0.7],[0.2,0.8], [0.3,0.7],0.6⟩
$\hbar_2$	⟨[0.4,0.6],[0.5,0.6],[0.6, 0.8],0.8⟩	⟨[0.5,0.7],[0.5,0.8],[0. 6,0.8],0.6⟩	⟨[0.4,0.8],[0.6,0.7],[0. 1,0.9],0.5⟩	⟨[0.6,0.8],[0.4,0.5], [0.3,0.8],0.6⟩
$\hbar_3$	⟨[0.2,0.8],[0.6,0.8],[0.5, 0.6],0.3⟩	⟨[0.7,0.8],[0.2,0.9],[0. 5,0.7], 0.6⟩	⟨[0.3,0.5],[0.4,0.8],[0. 2,0.8],0.6⟩	⟨[0.5,0.7],[0.7,0.9], [0.4,0.8],0.7⟩
$\hbar_4$	⟨[0.4,0.7],[0.7,0.9],[0.6, 0.8],0.5⟩	⟨[0.7,0.9],[0.5,0.6],[0. 8,0.9],0.4⟩	⟨[0.3,0.7],[0.4,0.9], [0.5,0.8],0.8⟩	⟨[0.4,0.7], [0.4,0.7], [0.5,0.8],0.6⟩
$\hbar_5$	⟨[0.3,0.7],[0.5,0.9], [0.5,0.7],0.6⟩	⟨[0.4,0.8],[0.5,0.9],[0. 3,0.7],0.6⟩	⟨[0.4,0.6],[0.5,0.8], [0.3,0.8],0.4⟩	⟨[0.3,0.6], [0.4,0.8], [0.2,0.8],0.6⟩

Table 3. Tabular representation of water treatment.

$(C_2,Ғ)$	E_1	E_2	E_3	E_4
$ℏ_1$	⟨[0.1,0.8], [0.7, 0.9], [0.2, 0.6], 0.3⟩	⟨[0.4,0.8],[0.5,0.7],[0.3,0.8],0.7⟩	⟨[0.5,0.9], [0.6.0.9], [0.4,0.7],0.8⟩	⟨[0.6,0.9], [0.3,0.7], [0.4,0.8],0.6⟩
$ℏ_2$	⟨[0.4,0.7], [0.6, 0.9], [0.3, 0.7], 0.5⟩	⟨[0.4,0.8],[0.5,0.8],[0.1,0.9], 0.6⟩	⟨[0.5,0.8], [0.1,0.8], [0.5,0.8],0.5⟩	⟨[0.5,0.8], [0.4,0.8], [0.4,0.9],0.6⟩
$ℏ_3$	⟨[0.4,0.8], [0.2, 0.8], [0.3, 0.5], 0.3⟩	⟨[0.3,0.7],[0.3,0.7],[0.6,0.8],0.7⟩	⟨[0.5,0.7], [0.4,0.9], [0.5,0.8],0.6⟩	⟨[0.5,0.9], [0.2,0.6], [0.2,0.8],0.3⟩
$ℏ_4$	⟨[0.4,0.7], [0.2, 0.8], [0.1, 0.9], 0.5⟩	⟨[0.3,0.8],[0.6,0.8],[0.6,0.9], 0.7⟩	⟨[0.4,0.9], [0.7,0.9], [0.5,0.8],0.5⟩	⟨[0.5,0.8], [0.4,0.8], [0.7,0.8],0.7⟩
$ℏ_5$	⟨[0.6, 0.8],[0.2, 0.7], [0.4, 0.7], 0.5⟩	⟨[0.3,0.7],[0.7,0.9],[0.2,0.8],0.5⟩	⟨[0.4,0.7], [0.5,0.8], [0.3,0.9],0.6⟩	⟨[0.7,0.8], [0.1,0.8], [0.4,0.6],0.6⟩

Table 4. Tabular representation of water treatment.

$(C_3,Ғ)$	E_1	E_2	E_3	E_4
$ℏ_1$	⟨[0.4,0.7], [0.1, 0.7], [0.5, 0.8], 0.6⟩	⟨[0.7,0.9],[0.3,0.6],[0.5,0.7], 0.6⟩	⟨[0.4,0.7], [0.3,0.8], [0.5,0.8],0.5⟩	⟨[0.4,0.7], [0.1,0.6], [0.6,0.9],0.7⟩
$ℏ_2$	⟨[0.6,0.9], [0.4, 0.8], [0.1,0.9],0.7⟩	⟨[0.5,0.9],[0.7,0.9],[0.2,0.5],0.3⟩	⟨[0.4,0.9], [0.4,0.7], [0.1,0.8],0.6⟩	⟨[0.6,0.9], [0.3,0.6], [0.1,0.8],0.6⟩
$ℏ_3$	⟨[0.1,0.7], [0.5, 0.8], [0.8, 0.9], 0.7⟩	⟨[0.6,0.8],[0.4,0.8],[0.1,0.9],0.6⟩	⟨[0.4,0.7], [0.3,0.8], [0.1,0.6],0.4⟩	⟨[0.2,0.8], [0.3,0.9], [0.6,0.8],0.7⟩
$ℏ_4$	⟨[0.5,0.8], [0.5, 0.9], [0.1, 0.9], 0.7⟩	⟨[0.5,0.8],[0.3,0.6],[0.4,0.7],0.4⟩	⟨[0.5,0.7], [0.1,0.6], [0.4,0.9],0.5⟩	⟨[0.3,0.7], [0.2,0.6], [0.1,0.5],0.6⟩
$ℏ_5$	⟨[0.4,0.8], [0.4, 0.6], [0.4, 0.8], 0.7⟩	⟨[0.5,0.8],[0.7,0.9],[0.2,0.6],0.7⟩	⟨[0.3,0.8], [0.1,0.6], [0.6,0.8],0.4⟩	⟨[0.4,0.7], [0.3,0.7], [0.3,0.6],0.4⟩

Table 5. Results of hamming distance and normalized hamming distance.

	Hamming Distance	**Normalized Hamming**
$(C_1,Ғ)$	0.7178_1	0.1435
$(C_2,Ғ)$	0.93571	0.18714
$(C_3,Ғ)$	0.8464	0.1692

5.2 *Comparison and Distinctiveness*

The distinctiveness of our proposed work is demonstrated in Table 6. This observation holds true for the corresponding structures as well.

Table 6. Comparing the IVNFSS with existing structures.

Structures	Membership Function	Indeterminate Function	Non-membership Function	Interval Value	Attributes	Fuzzy Value
FS [1]	Yes	No	No	No	No	No
NS [13]	Yes	Yes	Yes	No	No	No
SVNS [14]	Yes	Yes	Yes	No	No	No
IVNS [42]	Yes	Yes	Yes	Yes	No	No
IVNSS [21]	Yes	Yes	Yes	Yes	Yes	No
IVNFSS (Proposed)	Yes	Yes	Yes	Yes	Yes	Yes

6. Conclusion

Neutrosophic fuzzy sets may not be sufficient to handle complexity since every person's decision-making is unpredictable, especially when they are faced with many and split attributes under ambiguous and uncertain set tings. Due to this, the IVNFSS environments are described, and we can observe how carefully this theory handles uncertainty in a dynamically changing environment using various instances. To tackle this investigation, explore interval-valued neutrosophic fuzzy soft sets and distance measures for wastewater treatment in detail. This article determines that it has diligently built a solid theoretical framework, addressing flaws in current approaches and highlighting the requirement for IVNFSS. With the aid of a few examples, various concepts, including subsets, were presented, created a few operations, such as union, intersection, and complement. The use of IVNFSS in conjunction with distance measurements caught our attention as we looked to determine if the treated wastewater is safe to drink by accurately capturing and modeling uncertainty, ambiguity, and imprecision in data. Since IVNFSS is based on a mathematical foundation, it is easier to carry out in-depth analyses, construct algorithms, and enhance theory and practice. In the future, more distance measures can be proposed along with trigonometric similarities, and many real-life decision-making problems can be solved.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest
The authors declare that there is no conflict of interest in the research.

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Foundation of Appurtenance and Inclusion Equations for Constructing the Operations of Neutrosophic Numbers Needed in Neutrosophic Statistics

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Abstract: We introduce for the first time the appurtenance equation and inclusion equation, which help in understanding the operations with neutrosophic numbers within the frame of neutrosophic statistics. The way of solving them resembles the equations whose coefficients are sets (not single numbers).

Keywords: Real Neutrosophic Numbers; Neutrosophic Statistics; True Value; True Set-Value; Appurtenance Relationship; Appurtenance Equation; Inclusion Relationship; Inclusion Equation; Equality Equation with Set-Coefficients; NonAppurtenance Equation; NonInclusion Equation; NonEquality Equation; Operations with Neutrosophic Numbers.

1. Introduction

In neutrosophic statistics, from the fact that the single true value v is in I, it does not result that v is in $a + bI = N$ as well, but: $a + bv \in a + bI$. That's why the appurtenance relationship and equation must be introduced and studied.

Even more, if one has a set of true values, from the fact that the set of true values V is included in I, it does not mean that V is included in $a + bI$ too, but $a + bV \subset a + bI$ (or $a + bV \subseteq a + bI$). That's why the inclusion relationship and equation must be introduced [1, 2].

In the same way as the "=" symbol is used for an equality relationship or an equality equation, we use the symbol "$\in$" {belong(s) to} for an *appurtenance relationship* or *appurtenance equation* of a number to a set, respectively the symbol $\subset$ (or $\subseteq$) {included in, or included in or equal to} for an *inclusion relationship* or *inclusion equation*.

We use in this paper the tautological denomination Equality Equation with Set-Coefficients (=), in order to distinguish it from the Appurtenance Equation ($\in$) and Inclusion Equation {$\subset$ or $\subseteq$}.

Whatever operation we do on the left-hand side of an *appurtenance relationship* or *appurtenance equation* (respectively, *inclusion relationship* or *inclusion equation*), we must do the same on the right-hand side as well of the *appurtenance relationship* or *appurtenance equation* (respectively, *inclusion relationship* or *inclusion equation*).

In addition, we also present their complementary *NonAppurtenance Equation*, *NonInclusion Equation*, and the elementary *NonEquality Equation* respectively.

2. Definition of the Real Neutrosophic Number

A *Real Neutrosophic Number* (N) has the form:
$N = a + bI$, where a, b are real numbers, "a" is called the determinate part of N, while "bI" is the indeterminate part of N, while I is a real subset, $I \subset \mathbb{R}$.

They are mostly used in *Neutrosophic Statistics*.

The neutrosophic numbers frequently occur in our real world, where one often has imprecise, unclear data to deal with.

For example, let's have a right triangular shape land whose legs are 5 and 6 kilometers respectively, then the length of its hypotenuse is $\sqrt{5^2+6^2}=\sqrt{61}=7.81024967\ldots$ that we need to approximate with some required accuracy.

Another example, let's consider the real circular land of radius 10 km, compute its area $A=\pi\cdot 10^2=100\pi=314.159265\ldots\text{km}^2$, or to compute the volume or surface of a sphere, but π is a transcendental number (a number that is not the root of a non-zero polynomial with rational coefficients and of finite degree), having infinitely many decimals with no repeated pattern.

Similarly, the Euler's constant e = 2.71828182…is a transcendental number and occurs in many formulas.

In the same way when, from real world applications, one arises inexact results (radicals, exponential or logarithmic or trigonometric equations, differential equations, transcendental functions, etc.).

Examples of Real Neutrosophic Numbers.

(i). $N_1=2+3I$, where $I_1=[0,1]$ is an interval.

Or

$N_1=2+3\cdot[0,1]=2+[3.0,3.1]=2+[0,3]=[2+0,2+3]=[2,5]$

(ii). Let $I_2=\{0.6,0.8,0.9\}$, which is a finite discrete set of three elements.

Then:

$$\begin{aligned}N_2=2+3I_2&=2+3\cdot\{0.6,0.8,0.9\}=2+\{3\cdot(0.6),3\cdot(0.8),3\cdot(0.9)\}=2+\{1.8,2.4,2.7\}\\&=\{2+1.8,2+2.4,2+2.7\}=\{3.8,4.4,4.7\}\subset[2,5].\end{aligned}$$

(iii). Let $I_3=\left\{\frac{1}{n},1\leq n\leq\infty,n\text{ is integer}\right\}$, which is an infinite discrete set.

Then:

$$\begin{aligned}N_3=2+3I_3&=\left\{2+3\cdot\frac{1}{n},1\leq n\leq\infty,n\text{ is integer}\right\}\\&=\left\{2+3\cdot\frac{1}{1},2+3\cdot\frac{1}{2},2+3\cdot\frac{1}{3},\ldots,2+3\cdot\frac{1}{n},\ldots\right\}=\left\{5,3.5,3,\ldots,2+3\cdot\frac{1}{n},\ldots\right\}\\&\subset[2,5].\end{aligned}$$

3. Foundation of Appurtenance Relationship and Appurtenance Equation

The below Theorems 1 and 2 allow us to do operations on both sides of an appurtenance relationship and appurtenance equation respectively.

3.1 Theorem 1

Let A and B be real sets.

If $a\in A$ and $b\in B$, then:

- *Addition* $a+b\in A+B$
- *Subtraction* $a-b\in A-B$
- *Multiplication* $a\times b\in A\times B$
- *Division* $\frac{a}{b}\in\frac{A}{B}$
- *Power* $a^b\in A^B$

Proof

Let $\star$ be any of the above operation, then:
$A \star B = \{x \star y; \text{where } x \in A, y \in B\}$, and the operation $\star$ is well-defined.

If we let $x = a \in A$, and $y = b \in B$ into the above definition, then:
$a \star b = A \star B$.

3.2 Theorem 1

Let A be a real set, $a \in A$, β be a real scalar, and m, n be (positive) integers. Then:

- Scalar Multiplication of a Set $\beta \cdot a \in \beta \cdot A$ (1)
- Raising to a Power of a Set $a^n \in A^n$
- Root Index of a Set $\sqrt[n]{a} \in \sqrt[n]{A}$
- Negative Exponent of a Set $a^{-n} \in A^{-n}$
- Rational Exponent of a Set $a^{\frac{m}{n}} \in A^{\frac{m}{n}}$

Proof

Similarly.

$$\beta \cdot A = \left\{\beta \cdot x, x \in A\right\} \tag{2}$$

Letting $x = a \in A$ into Definition (2), it results that

$$\beta \cdot a \in \beta \cdot A^{\cdot}$$

For the next four appurtenance relationships, let p be any of the exponents, n, $\frac{1}{n}$, $-n$, or $\frac{m}{n}$, then:

$$A^p = \{x^p, x \in A\}. \tag{3}$$

Letting $x = a$ into the Definition (3), it results that: $a^p \in A^p$, for p being any of n, $\frac{1}{n}$, $-n$, or $\frac{m}{n}$.

3.3 Examples of Operations with Appurtenance Relationships

- Addition in an Appurtenance Relationship
  $(3 + 0.67) \in (3 + (0, 1))$ or $3.67 \in (3 + 0, 3 + 1)$ or $3.67 \in (3,4)$, which is true.
- Scalar Multiplication of an Appurtenance Relationship
  $2 \cdot (0.67) \in 2 \cdot (0, 1)$ or $1.34 \in (2.0, 2.1)$ or $1.34 \in (0, 2)$, which is true.
- Power
  $0.67^2 \in (0, 1)^2$ or $0.4489 \in (0^2, 1^2)$ or $00.4489 \in (0, 1)$, which is true.
- Division by a Scalar of an Appurtenance Relationship
  $\frac{0.67}{-2} \in \frac{(0,1)}{-2}$ or $-0.335 \in (-0.5, 0)$, which is true.

3.4 Appurtenance Equations

Its general form is defined as follows.

Let R be the set of real numbers, and f and g be real HyperFunctions {"hyper" stands for the fact that their domain and/or codomain are powersets *P(R)*},

$$f, g : P(R) \to P(R)$$

$$f(x) \in g(x)$$

All procedures done to solve a classical equation (but whose coefficients are sets, not single numbers) are similarly allowed to do for solving an *appurtenance equation*.

Because sometimes it is not clear what number, either a or b is bigger, we consider that:
$(a, b) \equiv (b, a)$ and $[a, b] \equiv [b, a]$.

3.5 Solution of an Appurtenance Equation

The solution of an appurtenance equation means a real set S, or $S \in P(R)$, to whom the unknown x belongs to, or $x \in S$.

3.6 Example 1 of Appurtenance Equation

Solve for x.

$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$.

Subtract 4 from both sides:

$(-5x) \in \big(-3 + 2(0.5, 0.8)\big)$.

We use parentheses ( ) in order to clearly distinguish between the left-hand side and the right-hand side of the appurtenance equation.

$(-5x) \in \big(-3 + (1, 1.6)\big)$

$(-5x) \in (-3 + 1, -3 + 1.6)$

Divide both sides by -5:

$\frac{(-5x)}{-5} \in \frac{(-3+1,-3+1.6)}{-5}$

$x \in \left(\frac{1.4}{5}, \frac{2}{5}\right)$

$x \in (0.28, 0.40)$.

There are infinitely many particular solutions of this appurtenance equation, i.e. all the numbers inside the open interval $(0.28, 0.40)$. We do not take the subsets of *(0.28, 0.40)* as particular solutions, since they are included ($\subset$ or $\subseteq$) in, not appurtenant ($\in$) to *(0.28, 0.40).*

Check the maximal solution of the appurtenance equation.

$4 - 5 \cdot x \in 1 + 2 \cdot (0.5, 0.8)$

$4 - 5 \cdot (0.28, 0.40) \overset{?}{\in} 1 + 2 \cdot (0.5, 0.8)$

where $\overset{?}{\in}$ means that we must check the appurtenance.

$4 - (1.4, 2.0) \overset{?}{\in} 1 + (1.0, 1.6)$

$(4 - 2.0, 4 - 1.4) \overset{?}{\in} (1 + 1.0, 1 + 1.6)$

or $(2, 2.6) \in (2, 2.6)$

Actually we have an equality here above, which means that any number x, and any subset inside of the left-hand side interval, are solutions.

Therefore, this appurtenance equation has infinitely solutions, $x \in (0.28, 0.40)$.

Let's check some of them, a particular solution as a single number:

$x = 0.35 \in (0.28, 0.40)$

The appurtenance equation:

$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ becomes, after substituting x,

$4 - 5 \cdot (0.35) \in 1 + 2(0.5, 0.8)$

$4 - (1.75) \in 1 + (1.0, 1.6)$

$2.25 \in (2.00, 2.60)$, which is true.

Let's check a particular solution-subset of (0.28, 0.40):

$x = (0.30, 0.34) \subset (0.28, 0.40)$.

The appurtenance equation: $4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ becomes:

$4 - 5 \cdot (0.30, 0.34) \in (2.00, 2.60)$

$4 - (5 \cdot 0.30, 5 \cdot 0.34) \in (2.00, 2.60)$

$4 - (1.50, 1.70) \in (2.00, 2.60)$

$(4 - 1.70, 4 - 1.50) \in (2.00, 2.60)$

$(2.30, 2.50) \in (2.00, 2.60)$,

Actually, the left-hand side is included into the right-hand side.

3.7 *Example of Equality Equation with Set-Coefficients*

The maximal solution-set $x = (0.28, 0.40)$ of this appurtenance equation becomes the set-solution of the following *equality equation with set-coefficients*:

$4 - 5x = 1 + 2 \cdot (0.5, 0.8)$.

This equation, whose one of the coefficients is a set, (0.5, 0.8), is solved in the same way:

- Subtract *4* from both sides: $-5x = -3 + 2 \cdot (0.5, 0.8)$,
- Then multiply and add the sets:
  $-5x = -3 + (1.0, 1.6)$
  $-5x = (-3 + 1.0, -3 + 1.6)$
  $-5x = (-2, -1.4)$,
- And divide by -5 to get:

  $x = \left(\frac{-1.4}{-5}, \frac{-2}{-5}\right)$

  $x = (0.28, 0.40)$, which is a set-solution.

4. Foundation of Inclusion Relationship and Inclusion Equation

Similarly for the *Inclusion* ($\subset$ or $\subseteq$) *Relationships* and *Inclusion Equation* as we did for Appurtenance Relationships and Appurtenance Equation respectively.

For the case where one has $\subseteq$, the below theorems will be the same, just using $\subseteq$ instead of $\subset$.

4.1 *Inclusion Equations*

Its general form is defined as follows.

Let *R* be the set of real numbers, and *f* and *g* be real HyperFunctions {"hyper" stands for the fact that their domain and/or codomain are powersets *P(R)*},

$f, g : P(R) \to P(R)$

Then, $f(x) \subset g(x)$ or $f(x) \subseteq g(x)$ are called inclusion equations.

4.2 *Theorem 3*

Let A and B be real sets, and A_1, B_1 also real sets, but: $A_1 \subset A$, and $B_1 \subset B$.

Then:

- Addition of Sets $A_1 + B_1 \subset A + B$
- Subtraction of Sets $A_1 - B_1 \subset A - B$
- Multiplication of Sets $A_1 \times B_1 \subset A \times B$
- Division of Sets $\frac{A_1}{B_1} \subset \frac{A}{B}$
- Power of Sets ${A_1}^{B_1} \subset A^B$

Proof:
In the same way, let $\star$ be any of above operations $+, -, \times, \div,$ ^ (power), then:

$$A \star B = \{x \star y; \text{where } x \in A, y \in B\} \tag{3}$$

and the operation $\star$ is well-defined.
We let $x = a_1 \in A_1 \subset A$, and $y = b_1 \in B_1 \subset B$ into the Definition (3), then:
$a_1 \star b_1 \in A \star B$, for all $a_1 \in A_1$ and $b_1 \in B_1$, which means that:
$A_1 \star B_1 \subset A \star B$.

4.3 *Theorem 4*

Let A and A_1 be real sets, with $A_1 \subset$ A, $\beta \neq 0$ a real number, and m, n positive integers. Then:

- Scalar Multiplication of a Set $\beta \cdot A_1 \subset \beta \cdot$ A
- Raising to the Power n of a Set $A_1^n \subset A^n$
- Root Index n of a Set $\sqrt[n]{A_1} \subset \sqrt[n]{A}$
- Negative Exponent of a Set $A_1^{-n} \subset A^{-n}$
- Rational Exponent of a Set $A_1^{\frac{m}{n}} \subset A^{\frac{m}{n}}$

Proof
Similarly to the previous theorem.

$\beta \cdot A = \left\{\beta \cdot x, x \in A\right\}$ and $\beta \cdot A_1 = \left\{\beta \cdot x, x \in A_1 \subset A\right\} \subset \left\{\beta \cdot x, x \in A\right\} = \beta \cdot A$.
For the other inclusion relationships, we let again p be any of the exponents n, $\frac{1}{n}$, $-n$, $\frac{m}{n}$, then:
$A_1^p = \{x^p, x \in A_1\} \subset \{x^p, x \in A\} = A$, since $A_1 \subset A$, where for any $x \in A$, and any p, all x^p operations are well-defined.
Analogously, these theorems 3 and 4 allow us to do many operations on both sides of an inclusion relationship or inclusion equation.

4.4 *Examples of Inclusion Relationships*

$(2, 3] \subset [0, 4]$

- *Let's add 1 in both sides:*
  $1 + (2, 3] \subset 1 + [0, 4]$ or $(1 + 2, 1 + 3] \subset [1 + 0, 1 + 4]$
  $(3, 4] \subset [1, 5]$, which is true.
- *Let's add an interval* $(-1, 5)$ *on both sides:*
  $(2, 3] + (-1, 5) \subset \{0, 4] + (-1, 5)$
  $(2 - 1, 3 + 5) \subset (0 - 1, 4 + 5)$
  $(1, 8) \subset (-1, 90)$, which is true.
- *Let's subtract 2 from both sides:*
  $(2, 3] - 2 \subset [0, 4] - 2$
  $(2 - 2, 3 - 2] \subset [0 - 2, 4 - 2]$
  $(0, 1] \subset [-2, 2]$, which is true.
- *Let's subtract a set [0.5, 0.6] from both sides.*
  $(2,3] - [0.5, 0.6] \subset [0, 4] - [0.5, 0.6]$
  $(2 - 0.6, 3 - 0.5] \subset [0 - 0.6, 4 - 0.5]$
  $(1.4, 2.5] \subset [-0.6, 3.5]$, which is true.
- Let's multiply both sides by a positive (non-zero) number 7:

$7 \cdot (2,3] \subset 7 \cdot [0,4]$

$(7.2, 7.3] \subset [7.0, 7.4]$

$(14, 21] \subset [0, 28]$, which is true.

- Let's multiply both sides by a negative (non-zero) number -5:

$-5 \cdot (2,3] \subset -5 \cdot [0,4]$

$(-5.3, -5.2] \subset [-5.4, -5.0]$

$(-15, -10] \subset [-20, 0]$, which is true.

- Let's multiply both sides with a set $(-1, 1)$.

$(-1,1) \cdot (2,3] \subset (-1,1) \cdot (0,4]$

$(-3,3) \subset (-4,4)$, which is true.

- Let's raise to the second power both sides:

$(2,3]^2 \subset [0,4]^2$

$(2^2, 3^2] \subset [0^2, 4^2]$

$(4, 9] \subset [0, 16]$, which is true.

- Let's divide by -5 both sides:

$\frac{(2,3]}{-5} \subset \frac{[0,4]}{-5}$

$\left[-\frac{3}{5}, -\frac{2}{5}\right) \subset \left[-\frac{4}{5}, -\frac{0}{5}\right]$

$[-0.6, -0.4) \subset [-0.8, 0]$, which is true.

- Let's divide each side by a real set [4, 5].

$\frac{(2,3]}{[4,5]} \subset \frac{[0,4]}{[4,5]}$

$\left(\frac{2}{5}, \frac{3}{4}\right] \subset \left[\frac{0}{5}, \frac{4}{4}\right]$

$(0.40, 0.75] \subset [0, 1]$, which is true.

4.5 Example 2 of Inclusion Equation

Solve for x.

$1 + x \cdot (1,2) \subset (0,5)$

$1 + (1, 2x) \subset (0,5)$

$(x+1, 2x+1) \subset (0,5)$

whence $0 < x+1 < 5$ or $-1 < x < 4$ and $0 < 2x+1 < 5$ or $-1 < 2x < 4$ or $-0.5 < x < 2$

whence $(-1,4) \cap (-0.5,2) = (-0.5,2)$

So $x = (-0.5, 2)$ is the maximal solution. All subsets of $(-0.5, 2)$ are particular solutions – therefore one has infinitely many particular solutions.

Check it:

$1 + x(1,2) \subset (0,5)$

$1 + (-0.5,2) \cdot (1,2) \subset (0,5)$

$1 + (-1,2) \subset (0,5)$

$(1-1, 1+\ 2) \subset (0,5)$

$(0,3) \subset (0,5)$, which is true.

4.6 Another Example of Inclusion Equation

Solve for x.

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + [1 \cdot x, 2 \cdot x] \subseteq [6, 10]$

$(4, 5) + [x, 2x] \subseteq [6, 10]$, for $x \geq 0$, one gets: $(4 + x, 5 + 2x) \subseteq [6, 10]$

Hence:

$6 \leq 4 + x \leq 10$,

whence $2 \leq x \leq 6$

and $6 \leq 5 + 2x \leq 10$

or $1 \leq 2x \leq 5$

or $1.5 \leq x \leq 2.5$

Thus, solution for $x \geq 0$ is $[2, 6] \cap [1.5, 2.5] = [2, 2.5]$.

For $x < 0$

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + [2x, x] \subseteq [6, 10]$

$(4 + 2x, 5 + x) \subseteq [6, 10]$

Whence

$6 \leq 4 + 2x \leq 10$, or $2 \leq 2x \leq 6$ or $1 \leq x \leq 3$

and $6 \leq 5 + x \leq 10$, or $1 \leq x \leq 5$.

But x must be negative, therefore this situation doesn't produce any solution.

The maximum solution is $x = [2, 2.5]$.

Let's check the maximal solution.

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

Then:

$(4, 5) + [2, 2.5] \subseteq [6, 10]$

$(4 + 2, 5 + 2.5) \subseteq [6, 10]$

$(6, 7.5) \subseteq [6, 10]$, which is true.

As particular solutions are all subsets of the maximal solution [2, 2.5], therefore infinitely many.

Verifications:

Let $x = 2 \in [2, 2.5]$. We may also write x as a set, $x = [2, 2] \subseteq [2, 2.5]$.

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + 2 \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + [2, 4] \subseteq [6, 10]$

$(6, 9) \subseteq [6, 10]$, which is true.

Let $x = 2.3 \in [2, 2.5]$.

Then: $(4, 5) + 2.3[1.2] \subseteq [6, 10]$

$(4, 5) + [2.3, 4.6] \subseteq [6, 10]$

$(6.3, 9.6) \subseteq [6, 10]$, which is true.

Let $x = [2.1, 2.4) \subseteq \{2, 2.5]$.

Then $(4,5) + x \cdot [1,2] \subseteq [6,10]$

$(4,5) + [2.1, 2.4) \cdot [1,2] \subseteq [6,10]$

$(4,5) + [2.1, 4.8] \subseteq [6,10]$

$(6.1, 9.8) \subseteq [6,10]$, which is true.

In conclusion, this inclusion equation has one maximal solution and infinitely many particular solutions which are actually included into the maximal solution.

In order to deal with inclusion only, in this problem, since it is an *inclusion* equation, we take as solutions only the subsets of the maximal solution, since: $subset \subseteq maximal\_solution$, not the single numbers, since: $number \in maximal\_solution$ (not$\subseteq$); this should better be adjusted as [*number, number*] $\subseteq$ *maximal_solution*, for example$[2.1, 2.1] \subseteq [2, 2.5]$.

4.7 Example of Inclusion Equation which has No Solution

Solve for x.

$(1,2) - 2x \subseteq (0, 0.5)$.

Hence:

$(1-2x, 2-2x) \subseteq (0, 0.5)$,

whence $0 \le 1 - 2x \le 0.5$ and $0 \le 2 - 2x \le 0.5$,

or $-1 \le -2x \le -0.5$ and $-2 \le -2x \le -1.5$,

or $0.25 \le x \le 0.50$ and $0.75 \le x \le 1$.

But $[0.25, 0.50] \cap [0.75, 1] = \emptyset$, therefore there is no solution x.

4.8 Inclusion Equation which has only One Solution

Solve for x.

$(1,2) - 2x \subseteq (0,1)$.

Hence: $(1-2x, 2-2x) \subseteq (0,1)$,

whence $0 \le 1 - 2x \le 1$, and $0 \le 2 - 2x \le 1$,

or $-1 \le -2x \le 0$, and $-2 \le -2x \le -1$,

or $0 \le x \le 0.5$, and $0.5 \le x \le 1$.

From $[0, 0.5] \cap [0.5, 1] = 0.5$, one gets that the only solution is $x = 0.5$ = [0.5, 0.5].

Let's check the inclusion solution:

$(1,2) - 2x \subseteq (0,1)$

$(1,2) - 2 \cdot 0.5 \subseteq (0,1)$

$(1,2) - 1 \subseteq (0,1)$

$(1-1, 2-1) \subseteq (0,1)$

$(0,1) \subseteq (0,1)$, which is true.

5. First Method of Operating with Real Neutrosophic Numbers used in Neutrosophic Statistics

5.1 The True Value v is a Single Number

A Neutrosophic Real Number has the form $N = a + bI$, where a and b are real numbers, while "I" is a real set. The determinate part of N is "a" and indeterminate (unclear) part of N is "bI".

Let's consider the real true value being the single value number v, that we are looking for in statistical problems where indeterminate, unclear, partially unknown data occur, where this single number v belongs to the real set I, or $v \in I$.

From the fact that the single true value v is in I, it does not result that v is in $a + bI = N$ as well, but: $a + bv \in a + bI$.

Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two real neutrosophic numbers, where $a_1, b_1, a_2, b_2 \in R$ and I is a subset (not necessarily interval) of real numbers.

Let the true value, we are looking for in statistics, under indeterminate (unclear, vague) data, be $v \in I$. Then:

$$a_1 + b_1 v \in a_1 + b_1 I = N_1$$
$$a_2 + b_2 v \in a_2 + b_2 I = N_2$$

The previous Theorems 1 and 2 allow us to do straightforward operations with real neutrosophic numbers.

- *Addition of Real Neutrosophic Numbers*

$$N_1 + N_2 = (a_1 + a_2) + (b_1 + b_2) I$$

Proof:

According to Theorem 1, since

$$a_1 + b_1 v \in a_1 + b_1 I$$

$$a_2 + b_2 v \in a_2 + b_2 I$$

We add, the left-hand sides, then the right-hand sides, referred to the appurtenance symbol ($\in$), and we get:

$$(a_1 + b_1 v) + (a_2 + b_2 v) \in (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2$$

Therefore:

$$(a_1 + a_2) + (b_1 + b_2) v \in N_1 + N_2 .$$

By similar proofs we can do the next operations with real neutrosophic numbers.

- *Subtraction of Real Neutrosophic Numbers*

$$N_1 - N_2 = (a_1 - a_2) + (b_1 - b_2) I$$

- *Scalar Multiplication of Real Neutrosophic Numbers*

Let $\beta \neq 0$ be a real scalar. Then:

$$\beta \cdot N_1 = \beta \cdot (a_1 + b_1 I) = \beta \cdot a_1 + \beta \cdot b_1 I$$

- *Multiplication of Real Neutrosophic Numbers*

$$N_1 \cdot N_2 = (a_1 + b_1 I) \cdot (a_2 + b_2 I) = a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2$$

- *Square of Real Neutrosophic Numbers*

$$N^2 = (a + bI)^2 = a^2 + 2abI + b^2 I^2$$

5.2 The True Value V is a Set

Let's consider a set of true values V, that we are looking for in statistical problems where indeterminate, unclear, partially unknown data occur, where V is included in I, or $V \subset I$ (or $V \subseteq I$).

From the fact that the set of true values V is in I, it does not result that V is included in $a + bI$ = N as well, but: $a + bV \subset a + bI$.

Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two real neutrosophic numbers, where $a_1, b_1, a_2, b_2 \in R$ and I is a subset (not necessarily interval) of real numbers.

Let the set of true values, we are looking for in statistics, under indeterminate (unclear, vague) data, be $V \subset I$. Then:

$$a_1 + b_1 V \subset a_1 + b_1 I = N_1$$

$$a_2 + b_2 V \subset a_2 + b_2 I = N_2$$

Whence:

$$(a_1 + b_1 V) + (a_2 + b_2 V) \subset (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2 .$$

Similarly, for subtraction, scalar multiplication, multiplication, etc.

$$(a_1 + b_1 V) - (a_2 + b_2 V) \subset (a_1 + b_1 I) - (a_2 + b_2 I) = N_1 - N_2 .$$

$$\beta \cdot (a_1 + b_1 V) \subset \beta \cdot (a_1 + b_1 I) = \beta \cdot N_1 .$$

$$(a_1 + b_1 V) \cdot (a_2 + b_2 V) \subset (a_1 + b_1 I) \cdot (a_2 + b_2 I) = a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2$$

$$= N_1 \cdot N_2 .$$

The previous Theorems 1, 2, 3, and 4 allow us to do straightforward operations with real neutrosophic numbers (for both cases: a single true value v, or a set of true values V).

6. **Second Method of Operating with Neutrosophic Numbers**

This method is to transform each real neutrosophic number into a real set:

$N = a + bI = \{a + b \cdot x, x \in I\}$ and do operations using sets as below.

In this case it is not necessarily to have the same indeterminate real set "I".

Firstly, we need to recall the operations with real sets.

6.1 Operations with Sets

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathcal{M}$ the set of other types of numbers.

Let A and B be two real or complex, or other type of number sets.

One or both may also be a scalar, because a scalar $\alpha \in \mathbb{R}$ may be written as a set, $[\alpha, \alpha]$.

Then, $A \star B = \{a \star b$, where $a \star b$ is well defined; $a \in A, b \in B\}$, where $\star$ means any operations: addition, subtraction, scalar multiplication, multiplication, division, power, radical (root).

Afterwards, one computes *min/inf* and *max/sup* of $A \star B$.

In the next sections we are referring only to the sets of real numbers, since they are needed in *Neutrosophic Statistics*, but for the other types of sets the research is similar.

- Addition of Sets

  $A + B = \{a + b; a \in A, b \in B\}$

  Examples:

  $A = (2,3),\ B = (0,1)$

  $A + B = (2,3) + (0,1) = (2 + 0, 3 + 1)$

  $A + A = (2,3) + (2,3) = (2 + 2, 3 + 3) = (4,6) = 2 \cdot (2,3) = 2A$

  The last one is similar to the addition of a number to itself, for example:

$5 + 5 = 2 \cdot 5$.

- *Subtraction of Sets*

  $A - B = \{a - b; a \in A, b \in B\}$

  Examples:

  $A = (2, 3),\ B = (0, 1)$

  $A - B = (2, 3) - (0, 1) = (2 - 1, 3 - 0) = (1, 3)$

  $A - A = (2, 3) - (2, 3) = (2 - 3, 3 - 2) = (-1, 1)$

  Therefore: $A - A \neq 0$ (zero) and $A - A \neq \emptyset$ (empty set), contrarily to the subtraction of a number from itself (for example, 5 - 5 = 0).

- *Scalar Multiplication of Sets*

  Let the scalar $\beta \in R$, then:

  $\beta \cdot A = \left\{\beta \cdot a; a \in A\right\}$.

  Examples:

  (i). $\beta = 6$, $A = (2, 3)$, then: $\beta \cdot A = 6 \cdot (2, 3) = (6 \cdot 2, 6 \cdot 3) = (12, 18)$.

  (ii). $\beta = 0$, $A = (2, 3)$, then: $\beta \cdot A = 0 \cdot (2, 3) = (0 \cdot 2, 0 \cdot 3) = (0, 0) = \emptyset$ (empty set).

  (iii). $\beta = 0$, $B = [2, 3]$, then: $\beta \cdot B = 0 \cdot [2, 3] = [0 \cdot 2, 0 \cdot 3] = [0, 0] =$

  $\{0\}$(a set that has only one element, 0).

- *Multiplication of Sets*

  $A \cdot B = \{a \cdot b; a \in A, b \in B\}$

  Examples:

  $A = (2, 3),\ B = (0, 1)$

  $A \cdot B = (2, 3) \cdot (0, 1) = (2 \cdot 0, 3 \cdot 1) = (0, 3)$

  $A \cdot A = (2, 3) \cdot (2, 3) = (2 \cdot 2, 3 \cdot 3) = (4, 9) = A^2$.

- *Division of Sets*

  $A \div B = \frac{A}{B} = \{a \div b; a \div b \text{ is well defined}, a \in A, b \in B\}$

  Examples:

  $A = (2, 3),\ B = (0, 1)$

  (i). For (A, B) intervals, one has $A \div B = \left(\frac{minA}{maxB}, \frac{maxA}{minB}\right) = \left(\frac{2}{1}, \frac{3}{0}\right) \to (2, +\infty)$ since the

  undefined $\frac{3}{0} \to +\infty$ (not $-\infty$, because B has only positive elements).

  (ii). Let $A = (2, 3)$, and $C = (-1, 0)$ be two intervals. Then:

  $A \div C = \left(\frac{minA}{maxC}, \frac{maxA}{minC}\right) = \left(\frac{2}{0}, \frac{3}{-1}\right) = \left(\frac{2}{0}, -3\right) \to (-\infty, -3)$, We take $\frac{2}{0}$ as $-\infty$, because the set C contains only negative elements.

(iii). $A \div A = \left(\frac{minA}{maxA}, \frac{maxA}{minA}\right) = \left(\frac{2}{3}, \frac{3}{2}\right)$.

Therefore, $A \div A \neq 1$, contrarily to the division of real numbers, where a non-zero number divided by itself is equal to 1, for example: $\frac{5}{5} = 1$.

(iv). $B \div B = \left(\frac{minB}{maxB}, \frac{maxB}{minB}\right) = \left(\frac{0}{1}, \frac{1}{0}\right) \rightarrow (0, +\infty) \neq 1$

- *Power and Root of Sets*

Let r be a rational number, i.e. $r = \frac{m}{n}$, where m, n are integers, $n \neq 0$.

$A^r = \{a^r;$ where a^r is well defined, $a \in A\}$.

(i). Positive integer power

$A = (2,3),\ r = 4$

$A^4 = (2,3)^4 = (2^4, 3^4) = (16, 81)$.

Let $E = (-2,3)$.

$E^2 = (-2,3) \cdot (-2,3) = (-2 \cdot 3, 3 \cdot 3) = (-6,9) \neq ((-2)^2, 3^2) = (4,9)$.

(ii). Power zero

$A = (2,3),\ r = 0$

$A^0 = (2,3)^0 = (2^0, 3^0) = (1,1) = \emptyset$ (Empty set).

Therefore, $A^0 \neq 1$, contrarily to the real numbers, where for example $7^0 = 1$.

Let $D = [2,3]$, then $D^0 = [2^0, 3^0] = [1,1] \equiv \{1\}$, as for real numbers, where for example $7^0 = 1$.

(iii). Square Root

$\sqrt{A} = \sqrt{(2,3)} = (\sqrt{2}, \sqrt{3})$.

(iv). *Partial Square Root*

Let the set $D = (-2,3)$, then:

$\sqrt{D} = [\sqrt{0}, \sqrt{3}) = [0, \sqrt{3})$, since in the set of real numbers one cannot compute square root of the negative numbers from the interval $(-2,0)$. We have only computed a partial square root of D.

(v). Negative Power

$A = (2,3), r = -2$.

$A^{-2} = (2,3)^{-2} = (2^{-2}, 3^{-2}) = \left(\frac{1}{4}, \frac{1}{9}\right) \equiv \left(\frac{1}{9}, \frac{1}{4}\right)$.

Now, the operations with the Real Neutrosophic Numbers follow the rules of operations with real sets presented above, because a Real Neutrosophic Number is equivalent to a real subset:

Let I be a real subset, $I \subset \mathbb{R}$.

$N = a + bI = \{a + b \cdot x,$ where $x \in I\}$, which is a real subset of the form as of I.

Actually N is the enlarged subset I.

If I is an interval of the form $I = (c, d)$, or $[c, d)$, or $(c, d]$, or $[c, d]$, then N will also be an interval of the same corresponding open/closed form.

If $I = \{c_1, c_2, \dots, c_n\}$ is a real discrete subset, of cardinal n, $1 \le n \le \infty$, then N will also be a real discrete subset of cardinal n.

It is a union of several subsets, $I = I_1 \cup I_2 \cup \dots \cup I_m$, then N will also be a union of corresponding subsets:

$N = N_1 \cup N_2 \cup \dots \cup N_m = \bigcup_{k=1}^{m} N_k$,

Where

$N_k = a + b \cdot I_k = \{a + bx, \text{where } x \in I_k\}$.

6.2 Second Method of Operations with Real Neutrosophic Numbers is the following.

Transform each real neutrosophic number into an equivalent real subset, especially when the indeterminacy (I) are not the same.

Examples:

$N_1 = 1 + 2I_1$, where $I_1 = \{02., 0.5, 0.8\}$

$N_2 = 3 - I_2$, where $I_2 = [0, 1)$

Then:

$N_1 = 1 + 2 \cdot \{0.2, 0.5, 0.8\} = 1 + \{0.4, 1.0, 1.6\} = \{1.4, 2.0, 2.6\}$

and

$N_2 = 3 - [0, 1) = (3 - 1, 3 - 0] = (2, 3]$

- *Addition of Real Neutrosophic Numbers*

$$N_1 + N_2 = \{1.4, 2.0, 2.6\} + (2, 3] = \{1.4 + (2, 3]\} \cup \{2.0 + (2, 3]\} \cup \{2.6 + (2, 3]\}$$
$$= (1.4 + 2, 1.4 + 3] \cup (2.0 + 2, 2.0 + 3] \cup (2.6 + 2, 2.6 + 3]$$
$$= (3.4, 4.4] \cup (4, 5] \cup (4.6, 5.6] = (3.4, 5.6].$$

- *Addition of a Scalar with a Neutrosophic Set*

$0.9 + N_1 = \{1.4, 2.0, 2.6\} + 0.9 = \{1.4 + 0.9,\ 2.0 + 0.9,\ 2.6 + 0.9\} = \{2.3, 2.9, 3.5\}$.

$0.9 + N_2 = 0.9 + (2, 3] = (0.9 + 2, 0.9 + 3] = (2.9, 3.9]$.

- *Subtraction of Real Neutrosophic Numbers*

$$N_1 - N_2 = \{1.4, 2.0, 2.6\} - (2, 3] = \{1.4 - (2, 3]\} \cup \{2.0 - (2, 3]\} \cup \{2.6 - (2, 3]\}$$
$$= [1.4 - 3, 1.4 - 2) \cup [2.0 - 3, 2.0 - 2) \cup [2.6 - 3, 2.6 - 2)$$
$$= [-1.6, -0.6) \cup [-1, 0) \cup [-0.4, 0.6) = [-1.6, 0.6).$$

- *Multiplication of Real Neutrosophic Numbers*

$$N_1 \cdot N_2 = \{1.4, 2.0, 2.6\} \cdot (2, 3] = \{1.4 \cdot (2, 3]\} \cup \{2.0 \cdot (2, 3]\} \cup \{2.6 \cdot (2, 3]\}$$
$$= (1.4 \cdot 2, 1.4 \cdot 3] \cup (2.0 \cdot 2, 2.0 \cdot 3] \cup (2.6 \cdot 2, 2.6 \cdot 3]$$
$$= (2.8, 4.2] \cup (4, 6] \cup (5.2, 7.8] = (2.8, 7.8].$$

- *Multiplication of a Scalar with a Neutrosophic Number*

$4 \cdot N_1 = 4 \cdot \{1.4, 2.0, 2.6\} = \{4 \cdot (1.4), 4 \cdot (2.0), 4 \cdot (2.6)\} = \{5.6, 8.0, 10.4\}$.

- Division of Real Neutrosophic Numbers

$$\frac{N_1}{N_2} = \frac{\{1.4, 2.0, 2.6\}}{(2, 3]} = \frac{1.4}{(2, 3]} \cup \frac{2.0}{(2, 3]} \cup \frac{2.6}{(2, 3]} = \left(\frac{1.4}{3}, \frac{1.4}{2}\right] \cup \left(\frac{2.0}{3}, \frac{2.0}{2}\right] \cup \left(\frac{2.6}{3}, \frac{2.6}{2}\right]$$
$$= (0.4\overline{6}, 0.7] \cup (0.\overline{6}, 1] \cup (0.8\overline{6}, 1.3] = (0.4\overline{6}, 1.3].$$

- *Another Example of Division of Neutrosophic Numbers*

Let $N_1 = 2 - 3I_1$ where $I_1 = [4, 5]$, and $N_2 = 1 + 4I_2$ where $I_2 = \{-1, 3, 5\}$.

Then:

$$N_1 = 2 - 3I_1 = 2 - 3 \cdot [4,5] = 2 - [3 \cdot 4, 3 \cdot 5] = 2 - [12,15] =$$
$$= [2-15, 2-12] = [-13,-10]$$

$$N_2 = 1 + 4I_2 = 1 + 4 \cdot \{-1,3,5\} = 1 + \{4 \cdot (-1), 4 \cdot 3, 4 \cdot 5\} =$$
$$= 1 + \{-4,12,20\} = \{1+(-4), 1+12, 1+20\} = \{-3,13,21\}$$

$$\frac{N_1}{N_2} = \frac{[-13,-10]}{\{-3,13,21\}} = [\frac{-13}{-3}, \frac{-10}{-3}] \cup [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] =$$
$$= [\frac{10}{3}, \frac{13}{3}] \cup [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] =$$
$$= [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] \cup [\frac{10}{3}, \frac{13}{3}]$$

- *Division between a Real Neutrosophic Number and a Scalar*

$\frac{N_1}{4} = \frac{\{1.4,2.0,2.6\}}{4} = \left\{\frac{1.4}{4}, \frac{2.0}{4}, \frac{2.6}{4}\right\} = \{0.35, 0.50, 0.65\}.$

$\frac{4}{N_1} = \frac{4}{\{1.4,2.0,2.6\}} = \left\{\frac{4}{1.4}, \frac{4}{2.0}, \frac{4}{2.6}\right\} = \{2.857, 2.000, 1.538\}.$

$\frac{4}{N_2} = \frac{4}{(2,3]} = \left[\frac{4}{3}, \frac{4}{2}\right) \approx [1.333, 2.000)$

$\frac{N_2}{4} = \frac{(2,3]}{4} = \left[\frac{2}{4}, \frac{3}{4}\right) = [0.50, 0.75).$

- *Power of Real Neutrosophic Numbers*

$${N_1}^{N_2} = \{1.4, 2.0, 2.6\}^{(2,3]} = 1.4^{(2,3]} \cup 2.0^{(2,3]} \cup 2.6^{(2,3]} = (1.4^2, 1.4^3] \cup (2.0^2, 2.0^3] \cup (2.6^2, 2.6^3]$$
$$= (1.960, 2.744] \cup (4, 8] \cup (6.760, 17.576] = (1.960, 2.744] \cup (4.000, 17.576].$$

*

$${N_2}^{N_1} = (2,3]^{\{1.4,2.0,2.6\}} = \{(2^{1.4}, 3^{1.4}], (2^{2.0}, 3^{2.0}], (2^{2.6}, 3^{2.6}]\}$$
$$\approx \{(2.639, 4.656], (4.0, 9.0], (6.063, 17.399]\}$$
$$\equiv \{2.639, 4.656] \cup (4.0, 9.0] \cup (6.063, 17.399]$$
$$= (2.639, 17.399].$$

- *Power of a Neutrosophic Real Numbers to a Scalar*

$(N_1)^4 = \{1.4, 2.0, 2.6\}^4 = \{1.4^4, 2.0^4, 2.6^4\} = \{3.8416, 16.000, 45.6976\}$; and $4^{N_1} = 4^{\{1.4,2.0,2.6\}} = \{4^{1.4}, 4^{2.0}, 4^{2.6}\} \approx \{6.9644, 16.000, 36.7583\}.$

- *Real Root of a Neutrosophic Real Number*

$\sqrt{N_1} = \sqrt{\{1.4, 2.0, 2.6\}} = \{\sqrt{1.4}, \sqrt{2.0}, \sqrt{2.6}\} \approx \{1.183, 1.414, 1.612\}$
$\sqrt[3]{N_2} = \sqrt[3]{(2,3]} = (\sqrt[3]{2}, \sqrt[3]{3}) \simeq (1.260, 1.442].$

- *Real Neutrosophic Root of a Neutrosophic Real Number*

$$\sqrt[N_1]{N_2} = N_2^{\frac{1}{N_1}} = (2,3]^{\frac{1}{\{1.4,2.0,2.6\}}} = \left\{(2,3]^{\frac{1}{1.4}}, (2,3]^{\frac{1}{2.0}}, (2,3]^{\frac{1}{2.6}}\right\} = \left\{\left(2^{\frac{1}{1.4}}, 3^{\frac{1}{1.4}}\right], \left(2^{\frac{1}{2.0}}, 3^{\frac{1}{2.0}}\right], \left(2^{\frac{1}{2.6}}, 3^{\frac{1}{2.6}}\right]\right\}$$

$$\approx \{(1.641, 2.192], (1.414, 1.732], (1.306, 1.526]\} \equiv (1.306, 2.192]$$

*

$$\sqrt[N_2]{N_1} = N_1^{\frac{1}{N_2}} = \{1.4, 2.0, 2.6\}^{\frac{1}{(2,3]}} = \{1.4, 2.0, 2.6\}^{[\frac{1}{3},\frac{1}{2})} = \left\{1.4^{[\frac{1}{3},\frac{1}{2})}, 2.0^{[\frac{1}{3},\frac{1}{2})}, 2.6^{[\frac{1}{3},\frac{1}{2})}\right\}$$

$$= \left\{\left[1.4^{\frac{1}{3}}, 1.4^{\frac{1}{2}}\right), \left[2.0^{\frac{1}{3}}, 2.0^{\frac{1}{2}}\right), \left[2.6^{\frac{1}{3}}, 2.6^{\frac{1}{2}}\right)\right\}$$

$$\approx \{[1.119, 1.183), [1.260, 1.414), [1.375, 1.612)\} \equiv [1.119, 1.183) \cup [1.260, 1.612).$$

7. Literal Neutrosophic Numbers

The Literal Neutrosophic Numbers (LNN) have the form: $LNN = a + bI$, where a, b are real or complex numbers, and $I =$ literal indeterminacy, where $I^2 = I$, and $I/I =$ undefined.

Their Addition, Subtraction, Scalar Multiplication, Multiplication, Division, Power, Radical are straightforward. The Literal Neutrosophic Numbers are not used in Neutrosophic Statistics, but in Neutrosophic Algebraic Structures, that's why we do not present their operations herein.

8. NonAppurtenance Equation, NonInclusion Equation, and NonEquality Equation

They are complementarians of the Appurtenance Equation, Inclusion Equation, and Equality Equation respectively.

We present them as a curiosity, or as recreational mathematics.

(i). The Appurtenance Equation from previous Example 1 was:
$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ whose solutions are all real numbers $x \in (0.28, 0.40)$.
Its corresponding NonAppurtenance Equation is:
$4 - 5x \notin 1 + 2 \cdot (0.5, 0.8)$ whose solutions are all real numbers $x \notin (0.28, 0.40)$, or all real numbers $x \in R - (0.28, 0.40)$.

(ii). The Inclusion Equation from previous Example 2 was:

$1 + x \cdot (1, 2) \subset (0, 5)$, whose maximal solution is $x = (-0.5, 2)$.

Its corresponding NonInclusion Equation is:

$1 + x \cdot (1, 2) \not\subset (0, 5)$, whose maximum solution is R - (- 0.5, 2).

(iii). An elementary Equality Equation

3x + 4 = 7, has the unique solution x = 1.

Its corresponding NonEquality Equation is:

$3x + 4 \neq 7$ has, of course, infinitely many solutions $x \in R - \{1\}$.

9. Conclusions

In neutrosophic statistics, from the fact that the single true value v is in I, it does not result that v is in $a + bI = N$ as well, but: $a + bv \in a + bI$. That's why the appurtenance relationship and equation must be introduced and studied.

Even more, if one has a set of true values, from the fact that the set of true values V is included in I, it does not mean that V is included in $a + bI$ too, but $a + bV \subset a + bI$ (or $a + bV \subseteq a + bI$). That's why the inclusion relationship and equation must be introduced.

In the same way as the "=" symbol is used for an equality relationship or an equality equation, we use the symbol "∈" {belong(s) to} for an *appurtenance relationship* or *appurtenance equation* of a number to a set, respectively the symbol $\subset$ (or $\subseteq$) {included in, or included in or equal to} for an *inclusion relationship* or *inclusion equation*.

We just introduced for the first time the Appurtenance Equation and Inclusion Equation, which help in understanding the operations with neutrosophic numbers within the frame of neutrosophic statistics. The way of solving them resembles the equations whose coefficients are sets (no single numbers).

In addition, we also presented their complementary NonAppurtenance Equation, NonInclusion Equation, and the elementary NonEquality Equation respectively.

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The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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A Multi-Criteria Decision Making Model for Sustainable and Resilient Supplier Selection and Management

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Abstract: Sustainable and resilient supplier selection and management are essential to building environmentally responsible and resilient supply chains. Selecting sustainable and resilient suppliers enables organizations to ensure the long-term viability of the supply chain. To do that, there is a need to identify suitable suppliers that align with sustainability and resilience goals. This study aims to provide an overview of the requirements and criteria for selecting and managing sustainable and resilient suppliers and emphasizes the significance of integrating sustainability and resilience principles throughout the supply chain. Multi-criteria decision-making (MCDM) is used to deal with various criteria. The double normalization-based multi-aggregation (DNMA) method ranks the alternatives. This study used 18 criteria and 12 alternatives to select the best one. By employing these criteria, organizations can identify suppliers that align with sustainability and resilience goals. The results show that alternative 3 is the best and alternative 11 is the worst. Also, the results show that the proposed method can provide a new method to rank alternatives. Moreover, the proposed method can introduce a more simple and flexible method for selecting a suitable supplier and ensuring the long-term viability of the supply chain.

Keywords: Double Normalization Based Multi-Aggregation Method; MCDM; Resilient Supplier Selection; Supply Chain.

1. Introduction

In today's rapidly changing business landscape, organizations increasingly recognize the importance of sustainability and resilience in their supply chains. A key aspect of achieving sustainability and resilience is selecting and managing suppliers who share the same values and commitment to these principles. Sustainable and resilient suppliers play a critical role in ensuring the long-term viability of supply chains, minimizing environmental impact, promoting social responsibility, and effectively managing risks and disruptions [1]. Sustainable suppliers are those that prioritize environmentally friendly practices, resource conservation, and the reduction of carbon footprints [2]. They seek to minimize waste, adopt renewable energy sources, and implement sustainable manufacturing processes. On the other hand, resilient suppliers demonstrate the ability to withstand and recover from disruptions such as natural disasters, geopolitical instability, or supply chain breakdowns. They have robust business continuity plans, diversified sourcing strategies, and proactive risk management practices [3].

Organizations must identify and assess suppliers based on specific criteria that align with their sustainability and resilience goals. These criteria encompass various dimensions, including environmental sustainability, social responsibility, supply chain transparency, resilience and business continuity, innovation and adaptability, financial stability, collaboration and communication, compliance and certifications, risk management, ethical practices, energy efficiency, water management, circular economy practices, community engagement, reporting and transparency, innovation and collaboration, and supplier diversity [4]. The selection and management of sustainable and resilient suppliers require a comprehensive and strategic approach. So, this study aims to provide an overview of the requirements and criteria for selecting and managing sustainable and resilient suppliers and emphasizes the significance of integrating sustainability and resilience principles throughout the supply chain. Multi-criteria decision-making (MCDM) is used to deal with various criteria. The double normalization-based multi-aggregation (DNMA) method ranks the alternatives. This study used 18 criteria and 12 alternatives to select the best one. The results show that alternative 3 is the best and alternative 11 is the worst. By employing these criteria, organizations can identify suppliers that align with sustainability and resilience goals, ensuring the long-term viability of the supply chain

This work is arranged as follows: the first section gives the introduction; the second section introduces motivations for evaluating suppliers; the third section introduces the concept of sustainable and resilient supplier; the fourth section represents the requirements for evaluating and selecting a suitable supplier. The fifth section gives the proposed Double Normalization Based Multi-Aggregation (DNMA) Method; the sixth section applies the proposed method with a numerical example; the seventh section discusses the results; the eighth section gives the conclusion of this work; and finally, it gives references.

2. Motivations for Evaluating Suppliers

By carefully evaluating suppliers against these criteria, organizations can make informed decisions that support their sustainability and resilience objectives. Sustainable and resilient suppliers contribute to the overall sustainability and resilience of the supply chain by reducing environmental impacts, promoting fair labour practices, ensuring a stable supply of goods and services, and actively managing risks. They also foster collaboration, innovation, and knowledge sharing, driving continuous improvement and the development of more sustainable and resilient practices [5].

Furthermore, selecting and managing sustainable and resilient suppliers are not isolated activities but require continuous monitoring and evaluation. Organizations should maintain open lines of communication with suppliers, engage in regular performance assessments, and collaborate on sustainability initiatives. This ongoing relationship-building and collaboration contribute to developing a robust and dynamic supply chain that can adapt to changing market conditions, regulatory requirements, and societal expectations. By integrating sustainability and resilience principles into supplier selection processes, organizations can ensure that their supply chains are profitable, environmentally sustainable, socially responsible, and equipped to withstand disruptions [6]. This proactive approach to supplier selection and management can support the organization's overall sustainability goals and help create a more sustainable and resilient future.

3. The Concept of Sustainable and Resilient Supplier

The concept of sustainability and sustainable development was introduced by the World Commission on Environment and Development (WCED) in 1987 [7]. Sustainable and resilience concepts can improve the overall performance of organizations and can decrease disruption propagation in the form of supply chain quantity downscaling [8]. Sustainable and resilient can be defined as follows in Table 1:

Table 1. Sustainable and resilient.

Sustainable	Sustainable suppliers are those that prioritize environmentally friendly practices, resource conservation, and the reduction of carbon footprints [2].
Resilient	Resilience is defined as the intrinsic ability of an organization to return to or recover from a steady state facing a disruptive event [9]. Can be defined as the ability to resist disruptions and to recover operational capability after disruptions have occurred [10]. Suppliers have robust business continuity plans, diversified sourcing strategies, and proactive risk management practices [3].

4. The Requirements for Evaluating and Selecting the Suitable Supplier

Supplier selection is a complex process that needs to evaluate different types of criteria in order to select consistent suppliers [11]. Supplier selection is divided into two main types, as shown in the following Figure 1 [12]:

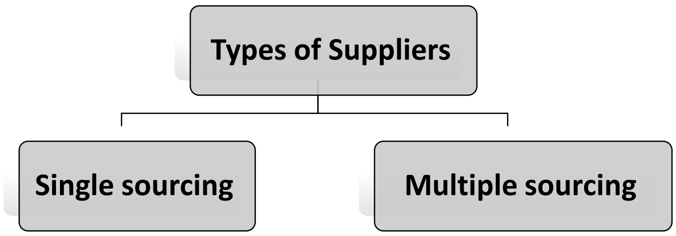

Figure 1. Types of suppliers.

- Single sourcing only one supplier is able to fulfill an organization's demands. The decision makers need to select only one supplier
- Multiple sourcing more than one supplier is selected as no one supplier is single-handedly capable of meeting the demand requirements of the enterprise. The decision makers face more challenges as they need to allocate optimal quantities to each supplier in order to create an environment of fair play and genuine competition, while maximizing returns for their own organization at the same time.

The requirements and criteria include environmental sustainability, social responsibility, supply chain transparency, resilience and business continuity, innovation and adaptability, financial stability, collaboration and communication, compliance and certifications, risk management, ethical practices, energy efficiency, water management, circular economy practices, community engagement, reporting and transparency, innovation and collaboration, and supplier diversity. By employing these criteria, organizations can identify suppliers that align with sustainability and resilience goals, ensuring the long-term viability of the supply chain. The criteria used in this study are organized as [13]:

- *Environmental Sustainability*: Assess the supplier's environmental practices and policies. Look for suppliers who demonstrate a commitment to environmental sustainability by implementing eco-friendly practices such as resource conservation, waste reduction, recycling, and pollution prevention. Consider their track record in minimizing carbon footprint and adherence to relevant environmental certifications or standards.
- *Social Responsibility:* Evaluate the supplier's social responsibility practices. This includes assessing their labour practices, human rights policies, and commitment to fair and ethical treatment of employees. Look for suppliers that promote diversity and inclusion, provide safe working conditions, and ensure fair wages and benefits for their workers.
- *Supply Chain Transparency:* Consider the supplier's level of transparency in their supply chain. Suppliers should be able to provide information on the origin of their materials, their supply

chain partners, and any potential risks or vulnerabilities in their supply chain. Transparency helps identify potential environmental and social risks and enables proactive risk management.

- *Resilience and Business Continuity:* Evaluate the supplier's resilience and business continuity plans. Assess their ability to mitigate and respond to disruptions such as natural disasters, supply chain disruptions, or other unforeseen events. Look for suppliers who have implemented measures to ensure continuity of supply, such as backup manufacturing facilities, diversified sourcing, or robust risk management strategies.
- *Innovation and Adaptability*: Consider the supplier's commitment to innovation and adaptability. Look for suppliers who invest in research and development, embrace new technologies, and continuously improve their processes to enhance sustainability and resilience. Suppliers demonstrating a forward-thinking approach and agility adapting to changing market demands are more likely to contribute to long-term sustainability and resilience goals.
- *Financial Stability:* Assess the financial stability of the supplier. A financially stable supplier is better positioned to invest in sustainable and resilient practices, maintain high-quality standards, and provide consistent supply. Financial stability ensures that the supplier can withstand economic fluctuations and continue to deliver products and services without compromising sustainability and resilience commitments.
- *Collaboration and Communication:* Evaluate the supplier's willingness to collaborate and communicate openly. Look for suppliers who are proactive in engaging in sustainability discussions, participate in industry collaborations, and are responsive to inquiries and feedback. Effective communication channels and cooperation foster a strong partnership and facilitate shared sustainability and resilience goals.
- *Compliance and Certifications:* Consider the supplier's compliance with relevant regulations and certifications. Look for suppliers who comply with environmental, social, and labour laws and regulations.
- *Long-Term Relationship Potential:* Assess the long-term relationship potential with the supplier. Look for suppliers who align with your organization's values, goals, and sustainability strategies. Building long-term partnerships allows for collaborative efforts in driving sustainable and resilient practices throughout the supply chain.
- *Risk Management:* Evaluate the supplier's risk management practices. Assess their ability to identify and mitigate potential risks and disruptions to their operations and supply chain. Look for suppliers with robust risk management strategies, including contingency plans, supply chain mapping, and proactive monitoring of potential risks such as climate change impacts, geopolitical instability, or regulatory changes.
- *Ethical Practices:* Consider the supplier's commitment to ethical practices. This includes evaluating their stance on anti-corruption, anti-bribery, and fair trade issues. Look for suppliers with clear policies and procedures to ensure ethical behaviour throughout their operations and supply chain.
- *Energy Efficiency:* Assess the supplier's energy efficiency initiatives. Look for suppliers seeking to reduce energy consumption, implement energy-efficient technologies, and invest in renewable energy sources. Energy-efficient practices contribute to sustainability while enhancing resilience by reducing reliance on fossil fuels and mitigating the impact of energy price fluctuations.
- *Water Management:* Consider the supplier's water management practices. Suppliers prioritising responsible water usage, implementing water conservation measures, and addressing water pollution concerns demonstrate a commitment to environmental sustainability. Effective water management is crucial for ensuring the availability of this vital resource and minimizing water-related risks.

- *Circular Economy Practices:* Assess the supplier's adoption of circular economy principles. Look for suppliers who embrace product lifecycle extension, recycling, reusing, and waste reduction practices. Suppliers promoting circularity contribute to resource conservation, waste reduction, and a more sustainable and resilient economy.
- *Community Engagement:* Evaluate the supplier's involvement in local communities. Look for suppliers who actively engage with local communities, support social initiatives, and contribute to local economic development. Suppliers with strong community relationships are more likely to positively impact the social resilience of the communities in which they operate.
- *Reporting and Transparency:* Assess the supplier's reporting and transparency practices. Look for suppliers who provide comprehensive sustainability reports, disclose their environmental and social impacts, and engage in third-party audits or certifications. Transparent reporting allows for accountability and facilitates better understanding and monitoring of the supplier's sustainability and resilience performance.
- *Innovation and Collaboration:* Consider the supplier's ability to innovate and collaborate on sustainability and resilience initiatives. Look for suppliers who actively seek opportunities for joint projects, knowledge sharing, and innovation in sustainable practices. Collaborative partnerships foster continuous improvement and the exchange of best practices, driving sustainability and resilience efforts forward.
- *Supplier Diversity:* Assess the supplier's commitment to supplier diversity and inclusion. Look for suppliers who promote diversity in their supply chain by engaging with minority-owned, women-owned, and small businesses. Supplier diversity enhances social resilience, fosters economic development, and promotes a more inclusive and equitable business environment.

5. The proposed Double Normalization Based Multi-Aggregation (DNMA) Method

This section introduces the steps of the DNMA method to rank the alternatives [14]. The MCDM can be used in this section [15]. The proposed DNMA method includes seven steps that are organized as Figure 2:

Step 1. Build the decision matrix.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

Step 2. Compute the target based linear normalization.

$$r_{ij}^1 = 1 - \frac{\left| a_{ij} - \max_i a_{ij} \right|}{\max\left\{ \max_i a_{ij} \max_i a_{ij} \right\} - \min\left\{ \min_i a_{ij} \max_i a_{ij} \right\}} \tag{2}$$

Step 3. Compute the target based vector normalization.

$$r_{ij}^2 = 1 - \frac{\left| a_{ij} - \max_i a_{ij} \right|}{\sqrt{\sum_{i=1}^{m} \left(a_{ij} \right)^2 + \left(\max_i a_{ij} \right)^2}} \tag{3}$$

Step 4. Compute the adjusted criteria weights.

$$d_j = \sqrt{\frac{\sum_{i=1}^{m} \left(\frac{a_{ij}}{\max_i a_{ij}} - \frac{1}{m} \sum_{i=1}^{m} \left(\frac{a_{ij}}{\max_i a_{ij}} \right) \right)^2}{m}} \tag{4}$$

$$w_j^d = \frac{d_j}{\sum_{j=1}^{n} dj} \tag{5}$$

$$w_j = \frac{\sqrt{w_j^d w_j}}{\sum_{j=1}^{n} \sqrt{w_j^d w_j}} \tag{6}$$

Step 5. Compute the values of complete compensatory, and incomplete compensatory.

$$q_1 = \sum_{j=1}^{n} w_j r_{ij}^1 / \max_i r_{ij}^1 \tag{7}$$

$$q_2 = \max_j w_j \left(1 - r_{ij}^1 / \max_i r_{ij}^1\right) \tag{8}$$

$$q_3 = \prod_j \left(r_{ij}^2 / \max_i r_{ij}^2\right)^{w_j} \tag{9}$$

Step 6. Compute the value of S_i.

$$S_i = \left\{ \begin{array}{c} \frac{1}{3}\sqrt{\beta\left(\frac{q_1}{\max_i q_1}\right)^2 - (1-\beta)\left(\frac{m-q_1+1}{m}\right)^2} + \\ \frac{1}{3}\sqrt{\beta\left(\frac{q_2}{\max_i q_2}\right)^2 - (1-\beta)\left(\frac{q_2}{m}\right)^2} \\ + \frac{1}{3}\sqrt{\beta\left(\frac{q_3}{\max_i q_3}\right)^2 - (1-\beta)\left(\frac{m-q_3+1}{m}\right)^2} \end{array} \right\} \tag{10}$$

Step 7. Rank the alternatives based on the largest value in S_i.

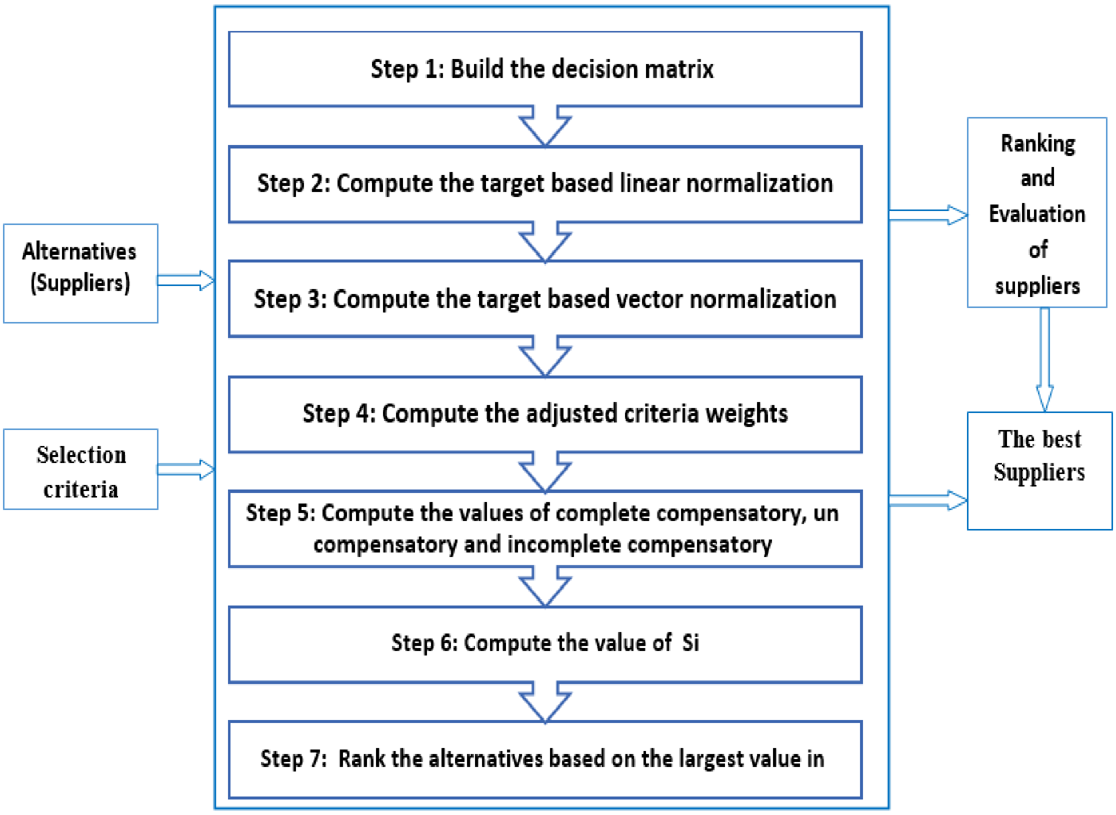

Figure 2. The proposed double normalization-based multi-aggregation (DNMA) method.

6. Applying the Proposed Method with Numerical Example

This section provides the application of the proposed method. We suggested this method for selecting the best supplier. We used the 18 criteria and 15 alternatives in this section. The criteria used in this study are organized as [13]:

- Environmental Sustainability
- Social Responsibility
- Supply Chain Transparency
- Resilience and Business Continuity
- Innovation and Adaptability
- Financial Stability
- Collaboration and Communication
- Compliance and Certifications
- Long-Term Relationship Potential
- Risk Management
- Ethical Practices
- Energy Efficiency
- Water Management
- Circular Economy Practices
- Community Engagement
- Reporting and Transparency
- Innovation and Collaboration
- Supplier Diversity

Step 1. Build the decision matrix between factors and suppliers by Eq. (1). The experts used the scale between 1 and 9 to evaluate the criteria and alternatives.

Step 2. Compute the target based linear normalization by Eq. (2) as shown in Table 2.

Step 3. Compute the target based vector normalization by Eq. (3) as shown in Table 3.

Step 4: Compute the adjusted criteria weights by Eqs. (4-6) as shown in Figure 3.

Step 5. Compute the values of complete compensatory, uncompensatory and incomplete compensatory by Eqs. (7-9)

Step 6. Compute the value of S_i by Eq. (10)

Step 7. Rank the alternatives based on the largest value in S_i as shown in Figure 4. We show the alternative 3 is the best and alternative 11 is the worst.

We change the value in β parameter to ensure the stable of the results. Figure 5 shows the different ranks under different value between 0.1 and 1 in β parameter.

Table 2. The target based linear normalization.

	TLA_1	TLA_2	TLA_3	TLA_4	TLA_5	TLA_6	TLA_7	TLA_8	TLA_9	TLA_{10}	TLA_{11}	TLA_{12}
TLC_1	1.047619	1.031746	1.015873	1	1.047619	1.095238	1.111111	1.063492	1.015873	1	1.047619	1.095238
TLC_2	1	1.063492	1.015873	1	1.063492	1.015873	1	1.063492	1.047619	1.063492	1.111111	1.063492
TLC_3	1.015873	1.079365	1.079365	1.047619	1.079365	1.063492	1.031746	1.015873	1	1.047619	1.063492	1.111111
TLC_4	1.027778	1.111111	1.055556	1.097222	1.055556	1.013889	1	1.041667	1.041667	1.041667	1.041667	1.041667
TLC_5	1.083333	1.166667	1	1.041667	1.041667	1	1.125	1.166667	1.041667	1.041667	1.166667	1.041667
TLC_6	1.0625	1.104167	1.125	1.041667	1.125	1.104167	1.104167	1	1.083333	1.125	1.104167	1.083333
TLC_7	1.047619	1.047619	1.095238	1.111111	1.095238	1.063492	1.047619	1	1.015873	1.063492	1.063492	1.031746
TLC_8	1.111111	1.111111	1	1.095238	1.047619	1.111111	1.063492	1.079365	1.047619	1	1.015873	1.015873
TLC_9	1.095238	1.063492	1.063492	1.047619	1.015873	1.047619	1.031746	1.111111	1.095238	1.047619	1	1
TLC_{10}	1.047619	1	1.031746	1.063492	1.031746	1	1.015873	1.063492	1.111111	1.063492	1.047619	1.047619
TLC_{11}	1.088889	1.022222	1.111111	1	1.088889	1.088889	1.088889	1.066667	1.066667	1.044444	1.088889	1.088889
TLC_{12}	1.079365	1.047619	1.015873	1.015873	1.047619	1.111111	1	1	1.063492	1.015873	1.111111	1
TLC_{13}	1.013889	1.055556	1	1.055556	1.097222	1.097222	1.041667	1.041667	1.027778	1.041667	1.111111	1.013889
TLC_{14}	1	1.079365	1.047619	1.111111	1.047619	1.095238	1.095238	1.047619	1.015873	1.111111	1.079365	1.079365
TLC_{15}	1.097222	1.111111	1.097222	1.041667	1.041667	1.055556	1.097222	1.083333	1.041667	1	1.097222	1.041667
TLC_{16}	1.095238	1.111111	1.063492	1.047619	1.063492	1.095238	1.111111	1.047619	1	1.015873	1.015873	1.063492
TLC_{17}	1.111111	1.095238	1.047619	1	1.063492	1.047619	1.063492	1.111111	1.095238	1.111111	1	1.111111
TLC_{18}	1.055556	1.041667	1	1.013889	1.055556	1.083333	1.097222	1.055556	1.069444	1.111111	1.041667	1.083333

Table 3. The target based vector normalization.

	TLA_1	TLA_2	TLA_3	TLA_4	TLA_5	TLA_6	TLA_7	TLA_8	TLA_9	TLA_{10}	TLA_{11}	TLA_{12}
TLC_1	0.974359	0.984615	0.993103	1	0.974359	0.933333	0.917647	0.962264	0.993103	1	0.974359	0.933333
TLC_2	1	0.962264	0.993103	1	0.962264	0.993103	1	0.962264	0.974359	0.962264	0.917647	0.962264
TLC_3	0.993103	0.948454	0.948454	0.974359	0.948454	0.962264	0.984615	0.993103	1	0.974359	0.962264	0.917647
TLC_4	0.984615	0.902439	0.962264	0.917647	0.962264	0.993103	1	0.974359	0.974359	0.974359	0.974359	0.974359
TLC_5	0.961538	0.9	1	0.983607	0.983607	1	0.933333	0.9	0.983607	0.983607	0.9	0.983607
TLC_6	0.966292	0.931507	0.911765	0.98	0.911765	0.931507	0.931507	1	0.95	0.911765	0.931507	0.95
TLC_7	0.974359	0.974359	0.933333	0.917647	0.933333	0.962264	0.974359	1	0.993103	0.962264	0.962264	0.984615
TLC_8	0.917647	0.917647	1	0.933333	0.974359	0.917647	0.962264	0.948454	0.974359	1	0.993103	0.993103
TLC_9	0.933333	0.962264	0.962264	0.974359	0.993103	0.974359	0.984615	0.917647	0.933333	0.974359	1	1
TLC_{10}	0.974359	1	0.984615	0.962264	0.984615	1	0.993103	0.962264	0.917647	0.962264	0.974359	0.974359
TLC_{11}	0.962264	0.993103	0.948454	1	0.962264	0.962264	0.962264	0.974359	0.974359	0.984615	0.962264	0.962264
TLC_{12}	0.948454	0.974359	0.993103	0.993103	0.974359	0.917647	1	1	0.962264	0.993103	0.917647	1
TLC_{13}	0.993103	0.962264	1	0.962264	0.917647	0.917647	0.974359	0.974359	0.984615	0.974359	0.902439	0.993103
TLC_{14}	1	0.948454	0.974359	0.917647	0.974359	0.933333	0.933333	0.974359	0.993103	0.917647	0.948454	0.948454
TLC_{15}	0.917647	0.902439	0.917647	0.974359	0.974359	0.962264	0.917647	0.933333	0.974359	1	0.917647	0.974359
TLC_{16}	0.933333	0.917647	0.962264	0.974359	0.962264	0.933333	0.917647	0.974359	1	0.993103	0.993103	0.962264
TLC_{17}	0.917647	0.933333	0.974359	1	0.962264	0.974359	0.962264	0.917647	0.933333	0.917647	1	0.917647
TLC_{18}	0.962264	0.974359	1	0.993103	0.962264	0.933333	0.917647	0.962264	0.948454	0.902439	0.974359	0.933333

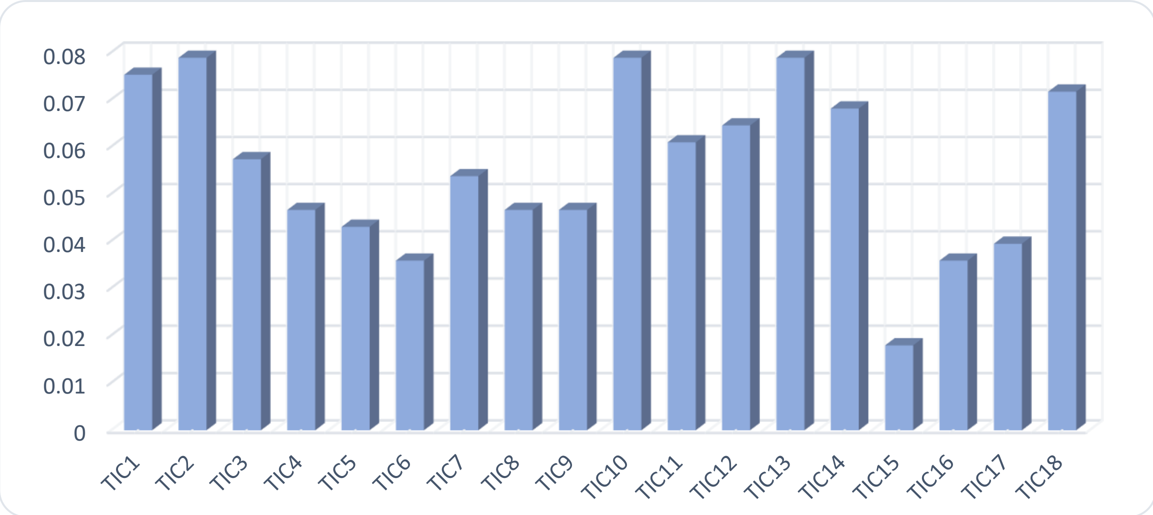

Figure 3. The weights of supplier selection criteria.

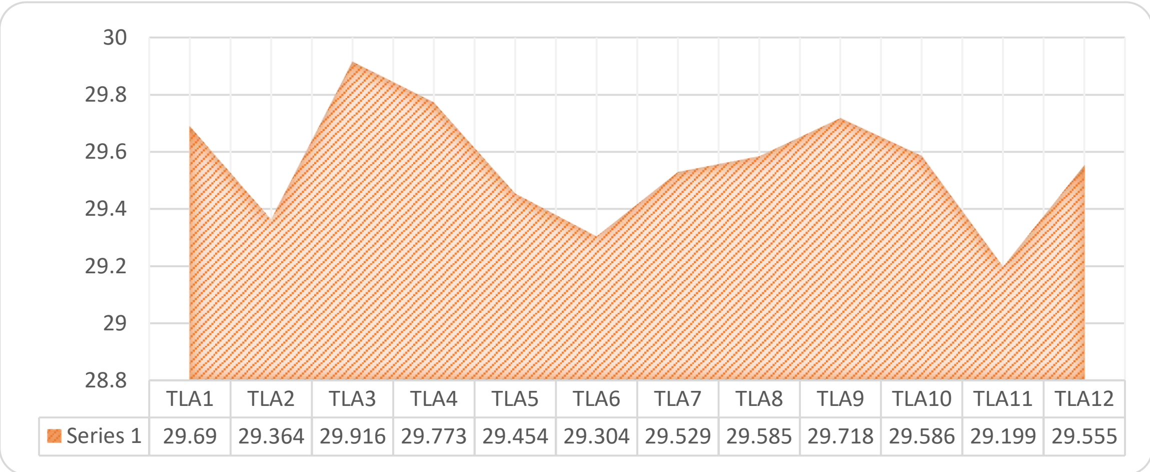

Figure 4. The value of S_i.

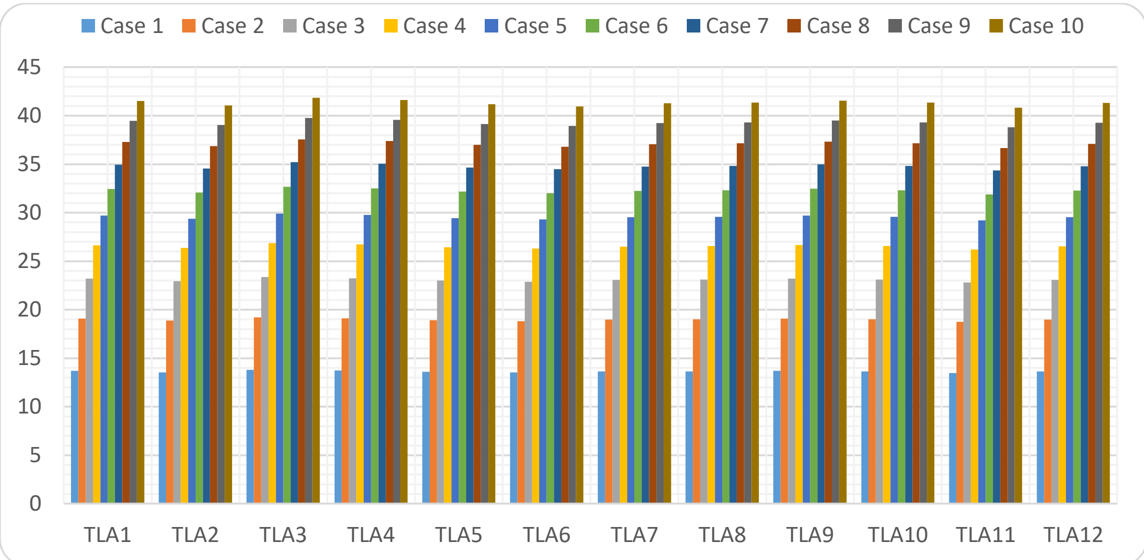

Figure 5. The rank of alternatives under β values.

7. Results Discussion

Selecting and managing sustainable and resilient suppliers are crucial in creating resilient and environmentally responsible supply chains. By considering the criteria discussed in this paper, organizations can make informed decisions that promote sustainability, mitigate risks, and enhance overall supply chain resilience. Environmental sustainability is a fundamental criterion, emphasizing the importance of suppliers' eco-friendly practices and commitment to reducing their ecological footprint. Social responsibility criteria ensure that suppliers uphold fair labour practices, human rights, and ethical treatment of employees, contributing to a socially sustainable supply chain. Supply chain transparency enables organizations to identify potential risks, vulnerabilities, and opportunities for improvement, fostering proactive risk management and resilience. By applying the proposed model the results show that:

- Integrating sustainability and resilience criteria into the selection and management processes, organizations can identify suppliers that align with their goals and contribute to long-term sustainability and resilience. The findings from these processes guide decision-making, facilitate risk management, and promote collaboration, ultimately leading to more sustainable, resilient, and responsible supply chains.
- The weights of the criteria are computed. Then, the alternatives are ranked. The results show that alternative 3 is the best and alternative 11 is the worst.
- By applying the MCDM to deal with various criteria and DNMA method to rank the alternatives; the proposed method can give more accurate results.
- The proposed method can introduce more simple and flexible method.
- The proposed method can handle any number of criteria and alternatives that give more reliability for results.

8. Conclusions

Sustainable and resilient supplier selection and management are essential for organizations striving to build environmentally responsible and resilient supply chains. This study aims to provide an overview of the requirements and criteria for selecting and managing sustainable and resilient suppliers and emphasizes the significance of integrating sustainability and resilience principles throughout the supply chain. Multi-criteria decision-making (MCDM) is used to deal with various criteria. The double normalization-based multi-aggregation (DNMA) method ranks the alternatives. This study used 18 criteria and 12 alternatives to select the best one. The weights of the criteria are computed. Then, the alternatives are ranked. The results show that alternative 3 is the best and alternative 11 is the worst. By employing these criteria, organizations can identify suppliers that align with sustainability and resilience goals, ensuring the long-term viability of the supply chain.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Neutrosophic Insights into Military Interventions: Assessing Legitimacy and Consequences in International Law

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Abstract: This scientific paper analyzes the legitimacy of military interventions within the framework of international law and their potential consequences. It highlights the need to support these interventions with robust legal and moral reasoning due to their complexity and controversy in the international community. The interpretation of legal and ethical principles can be subjective and lead to disagreements among states and international actors. The consequences of military interventions are explored, ranging from loss of life and infrastructure destruction to population displacement, political instability, and humanitarian crises. Legality and proportionality in interventions are essential to ensuring their legitimacy, and the potential consequences must be carefully assessed before undertaking an intervention. Adherence to the international legal framework is crucial to prevent military interventions from being deemed violations of international law. The neutrosophic DEMATEL methodology used in the study identifies cause-and-effect relationships among key criteria related to military actions in the realm of international law. The article highlights the importance of carefully considering the legitimacy and potential consequences of military interventions within the context of international law.

Keywords: Neutrosophic DEMATEL; Criteria; Cause and Effect; International Law; Military Interventions.

1. Introduction

The legitimacy of military interventions is often a subject of debate and controversy. Military interventions can lead to unforeseen consequences, such as the escalation of conflict, destruction of infra-structure, and loss of lives. Therefore, any military intervention must be carefully considered and supported by robust legal and moral reasons. The international community often discusses the legitimacy of such interventions and the need to balance state sovereignty with the protection of human rights.

International law establishes the fundamental principle of state sovereignty, prohibiting the use of force in international relations unless it is in legitimate self-defense or authorized by the United Nations Security Council. This is done to maintain international peace and security and prevent un-wanted armed conflicts [1].

However, in some cases, it has been argued that military interventions can be justified under certain circumstances, such as when a state fails to fulfill its obligation to protect its population from mass atrocities, such as genocide. Here, the principle of the Responsibility to Protect (R2P) plays a relevant role in international law, allowing for limited and proportionate interventions to prevent massive human suffering [2].

The legitimacy of military interventions is also related to respect for human rights. When severe and systematic violations of human rights occur in a country, it can be argued that intervention is necessary to stop human suffering and protect the fundamental rights of the population. However, the interpretation of these principles can be subjective and lead to disagreements among states and inter-national actors.

Military interventions in the context of international law are regulated by a legal framework that seeks to maintain international peace and security, as well as protect the sovereignty of states and human rights [3]. In this context, the following aspects are crucial:

- Legal Justification: Military intervention can be considered legitimate if it is supported by international law. Article 2(4) of the United Nations Charter prohibits the use of force in international relations unless in self-defense or authorized by the United Nations Security Council. Security Council resolutions can authorize military interventions in situations threatening international peace and security.
- Self-defense: A state has the right to use military force in self-defense when attacked or facing an imminent threat of armed attack. This is a fundamental principle of international law.
- Consent of the Affected State: In some cases, military intervention may be considered legitimate if the affected state requests military assistance from another state or an international coalition to address a crisis or internal conflict. The consent of the affected state is an important element for legitimacy.
- Responsibility to Protect (R2P): The Responsibility to Protect is an international norm that holds the international community responsible for intervening when a state cannot or is un-willing to protect its population from mass atrocities, such as genocide, crimes against humanity, or ethnic cleansing.
- Moral and Ethical Justification: In addition to legal justification, military interventions can also be evaluated from a moral and ethical perspective. Some argue that military intervention can be legitimate if it is the only way to stop massive human suffering or prevent atrocities.
- Legality and Proportionality: Any use of military force must be legal and proportional. This means it must comply with international law and must not cause unnecessary or disproportionate harm.
- Evaluation of Consequences: Before undertaking military intervention, states must carefully assess potential consequences, including humanitarian and political aspects, and consider whether intervention is the most appropriate option to address the situation.

It is important to note that military interventions without authorization from the UN Security Council or without a clear case of legitimate self-defense may be considered violations of international law. The consequences of unauthorized interventions can include international sanctions and criticism from the international community [4]. Therefore, respecting the international legal framework is crucial to ensure that military interventions are legal and legitimate [5].

Military interventions, whether carried out by a single state or a coalition of states, can have a range of consequences [6-14]. Some of these consequences can be complex and long-lasting, varying depending on the nature and purpose of the intervention, as well as how it is conducted. Some possible consequences of military interventions include [7, 8]:

- Loss of lives: Military interventions often involve the use of force, resulting in the loss of human lives, both military personnel and civilians. Armed conflicts can lead to significant casualties.
- Destruction of infrastructure: Military engagements can cause damage to civil infrastructure, including roads, bridges, hospitals, schools, and water and energy supply systems. This can have a long-term impact on a country's resilience.
- Population displacement: Military interventions can force people to leave their homes, often resulting in internal displacement or refugees. This can lead to humanitarian crises.

- Political instability: Military interventions can disrupt the political balance in a country, leading to instability and power struggles. This can prolong or worsen the conflict.
- Sectarianism and polarization: Military interventions can sometimes exacerbate existing ethnic, religious, or political tensions in a country, leading to increased sectarianism and polarization.
- Economic damage: War and resulting instability can cause significant harm to a country's economy, negatively impacting the lives of its inhabitants.
- Humanitarian crises: Military interventions can result in humanitarian crises, including a lack of food, water, and medical care, as well as exposure to diseases.
- Radicalization and terrorism: Military interventions can sometimes increase the recruitment of extremist groups and lead to the emergence of terrorist groups as some individuals radicalize in response to foreign military presence.
- Long-term impact on the region: Military interventions can also have lasting effects on the surrounding region, including the displacement of refugees to neighboring countries and the exacerbation of regional conflicts.
- Political and diplomatic repercussions: Military interventions can have repercussions on international relations, affecting diplomatic and political relationships between the involved countries.

The practice of military interventions has shown that they can have significant consequences, including loss of lives, destruction of infrastructure, and the creation of internally displaced persons and refugees. Therefore, it is crucial that any military intervention is supported by solid legal and moral reasons and is carried out proportionally with a focus on long-term reconstruction and peace consolidation.

It is important to note that the consequences of military interventions can vary widely depending on the nature of the conflict, how the intervention is conducted, and other factors. For this reason, military interventions are often subject to intense debates and evaluations before, during, and after their execution, aiming to minimize negative impacts and promote long-term stability and peace.

In summary, military interventions in International Law are a complex issue that involves a delicate balance between respecting state sovereignty, protecting human rights, and preserving international peace and security. The international community often faces the challenge of balancing these principles in a world where humanitarian crises and armed conflicts pose significant ethical and legal dilemmas. The work presented here analyzes, through multicriteria methods, the legitimacy and consequences of military interventions in the context of International Law.

2. Preliminaries

Definition 1. Suppose we have a nonempty space (or set) denoted as X, which is part of a larger universe of discourse called U. Let <A> represent an element, which could be a concept, attribute, idea, proposition, theory, etc., defined within the set X. Through a process called neutrosophication, we divide the set X into three distinct regions: two opposing ones <A> and <antiA>, and a neutral (indeterminate) region <neutA> positioned between them. These regions may or may not overlap, depending on the specific application, but they collectively cover the entire space.

A NeutroAlgebra is an algebraic structure that incorporates at least one NeutroOperation or one NeutroAxiom. A NeutroOperation is an operation that yields true results for some elements, indeterminate outcomes for others, and false results for yet another group of elements. This NeutroAlgebra concept is an extension of the Partial Algebra, which is an algebra that features at least one Partial Operation while all its Axioms are entirely true (classical axioms).

Definition 2. In the field of mathematics, a function denoted as f: X→Y is called a Partial Function when it exhibits a clear and precise behavior for certain elements within the set X while remaining undefined for all remaining elements in X. Consequently, there are particular elements denoted as 'a' within X for which the function f(a) is well-defined, and for all other elements, denoted as 'b' within X, the function f(b) remains undefined.

Definition 3. A function $f: X \rightarrow Y$ is called a NeutroFunction if it has elements in X for which the function is well-defined {degree of truth (T)}, elements in X for which the function is indeterminate {degree of indeterminacy (I)}, and elements in X for which the function is outer-defined {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the (Total) Function, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiFunction. Classification of Functions [9] (Figure 1).

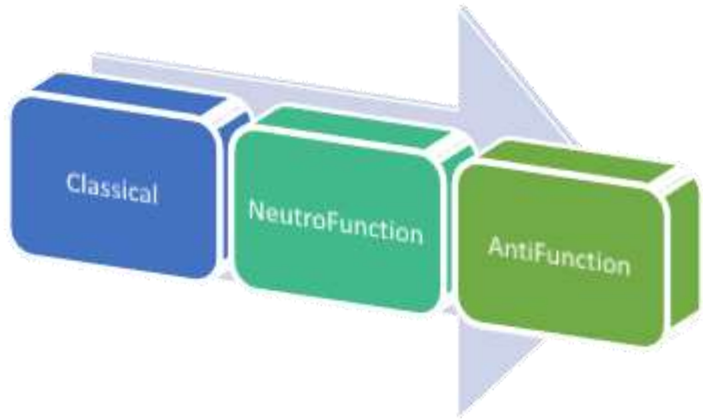

Figure 1. Classification of functions (Source: own elaboration).

Definition 4. A (classical) Algebraic Structure (or Algebra) is an unoccupied collection A that is equipped with certain (completely well-defined) operations (functions) on A, and it adheres to particular (classical) axioms (completely valid) - as per Universal Algebra.

Definition 5. A (classical) Partial Algebra is an algebra established on an unoccupied set PA that is furnished with some partial operations (or partial functions: partly well-defined, and partly undefined). However, the axioms (laws) established for a Partial Algebra are entirely (100%) true.

Definition 6. A NeutroAxiom (or Neutrosophic Axiom) defined on a nonempty set is an axiom that is true for some set of elements {degree of truth (T)}, indeterminate for other set of elements {degree of indeterminacy (I)}, or false for the other set of elements {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the (classical) Axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.
A (classical) Algebra is a nonempty set CA that is endowed with total operations (or total functions, i.e., true for all set elements) and (classical) Axioms (also true for all set elements). A NeutroAlgebra (or NeutroAlgebraic Structure) is a nonempty set NA that is endowed with at least one NeutroOperation (or NeutroFunction), or one NeutroAxiom that is referred to the set (partial-, neutro-, or total-) operations. An AntiAlgebra (or AntiAlgebraic Structure) is a nonempty set AA that is endowed with at least one AntiOperation (or AntiFunction) or at least one AntiAxiom.
Additionally, the PROSPECTOR function is defined in the MYCIN expert system in the following way: it is a mapping from $[-1, 1]^2$ into $[-1, 1]$ with formula:

$$P(x,y) = \frac{x+y}{1+xy} \quad (1)$$

This function is a uninorm, with neutral element 0, thus it fulfills commutatively, associativity, and monotonicity. Here we respect the condition that $P(-1,1)$ and $P(1,-1)$ are undefined.
Otherwise, for convenience $P(x,y)$ is extended to $\bar{P}(x,y)$ such that:
$\bar{P}(x,y) = P(x,y)$ for all $(x,y) \in [-1,1]^2 \setminus \{(-1,1),(1,-1)\}$,
$\bar{P}(-1,1) = \bar{P}(1,-1) = undefined$,
$\bar{P}(undefined, undefined) = undefined$.
$\bar{P}(undefined, x) = \bar{P}(x, undefined) = \begin{cases} \text{undefined, if x} > 0 \\ \text{x, if x} \leq 0 \end{cases}$.

Definition 7. Let S be a finite set defined as $S = \left\{(x,y): x, y \in \left\{\frac{k}{10}, undefined\right\},\ k \in \mathbb{Z} \cap [-10,10]\right\}$.
The operator $\odot$ is defined for every $(x,y) \in S$, such that:

- If $\bar{P}(x,y)$ is not undefined, then $\odot\, y = \frac{round(\bar{P}(x,y)*10)}{10}$, where *round* is the function that outputs the integer nearest to the argument.
- If $\bar{P}(x,y)$ is undefined then $x \odot y = undefined$.

Then $\odot$ is a finite NeutroAlgebra. This is because $\odot$ is commutative and associative for the subset of elements of S without any undefined component, but it is not associative otherwise.
E.g., if $a = -0.9$, $b = 0.8$, $c = undefined$, then $a \odot (b \odot c) = a$ and $(a \odot b) \odot c = -0.4 \neq a$, therefore associativity is a NeutroAxiom.
Function *round* is used for guarantying $\odot$ is an inner operator.
In this case, Cayley tables are used to generate data on the same scale as the input data. This is achieved by multiplying these elements by 10, allowing the obtainment of input values in a range between -10 and 10. Table 1 shows the results of this operation.

2.1 Neutrosophic DEMATEL using single-valued neutrosophic sets

Definition 8. Let X be a space of points (objects) with generic elements in X denoted by x. A single-valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $TA(x)$, indeterminacy-membership function $IA\,(x)$, and falsity membership function $FA\,(x)$. Then, an SVNS A can be denoted by $A = \{x, TA(x), IA(x), FA(x)\ x \in X\}$, where $TA\,(x), IA\,(x), FA\,(x) \in [0,1]$ for each point x in X. Therefore, the sum of $TA\,(x), IA\,(x)$ and $FA\,(x)$ satisfies the condition $0 \leq TA\,(x) + IA(x) + FA(x) \leq 3$.

Definition 9. Let $Ek = (T_k, I_k, F_k)$ be a neutrosophic number defined for the rating of the k-th decision-maker. Then, the weight of the k-th decision-maker can be written as:

$$\psi_k = \frac{1-\sqrt{[(1-T_k(x))^2+(I_k(x))^2+(F(x))^2]/3}}{\sum_{k=1}^{p}\sqrt{[(1-T_k(x))^2+(I_k(x))^2+(F(x))^2]/3}} \quad (2)$$

Further, in achieving a favorable solution, group decision-making is important in any decision-making process. In the group decision-making process, all the individual decision-maker assessments need to be aggregated into one aggregated neutrosophic decision matrix. This can be done by employing a single-valued neutrosophic weighted averaging (SVNWA) aggregation operator proposed by Ye [10-13].

Definition 10. [10] Let $D^{(k)}=(d_{ij}^{(k)})_{mxn}$ be the single-valued neutrosophic decision matrix of the k-th decision maker and $\psi = (\psi_1 \psi_2, \dots, \psi_p)^T$ be the weight vector of decision maker such that each $\psi_k \in [0,1], D = (d_{ij})_{mxn}$ where

$$d_{ij} = \langle 1 - \prod_{k=1}^{p} \left(1 - T_{ij}^{(p)}\right)^{\psi_k}, \prod_{k=1}^{p} \left(I_{ij}^{(p)}\right)^{\psi_k}, \prod_{k=1}^{p} \left(F_{ij}^{(p)}\right)^{\psi_k} \rangle \quad (3)$$

Table 1. Cayley's table of multiplication by 10. ⊙. (Source: own elaboration).

$x \odot y$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	I	1	2	3	4	5	6	7	8	9	10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	**I**
-9	-10	-10	-10	-10	-10	-10	-10	-9	-9	-9	-9	-9	-9	-9	-8	-8	-7	-7	-5	-4	0	10
-8	-10	-10	-10	-10	-9	-9	-9	-9	-9	-8	-8	-8	-8	-7	-7	-6	-5	-4	-2	0	4	10
-7	-10	-10	-10	-9	-9	-9	-9	-8	-8	-7	-7	-7	-6	-6	-5	-4	-3	-2	0	2	5	10
-6	-10	-10	-9	-9	-9	-8	-8	-8	-7	-7	-6	-6	-5	-5	-4	-3	-1	0	2	4	7	10
-5	-10	-10	-9	-9	-8	-8	-8	-7	-6	-6	-5	-5	-4	-3	-2	-1	0	1	3	5	7	10
-4	-10	-10	-9	-9	-8	-8	-7	-6	-6	-5	-4	-4	-3	-2	-1	0	1	3	4	6	8	10
-3	-10	-9	-9	-8	-8	-7	-6	-6	-5	-4	-3	-3	-2	-1	0	1	2	4	5	7	8	10
-2	-10	-9	-9	-8	-7	-6	-6	-5	-4	-3	-2	-2	-1	0	1	2	3	5	6	7	9	10
-1	-10	-9	-8	-7	-7	-6	-5	-4	-3	-2	-1	-1	0	1	2	3	4	5	6	8	9	10
I	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
0	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	0	1	2	3	4	5	6	7	8	9	10
1	-10	-9	-8	-6	-5	-4	-3	-2	-1	0	1	**I**	2	3	4	5	6	7	7	8	9	10
2	-10	-9	-7	-6	-5	-3	-2	-1	0	1	2	**I**	3	4	5	6	6	7	8	9	9	10
3	-10	-8	-7	-5	-4	-2	-1	0	1	2	3	**I**	4	5	6	6	7	8	8	9	9	10
4	-10	-8	-6	-4	-3	-1	0	1	2	3	4	**I**	5	6	6	7	8	8	9	9	10	10
5	-10	-7	-5	-3	-1	0	1	2	3	4	5	**I**	6	6	7	8	8	8	9	9	10	10
6	-10	-7	-4	-2	0	1	3	4	5	5	6	**I**	7	7	8	8	8	9	9	9	10	10
7	-10	-5	-2	0	2	3	4	5	6	6	7	**I**	7	8	8	9	9	9	9	10	10	10
8	-10	-4	0	2	4	5	6	7	7	8	8	**I**	8	9	9	9	9	9	10	10	10	10
9	-10	0	4	5	7	7	8	8	9	9	9	**I**	9	9	9	10	10	10	10	10	10	10
10	**I**	10	10	10	10	10	10	10	10	10	10	**I**	10	10	10	10	10	10	10	10	10	10

Definition 11. Deneutrosophication of SVNS Ñ can be defined as a process of mapping Ñ into a single crisp output $f: \tilde{N} \rightarrow \psi^*$ for $\in X$. If Ñ is a discrete set, then the vector of tetrads $\tilde{N} = \{(x \mid T\tilde{N}(x), I\tilde{N}(x), F\tilde{N}(x)) \mid x \in X\}$ is reduced to a single scalar quantity $\psi * \in X$ by deneutrosophication. The obtained scalar quantity $\psi * \in X$ best represents the aggregate distribution of three membership degrees of neutrosophic element $T\tilde{N}(x), I\tilde{N}(x), F\tilde{N}(x)$. Therefore, deneutrosophication can be obtained as follows.

$$\psi^* = 1 - \sqrt{[(1 - T_k(x))^2 + (I_k(x))^2 + (\mathrm{F}(x))^2]/3} \quad (4)$$

Decision-making normally involves human language, commonly referred to as linguistic variables. A linguistic variable simply represents words or terms used in human language. Therefore, this linguistic variable approach is a convenient way for decision-makers to express their assessments. Ratings of criteria can be expressed by using linguistic variables such as very influence (VI), influence (I), low influence (LI), no influence (NI), etc. Linguistic variables can be transformed into SVNSs as shown in Table 2.

Table 2. Linguistic variable and Single Valued Neutrosophic Numbers (SVNNs) [11].

Integer	Linguistic variable	SVNNs
0	No influence/Not important	(0.10,0.80,0.90)
1	Low influence/important	(0.35,0.60,0.70)
2	Medium influence/important	(0.50,0.40,0.45)
3	High influence/important	(0.80,0.20,0.15)
4	Very high influence/important	(0.90,0.10,0.10)

To carry out the DEMATEL method in its neutrosophic variant, follow the steps set out below [12] (Figure 2):

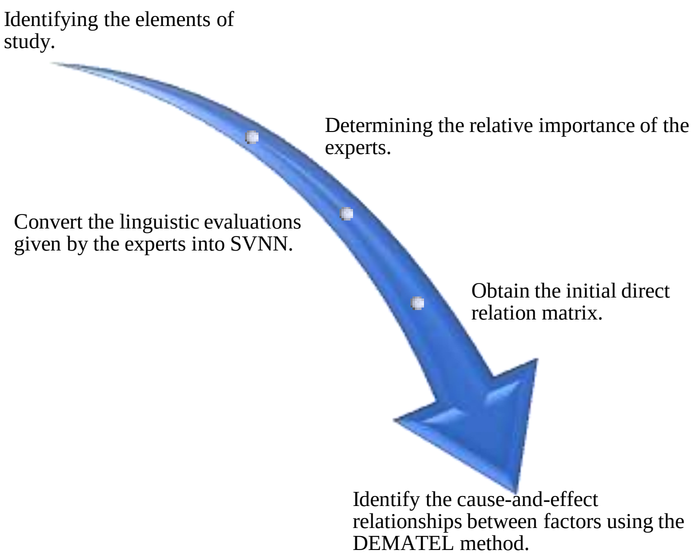

Figure 2. Steps of the neutrosophic DEMATEL method.

Through the application of semi-structured interviews to a population of interest and brainstorming, a set of influential factors in the subject under study is determined. Subsequently, experts are asked to assess the direct influence between factors through paired comparisons, using the scoring shown in Table 2.

The group of experts has their values of importance based on their level of experience and knowledge in the decision problem. Therefore, the weight of each decision-maker may be different from that of other decision-makers. The weight of each decision-maker is considered with linguistic variables and transmitted into SVNN to be later identified through Eq. (2).

From the individual crisp matrices obtained from the evaluations of the experts, individual neutrosophic matrices of decision-makers are constructed according to the indications in Table 2. To obtain the initial direct relation matrix, which is in the form of crisp numbers, the neutrosophic matrices of individual decision-makers must be aggregated and deneutrosophied using Eqs. (3) and (4) respectively. Based on the aggregated direct relation matrix A obtained in step 4, the total relation matrix T can be easily calculated using Eqs. (5-7) as shown below:

$$D = A * S \tag{5}$$

Where

$$S = \frac{1}{\max\limits_{l \le i \le n} \sum_{j=1}^{n} a_{ij}} \tag{6}$$

and
T = D * (ID) - 1 (7)
where I is the identity matrix. From this, the cause-effect relationship diagram $(ri + ci, ri - ci)$ is constructed.

- Analyze the cause-effect relationship diagram. The (ri −ci) indicates the importance of each factor while (ri −ci) is the net cause or effect group. The (ri + ci) is called "Prominence" and it measures the degree of the central role that the factor or criterion plays within the system.

3. Results

The analysis of international law and armed conflict is a complex task involving a series of key criteria and considerations. After collaborating with experts and examining reference documentation, the following criteria are chosen for analysis:

- Justification of the use of force: Evaluate whether the use of force in an armed conflict is consistent with international law, including the United Nations Charter and resolutions of the UN Security Council. The legality of the use of force may depend on factors such as self-defense, authorization from the UN Security Council, or the consent of the state in whose territory the conflict is taking place.
- Respect for the principle of proportionality: Analyze whether military actions are proportionate to the objectives pursued and whether damage to civilians and civilian infrastructure is minimized. Proportionality is a fundamental principle of humanitarian law.
- Protection of civilians: Verify whether the parties in conflict are fulfilling their obligation to distinguish between combatants and civilians and whether they are taking measures to protect civilians from unnecessary harm.
- Regional stability: International law seeks to promote peace and stability in world regions while providing a normative framework to address and resolve armed conflicts when they arise.
- International criminal responsibility: Consider whether serious violations of international humanitarian law and human rights are being investigated and whether those responsible are being brought to justice, either at the national level or through international tribunals.
- Humanitarian assistance: Evaluate whether safe and unrestricted access of humanitarian organizations to conflict-affected areas is allowed and whether humanitarian assistance is provided impartially to those in need.
- Rights of refugees and internally displaced persons: Consider whether the rights of people displaced by the conflict, including their right to seek asylum and their right to dignified and humanitarian treatment, are being respected.
- Application of sanctions and arms embargoes: Assess whether international sanctions and arms embargoes are being properly applied in conflicts to prevent the flow of arms to the parties involved.

These are some of the key criteria that can be relevant for a comprehensive analysis of international law and armed conflict. Evaluating an armed conflict based on these criteria can help determine whether the parties involved are fulfilling their legal and ethical obligations within the framework of international law.

Considering these elements, the DEMATEL methodology is employed to detect possible cause-and-effect relationships among these components. The result of this analysis simplifies the focus of the upcoming interview on the topics that have a more significant impact and relevance, which are the true triggers of the crisis, as indicated by the selected experts. In this process, a team of 5 specialists is involved.

Subsequently, a questionnaire is implemented targeting a group of professionals with experience in the field of study. The interviews conducted with these individuals use linguistic variables, contributing to a better understanding of the data and allowing for a more precise assessment by the participants.

Specifically, a group of 56 officials was chosen to respond to the designed study questions. Each of them is asked to rate the statements presented using a positive scale, assigning up to 10 points if they have a favorable opinion on the analyzed topic. Conversely, if they have an unfavorable opinion, they are to rate on a scale ranging from -10 to -1.

The notation $\mathrm{v_{ij}}$, where $i = 1,2,\ldots,56$; $j = 1,2,\ldots,n$ represents the evaluation of the i-th official on the j-th aspect.

Afterwards, the calculation of $\bar{v}_i = \left(\frac{\sum_{j=1}^{n^+} v_{ij}^+}{n^+}, \frac{\sum_{j=1}^{n^0} v_{ij}^0}{n^0}, \frac{\sum_{j=1}^{n^-} v_{ij}^-}{n^-}\right)$ is performed as follows: the positive responses of the i-th official on the j-th aspects are treated as neutral responses, resulting in $\frac{\sum_{j=1}^{n^0} v_{ij}^0}{n^0} = 0$, and v_{ij}^- represents negative responses. Additionally, n^+, n^0, and n^- represent the numbers of positive, neutral, and negative responses, respectively. This novel approach ensures greater precision in the results than a simple arithmetic mean calculation. Subsequently, the calculation of $\hat{v}_i = round\left(\frac{\sum_{j=1}^{n^+} v_{ij}^+}{n^+}\right) \odot round\left(\frac{\sum_{j=1}^{n^-} v_{ij}^-}{n^-}\right)$ is performed. In cases where both $round\left(\frac{\sum_{j=1}^{n^+} v_{ij}^+}{n^+}\right) = 10$, and $round\left(\frac{\sum_{j=1}^{n^-} v_{ij}^-}{n^-}\right) = -10$, it is defined that $\hat{v}_i = -10$.

The decision-making process occurs in two different situations:

- If less than 30% of the respondents yield contradictory results for each fixed j, i.e., if there are 30 pairs or fewer of values of $(-10,10)$ or $(10,-10)$, these values are excluded for aggregation.
- Otherwise, the j-th aspect is assessed as "undefined," and a more detailed review is required to understand why such a contradiction exists.

In the first case, when aggregation is performed, $\hat{v}_i$ is calculated using the $\odot$ operator.

The implementation of the suggested approach allowed establishing the presence of a causal relationship among the originally examined elements. Thus, Table 3 provides a summary of the main elements of interest. This achieved clarity regarding the criteria evaluated concerning military actions in the realm of international law.

Table 3. Results of the application of the DEMATEL method.

Query elements	Ri + Ci	Ri - Ci
Justification of the use of force	8,572	0.106
Respect for the principle of proportionality	6,501	0.037
Protection of civilians	6.86	-0.254
International criminal responsibility	6,739	-0.021
Humanitarian assistance	7,449	0.979
Law of refugees and internally displaced persons	8.38	0.612
Application of sanctions and arms embargoes	6,094	-1,246
Justification of the use of force	5,329	-0.213

4. Discussion

As can be observed, in the studied system, the most related factors are the justification of the use of force, respect for the principle of proportionality, humanitarian assistance, and the rights of refugees and internally displaced persons. The relationship values indicate a strong connection when evaluating assessments of factors affecting military interventions.

Given the previous results, the interview with the selected officials will be strongly influenced by these four elements, allowing for a deeper exploration of causal factors whose elimination or reduction has a greater impact. In this regard, each of these four elements was broken down into five questions designed to determine the level of opinion of the interviewees. The results of the analysis showed an average value of 5. Although these results showed positivity, unfavorable responses were observed regarding humanitarian assistance and the rights of refugees and internally displaced persons.

Regarding the interview results concerning humanitarian assistance, it was noted that despite the existence of international norms and principles supporting humanitarian assistance in armed conflicts, the effectiveness and access to humanitarian aid can be hindered by various barriers. These include a lack of secure access to affected areas, a lack of cooperation from conflicting parties, and insufficient funds for assistance. To improve this, the following is proposed:

(i). Promote cooperation between conflicting parties to enable safe and unrestricted access for humanitarian organizations.

(ii). Increase funding and resources allocated to humanitarian assistance, both at the national and international levels.

(iii). Develop monitoring and accountability mechanisms to ensure that assistance reaches those in need and is used appropriately.

Regarding the rights of refugees and internally displaced persons, the discussion was expanded to address the persistent lack of access to essential services, discrimination, and the lack of guarantees for a safe return. The following strategies were proposed:

(i). Raise awareness within the international community about the importance of protecting the rights of refugees and internally displaced persons and exert pressure on conflicting parties to fulfill their legal obligations.

(ii). Facilitate the identification and registration of displaced individuals, ensuring proper documentation and access to basic services such as healthcare and education.

(iii). Promote durable solutions, such as the voluntary and safe return of internally displaced per-sons and refugees when possible, or local integration when necessary.

Both humanitarian assistance and the rights of refugees and internally displaced persons are essential elements within the framework of international law in armed conflicts. To enhance their effectiveness, it is crucial to address existing barriers and work on strategies that promote their fulfillment and protection in conflict situations.

5. Conclusions

This scientific article has explored the legitimacy of military interventions in the context of International Law and analyzed the potential consequences of such interventions. The legitimacy of military interventions is a complex and debated topic in the international community, emphasizing the importance of supporting any military intervention with strong legal and moral reasons.

The interpretation of legal and ethical principles related to military interventions can be subjective, leading to disagreements among states and international actors. Military interventions can have various consequences, ranging from loss of lives and infrastructure destruction to population displacement, political instability, and humanitarian crises.

The legality and proportionality of military interventions are crucial to ensuring their legitimacy. Potential consequences must be carefully evaluated before undertaking an intervention. Adherence to the international legal framework is essential to prevent military interventions from being considered violations of International Law, which could lead to international sanctions and criticism from the global community.

The neutrosophic DEMATEL methodology used in this study has allowed the identification of cause-and-effect relationships among key criteria related to military actions in the field of international law. In summary, this article highlights the importance of carefully considering the legitimacy and potential consequences of military interventions in the context of International Law. The complexity of this issue and the need to balance state sovereignty, protection of human rights, and preservation of international peace and security make it crucial to conduct comprehensive assessments and ongoing debates in the international community.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

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q-Rung Neutrosophic Sets and Topological Spaces

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Abstract: The concept of q-rung orthopair neutrosophic set is introduced in this paper and fundamental properties of it are studied. Also the ordinary notion of topological space is extended to q-rung orthopair neutrosophic environment, as well as the fundamental concepts of convergence, continuity, compactness and Hausdorff topological space. All these generalizations are illustrated by suitable examples.

Keywords: Fuzzy Set; Intuitionistic Fuzzy Set; Neutrosophic Set; q-Rung Orthopair Fuzzy Set; q-Rung Orthopair Neutrosophic Set; q-Rung Orthopair Neutrosophic Topological Space.

1. Introduction

Zadeh, in 1965, extended the concept of a crisp set to that of a *fuzzy set* [1], on the purpose of tackling mathematically the existing in everyday life partial truths, as well as the definitions having no clear boundaries, like “high mountains”, “clever people”, “good players”, etc. Zadeh’s idea was to replace the objective function of crisp sets with the membership function in fuzzy sets taking values in the interval [0, 1]. In this way a membership degree between 0 and 1 is assigned to each element of the universal set within the corresponding fuzzy set.

Before the introduction of fuzzy sets, probability used to be the unique mathematical tool in hands of the experts for managing the existing in real world uncertainty, caused by the shortage of knowledge about an observed phenomenon. Several types of uncertainty appear in everyday life, including *randomness, imprecision, vagueness, ambiguity, inconsistency*, etc. [2]**.** The uncertainty due to randomness is related to well-defined events whose outcomes cannot be predicted in advance, like the turn of a coin, the throwing of a die, etc. Imprecision occurs when the corresponding events are well defined, but the possible outcomes cannot be expressed with an exact numerical value; e.g. “The temperature tomorrow will be over 30^{o} C”. Vagueness is created when one is unable to clearly differentiate between two properties, like a good and a mediocre student. In case of ambiguity the existing information leads to several interpretations by different observers. For example, the phrase “Boy no girl” written as “Boy, no girl” means boy, but written as “Boy no, girl” means girl. Inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, “The chance of raining tomorrow is 80%, but this does not mean that the chance of not raining is 20%, because they might be hidden weather factors”.

Probability, however, was proved to be effective only for tackling the uncertainty due to randomness, in contrast to fuzzy sets which were proved to be suitable for tackling other forms of uncertainty as well, and in particular the uncertainty due to vagueness [3]. Following the introduction of fuzzy sets, several generalizations of them and theories related to them have been proposed on the purpose of tackling more effectively all the forms of the existing uncertainty, e.g. see [4].None of these generalizations or theories, however, was proved to be suitable for tackling all the forms of the uncertainty alone, but the synthesis of all of them forms an effective framework for this purpose.

In 1986, Atanassov expanded the definition of fuzzy set to *intuitionistic fuzzy set* by adding the degree of non-membership to Zadeh's degree of membership [5]. Intuitionistic fuzzy sets can be used everywhere the ordinary fuzzy sets can be applied, but this is not always necessary. An example where the use of intuitionistic fuzzy sets is necessary is the case of an election, where a candidate can have voted for (membership), or voted against (non-membership) by the electoral vote. The intuitionistic fuzzy sets are suitable for tackling the uncertainty due to imprecision, which appears frequently in human reasoning [6].

Yager introduced later the concept of the *q-rung orthopair fuzzy set* with q a positive integer, in which the sum of the q-th powers of the membership and non-membership degrees of the elements of the universal set is bounded by 1 [7]. When q=1 we have an intuitionistic fuzzy set, when q=2 a *Pythagorean fuzzy set* [8] and when q=3 a *Fermatean fuzzy set* [9]. It has been established that Pythagorean and Fermatean fuzzy sets have stronger ability than intuitionistic fuzzy sets for tackling the uncertainty in decision making problems [10].

In 1995, Smarandache proposed the concept of the degree of indeterminacy or neutrality and further expanded the idea of intuitionistic fuzzy set to that of *neutrosophic set* [11], drawing inspiration from the neutralities that frequently surface in everyday life, like <friend, neutral, enemy>, <win, draw, defeat>, <high, medium, short>, etc. Neutrosophic sets are effective for tackling the uncertainty due to inconsistency and ambiguity.

In 2019 the concept of Pythagorean fuzzy set was extended to neutrosophic environments [12] and in 2021 the same happened with the Fermatean fuzzy set [13].

In this work we introduce the concept of the *q-rung orthopair neutrosophic set* focusing on fundamental properties of this kind of set and on *q-rung orthopair neutrosophic topological spaces.* The rest of the paper is formulated as follows: Section 2 contains the necessary mathematical background for the understanding of the paper. The concept of the q-rung orthopair neutrosophic set is presented in Section 3 together with basic properties of these sets. In Section 4 the classical notion of topological space is extended to *q-rung orthopair neutrosophic topological spaces* together with fundamental properties and concepts like convergence, continuity, compactness and Hausdorff topological spaces. The paper closes with the final conclusions and some hints for further research included in its last Section 5.

2. Mathematical Background

The exact definition of a fuzzy set [1] is the following:

Definition 1: A fuzzy set A in the universal set of the discourse U is a set of ordered pairs of the form:

$$A = \{(x, m(x)): x \in U, m(x) \in [0, 1]\} \tag{1}$$

In Eq. (1) m: U → [0, 1] is the membership function of A, and the real value y = m(x) is the membership degree of x in A, for all x in U. The greater m(x), the better x satisfies the characteristic property of A.

The definition of the membership function of A is not unique depending on the personal goals of each observer. For example, if A is the fuzzy set of "tall men", one may consider all the men with heights greater than 1.90 m tall and another one all those with heights greater than 2 m. As a result, the first observer will assign membership degree 1 to all men with heights between 1.90 m and 2 m, whereas the second one will assign membership degrees <1 to them. The only restriction for the definition of the membership function is to be compatible with common sense; otherwise it does not give a creditable description of the real situation represented by the corresponding fuzzy set. This could happen for instance in the previous example, if men with heights less than 1.60 m had membership degrees > 0.5. Most of the properties and operations of crisp sets can be extended in a natural way to fuzzy sets; e.g. see [2].

Atanassov extended the concept of fuzzy set to that of intuitionistic fuzzy set [5] as follows:

Definition 2: An intuitionistic fuzzy set A in the universal set U is of the form

$A = \{(x, m(x), n(x)): x \in U, m(x), n(x) \in [0, 1], \quad 0 \leq m(x) + n(x) \leq 1\}$ (2)

In Eq. (2) m: U → [0, 1] is the membership function and n: U → [0, 1] is the non-membership function of A and m(x), n(x) are the degrees of membership and non-membership respectively for each x in A. For simplicity we write A = <m, n>. Further, h(x) = 1 - m(x) - n(x), is said to be the degree of hesitation of x in A. If h(x) = 0, then A is a fuzzy set.

The term intuitionistic fuzzy set is due to Atanassov's collaborator Gargov [14] in analogy to the idea of intuitionism introduced by Brouwer at the beginning of the last century [15]. It is recalled that intuitionism rejects Aristotle's law of the excluded middle by stating that a proposition is either true, or not true, or we do not know if it is true or not true. The first part of this statement corresponds to Zadeh's membership degree, the second to Atanassov's non-membership and the third to the degree of hesitation.

Example 1: Let U be the set of the students of a class, let A be the intuitionistic fuzzy set of the good students of the class. Then each student x of U is characterized by an *intuitionistic fuzzy pair* (m, n) with respect to A, with m, n in [0, 1].For example, if (x, 0.5, 0.3) is in A, then, there is a 50% belief that x is a good student, but also a 30% belief that he is not a good student and a 20% hesitation to characterize him as a good student or not.

The properties and operations of fuzzy sets can be extended to intuitionistic fuzzy sets; e.g. see [6].

When defining the membership and non-membership degrees of the elements of the universal set U, it could happen that m(x) + n(x) >1. In such cases the corresponding structure cannot be treated as an intuitionistic fuzzy set. This motivated Yager to define the wider class of q-rung orthopair fuzzy sets [7] as follows:

Definition 3: A q-rung orthopair fuzzy set A in the universal set U, where q is a positive integer, is of the fo

$A = \{(x, m(x), n(x)): x \in U, m(x), n(x) \in [0, 1], \quad 0 \leq [m(x)]^q + [n(x)]^q \leq 1\}$ (3)

In Eq. (3) m(x) is the membership and n(x) is the non-membership degree of x in A respectively.

For q = 1, a 1-rung orthopair fuzzy set is an ordinary intuitionistic fuzzy set. Further, a 2-rung orthopair fuzzy set is referred to as a Pythagorean fuzzy set [8] and a 3-rung orthopair fuzzy set is referred to as a Fermatean fuzzy set [9].

The following Proposition helps to clarify the Yager's motivation for introducing the notion of the q-rung orthopair fuzzy set:

Proposition 1: Let q_1, q_2 be positive integers, with $q_2 > q_1$. Then the set $\{(m, n): m, n \in [0, 1], 0 \leq m^{q_2} + n^{q_2} \leq 1\}$ is larger than the set $\{(m, n): m, n \in [0, 1], 0 \leq m^{q_1} + n^{q_1} \leq 1\}$.

Proof: Since m, n ∈ [0, 1], is $m^{q_2} + n^{q_2} < m^{q_1} + n^{q_1}$. Consequently, if $m^{q_1} + n^{q_1} \leq 1$, it is also $m^{q_2} + n^{q_2} \leq 1$ and the result follows.

Example 2: Let (x, 0.8, 9.7) be an element of the q-rung orthopair fuzzy set A. Then, since 0.8 + 0.7 >1, A is not an intuitionistic fuzzy set, Also, since $(0.8)^2 + (0.7)^2 = 0.64 + 0.49 > 1$, A is not a Pythagorean fuzzy set too. But $(0.8)^3 + (0.7)^3 = 0.512 + 0.343 < 1$. Thus A could be a Fermatean fuzzy set, this depending on the form of its other elements.

Proposition 2: Let A be q_1-rung and B be q_2-rung orthopair fuzzy sets respectively, with $q_2 > q_1$. Then A is also a q_2-rung orthopair fuzzy set.

Proof: Let x(m, n) be an element of A. Then $0 \leq m^{q_1} + n^{q_1} \leq 1$, with m, n ∈ [0, 1]. But $q_2 > q_1$, therefore, $0 \leq m^{q_2} + + n^{q_2} \leq m^{q_1} + n^{q_1} \leq 1$ and the result follows.

In particular, an intuitionistic fuzzy set is a Pythagorean fuzzy set, which is a Fermatean fuzzy set.

The simplest form of a neutrosophic set is the *single valued neutrosophic set*, which is defined as follows [16]:

Definition 4: A single valued neutrosophic set A in the universe U is of the form

$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m(x)+i(x)+n(x) \leq 3\}$ (4)

In Eq. (4) m(x), i(x), n(x) are the degrees of membership (or truth), indeterminacy (or neutrality) and non-membership (or falsity) with respect to A, for all x in U, referred to as the neutrosophic components of x. For simplicity, we write A = <m, i, n>.

The etymology of the term "neutrosophy" comes from the adjective "neutral" and the Greek word "sophia" (wisdom) and, according to Smarandanche, who introduced it, means the "knowledge of the neutral thought".

Example 3: Let U be the set of the players of a football team and let A be the single valued neutrosophic set of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (m, i, n) with respect to A, with m, i, n in [0, 1]. For example, x(0.6, 0.2, 0.4) ∈ A, means that there is a 60% belief that x is a good player, but simultaneously a 20% doubt about and a 40% belief that x may not be a good player. In particular, x(0,1,0) ∈ A means that we do not know absolutely nothing about x's affiliation with A.

The concepts and operations defined on intuitionistic fuzzy sets can be extended in a natural way to neutrosophic sets [11].

Remark 1:

(i). Indeterminacy is understood to be everything which is between the opposites of truth and falsity [17]. In an intuitionistic fuzzy set the indeterminacy is equal by default with the hesitancy, i.e. we have i(x)=1- m(x) – n(x). Also, in a fuzzy set is i(x) = 0 and n(x) = 1 – m(x), whereas in a crisp set it is m(x) =1 (or 0) and n(x)= 0 (or 1). In other words, crisp sets, fuzzy sets and intuitionistic fuzzy sets are special cases of single valued neutrosophic sets.

(ii). When the sum m(x) + i(x) + n(x) of the neutrosophic components of x ∈ U in a neutrosophic set in U is < 1, then it leaves room for incomplete information about x, when it is equal to 1 for complete information and when is greater than 1 for inconsistent (i.e. contradiction tolerant) information about x. A single valued neutrosophic set may contain simultaneously elements leaving room for all the previous types of information.

(iii). When m(x) + i(x) + n(x) < 1, ∀ x ∈ U, then the corresponding single valued neutrosophic set is usually referred to as *picture fuzzy set* [18]. In this case 1- m(x) -i(x) -n(x) is called the degree of refusal membership of x in A. The picture fuzzy sets based models are suitable for describing situations where we face human opinions involving answers of types yes, abstain, no and refusal to express an opinion. Voting is a representative example of such a situation.

(iv). The difference between the general definition of a neutrosophic set and of the previously given definition of a single valued neutrosophic set is that in the general definition m(x), i(x) and n(x) may take values in the non-standard unit interval]−0, 1+[including values < 0 or > 1. This is something that can happen in everyday life situations; e.g. see an example, in [11].

3. Extending the Concept of Orthopair Fuzzy Set to Neutrosophic Environment

The concept of Pythagorean fuzzy set was extended to neutrosophic environment [12] as follows:

Definition 5: A Pythagorean neutrosophic set A in the universe U is of the form:

$$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^2(x) + n^2(x) \leq 1\} \quad (5)$$

In Eq. (5) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. Since $0 \leq m^2(x)+n^2(x) \leq 1$, the neutrosophic components m(x) and n(x) are dependent and the component i(x) is independent.

Proposition 3: Let A be a Pythagorean neutrosophic set, then $0 \leq m^2(x) + i^2(x) + n^2(x) \leq 2$, for all x in U.

Proof: Since $i(x) \in [0, 1]$, is $0 \leq i^2(x) \leq 1$ and the result follows by $0 \leq m^2(x) + n^2(x) \leq 1$.

Remark 2: By Corollary 1 an intuitionistic fuzzy set is a Pythagorean fuzzy set too. A neutrosophic set, however, need not be a Pythagorean neutrosophic set. For example, let A be a neutrosophic set and let x(0.8, 0.3. 0.7) be an element of A. Then $(0.8)^2 + (0.7)^2 = 0.64 + 0.49 > 1$, therefore A is not a Pythagorean neutrosophic set.

The concept of Fermatean neutrosophic set has also been defined [13] as follows:

Definition 6: A Fermatean neutrosophic set A in the universe U is of the form

$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^3(x) + n^3(x) \leq 1\}$ (6)

In Eq. (6) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. The components m(x) and n(x) are dependent and the component i(x) is independent. Similarly with Proposition 3 it can be shown that $0 \leq m^3(x) + i^3(x) + n^3(x) \leq 2$, for all x in U.

Pythagorean and Fermatean neutrosophic sets have found some interesting applications; e.g. see [12, 19, 20-23], etc.

Here we extend the concept of a Pythagorean (Fermatean) neutrosophic set as follows:

Definition 7: A *q-rung orthopair neutrosophic set* A, with q a positive integer, is of the form

$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^q(x) + n^q(x) \leq 1\}$ (7)

In Eq. (7) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. The components m(x) and n(x) are dependent and the component i(x) is independent. Similarly with Proposition 3 it can be shown that $0 \leq m^q(x) + i^q(x) + n^q(x) \leq 2$, for all x in U. For simplicity we write $A = <m, i, n>$.

A 1-rung orthopair neutrosophic set is referred to as *intuitionistic neutrosophic set*; e.g. see Table 1of [19]. Also, for q = 2 we have a Pythagorean neutrosophic set and for q = 3 a Fermatean neutrosophic set. Following the proof of Proposition 2 one can show:

Proposition 4: Let A be q_1-rung and B be q_2-rung orthopair neutrosophic sets respectively, with $q_2 > q_1$, then A is also a q_2-rung orthopair neutrosophic set.

Proof: The same with the proof of Proposition 2.

In particular, an intuitionistic neutrosophic set is a Pythagorean neutrosophic set, which is a Fermatean neutrosophic set.

The classical operations on crisp sets can be generalized for q-rung orthopair neutrosophic sets. Here we define the subset and the complement of a q-rung orthopair neutrosophic set, as well as the union and intersection of two such sets.

Definition 8: Let $A = <m_A, i_A, n_A>$ and $B = <m_B, i_B, n_B>$ be two q-rung orthopair neutrosophic sets in the universe U. Then:

i. A is called a *subset* of B ($A \subseteq B$), if, and only if, $m_A(x) \leq m_B(x)$, $i_A(x) \leq i_B(x)$ and $n_A(x) \geq n_B(x)$, $\forall x \in U$. If we have simultaneously $A \subseteq B$ and $B \subseteq A$, then A and B are called *equal* q-rung orthopair neutrosophic sets (A=B).
ii. The *complement* of $A = <m_A, i_A, n_A>$ is the q-rung orthopair neutrosophic set $A^C = <n_A, 1-i_A, m_A>$ in U.
iii. The *union* $A \cup B$ is the q-rung orthopair neutrosophic set $C = <m_C, i_C, n_C>$ in U with $m_C = \max\{m_A, m_B\}$, $i_C = \max\{i_A, i_B\}$ and $n_C = \min\{n_A, n_B\}$.
iv. The *intersection* $A \cap B$ is the q-rung orthopair neutrosophic set $D = <m_D, i_D, n_D>$ in U with $m_D = \min\{m_A, m_B\}$, $i_C = \min\{i_A, i_B\}$ and $n_C = \max\{n_A, n_B\}$.

Remark 3:

(i). It is easy to check that all the above relations are well defined. For the union, for example, set $m = \max\{m_A, m_B\}$ and $n = \min\{n_A, n_B\}$. If $m = m_A$ and $n = n_B$, then $0 \leq m^q + n^q = (m_A)^q + (n_B)^q \leq (m_A)^q + (n_A)^q \leq 1$. In an analogous way one can show that we always have $m^q + n^q \leq 1$ for all the other possible combinations, which means that $A \cup B$ is a q-rung orthopair neutrosophic set.

(ii). When A and B are crisp sets, it is straightforward to check that the previous definitions are reduced to the corresponding ordinary definitions for crisp sets.

(iii). With the help of the previous definitions it is straightforward to check that most of the laws and properties of crisp sets are also true for q-rung orthopair neutrosophic sets, like the commutative and associative laws for the union and intersection, the distributive law of the union with respect to intersection and vice versa, the double complement property $(A^C)^C = A$, etc.

Example 4: Let U= {x_1, x_2, x_3} be the universal set and let A={(0.3,0.3,0.6,x_1),(0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)}and B={(0.6,0.1,0.2,x_1), (0.3,0.2,0.5, x_2), (0.3,0.1,0.6,x_3)}be two q-rung orthopair neutrosophic set in U, q>1. Then:

(i). Neither $A \subseteq B$, nor $B \subseteq A$

(ii). c(A)={(0.6,0.7,0.3,x_1),(0.4,0.7,0.5,x_2), (0.5,0.8, 0.7,x_3)} and c(B)={(0.2,0.9,0.6,x_1),0.5,0.8,0.3,x_2) (0.6,0.9,0.3..,x_3)}.

(iii). A∪B={(0.6,0.3,0.2,x_1),(0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)}

(iv). A∩B={(0.3,0.1,0.6,x_1),(0.3,0.2,0.5,x_2), (0.3,0.1,0.6,x_3)}

Definition 9:

(i). The *empty* q-rung orthopair neutrosophic set in the universe U is defined to be $\emptyset_U$ = {(x, 0, 0, 1): x ∈ U}.

(ii). The *universal* q-rung orthopair neutrosophic set in U is defined to be I_U = {(x, 1, 1, 0): x ∈ U}. It is straightforward to check that for each q-rung orthopair neutrosophic set A in U is $A \cup I_U = I_U$, $A \cap I_U = A$, $A \cup \emptyset_U = A$ and $A \cap \emptyset_U = \emptyset_U$.

4. q-Rung Orthopair Neutrosophic Topological Spaces

Topological spaces are the most general category of mathematical spaces, on which fundamental properties like convergence, continuity, compactness, etc. are defined [24]. The ordinary notion of topological space has been extended to fuzzy [25], to intuitionistic fuzzy [26], to soft [27], to neutrosophic topological space [26], etc. Here we generalize the notion of topological space to the notion to *q-rung orthopair neutrosophic topological space* and we study the previously mentioned properties on such kind of spaces.

Definition 10: A *q-rung orthopair neutrosophic topology* T on a non-empty set U is defined as a collection of q-rung orthopair neutrosophic sets in U such that:

1 I_U and $\emptyset_U$ belong to T.

2 The intersection of any two elements of T belongs to T.

3 The union of any number (finite or infinite) of elements of T belongs also to T.

Trivial examples are the *discrete q-rung neutrosophic topology* of all q-rung orthopair neutrosophic sets in U and the *non-discrete q-rung neutrosophic topology* T= {I_U, $\emptyset_U$}.

The elements of a q-rung neutrosophic topology Ton U are called *open* q-rung orthopair neutrosophic sets of U and their complements are called *closed* q-rung orthopair neutrosophic sets of U. The pair (U, T) is referred to as a *q-rung neutrosophic topological space* on U.

Example 5: Let U = {x} and let A = {(x ,0.5,0.5,0.4)}, B = {(x ,0.4,0.6,0.8)}, C = {(x ,0.5,0.6,0.4)}, D ={(x ,0.4,0.5,0.8)} be q-rung orthopair neutrosophic sets in U, q > 1. Then it is straightforward to check that the collection T = {$\emptyset_U$, I_U, A, B, C, D} is a q-rung orthopair neutrosophic topology on U.

We close this work by extending the concepts of *convergence*, *continuity*, *compactness* and Hausdorff topological space to q-rung orthopair neutrosophic topological spaces.

Definition 11: Given two q-rung orthopair neutrosophic sets A and B of the q-rung neutrosophic topological space (U, T), B is said to be a *neighborhood* of A, if there exists an open q-rung orthopair neutrosophic set Q such that $A \subseteq Q \subset B$. Further, we say that a sequence {A_n} of q-rung orthopair neutrosophic sets of (U, T) *converges* to the q-rung orthopair neutrosophic set A of (U, T), if there exists a positive integer m such that for each integer n≥m and each neighborhood B of A we have that $A_n \subset$ B.

The following Proposition generalizes Zadeh's extension principle for fuzzy sets (see [2], pp. 20-21) to q-rung orthopair neutrosophic sets.

Proposition 5: Let U and V be two non-empty crisp sets and let g: U $\rightarrow$ V be a function. Then g can be extended to a function G mapping q-rung orthopair neutrosophic sets in U to q-rung orthopair neutrosophic sets in V.

Proof: Let A<m_A, i_A, n_A> be a q-rung orthopair neutrosophic set in U. Then its image G(A) is a q-rung orthopair neutrosophic set B in V, whose neutrosophic components are defined as follows: Given y in V, consider the set $g^{-1}(y)=\{x \in U: g(x)=y\}$. If $g^{-1}(y) = \emptyset$, then $m_B(y)=0$, and if $g^{-1}(v)\neq\emptyset$, then $m_B(y)$ is equal to the maximal value of all $m_A(x)$ such that $x \in g^{-1}(y)$. Conversely, the inverse image $G^{-1}(B)$ is the q-rung orthopair neutrosophic set A in U with membership function $m_A(x)=m_B(g(ux)$, for each x $\in$ U. In an analogous way one can determine the neutrosophic components i_B and n_B of B.

Definition 12: Let (U, T) and (V, S) be two q-rung neutrosophic topological spaces on the non-empty crisp sets U and V respectively and let g be a function g: U $\rightarrow$ V. Then, according to Proposition 5, g can be extended to a function G which maps q-rung orthopair neutrosophic sets of U to q-rung orthopair neutrosophic sets of V. We say then that g is a *q-rung orthopair neutrosophical continuous* function, if, and only if, the inverse image of each open q-rung orthopair neutrosophic set of V through G is an open q-rung orthopair neutrosophic set of U.

Definition 13: A family A = {A_i,i∈I} of q-rung orthopair neutrosophic sets of the q-rung orthopair neutrosophic topological space (U, T) is called a *cover* of U, if U =.$\bigcup_{i \in I} A_i$. If the elements of A are open q-rung orthopair neutrosophic sets, then A is called an *open cover* of U. Also, each q-rung orthopair neutrosophic subset of A which is also a cover of U is called a sub-cover of A. The q-rung orthopair neutrosophic topological space (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.

Definition 14: A q-rung orthopair neutrosophic topological space (U, T) is called a *T_1- q-rung orthopair neutrosophic topological space* if, and only if, for each pair of elements x_1, x_2 of U with $x_1 \neq x_2$, there exist at least two open q-rung orthopair neutrosophic sets Q_1 and Q_2 such that $x_1 \in Q_1$, $x_2 \notin O_1$ and $x_2 \in Q_2$, $x_1 \notin O_2$.

Definition 15: A q-rung orthopair neutrosophic topological space (U, T) is called a *T_2- q-rung orthopair neutrosophic topological space* if, and only if, for each pair of elements x_1, x_2 of U with $x_1 \neq x_2$, there exist at least two open q-rung orthopair neutrosophic sets Q_1 and Q_2 such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset_U$.

A T_2-q-rung orthopair neutrosophic topological space is also called a *Hausdorff* or a *separable* q-rung orthopair neutrosophic topological space. Obviously a T_2-rung orthopair neutrosophic topological space is always a T_1- q-rung orthopair neutrosophic topological space.

5. Conclusions

In this work we introduced the concept of q-rung orthopair neutrosophic set and we extended the classical notion of topological space and the fundamental properties of convergence, continuity, compactness and Hausdorff space defined in it to q-rung orthopair neutrosophic topological spaces. Examples were also given to illustrate our results.

It looks that proper combinations of the theories developed for tackling the existing in real life uncertainty is a promising tool for obtaining better results in a variety of human activities characterized by uncertainty (see also [29, 30]). This is, therefore, a fruitful area for future research

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

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This article does not contain any studies with human participants or animals performed by any of the authors.

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