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The submitted papers should be professional, and in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e., notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory, every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjointed two by two. But, since in many cases, the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and intuitionistic fuzzy logic). In neutrosophic logic, a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of classical statistics.

What distinguishes neutrosophic from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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MAGDM Model Using Single-Valued Neutrosophic Credibility Matrix Energy and Its Decision-Making Application

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Abstract: This paper aims to develop a MAGDM model using single-valued neutrosophic credibility matrix (SVNCM) energy in a SVNCM scenario. To do it, first, SVNCM energy and its score function are presented as a conceptual extension of existing single-valued neutrosophic matrix (SVNM) energy. Then, a MAGDM model is developed in terms of SVNCM energy and its score function in a SVNCM scenario and also its decision algorithm is provided to solve MAGDM problems with SVNCMs. Finally, the developed MAGDM model is applied in the school site selection problem as an actual example, then the comparative investigation of the decision results in the SVNM and SVNCM scenarios indicates the superiority of the developed model over existing MAGDM model.

Keywords: Single-Valued Neutrosophic Credibility Matrix; Single-Valued Neutrosophic Credibility Matrix Energy; Score Function; Group Decision Making.

1. Introduction

Matrix energy (ME) is one of important mathematical tools in the representation and processing of collective data, it is usually used in group decision making (GDM) applications. Bravo et al. [1] introduced ME as a generalization of graph energy and provided the upper and lower bounds of ME. Donbosco et al. [2] introduced rough neutrosophic ME as a generalization of ME and established its MAGDM method for handling multiple attribute group decision making (MAGDM) problems with rough neutrosophic matrix information, and then applied it to the optimal choice of building sites. After that, Li and Ye [3] proposed intuitionistic fuzzy matrix (IFM) energy and its MAGDM model for the best selection of hospital sites in a complete IFM scenario. Yong et al. [4] further presented the linguistic neutrosophic ME and its MAGDM model to solve the MAGDM problems in the scenario of full linguistic neutrosophic matrices. Jeni Seles Martina and Deepa [5] gave the concepts of multi-valued neutrosophic ME and neutrosophic hesitant ME and used them for MAGDM problems. However, the aforementioned neutrosophic ME lacks the credibility measures of true, false, and uncertain membership values in inconsistent and uncertain scenarios so that it is difficult to guarantee its decision credibility level in uncertain and ambiguous MAGDM environments.

In general, neutrosophic sets (NSs) [6] are not only the extended form of fuzzy sets (FSs) [7] and intuitionistic FSs [8], but also independently depict inconsistent, uncertain, and incomplete information though the true, false, and uncertain membership values, which FSs and intuitionistic FSs cannot do. Although existing fuzzy, intuitionistic fuzzy, and neutrosophic decision making methods and applications [9-20] have contained a lot of studies in existing literature, but they do not consider the credibility measures of various evaluation values in uncertain and ambiguous setting. To guarantee the credibility degrees of fuzzy values in uncertain and ambiguous environments, Ye et al. [21] first proposed fuzzy credibility values and their aggregation operators to perform the multiple attribute decision making (MADM) application in the selection of slope design schemes. Then, Ye et al. [22] further introduced intuitionistic fuzzy credibility sets and their similarity measures and applied them to the performance assessment of industrial robots. Ye et al. [23] also proposed single-valued neutrosophic credibility sets/values (SVNCSs/SVNCVs) to ensure the credibility degrees of true, false and uncertain membership values, and then developed their trigonometric aggregation operators and their MADM application in the selection of slope design schemes, but the MADM model [23] cannot tackle MAGDM problems in the scenario of full single-valued neutrosophic credibility matrices (SVNCMs). In this case, the existing MADM model [23] implies its obvious insufficiency and research gap in full SVNCM setting. Therefore, it is necessary to develop a MAGDM model using the SVNCM energy and score function in a SVNCM scenario to fill the research gap.

In general, this study mainly contains the following original contributions:

- SVNCM energy is defined as a generalization of neutrosophic ME.
- A score function for the SVNCM energy is presented to rank the SVNCM energy.
- A MAGDM model using the SVNCM energy and score function is developed to solve MAGDM problems in the full SVNCM scenario.
- The developed MAGDM model is applied in the actual example on the selection of primary school sites in Shaoxing, China.

The rest of the paper includes the following content. Section 2 introduces some concepts of SVNCSs, SVNCVs, and single-valued neutrosophic matrix (SVNM) energy as the preliminaries of this study. Section 3 proposes SVNCM energy and the score function and ranking rules of SVNCM energy. In Section 4, we develop a MAGDM model based on the SVNCM energy and score function. A MAGDM example on the selection of primary school sites and a comparative investigation are provided in Section 5. Section 6 remarks conclusions and future work.

2. Preliminaries

2.1 Some Concepts of SVNCSs and SVNCVs

Wang et al. [8] introduced the SVNS $N_S = \{<y, V_T(y), V_U(y), V_F(y)> \mid y \in Y\}$ in a universe set Y, where $V_T(y)$, $V_U(y)$, $V_F(y) \in [0, 1]$ for $y \in Y$ are the true, uncertain, and false membership values. Then, each element $<y, V_T(y), V_U(y), V_F(y)>$ in N_S can be simply denoted by the single-valued neutrosophic value (SVNV) $n_S = <V_T, V_U, V_F>$.

To measure the credibility level of SVNV, Ye et al. [23] proposed a SVNCS in Y, which is represented by

$$N_C = \left\{ \left\langle y, \left(V_T(y), C_T(y)\right), \left(V_U(y), C_U(y)\right), \left(V_F(y), C_F(y)\right) \right\rangle \mid y \in Y \right\}, \tag{1}$$

where $(V_T(y), C_T(y))$, $(V_U(y), C_U(y))$ and $(V_F(y), C_F(y))$ are the true, false and uncertain fuzzy credibility values, then their true, false and uncertain membership values and their corresponding credibility values are $V_T(y)$, $V_U(y)$, $V_F(y) \in [0, 1]$ and $C_T(y)$, $C_U(y)$, $C_F(y) \in [0, 1]$, respectively, such that $0 \le V_T(y) + V_U(y) + V_F(y) \le 3$ and $0 \le C_T(y) + C_U(y) + C_F(y) \le 3$ for $y \in Y$. For ease of expression, any element $<y, (V_T(y), C_T(y)), (V_U(y), C_U(y)), (V_F(y), C_F(y))>$ in N_C can be expressed as a simplified form of the SVNCV $n_c = <(V_T, C_T), (V_U, C_U), (V_F, C_F)>$.

It is worth noting that when one does not consider the credibility values in the SVNCV n_c, n_C becomes SVNV. Therefore, the credibility values contained in the SVNCV n_C can guarantee the credibility degree of SVNV.

For any two SVNCVs $n_{C1} = <(V_{T1}, C_{T1}), (V_{U1}, C_{U1}), (V_{F1}, C_{F1})>$ and $n_{C2} = <(V_{T2}, C_{T2}), (V_{U2}, C_{U2}), (V_{F2}, C_{F2})>$, their operation laws are presented as follows:

(1) $n_{C1} \subseteq n_{C2} \Leftrightarrow V_{T1} \le V_{T2}, C_{T1} \le C_{T2}, V_{U1} \ge V_{U2}, C_{U1} \ge C_{U2}, V_{F1} \ge V_{F2}, C_{F1} \ge C_{F2}$;

(2) $n_{C1} = n_{C2} \Leftrightarrow n_{C1} \subseteq n_{C2}, n_{C2} \subseteq n_{C1}$;

(3) $n_{C1} \cup n_{C2} = \langle (V_{T1} \vee V_{T2}, C_{T1} \vee C_{T2}), (V_{U1} \wedge V_{U2}, C_{U1} \wedge C_{U2}), (V_{F1} \wedge V_{F2}, C_{F1} \wedge C_{F2}) \rangle$;

(4) $n_{C1} \cap n_{C2} = \langle (V_{T1} \wedge V_{T2}, C_{T1} \wedge C_{T2}), (V_{U1} \vee V_{U2}, C_{U1} \vee C_{U2}), (V_{F1} \vee V_{F2}, C_{F1} \vee C_{F2}) \rangle$;

(5) $(n_{C1})^c = \langle (V_{F1}, C_{F1}), (1 - V_{U1}, 1 - C_{U1}), (V_{T1}, C_{T1}) \rangle$ (Complement of n_{C1});

(6) $n_{C1} \oplus n_{C2} = \left\langle \begin{matrix} (V_{T1} + V_{T2} - V_{T1}V_{T2}, C_{T1} + C_{T2} - C_{T1}C_{T2}), \\ (V_{U1}V_{U2}, C_{U1}C_{I2}), (V_{F1}V_{F2}, C_{F1}C_{F2}) \end{matrix} \right\rangle$;

(7) $n_{C1} \otimes n_{C2} = \left\langle \begin{matrix} (V_{T1}V_{T2}, C_{T1}C_{T2}), (V_{U1} + V_{U2} - V_{U1}V_{U2}, C_{U1}C_{U2} - C_{U1}C_{U2}), \\ (V_{F1} + V_{F2} - V_{F1}V_{F2}, C_{F1} + C_{F2} - C_{F1}C_{F2}) \end{matrix} \right\rangle$;

(8) $\zeta n_{C1} = \left\langle \begin{matrix} \left(1 - (1 - V_{T1})^{\zeta}, 1 - (1 - C_{T1})^{\zeta}\right), \\ (V_{U1}^{\zeta}, C_{U1}^{\zeta}), (V_{F1}^{\zeta}, C_{F1}^{\zeta}) \end{matrix} \right\rangle, \zeta > 0$;

(9) $n_{C1}^{\zeta} = \left\langle \begin{matrix} (V_{T1}^{\zeta}, C_{T1}^{\zeta}), \left(1 - (1 - V_{U1})^{\zeta}, 1 - (1 - C_{U1})^{\zeta}\right), \\ \left(1 - (1 - V_{F1})^{\zeta}, 1 - (1 - C_{F1})^{\zeta}\right) \end{matrix} \right\rangle, \zeta > 0$.

2.1 Matrix Energy

Set $M(d_{jl})$ for $d_{jl} \in \Re$ (all real numbers) (j, l = 1, 2, …, b) as a $b \times b$ matrix, which is represented as

$$M(d_{jl}) = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1b} \\ d_{21} & d_{22} & \cdots & d_{2b} \\ \vdots & \vdots & \vdots & \vdots \\ d_{b1} & d_{b2} & \cdots & d_{bb} \end{bmatrix}. \tag{2}$$

Then, ME of $M(d_{jl})$ is introduced below [1]:

$$E(M(d_{jl})) = \sum_{j=1}^{b} \left| \delta_j - \frac{1}{b} \sum_{j=1}^{b} \delta_j \right|, \tag{3}$$

where δ_j (j = 1, 2, …, b) are the eigenvalues of $M(d_{jl})$.

(4)

Set the SVNM $M(n_{Sjl})$ (j, l = 1, 2, …, b) as a $b \times b$ matrix [5]:

$$M(n_{Sjl}) = \begin{bmatrix} n_{S11} & n_{S12} & \cdots & n_{S1b} \\ n_{S21} & n_{S22} & \cdots & n_{S2b} \\ \vdots & \vdots & \vdots & \vdots \\ n_{Sb1} & n_{Sb2} & \cdots & n_{Sbb} \end{bmatrix}, \tag{5}$$

where n_{Sjl} is the SVNV $n_{Sjl} = \langle V_{Tjl}, V_{Ujl}, V_{Fjl} \rangle$ (j, l = 1, 2, …, b) that consists of the true, uncertain, and false membership values $V_{Tjl}, V_{Ujl}, V_{Fjl} \in [0, 1]$. Then, the SVNM $M(n_{Sjl})$ can be divided into the true matrix $M(V_{Tjl})$, the uncertain matrix $M(V_{Ujl})$, and the false matrix $M(V_{Fjl})$, which is also represented as the following SVNM form:

$$M(n_{Sjl}) = \langle M(V_{Tjl}), M(V_{Ujl}), M(V_{Fjl}) \rangle = \left\langle \begin{bmatrix} V_{T11} & V_{T12} & \cdots & V_{T1b} \\ V_{T21} & V_{T22} & \cdots & V_{T2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Tb1} & V_{Tb2} & \cdots & V_{Tbb} \end{bmatrix}, \begin{bmatrix} V_{U11} & V_{U12} & \cdots & V_{U1b} \\ V_{U21} & V_{U22} & \cdots & V_{U2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Ub1} & V_{Ub2} & \cdots & V_{Ubb} \end{bmatrix}, \begin{bmatrix} V_{F11} & V_{F12} & \cdots & V_{F1b} \\ V_{F21} & V_{F22} & \cdots & V_{F2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Fb1} & V_{Tb2} & \cdots & V_{Fbb} \end{bmatrix} \right\rangle. \quad (6)$$

In terms of the concepts of true, uncertain and false ME, the energy of the SVNM $M(n_{Sjk})$ is introduced below [5]:

$$E(M(n_{Sjl})) = \langle E[M(V_{Tjl})], E[M(V_{Ujl})], E[M(V_{Fjl})] \rangle = \left\langle \sum_{j=1}^{b} \left|\mu_{Tj} - \mu_{MT}\right|, \sum_{j=1}^{b} \left|\mu_{Uj} - \mu_{MU}\right|, \sum_{j=1}^{b} \left|\mu_{Fj} - \mu_{MF}\right| \right\rangle, \quad (7)$$

where μ_{Tj}, μ_{Uj}, and μ_{Fj} (j = 1, 2, ..., b) are the eigenvalues corresponding to the three matrices $M(V_{Tjl})$, $M(V_{Ujl})$, and $M(V_{Fjl})$ and μ_{MT}, μ_{MU}, and μ_{MF} are the average values corresponding to the eigenvalues μ_{Tj}, μ_{Uj}, and μ_{Fj} (j = 1, 2, ..., b). Then, there are the following equations [5]:

(1) $\sum_{j=1}^{b} (\mu_{Tj} - \mu_{MT}) = \sum_{j=1}^{b} (V_{Tjj} - \mu_{MT}) = 0$;

(2) $\sum_{j=1}^{b} (\mu_{Uj} - \mu_{MU}) = \sum_{j=1}^{b} (V_{Ujj} - \mu_{MU}) = 0$;

(3) $\sum_{j=1}^{b} (\mu_{Fj} - \mu_{MF}) = \sum_{j=1}^{b} (V_{Fjj} - \mu_{MF}) = 0$;

(4) $\sum_{j=1}^{b} (\mu_{Tj} - \mu_{MT})^2 = \sum_{j=1}^{b} V_{Tjj}^2 + 2 \sum_{1 \le j < l \le b} V_{Tjl} V_{Tlj} - b\mu_{MT}^2$;

(5) $\sum_{j=1}^{b} (\mu_{Uj} - \mu_{MU})^2 = \sum_{j=1}^{b} V_{Ujl}^2 + 2 \sum_{1 \le j < l \le b} V_{Ujl} V_{Ulj} - b\mu_{MU}^2$;

(6) $\sum_{j=1}^{b} (\mu_{Fj} - \mu_{MF})^2 = \sum_{j=1}^{b} V_{Fjj}^2 + 2 \sum_{1 \le j < l \le b} V_{Fjl} V_{Flj} - b\mu_{MF}^2$.

The lower and upper bounds of the true, uncertain, and false MEs and the true, uncertain, and false credibility MEs are implied below [5]:

(1) $$\sqrt{\left(\sum_{j=1}^{b} \left|\mu_{Tj} - \mu_{MT}\right|\right)^2 - 2 \sum_{1 \le j < l \le b} \left|\mu_{Tj} - \mu_{MT}\right| \left|\mu_{Tl} - \mu_{MT}\right| + b(b-1) \left|M(V_{Tjl}) - \mu_{MT}\right|^{2/b}} \le E[M(V_{Tjl})] \le \sqrt{b\left(\left(\sum_{j=1}^{b} \left|\mu_{Tj} - \mu_{MT}\right|\right)^2 - 2 \sum_{1 \le j < l \le b} \left|\mu_{Tj} - \mu_{MT}\right| \left|\mu_{Tl} - \mu_{MT}\right|\right)};$$

(2) $$\sqrt{\left(\sum_{j=1}^{b}\left|\mu_{Uj}-\mu_{MU}\right|\right)^{2}-2\sum_{1\le j<l\le b}\left|\mu_{Uj}-\mu_{MU}\right|\left|\mu_{Ul}-\mu_{MU}\right|+b(b-1)\left|M(V_{Ujl})-\mu_{MU}\right|^{2/b}}\le E[M(V_{Ujl})]\le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\mu_{Uj}-\mu_{MU}\right|\right)^{2}-2\sum_{1\le j<l\le b}\left|\mu_{Uj}-\mu_{MU}\right|\left|\mu_{Ul}-\mu_{MU}\right|\right)};$$

(3) $$\sqrt{\left(\sum_{j=1}^{b}\left|\mu_{Fj}-\mu_{MF}\right|\right)^{2}-2\sum_{1\le j<l\le b}\left|\mu_{Fj}-\mu_{MF}\right|\left|\mu_{Fl}-\mu_{MF}\right|+b(b-1)\left|M(V_{Fjl})-\mu_{MF}\right|^{2/b}}\le E[M(V_{Fjl})]\le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\mu_{Fj}-\mu_{MF}\right|\right)^{2}-2\sum_{1\le j<l\le b}\left|\mu_{Fj}-\mu_{MF}\right|\left|\mu_{Fl}-\mu_{MF}\right|\right)}.$$

To compare SVNM energy magnitudes, the ranking values are given by a SVNME score function [5]:

$$H\left\{E\left[M\left(n_{Sjk}\right)\right]\right\}=2E\left[M\left(V_{Tjk}\right)\right]+E\left[M\left(V_{Ujk}\right)\right]-E\left[M\left(V_{Fjk}\right)\right]. \quad (8)$$

In view of the score values of Eq. (8), the ranking rules between $E[M(n_{S2l})]$ and $E[M(n_{S1l})]$ are presented below:

(a) If $H\{E[M(n_{S1k})]\} > H\{E[M(n_{S2k})]\}$, then $E[M(n_{S1k})] > E[M(n_{S2k})]$;

(b) If $H\{E[M(n_{Sj1})]\} < H\{E[M(n_{Sj2})]\}$, then $E[M(n_{Sj1})] < E[M(n_{Sj2})]$;

(c) If $H\{E[M(n_{Sj1})]\} = H\{E[M(n_{Sj2})]\}$, then $E[M(n_{Sj1})] = E[M(n_{Sj2})]$.

3. SVNCM Energy

This section presents the concepts of SVNCM and SVNCM energy based on the energy of the true, false, and uncertain fuzzy credibility matrices in the setting of SVNCMs.

Definition 1. Set the SVNCM $M(n_{Cjl})$ (j, l = 1, 2, …, b) as a $b \times b$ matrix:

$$M(n_{Cjl})=\begin{bmatrix} n_{C11} & n_{C12} & \cdots & n_{C1b} \\ n_{C21} & n_{C22} & \cdots & n_{C2b} \\ \vdots & \vdots & \vdots & \vdots \\ n_{Cb1} & n_{Cb2} & \cdots & n_{Cbb} \end{bmatrix}, \quad (9)$$

where n_{Cjl} is the SVNCV n_{Cjl} = <(V_{Tjl}, C_{Tjl}), (V_{Ujl}, C_{Ujl}), (V_{Fjl}, C_{Fjl})> (j, l = 1, 2, …, b) that consists of the true, uncertain, and false membership values V_{Tjl}, V_{Ujl}, $V_{Fjl} \in [0, 1]$ and the true, uncertain, and false credibility values C_{Tjl}, C_{Ujl}, $C_{Fjl} \in [0, 1]$. Then, the SVNCM $M(n_{Sjl})$ can be divided into the true matrix $M(V_{Tjl})$, the uncertain matrix $M(V_{Ujl})$, and the false matrix $M(V_{Fjl})$ and the true credibility matrix $M(C_{Tjl})$, the uncertain credibility matrix $M(C_{Ujl})$, and the false credibility matrix $M(C_{Fjl})$, which is also represented as the following SVNCM form:

$$M(n_{Cjl})=\left\langle \left(M(V_{Tjl}),M(C_{Tjl})\right),\left(M(V_{Ujl}),M(C_{Ujl})\right),\left(M(V_{Fjl}),M(C_{Fjl})\right)\right\rangle$$
$$=\left\langle \begin{array}{l} \left(\begin{bmatrix} V_{T11} & V_{T12} & \cdots & V_{T1b} \\ V_{T21} & V_{T22} & \cdots & V_{T2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Tb1} & V_{Tb2} & \cdots & V_{Tbb} \end{bmatrix}, \begin{bmatrix} C_{T11} & C_{T12} & \cdots & C_{T1b} \\ C_{T21} & C_{T22} & \cdots & C_{T2b} \\ \vdots & \vdots & \vdots & \vdots \\ C_{Tb1} & C_{Tb2} & \cdots & C_{Tbb} \end{bmatrix}\right), \\ \left(\begin{bmatrix} V_{U11} & V_{U12} & \cdots & V_{U1b} \\ V_{U21} & V_{U22} & \cdots & V_{U2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Ub1} & V_{Ub2} & \cdots & V_{Ubb} \end{bmatrix}, \begin{bmatrix} C_{U11} & C_{U12} & \cdots & C_{U1b} \\ C_{U21} & C_{U22} & \cdots & C_{U2b} \\ \vdots & \vdots & \vdots & \vdots \\ C_{Ub1} & C_{Ub2} & \cdots & C_{Ubb} \end{bmatrix}\right), \\ \left(\begin{bmatrix} V_{F11} & V_{F12} & \cdots & V_{F1b} \\ V_{F21} & V_{F22} & \cdots & V_{F2b} \\ \vdots & \vdots & \vdots & \vdots \\ V_{Fb1} & V_{Tb2} & \cdots & V_{Fbb} \end{bmatrix}, \begin{bmatrix} C_{F11} & C_{F12} & \cdots & C_{F1b} \\ C_{F21} & C_{F22} & \cdots & C_{F2b} \\ \vdots & \vdots & \vdots & \vdots \\ C_{Fb1} & C_{Tb2} & \cdots & C_{Fbb} \end{bmatrix}\right) \end{array} \right\rangle . \quad (10)$$

Definition 2. Let the SVNCM $M(n_{Cjl})$ (j, l = 1, 2, ..., b) be a $b \times b$ matrix, which can be expressed as $M(n_{Cjl})$ = ⟨($M(V_{Tjl})$, $M(C_{Tjl})$), ($M(V_{Ujl})$, $M(C_{Ujl})$), ($M(V_{Fjl})$, $M(C_{Fjl})$)⟩, including the true, uncertain and false matrices $M(V_{Tjl})$, $M(V_{Ujl})$ and $M(V_{Fjl})$ and the true, uncertain and false credibility matrices $M(C_{Tjl})$, $M(C_{Ujl})$ and $M(C_{Fjl})$. Then ME of $M(n_{Cjl})$ can be represented below:

$$E\left[M\left(n_{Cjk}\right)\right]=\left\langle \begin{array}{l} \left(E\left[M\left(V_{Tjk}\right)\right],E\left[M\left(C_{Tjk}\right)\right]\right), \\ \left(E\left[M\left(V_{Ujk}\right)\right],E\left[M\left(C_{Ujk}\right)\right]\right), \\ \left(E\left[M\left(V_{Fjk}\right)\right],E\left[M\left(C_{Fjk}\right)\right]\right) \end{array} \right\rangle$$
$$=\left\langle \begin{array}{l} \left(\sum_{j=1}^{b}\left|\mu_{Tj}-\frac{1}{b}\sum_{j=1}^{b}\mu_{Tj}\right|,\sum_{j=1}^{b}\left|\rho_{Tj}-\frac{1}{b}\sum_{j=1}^{b}\rho_{Tj}\right|\right), \\ \left(\sum_{j=1}^{b}\left|\mu_{Uj}-\frac{1}{b}\sum_{j=1}^{b}\mu_{Uj}\right|,\sum_{j=1}^{b}\left|\rho_{Uj}-\frac{1}{b}\sum_{i=1}^{b}\rho_{Uj}\right|\right), \\ \left(\sum_{j=1}^{b}\left|\mu_{Fj}-\frac{1}{b}\sum_{j=1}^{b}\mu_{Fj}\right|,\sum_{j=1}^{b}\left|\rho_{Fj}-\frac{1}{b}\sum_{i=1}^{b}\rho_{Fj}\right|\right) \end{array} \right\rangle = \left\langle \begin{array}{l} \left(\sum_{j=1}^{b}\left|\mu_{Tj}-\mu_{MT}\right|,\sum_{j=1}^{b}\left|\rho_{Tj}-\rho_{MT}\right|\right), \\ \left(\sum_{j=1}^{b}\left|\mu_{Uj}-\mu_{MU}\right|,\sum_{j=1}^{b}\left|\rho_{Uj}-\rho_{MU}\right|\right), \\ \left(\sum_{j=1}^{b}\left|\mu_{Fj}-\mu_{MF}\right|,\sum_{j=1}^{b}\left|\rho_{Fj}-\rho_{MF}\right|\right) \end{array} \right\rangle , \quad (11)$$

where μ_{Tj}, μ_{Uj}, and μ_{Fj} (j = 1, 2, ..., b) are the eigenvalues corresponding to the three matrices $M(V_{Tjl})$, $M(V_{Ujl})$, $M(V_{Fjl})$; ρ_{Tj}, ρ_{Uj} and ρ_{Fj} (j = 1, 2, ..., b) are the eigenvalues corresponding to the three credibility matrices $M(C_{Tjl})$, $M(C_{Ujl})$, $M(C_{Fjl})$; μ_{MT}, μ_{MU}, and μ_{MF} are the average values corresponding to the eigenvalues μ_{Tj}, μ_{Uj}, and μ_{Fj} (j = 1, 2, ..., b) and ρ_{MT}, ρ_{MU} and ρ_{MF} are the average values corresponding to the eigenvalues ρ_{Tj}, ρ_{Uj} and ρ_{Fj} (j = 1, 2, ..., b).

Especially when one does not consider the credibility values in the SVNCM $M(n_{Cjl})$, $E[M(n_{Cjl})]$ is reduced to the SVNM energy of Eq. (3).

In terms of similar properties corresponding to SVNM [5], the SVNCM $M(n_{Cjl})$ (j, l = 1, 2, ..., b) also contains the following equations:

(1) $\sum_{j=1}^{b}(\mu_{Tj}-\mu_{MT})=\sum_{j=1}^{b}(V_{Tjj}-\mu_{MT})=0$;

(2) $\sum_{j=1}^{b}(\mu_{Uj}-\mu_{MU})=\sum_{j=1}^{b}(V_{Ujj}-\mu_{MU})=0$;

(3) $\sum_{j=1}^{b}(\mu_{Fj}-\mu_{MF})=\sum_{j=1}^{b}(V_{Fjj}-\mu_{MF})=0$;

(4) $\sum_{j=1}^{b}(\mu_{Tj}-\mu_{MT})^2=\sum_{j=1}^{b}V_{Tjj}^2+2\sum_{1\le j<l\le b}V_{Tjl}V_{Tlj}-b\mu_{MT}^2$;

(5) $\sum_{j=1}^{b}(\mu_{Uj}-\mu_{MU})^2=\sum_{j=1}^{b}V_{Ujl}^2+2\sum_{1\le j<l\le b}V_{Ujl}V_{Ulj}-b\mu_{MU}^2$;

(6) $\sum_{j=1}^{b}(\mu_{Fj}-\mu_{MF})^2=\sum_{j=1}^{b}V_{Fjj}^2+2\sum_{1\le j<l\le b}V_{Fjl}V_{Flj}-b\mu_{MF}^2$;

(7) $\sum_{j=1}^{b}(\rho_{Tj}-\rho_{MT})=\sum_{j=1}^{b}(C_{Tjj}-\rho_{MT})=0$;

(8) $\sum_{j=1}^{b}(\rho_{Uj}-\rho_{MU})=\sum_{j=1}^{b}(C_{Ujj}-\rho_{MU})=0$;

(9) $\sum_{j=1}^{b}(\rho_{Fj}-\rho_{MF})=\sum_{j=1}^{b}(C_{Fjj}-\rho_{MF})=0$;

(10) $\sum_{j=1}^{b}(\rho_{Tj}-\rho_{MT})^2=\sum_{j=1}^{b}C_{Tjj}^2+2\sum_{1\le j<l\le b}C_{Tjl}C_{Tlj}-b\rho_{MT}^2$;

(11) $\sum_{\mu=1}^{e}(\rho_{Uj}-\rho_{MU})^2=\sum_{j=1}^{b}C_{Ujl}^2+2\sum_{1\le j<l\le b}C_{Ujl}C_{Ulj}-b\rho_{MU}^2$;

(12) $\sum_{j=1}^{b}(\rho_{Fj}-\rho_{MF})^2=\sum_{j=1}^{b}C_{Fjj}^2+2\sum_{1\le j<l\le b}C_{Fjl}C_{Flj}-b\rho_{MF}^2$.

Furthermore, the lower and upper bounds of the true, uncertain, and false MEs are introduced below:

(1)
$$\sqrt{\left(\sum_{j=1}^{b}\left|\mu_{Tj}-\mu_{MT}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Tj}-\mu_{MT}\right|\left|\mu_{Tl}-\mu_{MT}\right|+b(b-1)\left|M(V_{Tjl})-\mu_{MT}\right|^{2/b}}\le E[M(V_{Tjl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\mu_{Tj}-\mu_{MT}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Tj}-\mu_{MT}\right|\left|\mu_{Tl}-\mu_{MT}\right|\right)};$$

(2)
$$\sqrt{\left(\sum_{j=1}^{b}\left|\mu_{Uj}-\mu_{MU}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Uj}-\mu_{MU}\right|\left|\mu_{Ul}-\mu_{MU}\right|+b(b-1)\left|M(V_{Ujl})-\mu_{MU}\right|^{2/b}}\le E[M(V_{Ujl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\mu_{Uj}-\mu_{MU}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Uj}-\mu_{MU}\right|\left|\mu_{Ul}-\mu_{MU}\right|\right)};$$

(3) $$\sqrt{\left(\sum_{j=1}^{b}\left|\mu_{Fj}-\mu_{MF}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Fj}-\mu_{MF}\right|\left|\mu_{Fl}-\mu_{MF}\right|+b(b-1)\left|M(V_{Fjl})-\mu_{MF}\right|^{2/b}}\le E[M(V_{Fjl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\mu_{Fj}-\mu_{MF}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\mu_{Fj}-\mu_{MF}\right|\left|\mu_{Fl}-\mu_{MF}\right|\right)};$$

(4) $$\sqrt{\left(\sum_{j=1}^{b}\left|\rho_{Tj}-\rho_{MT}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Tj}-\rho_{MT}\right|\left|\rho_{Tl}-\rho_{MT}\right|+b(b-1)\left|M(C_{Tjl})-\rho_{MT}\right|^{2/b}}\le E[M(C_{Tjl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\rho_{Tj}-\rho_{MT}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Tj}-\rho_{MT}\right|\left|\rho_{Tl}-\rho_{MT}\right|\right)};$$

(5) $$\sqrt{\left(\sum_{j=1}^{b}\left|\rho_{Uj}-\rho_{MU}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Uj}-\rho_{MU}\right|\left|\rho_{Ul}-\rho_{MU}\right|+b(b-1)\left|M(C_{Ujl})-\rho_{MU}\right|^{2/b}}\le E[M(C_{Ujl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\rho_{Uj}-\rho_{MU}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Uj}-\rho_{MU}\right|\left|\rho_{Ul}-\rho_{MU}\right|\right)};$$

(6) $$\sqrt{\left(\sum_{j=1}^{b}\left|\rho_{Fj}-\rho_{MF}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Fj}-\rho_{MF}\right|\left|\rho_{Fl}-\rho_{MF}\right|+b(b-1)\left|M(C_{Fjl})-\rho_{MF}\right|^{2/b}}\le E[M(C_{Fjl})] \le\sqrt{b\left(\left(\sum_{j=1}^{b}\left|\rho_{Fj}-\rho_{MF}\right|\right)^2-2\sum_{1\le j<l\le b}\left|\rho_{Fj}-\rho_{MF}\right|\left|\rho_{Fl}-\rho_{MF}\right|\right)}.$$

To compare two SVNCM energy magnitudes, we present the score function of the SVNCM energy $E(M(n_{Cijl}))$ ($j, l = 1, 2, ..., b; i = 1, 2$):

$$Z\left\{E\left(M\left(n_{Cijk}\right)\right)\right\}=2E\left[M\left(V_{Tijk}\right)\right]E\left[M\left(C_{Tijk}\right)\right]+E\left[M\left(V_{Uijk}\right)\right]E\left[M\left(C_{Uijk}\right)\right]-E\left[M\left(V_{Fijk}\right)\right]E\left[M\left(C_{Fijk}\right)\right]. \quad (12)$$

In view of the score values of Eq. (12), the ranking rules between $E(M(n_{C1jl}))$ and $E(M(n_{C2jl}))$ are presented below:

(a) If $Z\{E[M(n_{C1jl})]\} > Z\{E[M(n_{C2jl})]\}$, then $E[M(n_{C1jl})] > E[M(n_{C2jl})]$;

(b) If $Z\{E[M(n_{C1jl})]\} < Z\{E[M(n_{C2jl})]\}$, then $E[M(n_{C1jl})] < E[M(n_{C2jl})]$;

(c) If $Z\{E[M(n_{C1jl})]\} = Z\{E[M(n_{C2jl})]\}$, then $E[M(n_{C1jl})] = E[M(n_{C2jl})]$.

Example 1. Assume that there are two SVNCMs:

$$M(n_{C1jl})=\begin{bmatrix} <(0.6,0.7),(0.3,0.7),(0.2,0.7)> & <(0.5,0.6),(0.5,0.8),(0.3,0.6)> & <(0.7,0.6),(0.1,0.5),(0.3,0.9)> \\ <(0.8,0.7),(0.2,0.8),(0.1,0.8)> & <(0.8,0.8),(0.2,0.8),(0.4,0.6)> & <(0.3,0.8),(0.2,0.6),(0.1,0.6)> \\ <(0.7,0.9),(0.1,0.9),(0.3,0.8)> & <(0.7,0.5),(0.2,0.6),(0.1,0.9)> & <(0.8,0.5),(0.3,0.6),(0.5,0.8)> \end{bmatrix},$$

$$M(n_{C2jl})=\begin{bmatrix} <(0.5,0.6),(0.2,0.8),(0.3,0.8)> & <(0.6,0.7),(0.6,0.8),(0.2,0.8)> & <(0.6,0.6),(0.1,0.7),(0.2,0.8)> \\ <(0.7,0.7),(0.2,0.7),(0.2,0.9)> & <(0.7,0.7),(0.1,0.8),(0.3,0.7)> & <(0.2,0.7),(0.4,0.7),(0.3,0.7)> \\ <(0.6,0.8),(0.1,0.7),(0.1,0.8)> & <(0.6,0.6),(0.1,0.6),(0.1,0.7)> & <(0.7,0.6),(0.2,0.8),(0.4,0.5)> \end{bmatrix}.$$

Then, their SVNCM energy and ranking order are given by the following results:

Using Eq. (11), there are $E[M(n_{C1jl})]$ = <(2.4559, 2.7161), (0.8413, 2.8041), (0.8193, 3.1193)> and $E[M(n_{C2jl})]$ = (2.1708, 2.7372), (0.9355, 2.8000), (0.6916, 3.1601)>.

Using Eq. (12), since $Z\{E[M(n_{C1jl})]\}$ = 13.1444 > $Z\{E[M(n_{C2jl})]\}$ = 12.3177, there is $E[M(n_{C1jl})] > E[M(n_{C2jl})]$.

4. MAGDM Model

This section establishes a MAGDM model based on the SVNCM energy and score function in the setting of SVNCMs.

Considering a MADM problem, there are a group of alternatives and a group of attributes, denoted respectively by $Gs = \{Gs_1, Gs_2, \ldots, Gs_a\}$ and $Cs = \{Cs_1, Cs_2, \ldots, Cs_b\}$. A group of decision makers/experts, denoted as $Es = \{Es_1, Es_2, \ldots, Es_r\}$, is invited to assess the satisfiability levels of each alternative over the attributes and the weight vector of the decision makers/experts is specified as $\theta_{Cj} = \langle(\theta_{Tj}, \theta_{CTj}), (\theta_{Uj}, \theta_{CUj}), (\theta_{Fj}, \theta_{CFj})\rangle$ ($j = 1, 2, \ldots, r$).

In this MADM problem, the SVNCM energy can be used to build a MADM model in the following steps:

Step 1: The decision makers/experts specify the SVNCV weights of the attributes by $\lambda_{Cjk} = \langle(\lambda_{Tjk}, \lambda_{CTjk}), (\lambda_{Ujk}, \lambda_{CUjk}), (\lambda_{Fjk}, \lambda_{CFjk})\rangle$ ($j = 1, 2, \ldots, r$; $k = 1, 2, \ldots, b$) for $\lambda_{Tjk}, \lambda_{CTjk}, \lambda_{Ujk}, \lambda_{CUjk}, \lambda_{Fjk}, \lambda_{CFjk} \in [0, 1]$, and then they are constructed as the weight matrix of the attributes:

$$M(\lambda_{Cjk}) = \begin{matrix} \\ Es_1 \\ Es_2 \\ \vdots \\ Es_r \end{matrix} \begin{matrix} \begin{matrix} Cs_1 & Cs_2 & \cdots & Cs_b \end{matrix} \\ \begin{bmatrix} \lambda_{C11} & \lambda_{C12} & \cdots & \lambda_{C1b} \\ \lambda_{C21} & \lambda_{C22} & \cdots & \lambda_{C2b} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{Cr1} & \lambda_{Cr2} & \cdots & \lambda_{Crb} \end{bmatrix} \end{matrix}. \tag{13}$$

Step 2: Decision makers/experts evaluate their satisfiability levels of each alternative Gs_i over attributes Cs_k by providing the SVNCVs $n_{Cijk} = \langle(V_{Tijk}, C_{Tijk}), (V_{Uijk}, C_{Uijk}), (V_{Fijk}, C_{Fijk})\rangle$ ($i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, r$; $k = 1, 2, \ldots, b$), and then the i-th SVNCM for Gs_i can be built below:

$$M\left(n_{Cijk}\right) = \begin{bmatrix} n_{Ci11} & n_{Ci12} & \ldots & n_{Ci1b} \\ n_{Ci21} & n_{Ci22} & \ldots & n_{Ci2b} \\ \vdots & \vdots & \vdots & \vdots \\ n_{Cir1} & n_{Cir2} & \ldots & n_{Cirb} \end{bmatrix}. \tag{14}$$

Step 3: In view of the influence of the decision makers/experts' weights θ_{Cj} on the i-th SVNCM for Gs_i, the weighted SVNCM can be obtained below:

$$
M_E\left(\theta_{Cj} \otimes n_{Cijk}\right) =
\begin{bmatrix}
\left\langle \begin{matrix} \left(\theta_{T1}V_{Ti11}, \theta_{CT1}C_{Ti11}\right), \\ \left(\theta_{U1}+V_{Ui11}-\theta_{U1}V_{Ui11}, \theta_{CU1}+C_{Ui11}-\theta_{CU1}C_{Ui11}\right), \\ \left(\theta_{F1}+V_{Fi11}-\theta_{F1}V_{Fi11}, \theta_{CF1}+C_{Fi11}-\theta_{CF1}C_{Fi11}\right) \end{matrix} \right\rangle &
\left\langle \begin{matrix} \left(\theta_{T1}V_{Ti12}, \theta_{CT1}C_{Ti12}\right), \\ \left(\theta_{U1}+V_{Ui12}-\theta_{U1}V_{Ui12}, \theta_{CU1}+C_{Ui12}-\theta_{CU1}C_{Ui12}\right), \\ \left(\theta_{F1}+V_{Fi12}-\theta_{F1}V_{Fi12}, \theta_{CF1}+C_{Fi12}-\theta_{CF1}C_{Fi12}\right) \end{matrix} \right\rangle &
\cdots &
\left\langle \begin{matrix} \left(\theta_{T1}V_{Ti1b}, \theta_{CT1}C_{Ti1b}\right), \\ \left(\theta_{U1}+V_{Ui1b}-\theta_{U1}V_{Ui1b}, \theta_{CU1}+C_{Ui1b}-\theta_{CU1}C_{Ui1b}\right), \\ \left(\theta_{F1}+V_{Fi1b}-\theta_{F1}V_{Fi1b}, \theta_{CF1}+C_{Fi1b}-\theta_{CF1}C_{Fi1b}\right) \end{matrix} \right\rangle \\
\left\langle \begin{matrix} \left(\theta_{T2}V_{Ti21}, \theta_{CT2}C_{Ti21}\right), \\ \left(\theta_{U2}+V_{Ui21}-\theta_{U2}V_{Ui21}, \theta_{CU2}+C_{Ui21}-\theta_{CU2}C_{Ui21}\right), \\ \left(\theta_{F2}+V_{Fi21}-\theta_{F2}V_{Fi21}, \theta_{CF2}+C_{Fi21}-\theta_{CF2}C_{Fi21}\right) \end{matrix} \right\rangle &
\left\langle \begin{matrix} \left(\theta_{T2}V_{Ti21}, \theta_{CT2}C_{Ti21}\right), \\ \left(\theta_{U2}+V_{Ui22}-\theta_{U2}V_{Ui22}, \theta_{CU2}+C_{Ui22}-\theta_{CU2}C_{Ui22}\right), \\ \left(\theta_{F2}+V_{Fi22}-\theta_{F2}V_{Fi22}, \theta_{CF2}+C_{Fk22}-\theta_{CF2}C_{Fi22}\right) \end{matrix} \right\rangle &
\cdots &
\left\langle \begin{matrix} \left(\theta_{T2}V_{Ti2b}, \theta_{CT2}C_{Ti2b}\right), \\ \left(\theta_{U2}+V_{Ui2b}-\theta_{U2}V_{Ui2b}, \theta_{CU2}+C_{Ui2b}-\theta_{CU2}C_{Ui2b}\right), \\ \left(\theta_{F2}+V_{Fi2b}-\theta_{F2}V_{Fi2b}, \theta_{CF2}+C_{Fi2b}-\theta_{CF2}C_{Fi2b}\right) \end{matrix} \right\rangle \\
\vdots & \vdots & \vdots & \vdots \\
\left\langle \begin{matrix} \left(\theta_{Tr}V_{Tir1}, \theta_{CTr}C_{Tir1}\right), \\ \left(\theta_{Ur}+V_{Uir1}-\theta_{Ur}V_{Uir1}, \theta_{CUr}+C_{Uir1}-\theta_{CUr}C_{Uir1}\right), \\ \left(\theta_{Fr}+V_{Fir1}-\theta_{Fr}V_{Fir1}, \theta_{CFr}+C_{Fir1}-\theta_{CFr}C_{Fir1}\right) \end{matrix} \right\rangle &
\left\langle \begin{matrix} \left(\theta_{Tr}V_{Tir2}, \theta_{CTr}C_{Tir2}\right), \\ \left(\theta_{Ur}+V_{Uir2}-\theta_{Ur}V_{Uir2}, \theta_{CUr}+C_{Uir2}-\theta_{CUr}C_{Uir2}\right), \\ \left(\theta_{Fr}+V_{Fir2}-\theta_{Fr}V_{Fir2}, \theta_{CFr}+C_{Fir2}-\theta_{CFr}C_{Fir2}\right) \end{matrix} \right\rangle &
\cdots &
\left\langle \begin{matrix} \left(\theta_{Tr}V_{Tirb}, \theta_{CTr}C_{Tirb}\right), \\ \left(\theta_{Ur}+V_{Uirb}-\theta_{Ur}V_{Uirb}, \theta_{CUr}+C_{Uirb}-\theta_{CUr}C_{Uirb}\right), \\ \left(\theta_{Fr}+V_{Firb}-\theta_{Fr}V_{Firb}, \theta_{CFr}+C_{Firb}-\theta_{CFr}C_{Firb}\right) \end{matrix} \right\rangle
\end{bmatrix}. \quad (15)
$$

Step 4: In view of the influence of the attribute weights λ_{Cjk} on the *i*-th SVNCM for G_{Si}, the weighted SVNCM can be obtained below:

$$M_C\left(\lambda_{Cjk} \otimes n_{Cijk}\right) =$$

$$\begin{bmatrix}
\left\langle \begin{array}{l} \left(\lambda_{T11}V_{Ti11}, \lambda_{CT11}C_{Ti11}\right), \\ \left(\lambda_{U11}+V_{Ui11}-\lambda_{U11}V_{Ui11}, \lambda_{CU11}+C_{Ui11}-\lambda_{CU11}C_{Ui11}\right), \\ \left(\lambda_{F11}+V_{Fi11}-\lambda_{F1}V_{Fi11}, \lambda_{CF11}+C_{Fi11}-\lambda_{CF11}C_{Fi11}\right) \end{array} \right\rangle & \left\langle \begin{array}{l} \left(\lambda_{T12}V_{Ti12}, \lambda_{CT12}C_{Ti12}\right), \\ \left(\lambda_{U12}+V_{Ui12}-\lambda_{U12}V_{Ui12}, \lambda_{CU12}+C_{Ui12}-\lambda_{CU1}C_{Ui12}\right), \\ \left(\lambda_{F12}+V_{Fi12}-\lambda_{F12}V_{Fi12}, \lambda_{CF12}+C_{Fi12}-\lambda_{CF12}C_{Fi12}\right) \end{array} \right\rangle & \cdots & \left\langle \begin{array}{l} \left(\lambda_{T1b}V_{Ti1b}, \lambda_{CT1b}C_{Ti1b}\right), \\ \left(\lambda_{U1b}+V_{Ui1b}-\lambda_{U1b}V_{Ui1b}, \lambda_{CU1b}+C_{Ui1b}-\lambda_{CU1b}C_{Ui1b}\right), \\ \left(\lambda_{F1b}+V_{Fi1b}-\lambda_{F1b}V_{Fi1b}, \lambda_{CF1b}+C_{Fi1b}-\lambda_{CF1b}C_{Fi1b}\right) \end{array} \right\rangle \\
\left\langle \begin{array}{l} \left(\lambda_{T21}V_{Ti21}, \lambda_{CT2}C_{Ti21}\right), \\ \left(\lambda_{U21}+V_{Ui21}-\lambda_{U21}V_{Ui21}, \lambda_{CU21}+C_{Ui21}-\lambda_{CU21}C_{Ui21}\right), \\ \left(\lambda_{F21}+V_{Fi21}-\lambda_{F21}V_{Fi21}, \lambda_{CF21}+C_{Fi21}-\lambda_{CF21}C_{Fi21}\right) \end{array} \right\rangle & \left\langle \begin{array}{l} \left(\lambda_{T22}V_{Ti22}, \lambda_{CT22}C_{Ti22}\right), \\ \left(\lambda_{U22}+V_{Ui22}-\lambda_{U22}V_{Ui22}, \lambda_{CU22}+C_{Ui22}-\lambda_{CU22}C_{Ui22}\right), \\ \left(\lambda_{F22}+V_{Fi22}-\lambda_{F22}V_{Fi22}, \lambda_{CF22}+C_{Fi22}-\lambda_{CF22}C_{Fi22}\right) \end{array} \right\rangle & \cdots & \left\langle \begin{array}{l} \left(\lambda_{T2b}V_{Ti2b}, \lambda_{CT2b}C_{Ti2b}\right), \\ \left(\lambda_{I2b}+V_{Ui2b}-\lambda_{U2b}V_{Ui2b}, \lambda_{CU2b}+C_{Ui2b}-\lambda_{CU2b}C_{Ui2b}\right), \\ \left(\lambda_{F2b}+V_{Fi2b}-\lambda_{F2b}V_{Fi2b}, \lambda_{CF2b}+C_{Fi2b}-\lambda_{CF2b}C_{Fi2b}\right) \end{array} \right\rangle \\
\vdots & \vdots & \vdots & \vdots \\
\left\langle \begin{array}{l} \left(\lambda_{Tr1}V_{Tir1}, \lambda_{CTr1}C_{Tir1}\right), \\ \left(\lambda_{Ur1}+V_{Uir1}-\lambda_{Ur1}N_{Uir1}, \lambda_{CUr1}+C_{Uir1}-\lambda_{CUr1}C_{Uir1}\right), \\ \left(\lambda_{Fr1}+V_{Fir1}-\lambda_{Fr1}N_{Fir1}, \lambda_{CFr1}+C_{Fir1}-\lambda_{CFr1}C_{Fir1}\right) \end{array} \right\rangle & \left\langle \begin{array}{l} \left(\lambda_{Tr2}V_{Tir2}, \lambda_{CTr2}C_{Tir2}\right), \\ \left(\lambda_{Ur2}+V_{Uir2}-\lambda_{Ur2}V_{Uir2}, \lambda_{CUr2}+C_{Uir2}-\lambda_{CUr2}C_{Uir2}\right), \\ \left(\lambda_{Fr2}+V_{Fir2}-\lambda_{Fr2}V_{Fir2}, \lambda_{CFr2}+C_{Fir2}-\lambda_{CFr2}C_{Fir2}\right) \end{array} \right\rangle & \cdots & \left\langle \begin{array}{l} \left(\lambda_{Trb}V_{Tirb}, \lambda_{CTrb}C_{Tirb}\right), \\ \left(\lambda_{Urb}+V_{Uirb}-\lambda_{Urb}V_{Uirb}, \lambda_{CUrb}+C_{Uirb}-\lambda_{CUrb}C_{Uirb}\right), \\ \left(\lambda_{Frb}+V_{Firb}-\lambda_{Frb}V_{Firb}, \lambda_{CFrb}+C_{Firb}-\lambda_{CFrb}C_{Firb}\right) \end{array} \right\rangle
\end{bmatrix}. \quad (16)$$

Step 5: Based on the above weighted SVNCMs, we obtain the collective SVNCMs $M(n_{Cijk})$ = <($M(V_{Tijl})$, $M(C_{Tijl})$), ($M(V_{Uijl})$, $M(C_{Uijl})$), ($M(V_{Fijl})$, $M(C_{Fijl})$> (j, l = 1, 2, …, r; i = 1, 2, …, a) by calculating the true, false and uncertain squire matrices and the true, false and uncertain credibility squire matrices:

$$M\left(V_{Tijl}\right) = M_C\left(\lambda_{Tjk}V_{Tijk}\right) \times \left[M_E\left(\theta_{Tj}V_{Tijk}\right)\right]^T$$
$$= \begin{bmatrix} \lambda_{T11}V_{Ti11} & \lambda_{T12}V_{Ti12} & \cdots & \lambda_{T1b}V_{Ti1b} \\ \lambda_{T21}V_{Ti21} & \lambda_{T22}V_{Ti22} & \cdots & \lambda_{T2b}V_{Ti2b} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{Tr1}V_{Tir1} & \lambda_{Tr2}V_{Tir2} & \cdots & \lambda_{Trb}V_{Tirb} \end{bmatrix} \times \begin{bmatrix} \theta_{T1}V_{Ti11} & \theta_{T2}V_{Ti21} & \cdots & \theta_{Tr}V_{Tir1} \\ \theta_{T1}V_{Ti12} & \theta_{T2}V_{Ti22} & \cdots & \theta_{Tr}V_{Tir2} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{T1}V_{Ti1b} & \theta_{T2}V_{Ti2b} & \cdots & \theta_{Tr}V_{Tirb} \end{bmatrix}, \quad (17)$$

$$M\left(V_{Uijl}\right)=M_C\left(\lambda_{Ujk}+V_{Uijk}-\lambda_{Ujk}V_{Uijk}\right)\times\left[M_E\left(\theta_{Ui}+V_{Uijk}-\theta_{Uj}V_{Uijk}\right)\right]^T$$

$$=\begin{bmatrix}\lambda_{U11}+V_{Ui11}-\lambda_{U11}V_{Ui11} & \lambda_{U12}+V_{Ui12}-\lambda_{U12}V_{Ui12} & \cdots & \lambda_{U1b}+V_{Ui1b}-\lambda_{U1b}V_{Ui1b}\\ \lambda_{U21}+V_{Ui21}-\lambda_{U21}V_{Ui21} & \lambda_{U22}+V_{Ui22}-\lambda_{U22}V_{Ui22} & \cdots & \lambda_{U2b}+V_{Ui2b}-\lambda_{U2b}V_{Ui2b}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{Ur1}+V_{Uir1}-\lambda_{Ur1}V_{Uir1} & \lambda_{Ur2}+V_{Uir2}-\lambda_{Ur2}V_{Uir2} & \cdots & \lambda_{Urb}+V_{Uirb}-\lambda_{Urb}V_{Uirb}\end{bmatrix},\quad(18)$$

$$\times\begin{bmatrix}\theta_{U1}+V_{Ui11}-\theta_{U1}V_{Ui11} & \theta_{U2}+V_{Ui21}-\theta_{U2}V_{Ui21} & \cdots & \theta_{Ur}+V_{Uir1}-\theta_{Ur}V_{Uir1}\\ \theta_{U1}+V_{Ui12}-\theta_{U1}V_{Ui12} & \theta_{U2}+V_{Ui22}-\theta_{U2}V_{Ui22} & \cdots & \theta_{Ur}+V_{Uir2}-\theta_{Ur}V_{Uir2}\\ \vdots & \vdots & \vdots & \vdots\\ \theta_{U1}+V_{Ui1b}-\theta_{I1}V_{Ui1b} & \theta_{U2}+V_{Ui2b}-\theta_{I2}V_{Ui2b} & \cdots & \theta_{Ur}+V_{Uirb}-\theta_{Ur}V_{Uirb}\end{bmatrix}$$

$$M\left(V_{Fijl}\right)=M_C\left(\lambda_{Fjk}+V_{Fijk}-\lambda_{Fjk}V_{Fijk}\right)\times\left[M_E\left(\theta_{Fj}+V_{Fijk}-\theta_{Fj}V_{Fijk}\right)\right]^T$$

$$=\begin{bmatrix}\lambda_{F11}+V_{Fi11}-\lambda_{F11}V_{Fi11} & \lambda_{F12}+V_{Fi12}-\lambda_{F12}V_{Fi12} & \cdots & \lambda_{F1b}+V_{Fi1b}-w_{F1b}N_{Fi1b}\\ \lambda_{F21}+V_{Fi21}-\lambda_{F21}V_{Fi21} & \lambda_{F22}+V_{Fi22}-\lambda_{F22}V_{Fi22} & \cdots & \lambda_{F2b}+V_{Fi2b}-w_{F2b}N_{Fi2b}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{Fr1}+V_{Fir1}-\lambda_{Fr1}V_{Fir1} & \lambda_{Fr2}+V_{Fir2}-\lambda_{Fr2}V_{Fir2} & \cdots & \lambda_{Frb}+V_{Firb}-\lambda_{Frb}V_{Firb}\end{bmatrix},\quad(19)$$

$$\times\begin{bmatrix}\theta_{F1}+V_{Fi11}-\theta_{F1}V_{Fi11} & \theta_{F2}+V_{Fi21}-\theta_{F2}V_{Fi21} & \cdots & \theta_{Fr}+V_{Fir1}-\theta_{Fr}V_{Fir1}\\ \theta_{F1}+V_{Fi12}-\theta_{F1}V_{Fi12} & \theta_{F2}+V_{Fi22}-\theta_{F2}V_{Fi22} & \cdots & \theta_{Fr}+V_{Fir2}-\theta_{Fr}V_{Fir2}\\ \vdots & \vdots & \vdots & \vdots\\ \theta_{F1}+V_{Fi1b}-\theta_{F1}V_{Fi1b} & \theta_{F2}+V_{Fi2b}-\theta_{F2}N_{Fi2b} & \cdots & \theta_{Fr}+V_{Firb}-\theta_{Fr}V_{Firb}\end{bmatrix}$$

$$M\left(C_{Tijl}\right)=M_C\left(\lambda_{Tjk}C_{Tijk}\right)\times\left[M_E\left(\theta_{Tj}C_{Tijk}\right)\right]^T$$

$$=\begin{bmatrix}\lambda_{T11}C_{Ti11} & \lambda_{T12}C_{Ti12} & \cdots & \lambda_{T1b}C_{Ti1b}\\ \lambda_{T21}C_{Ti21} & \lambda_{T22}C_{Ti22} & \cdots & \lambda_{T2b}C_{Ti2b}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{Tr1}C_{Tir1} & \lambda_{Tr2}C_{Tir2} & \cdots & \lambda_{Trb}C_{Tirb}\end{bmatrix}\times\begin{bmatrix}\theta_{T1}C_{Ti11} & \theta_{T2}C_{Ti21} & \cdots & \theta_{Tr}C_{Tir1}\\ \theta_{T1}C_{Ti12} & \theta_{T2}C_{Ti22} & \cdots & \theta_{Tr}C_{Tir2}\\ \vdots & \vdots & \vdots & \vdots\\ \theta_{T1}C_{Ti1b} & \theta_{T2}C_{Ti2b} & \cdots & \theta_{Tr}C_{Tirb}\end{bmatrix},\quad(20)$$

$$M\left(C_{Uijl}\right)=M_C\left(\lambda_{Ujk}+C_{Uijk}-\lambda_{Ujk}C_{Uijk}\right)\times\left[M_E\left(\theta_{Uj}+C_{Uijk}-\theta_{Uj}C_{Uijk}\right)\right]^T$$

$$=\begin{bmatrix}\lambda_{I11}+C_{Ui11}-\lambda_{U11}C_{Ui11} & \lambda_{U12}+C_{Ui12}-\lambda_{U12}C_{Ui12} & \cdots & \lambda_{U1b}+C_{Ui1b}-\lambda_{U1b}C_{Ui1b}\\ \lambda_{U21}+C_{Ui21}-\lambda_{U21}C_{Ui21} & \lambda_{U22}+C_{Ui22}-\lambda_{U22}C_{Ui22} & \cdots & \lambda_{U2b}+C_{Ui2b}-\lambda_{U2b}C_{Ui2b}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{Ur1}+C_{Uir1}-\lambda_{Ur1}C_{Uir1} & \lambda_{Ur2}+C_{Uir2}-\lambda_{Ur2}C_{Uir2} & \cdots & \lambda_{Urb}+C_{Uirb}-\lambda_{Urb}C_{Uirb}\end{bmatrix},\quad(21)$$

$$\times\begin{bmatrix}\theta_{U1}+C_{Ui11}-\theta_{U1}C_{Ui11} & \theta_{U2}+C_{Ui21}-\theta_{U2}C_{Ui21} & \cdots & \theta_{Ur}+C_{Uir1}-\theta_{Ur}C_{Uir1}\\ \theta_{U1}+C_{Ui12}-\theta_{U1}C_{Ui12} & \theta_{U2}+C_{Ui22}-\theta_{U2}C_{Ui22} & \cdots & \theta_{Ur}+C_{Uir2}-\theta_{Ur}C_{Uir2}\\ \vdots & \vdots & \vdots & \vdots\\ \theta_{U1}+C_{Ui1b}-\theta_{U1}C_{Ui1b} & \theta_{U2}+C_{Ui2b}-\theta_{U2}C_{Ui2b} & \cdots & \theta_{Ur}+C_{Uirb}-\theta_{Ur}C_{Uirb}\end{bmatrix}$$

$$M\left(C_{Fijl}\right)=M_C\left(\lambda_{Fjk}+C_{Fijk}-\lambda_{Fjk}C_{Fijk}\right)\times\left[M_E\left(\theta_{Fj}+C_{Fijk}-\theta_{Fj}C_{Fijk}\right)\right]^T$$
$$=\begin{bmatrix}\lambda_{F11}+C_{Fi11}-\lambda_{F11}C_{Fk11} & \lambda_{F12}+C_{Fi12}-\lambda_{F12}C_{Fi12} & \cdots & \lambda_{F1b}+C_{Fi1b}-\lambda_{F1b}C_{Fi1b}\\ \lambda_{F21}+C_{Fi21}-\lambda_{F21}C_{Fk21} & \lambda_{F22}+C_{Fi22}-\lambda_{F22}C_{Fi22} & \cdots & \lambda_{F2b}+C_{Fi2b}-\lambda_{F2b}C_{Fi2b}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{Fr1}+C_{Fir1}-\lambda_{Fr1}C_{Fir1} & \lambda_{Fr2}+C_{Fir2}-\lambda_{Fr2}C_{Fir2} & \cdots & \lambda_{Frb}+C_{Firb}-\lambda_{Frb}C_{Firb}\end{bmatrix}. \quad (22)$$
$$\times\begin{bmatrix}\theta_{F1}+C_{Fi11}-\theta_{F1}C_{Fi11} & \theta_{F2}+C_{Fi21}-\theta_{F2}C_{Fi21} & \cdots & \theta_{Fr}+C_{Fir1}-\theta_{Fr}C_{Fir1}\\ \theta_{F1}+C_{Fi12}-\theta_{F1}C_{Fi12} & \theta_{F2}+C_{Fi22}-\theta_{F2}C_{Fi22} & \cdots & \theta_{Fr}+C_{Fir2}-\theta_{Fr}C_{Fir2}\\ \vdots & \vdots & \vdots & \vdots\\ \theta_{F1}+C_{Fi1b}-\theta_{F1}C_{Fi1b} & \theta_{F2}+C_{Fi2b}-\theta_{F2}C_{Fi2b} & \cdots & \theta_{Fr}+C_{Firb}-\theta_{Fr}C_{Firb}\end{bmatrix}$$

Step 6: The respective SVNCV matrix energy values for each alternative can be obtained by Eq. (11).
Step 7: The SVNCM energy score values of for each alternative G_{Si} (i = 1, 2, …, a) are calculated by Eq. (12).
Step 8: According to the score values, all alternatives are ranked in descending order and the alternative with the largest value is the best.

5. MAGDM Application in Primary School Site Selection

5.1 Actual Example of Primary School Site Selection

In recent years, Shaoxing's level of economic development has risen in China, and as the city's framework has been further expanded, the city's population has dispersed to multiple centers. It is necessary to build a new primary school in a suitable position of Shaoxing City in China. In this section, the feasibility and validity of the MAGDM model in a SVNCM environment are verified through an actual example of primary school site selection in Shaoxing.

By analyzing the city framework and population distribution in Shaoxing, the decision department provides four potential locations as a set of alternatives G_S = {G_{S1}, G_{S2}, G_{S3}, G_{S4}}. In the assessment issue of the alternatives, the four main requirements/attributes of the school site can be considered by construction cost (C_{S1}), regional population (C_{S2}), transport facilities (C_{S3}) and regional environment (C_{S4}). For this siting decision problem, a group of three experts E_S = {E_{S1}, E_{S2}, E_{S3}} is invited to evaluate the best alternative among them, and then the three experts' SVNCV weights are specified as θ_{C1} = ⟨(0.8, 0.7), (0.1, 0.8), (0.2, 0.7)⟩, θ_{C2} = ⟨(0.7, 0.6), (0.2, 0.7), (0.3, 0.7)⟩, and θ_{C3} = ⟨(0.6, 0.8), (0.2, 0.6), (0.1, 0.9)⟩.

The MAGDM model based on the SVNCM energy proposed in the above section can be applied to the site selection problem of this school in the following steps:
Step 1: The three experts specify the SVNCV weights of the attributes by λ_{Cjk} = ⟨(λ_{Tjk}, λ_{CTjk}), (λ_{Ujk}, λ_{CUjk}), (λ_{Fjk}, λ_{CFjk})⟩ (j = 1, 2, 3; k = 1, 2, 3, 4) for λ_{Tjk}, λ_{CTjk}, λ_{Ujk}, λ_{CUjk}, λ_{Fjk}, λ_{CFjk} ∈ [0, 1], and then they are constructed as the weight matrix of the attributes:

$$M(\lambda_{Cjk})=\begin{bmatrix} \langle(0.8,0.8),(0.1,0.7),(0.3,0.8)\rangle & \langle(0.6,0.9),(0.2,0.8),(0.1,0.7)\rangle & \langle(0.8,0.6),(0.4,0.9),(0.3,0.8)\rangle & \langle(0.7,0.7),(0.1,0.7),(0.2,0.6)\rangle \\ \langle(0.7,0.7),(0.2,0.6),(0.1,0.7)\rangle & \langle(0.6,0.8),(0.2,0.7),(0.2,0.6)\rangle & \langle(0.6,0.7),(0.1,0.8),(0.1,0.9)\rangle & \langle(0.9,0.6),(0.1,0.8),(0.2,0.9)\rangle \\ \langle(0.8,0.7),(0.3,0.7),(0.2,0.6)\rangle & \langle(0.7,0.9),(0.2,0.7),(0.2,0.7)\rangle & \langle(0.9,0.9),(0.2,0.6),(0.3,0.8)\rangle & \langle(0.8,0.8),(0.2,0.7),(0.1,0.8)\rangle \end{bmatrix}.$$

Step 2: Decision makers/experts evaluate their satisfiability levels of each alternative Gs_i over attributes Cs_k by providing the SVNCVs n_{Cijk} = <(V_{Tijk}, C_{Tijk}), (V_{Uijk}, C_{Uijk}), (V_{Fijk}, C_{Fijk})> (i, k = 1, 2, 3, 4; j = 1, 2, 3), and then SVNCMs for Gs_i for i = 1, 2, 3, 4 can be built below:

$$M\left(n_{C1jk}\right)=\begin{bmatrix} \langle(0.7,0.8),(0.2,0.7),(0.1,0.8)\rangle & \langle(0.6,0.7),(0.1,0.8),(0.3,0.7)\rangle & \langle(0.8,0.8),(0.1,0.8),(0.3,0.8)\rangle & \langle(0.9,0.8),(0.3,0.7),(0.2,0.6)\rangle \\ \langle(0.6,0.7),(0.1,0.8),(0.2,0.6)\rangle & \langle(0.7,0.9),(0.2,0.7),(0.3,0.8)\rangle & \langle(0.7,0.8),(0.2,0.7),(0.3,0.7)\rangle & \langle(0.8,0.7),(0.1,0.7),(0.2,0.6)\rangle \\ \langle(0.8,0.8),(0.4,0.7),(0.2,0.7)\rangle & \langle(0.8,0.7),(0.3,0.7),(0.2,0.8)\rangle & \langle(0.8,0.7),(0.3,0.7),(0.2,0.8)\rangle & \langle(0.6,0.8),(0.2,0.8),(0.1,0.9)\rangle \end{bmatrix},$$

$$M\left(n_{C2jk}\right)=\begin{bmatrix} \langle(0.7,0.7),(0.2,0.7),(0.3,0.6)\rangle & \langle(0.7,0.8),(0.2,0.7),(0.3,0.7)\rangle & \langle(0.8,0.9),(0.2,0.8),(0.3,0.7)\rangle & \langle(0.6,0.7),(0.1,0.7),(0.3,0.6)\rangle \\ \langle(0.8,0.6),(0.3,0.6),(0.2,0.7)\rangle & \langle(0.9,0.7),(0.3,0.7),(0.2,0.8)\rangle & \langle(0.9,0.7),(0.4,0.6),(0.3,0.7)\rangle & \langle(0.8,0.7),(0.1,0.8),(0.1,0.9)\rangle \\ \langle(0.7,0.8),(0.2,0.7),(0.3,0.8)\rangle & \langle(0.8,0.7),(0.1,0.8),(0.2,0.9)\rangle & \langle(0.8,0.6),(0.1,0.7),(0.2,0.8)\rangle & \langle(0.7,0.8),(0.2,0.8),(0.3,0.7)\rangle \end{bmatrix},$$

$$M\left(n_{C3jk}\right)=\begin{bmatrix} \langle(0.7,0.8),(0.2,0.6),(0.1,0.7)\rangle & \langle(0.9,0.8),(0.2,0.7),(0.3,0.6)\rangle & \langle(0.7,0.8),(0.2,0.7),(0.3,0.7)\rangle & \langle(0.6,0.8),(0.2,0.8),(0.3,0.8)\rangle \\ \langle(0.6,0.9),(0.2,0.7),(0.3,0.8)\rangle & \langle(0.8,0.8),(0.2,0.7),(0.1,0.9)\rangle & \langle(0.9,0.8),(0.2,0.6),(0.1,0.8)\rangle & \langle(0.8,0.9),(0.1,0.9),(0.3,0.7)\rangle \\ \langle(0.8,0.7),(0.1,0.8),(0.2,0.7)\rangle & \langle(0.7,0.9),(0.1,0.8),(0.3,0.8)\rangle & \langle(0.8,0.7),(0.1,0.7),(0.2,0.7)\rangle & \langle(0.7,0.8),(0.3,0.7),(0.1,0.8)\rangle \end{bmatrix},$$

$$M\left(n_{C4jk}\right)=\begin{bmatrix} \langle(0.9,0.7),(0.2,0.7),(0.1,0.8)\rangle & \langle(0.8,0.8),(0.2,0.8),(0.2,0.7)\rangle & \langle(0.7,0.8),(0.1,0.7),(0.3,0.7)\rangle & \langle(0.6,0.8),(0.2,0.8),(0.3,0.6)\rangle \\ \langle(0.7,0.8),(0.3,0.7),(0.2,0.9)\rangle & \langle(0.8,0.8),(0.3,0.7),(0.1,0.8)\rangle & \langle(0.9,0.7),(0.2,0.8),(0.1,0.8)\rangle & \langle(0.8,0.7),(0.4,0.8),(0.2,0.7)\rangle \\ \langle(0.8,0.9),(0.1,0.8),(0.1,0.8)\rangle & \langle(0.9,0.7),(0.1,0.8),(0.3,0.7)\rangle & \langle(0.7,0.8),(0.2,0.9),(0.2,0.9)\rangle & \langle(0.7,0.8),(0.3,0.7),(0.1,0.9)\rangle \end{bmatrix}.$$

Step 3: In view of the influence of the decision makers/experts' weights θ_{Cj} on the four SVNCMs for G_{Si} for i = 1, 2, 3, 4, the weighted SVNCMs using Eq. (15) can be obtained below:

$$M_E\left(\theta_{Cj}\otimes n_{C1jk}\right)=\begin{bmatrix} \langle(0.56,0.56),(0.28,0.94),(0.28,0.94)\rangle & \langle(0.48,0.49),(0.19,0.96),(0.44,0.91)\rangle & \langle(0.64,0.56),(0.19,0.96),(0.44,0.94)\rangle & \langle(0.72,0.56),(0.37,0.94),(0.36,0.88)\rangle \\ \langle(0.42,0.42),(0.28,0.94),(0.44,0.88)\rangle & \langle(0.49,0.54),(0.36,0.91),(0.51,0.94)\rangle & \langle(0.49,0.48),(0.36,0.91),(0.51,0.91)\rangle & \langle(0.56,0.42),(0.28,0.91),(0.44,0.88)\rangle \\ \langle(0.48,0.64),(0.52,0.88),(0.28,0.97)\rangle & \langle(0.48,0.56),(0.44,0.88),(0.28,0.98)\rangle & \langle(0.48,0.56),(0.44,0.88),(0.28,0.98)\rangle & \langle(0.36,0.64),(0.36,0.92),(0.19,0.99)\rangle \end{bmatrix},$$

$$M_E\left(\theta_{Cj}\otimes n_{C2jk}\right)=\begin{bmatrix} \langle(0.56,0.49),(0.28,0.94),(0.44,0.88)\rangle & \langle(0.56,0.56),(0.28,0.94),(0.44,0.91)\rangle & \langle(0.64,0.63),(0.28,0.96),(0.44,0.91)\rangle & \langle(0.48,0.49),(0.19,0.94),(0.44,0.88)\rangle \\ \langle(0.56,0.36),(0.44,0.88),(0.44,0.91)\rangle & \langle(0.63,0.42),(0.44,0.91),(0.44,0.94)\rangle & \langle(0.63,0.42),(0.52,0.88),(0.51,0.91)\rangle & \langle(0.56,0.42),(0.28,0.94),(0.37,0.97)\rangle \\ \langle(0.42,0.64),(0.36,0.88),(0.37,0.98)\rangle & \langle(0.48,0.56),(0.28,0.92),(0.28,0.99)\rangle & \langle(0.48,0.48),(0.28,0.88),(0.28,0.98)\rangle & \langle(0.42,0.64),(0.36,0.92),(0.37,0.97)\rangle \end{bmatrix},$$

$$M_E\left(\theta_{Cj}\otimes n_{C3jk}\right)=\begin{bmatrix} \langle(0.56,0.56),(0.28,0.92),(0.28,0.91)\rangle & \langle(0.72,0.56),(0.28,0.94),(0.44,0.88)\rangle & \langle(0.56,0.56),(0.28,0.94),(0.44,0.91)\rangle & \langle(0.48,0.56),(0.28,0.96),(0.44,0.94)\rangle \\ \langle(0.42,0.54),(0.36,0.91),(0.51,0.94)\rangle & \langle(0.56,0.48),(0.36,0.91),(0.37,0.97)\rangle & \langle(0.63,0.48),(0.36,0.88),(0.37,0.94)\rangle & \langle(0.56,0.54),(0.28,0.97),(0.51,0.91)\rangle \\ \langle(0.48,0.56),(0.28,0.92),(0.28,0.97)\rangle & \langle(0.42,0.72),(0.28,0.92),(0.37,0.98)\rangle & \langle(0.48,0.56),(0.28,0.88),(0.28,0.97)\rangle & \langle(0.42,0.64),(0.44,0.88),(0.19,0.98)\rangle \end{bmatrix},$$

$$M_E\left(\theta_{Cj}\otimes n_{C4jk}\right)=\begin{bmatrix} \langle(0.72,0.49),(0.28,0.94),(0.28,0.94)\rangle & \langle(0.64,0.56),(0.28,0.96),(0.36,0.91)\rangle & \langle(0.56,0.56),(0.19,0.94),(0.44,0.91)\rangle & \langle(0.48,0.56),(0.28,0.96),(0.44,0.88)\rangle \\ \langle(0.49,0.48),(0.44,0.91),(0.44,0.97)\rangle & \langle(0.56,0.48),(0.44,0.91),(0.37,0.94)\rangle & \langle(0.63,0.42),(0.36,0.94),(0.37,0.94)\rangle & \langle(0.56,0.42),(0.52,0.94),(0.44,0.91)\rangle \\ \langle(0.48,0.72),(0.28,0.92),(0.19,0.98)\rangle & \langle(0.54,0.56),(0.28,0.92),(0.37,0.97)\rangle & \langle(0.42,0.64),(0.36,0.96),(0.28,0.99)\rangle & \langle(0.42,0.64),(0.44,0.88),(0.19,0.99)\rangle \end{bmatrix}.$$

Step 4: In terms of the influence of the attribute weights λ_{Cjk} on the four SVNCMs for G_{Si} for i = 1, 2, 3, 4, the weighted SVNCMs using Eq. (16) can be obtained below:

$$M_C\left(\lambda_{Cjk}\otimes n_{C1jk}\right)=\begin{bmatrix}\langle(0.56,0.64),(0.28,0.91),(0.37,0.96)\rangle & \langle(0.36,0.63),(0.28,0.96),(0.37,0.91)\rangle & \langle(0.64,0.48),(0.46,0.98),(0.51,0.96)\rangle & \langle(0.63,0.56),(0.37,0.91),(0.36,0.84)\rangle \\ \langle(0.42,0.49),(0.28,0.92),(0.28,0.88)\rangle & \langle(0.42,0.72),(0.36,0.91),(0.44,0.92)\rangle & \langle(0.42,0.56),(0.28,0.94),(0.37,0.97)\rangle & \langle(0.72,0.42),(0.19,0.94),(0.36,0.96)\rangle \\ \langle(0.64,0.56),(0.58,0.91),(0.36,0.88)\rangle & \langle(0.56,0.63),(0.44,0.91),(0.36,0.94)\rangle & \langle(0.72,0.63),(0.44,0.88),(0.44,0.96)\rangle & \langle(0.48,0.64),(0.36,0.94),(0.19,0.98)\rangle\end{bmatrix},$$

$$M_C\left(\lambda_{Cjk}\otimes n_{C2jk}\right)=\begin{bmatrix}\langle(0.56,0.56),(0.28,0.91),(0.51,0.92)\rangle & \langle(0.42,0.72),(0.36,0.94),(0.37,0.91)\rangle & \langle(0.64,0.54),(0.52,0.98),(0.51,0.94)\rangle & \langle(0.42,0.49),(0.19,0.91),(0.44,0.84)\rangle \\ \langle(0.56,0.42),(0.44,0.84),(0.28,0.91)\rangle & \langle(0.54,0.56),(0.44,0.91),(0.36,0.92)\rangle & \langle(0.54,0.49),(0.46,0.92),(0.37,0.97)\rangle & \langle(0.72,0.42),(0.19,0.96),(0.28,0.99)\rangle \\ \langle(0.56,0.56),(0.44,0.91),(0.44,0.92)\rangle & \langle(0.56,0.63),(0.28,0.94),(0.36,0.97)\rangle & \langle(0.72,0.54),(0.28,0.88),(0.44,0.96)\rangle & \langle(0.56,0.64),(0.36,0.94),(0.37,0.94)\rangle\end{bmatrix},$$

$$M_C\left(\lambda_{Cjk}\otimes n_{C3jk}\right)=\begin{bmatrix}\langle(0.56,0.64),(0.28,0.88),(0.37,0.94)\rangle & \langle(0.54,0.72),(0.36,0.94),(0.37,0.88)\rangle & \langle(0.56,0.48),(0.52,0.97),(0.51,0.94)\rangle & \langle(0.42,0.56),(0.28,0.94),(0.44,0.92)\rangle \\ \langle(0.42,0.63),(0.36,0.88),(0.37,0.94)\rangle & \langle(0.48,0.64),(0.36,0.91),(0.28,0.96)\rangle & \langle(0.54,0.56),(0.28,0.92),(0.19,0.98)\rangle & \langle(0.72,0.54),(0.19,0.98),(0.44,0.97)\rangle \\ \langle(0.64,0.49),(0.37,0.94),(0.36,0.88)\rangle & \langle(0.49,0.81),(0.28,0.94),(0.44,0.94)\rangle & \langle(0.72,0.63),(0.28,0.88),(0.44,0.94)\rangle & \langle(0.56,0.64),(0.44,0.91),(0.19,0.96)\rangle\end{bmatrix},$$

$$M_C\left(\lambda_{Cjk}\otimes n_{C4jk}\right)=\begin{bmatrix}\langle(0.72,0.56),(0.28,0.91),(0.37,0.96)\rangle & \langle(0.48,0.72),(0.36,0.96),(0.28,0.91)\rangle & \langle(0.56,0.48),(0.46,0.97),(0.51,0.94)\rangle & \langle(0.42,0.56),(0.28,0.94),(0.44,0.84)\rangle \\ \langle(0.49,0.56),(0.44,0.88),(0.28,0.97)\rangle & \langle(0.48,0.64),(0.44,0.91),(0.28,0.92)\rangle & \langle(0.54,0.49),(0.28,0.96),(0.19,0.98)\rangle & \langle(0.72,0.42),(0.46,0.96),(0.36,0.97)\rangle \\ \langle(0.64,0.63),(0.37,0.94),(0.28,0.92)\rangle & \langle(0.63,0.63),(0.28,0.94),(0.44,0.91)\rangle & \langle(0.63,0.72),(0.36,0.96),(0.44,0.98)\rangle & \langle(0.56,0.64),(0.44,0.91),(0.19,0.98)\rangle\end{bmatrix}.$$

Step 5: Using Eqs. (17)–(22), we obtain the collective SVNCMs $M(n_{Cijk}) = <(M(V_{Tijl}), M(C_{Tijl}), (M(V_{Uijl}), M(C_{Uijl}), (M(V_{Fijl}), M(C_{Fijl})>$ (j, l = 1, 2, 3; i = 1, 2, 3, 4), where $M(V_{Tijl})$, $M(C_{Tijl})$, $(M(V_{Uijl})$, $M(C_{Uijl})$, and $(M(V_{Fijl})$, $M(C_{Fijl})$ are given as follows:

$$M\left(V_{T1jl}\right)=\begin{bmatrix}1.3496 & 1.0780 & 0.9756\\ 1.2240 & 1.9912 & 0.8640\\ 1.4336 & 1.1648 & 1.0944\end{bmatrix},\quad M\left(V_{T2jl}\right)=\begin{bmatrix}1.1600 & 1.2166 & 0.9204\\ 1.3072 & 1.3972 & 1.0560\\ 1.3568 & 1.4336 & 1.0848\end{bmatrix},$$

$$M\left(V_{T3jl}\right)=\begin{bmatrix}1.2176 & 1.1256 & 0.9408\\ 1.2288 & 1.1886 & 0.9648\\ 1.3832 & 1.3104 & 1.0938\end{bmatrix},\quad M\left(V_{T4jl}\right)=\begin{bmatrix}1.3408 & 1.2096 & 1.0164\\ 1.3080 & 1.2523 & 1.0236\\ 1.4856 & 1.3769 & 1.1472\end{bmatrix},$$

$$M\left(V_{U1jl}\right)=\begin{bmatrix}0.3559 & 0.4484 & 0.6044\\ 0.2703 & 0.3620 & 0.4956\\ 0.4628 & 0.5800 & 0.8184\end{bmatrix},\ M\left(V_{U2jl}\right)=\begin{bmatrix}0.3609 & 0.6052 & 0.4156\\ 0.4113 & 0.6796 & 0.4788\\ 0.3484 & 0.5632 & 0.4448\end{bmatrix},$$

$$M\left(V_{U3jl}\right)=\begin{bmatrix}0.4032 & 0.4960 & 0.4480\\ 0.3332 & 0.4132 & 0.3636\\ 0.3836 & 0.4580 & 0.4540\end{bmatrix},\ M\left(V_{U4jl}\right)=\begin{bmatrix}0.3450 & 0.5928 & 0.4680\\ 0.4284 & 0.7272 & 0.5496\\ 0.3736 & 0.6444 & 0.5052\end{bmatrix},$$

$$M\left(V_{F1jl}\right)=\begin{bmatrix}0.6204 & 0.7700 & 0.4184\\ 0.5644 & 0.6947 & 0.3736\\ 0.5212 & 0.6500 & 0.3609\end{bmatrix},\ M\left(V_{F2jl}\right)=\begin{bmatrix}0.8052 & 0.8101 & 0.5979\\ 0.5676 & 0.5739 & 0.4116\\ 0.7084 & 0.7133 & 0.5237\end{bmatrix},$$

$$M\left(V_{F3jl}\right)=\begin{bmatrix}0.6844 & 0.7387 & 0.4669\\ 0.5040 & 0.5870 & 0.3440\\ 0.5716 & 0.6061 & 0.4229\end{bmatrix},\ M\left(V_{F4jl}\right)=\begin{bmatrix}0.6224 & 0.6487 & 0.4003\\ 0.4212 & 0.4555 & 0.2784\\ 0.5140 & 0.5324 & 0.3753\end{bmatrix},$$

$$M\left(C_{T1jl}\right)=\begin{bmatrix}1.2495 & 1.0746 & 1.3896\\ 1.1760 & 1.0398 & 1.2992\\ 1.3335 & 1.1466 & 1.4736\end{bmatrix},\ M\left(C_{T2ii}\right)=\begin{bmatrix}1.2579 & 0.9366 & 1.3344\\ 1.0339 & 0.7686 & 1.0864\\ 1.2810 & 0.9618 & 1.3800\end{bmatrix},$$

$$M\left(C_{T3jl}\right)=\begin{bmatrix}1.3440 & 1.2240 & 1.5040\\ 1.3272 & 1.2078 & 1.4728\\ 1.4392 & 1.3014 & 1.6200\end{bmatrix},\ M\left(C_{T4jl}\right)=\begin{bmatrix}1.2600 & 1.0512 & 1.4720\\ 1.1424 & 0.9582 & 1.3440\\ 1.4231 & 1.1760 & 1.6768\end{bmatrix},$$

$$M\left(C_{U1jl}\right)=\begin{bmatrix}3.5732 & 3.4489 & 3.3452\\ 3.5244 & 3.4037 & 3.3024\\ 3.4574 & 3.3397 & 3.2408\end{bmatrix},\ M\left(C_{U2jl}\right)=\begin{bmatrix}3.5352 & 3.3740 & 3.3652\\ 3.4306 & 3.2793 & 3.2692\\ 3.4674 & 3.3142 & 3.3048\end{bmatrix},$$

$$M\left(C_{U3jl}\right)=\begin{bmatrix}3.5074 & 3.4216 & 3.3552\\ 3.4706 & 3.3891 & 3.3188\\ 3.4492 & 3.3679 & 3.3048\end{bmatrix},\ M\left(C_{U4jl}\right)=\begin{bmatrix}3.5912 & 3.4971 & 3.4788\\ 3.5248 & 3.4337 & 3.4132\\ 3.5620 & 3.4686 & 3.4520\end{bmatrix},$$

$$M\left(C_{F1jl}\right)=\begin{bmatrix}3.3721 & 3.3130 & 3.5954\\ 3.4210 & 3.3667 & 3.6562\\ 3.4474 & 3.3940 & 3.6858\end{bmatrix},\ M\left(C_{F2jl}\right)=\begin{bmatrix}3.2323 & 3.3628 & 3.5385\\ 3.3919 & 3.5359 & 3.7135\\ 3.3931 & 3.5344 & 3.7145\end{bmatrix},$$

$$M\left(C_{F3jl}\right)=\begin{bmatrix}3.3500 & 3.4580 & 3.5876\\ 3.5038 & 3.6187 & 3.7538\\ 3.3858 & 3.4962 & 3.6274\end{bmatrix},\ M\left(C_{F4jl}\right)=\begin{bmatrix}3.3251 & 3.4346 & 3.5857\\ 3.4944 & 3.6096 & 3.7735\\ 3.4471 & 3.5608 & 3.7247\end{bmatrix}.$$

Step 6: Using Eq. (11), the respective SVNCM energy values for all alternatives can be obtained below:

$E[M(n_{C1jl})] = $<(4.4771, 4.9817), (2.0120, 13.6191), (2.2225, 13.8893)>;

$E[M(n_{C2jl})] = $<(4.8330, 4.5180), (1.9188, 13.4854), (2.5293, 13.9709)>;

$E[M(n_{C3jl})] = $<(4.5915, 5.5376), (1.6398, 13.5913), (2.1478, 14.1255)>;

$E[M(n_{C4jl})] = $<(4.9048, 5.1673), (2.0910, 13.9646), (1.8518, 14.2063)>.

Step 7: Using Eq. (12), the SVNCM energy score values for each alternative G_{Si} (i = 1, 2, 3, 4) is calculated and given as follows:

$Z\{E[M(n_{C1jk})]\} = 41.1393$, $Z\{E[M(n_{C2jk})]\} = 34.2103$, $Z\{E[M(n_{C3jk})]\} = 42.8003$, and $Z\{E[M(n_{C4jk})]\} = 53.5819$.

Step 8: According to the score values, the ranking order of the four alternatives is $G_{S4} > G_{S3} > G_{S1} > G_{S2}$ and the best one is G_{S4}.

5.2 Comparative Investigation of the Decision Results Between SVNM and SVNCM Scenarios

Since the existing MAGDM model [5] introduced in the SVNM scenario cannot perform the school site selection problem in the SVNCM scenario, we must ignore all the credibility values in SVNCMs as a special case of the site selection problem. Thus, we can apply the existing MAGDM model based on SVNM energy in the above site section problem to compare the proposed model with the existing model in the SVNM and SVNCM scenarios.

Based on the MAGDM algorithm in [5], we can obtain the respective SVNM energy values for all alternatives G_{Si} (i = 1, 2, 3, 4):

$E[M(n_{1jk})] = $<4.4771, 2.0120, 2.2225>, $E[M(n_{2jk})] = $<4.8330, 1.9188, 2.5293>, $E[M(n_{3jk})] = $<4.5915, 1.6398, 2.1478>, and $E[M(n_{4jk})] = $<4.9048, 2.0910, 1.8518>.

Using Eq. (8) [5], the SVNM energy score values for all alternative G_{Si} (i = 1, 2, 3, 4) are calculated and given as follows:

$H\{E[M(n_{1jk})]\} = 8.7436$, $H\{E[M(n_{2jk})]\} = 9.0555$, $H\{E[M(n_{3jk})]\} = 8.6750$, and $H\{E[M(n_{4jk})]\} = 9.4150$.

According to the score values, the ranking order of the four alternatives is $G_{S4} > G_{S2} > G_{S1} > G_{S3}$ and the best one is G_{S4}.

For the comparative convenience of the decision results in the SVNM and SVNCM scenarios, all results are shown in Table 1.

Table 1. Decision results between SVNM and SVNCM scenarios

MAGDM model	Ranking	Best one	Information environment
Proposed model	$G_{S4} > G_{S3} > G_{S1} > G_{S2}$	G_{S4}	SVNCMs
Existing model [5]	$G_{S4} > G_{S2} > G_{S1} > G_{S3}$	G_{S4}	SVNMs

In terms of the decision results in Table 1, the ranking orders of the four alternatives between the SVNM and SVNCM scenarios are different, then the best one G_{S4} is the same in the school site selection problem. It is clear that the credibility measures with respect to true, false, and uncertain evaluation values reveal their importance in the neutrosophic MAGDM problem because they can affect the ranking order and decision credibility of the four alternatives. Furthermore, the proposed model is the generalization of the existing model [5] and more general and creditable than the existing model in neutrosophic MAGDM problems under uncertain and inconsistent environments.

6. Conclusions

Regarding an extension of SVNM energy, this study presented SVNCM energy and its properties. Then, a MAGDM model using the SVNCM energy was established in the SVNCM scenario, which can solve MAGDM problems and fill a research gap of MAGDM in the SVNCM scenario. Finally, the proposed MAGDM model was applied to the school site selection problem, then the comparative investigation of the decision results in the SVNM and SVNCM scenarios indicated that the proposed model was more general and creditable than the existing model in neutrosophic MAGDM problems under uncertain and inconsistent environments. Furthermore, the credibility measures with respect to true, false and uncertain evaluation values revealed their importance and necessity in the neutrosophic MAGDM problem and affected the ranking of the alternatives, then the decision credibility of the proposed model in the SVNCM scenario is significantly better than the existing model in the SVNM scenario.

However, the proposed SVNCM energy and MAGDM model can be further applied in image processing, clustering analysis, project risk evaluation, slope stability analysis/assessment, and so on in engineering fields, which are future research directions.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest
The authors declare that there is no conflict of interest in the research.

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Fixed Point Results in Complex Valued Neutrosophic b-Metric Spaces with Application

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Abstract: In this manuscript, we introduce the idea of complex-valued Neutrosophic b-metric spaces along with numerous significant illustrations. We provide fixed-point results for contraction maps. To support the main result, we establish the existence and uniqueness of solutions for nonlinear integral equations after the work.

Keywords: Fuzzy Metric; Complex Valued Neutrosophic Metric Space; Fixed Point; Contractive Map; Unique Solution.

1. Introduction

Azam et al. [1] pioneered the idea of complex-valued metric spaces in 2011. Rouzkard et al. [2] studied and extended the conclusions of [1] by investigating numerous common fixed point theorems in this space. Many standard fixed point solutions in such space for mappings satisfying rational expressions on a closed ball were examined by Ahmad et al. [3]. Common fixed point theorem in complex-valued b-metric established by Rao et al. [4]. Following the development of this concept, Mukheimer [5] discovered common fixed point outcomes of a pair of self-mappings meeting a rational inequality in complex-valued b-metric space. Zadeh [6] established the basis for fuzzy mathematics in 1965. Kramosil and Michalek [7] initially brought up the concept of fuzzy metric-like space and then modified it by George and Veeramani [8]. Atanassov [9] stirred things up by adding the idea of a non-membership grade of fuzzy set theory. Fuzzy metric space has been widened to Intuitionistic fuzzy metric space by Park [10]. Park used continuous triangular norm as well as continuous triangular conorm to describe this idea. Smarandache [11] described the concept of neutrosophic logic and neutrosophic sets in 1998.

This study aims to present the concept of Complex Valued Neutrosophic b-metric Space. In addition, this research expands on previous fixed-point findings over contractions. To strengthen, we finish our work with an application to integral equations and an example illustrating the applicability of our main results.

2. Preliminaries

This study will require the following definitions and results.

$\mathbb{C}$ denotes the set of complex numbers.

We set $\mathfrak{H} = \{(\wp, q): 0 \leq \wp < \infty, 0 \leq q < \infty\} \subset \mathbb{C}$.

A partial ordering $\precsim$ on $\mathbb{C}$ is defined by $\tau_1 \precsim \tau_2$ (equivalently, $\tau_2 \precsim \tau_1$) $\Leftrightarrow$ $Re(\tau_1) \leq Re(\tau_2)$ and $Im(\tau_1) \leq Im(\tau_2)$. The closed unit complex interval is defined as $\mathfrak{F} = \{(\wp, q): 0 \leq \wp < 1, 0 \leq q < 1\}$ and the open unit complex interval by $\mathfrak{F}_{\ddot{o}} = \{(\wp, q): 0 < \wp < 1, 0 < q < 1\}$.

The set $\{(\wp, q): 0 < \wp < \infty, 0 < q < \infty\}$ denoted by $\mathfrak{H}_{\ddot{o}}$. The elements $(1, 1), (0,0) \in \mathfrak{H}$ are indicated by ℓ and $\ddot{o}$, respectively.

Remark 2.1[12]. Let $\{\tau_\iota\}$ be a sequence in $\mathfrak{H}$. Then,

(i) If $\{\tau_\iota\}$ is monotonic in $\mathfrak{H}$ and there exists $\rho, \sigma \in \mathfrak{H}$ such that $\rho \precsim \tau_\iota \precsim \sigma$, for every $\iota \in \mathrm{N}$, then there exists a $\tau \in \mathfrak{H}$ such that $\lim_{\iota \to \infty} \tau_\iota = \tau$.

(ii) $\Theta \subset \mathbb{C}$ is that there exists $\rho, \sigma \in \mathbb{C}$ with $\rho \precsim \mathbb{C} \precsim \sigma$ for all $\theta \in \Theta$, then $\inf \Theta$ and $\sup \Theta$ both exist.

Remark 2.2 [12]. Let $\tau_\iota, \tau'_\iota, \eta \in \mathfrak{H}$ for every $\iota \in \mathrm{N}$. Then,

(i) If $\tau_\iota \precsim \tau'_\iota \precsim \ell$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota \to \infty} \tau_\iota = \ell$, then $\lim_{\iota \to \infty} \tau'_\iota = \ell$.

(ii) If $\tau_\iota \precsim \eta$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota \to \infty} \tau_\iota = \tau \in \mathfrak{H}$, then $\iota \precsim \eta$.

(iii) If $\eta \precsim \tau_\iota$ for every $\iota \in \mathrm{N}$ and $\lim_{\iota \to \infty} \tau_\iota = \tau \in \mathfrak{H}$, then $\eta \precsim \iota$.

Definition 2.3 [12]. Let $\{\tau_\iota\}$ be a sequence in $\mathfrak{H}$. If for all $\tau \in \mathfrak{H}$ there exists an $\iota_0 \in \mathrm{N}$ such that $\tau \precsim \tau_\iota$ for all $\iota > \iota_0$. Then $\{\tau_\iota\}$ is named to be diverged to ∞ as $\iota \to \infty$, and we write $\lim_{\iota \to \infty} \tau_\iota = \infty$.

Definition 2.4 [12]. A binary operation $* : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ is named a complex-valued t-norm, if for all $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathfrak{F}$

(i) $\tau_1 * \tau_2 = \tau_2 * \tau_1$;

(ii) $\tau * \ddot{o} = \ddot{o}$, $\tau * \ell = \tau$;

(iii) $\tau_1 * (\tau_2 * \tau_3) = (\tau_1 * \tau_2) * \tau_3$;

(iv) $\tau_1 * \tau_2 \precsim \tau_3 * \tau_4$ whenever $\tau_1 \precsim \tau_3$, $\tau_2 \precsim \tau_4$.

Example 2.5 [12].

(i) $\tau_1 * \tau_2 = (\wp_1 \wp_2, q_1 q_2)$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(ii) $\tau_1 * \tau_2 = (\min\{\wp_1, \wp_2\}, \min\{q_1, q_2\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(iii) $\tau_1 * \tau_2 = (\max\{\wp_1 + \wp_2 - 1, 0\}, \max\{q_1 + q_2 - 1, 0\})$,
for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$.
These are examples of complex-valued t-norm.

Example 2.6 [12]. The following are examples of complex-valued t-conorm:

(i) $\tau_1 \star \tau_2 = (\max\{\wp_1, \wp_2\}, \max\{q_1, q_2\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$,

(ii) $\tau_1 \star \tau_2 = (\min\{\wp_1 + \wp_2, 1\}, \min\{q_1 + q_2, 1\})$, for all $\tau_1 = (\wp_1, q_1)$, $\tau_2 = (\wp_2, q_2) \in \mathfrak{F}$.

Definition 2.7. Let Ξ be a nonvoid set, $*$, $\star$ are complex-valued continuous t-norm and t-conorm, $\widetilde{\mathfrak{P}}$, $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{Q}}$ are complex fuzzy sets on $\Xi^2 \times \mathfrak{H}_{\ddot{o}}$ fulfilling the following assertions:

(1) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) + \widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) + \widetilde{\mathfrak{Q}}(\mathfrak{u}, \mathfrak{v}, \tau) \precsim 3$;

(2) $\ddot{o} \prec \widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau)$;

(3) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \ell$ for every $\tau \in \mathfrak{H}_{\ddot{o}} \Leftrightarrow$ if $\mathfrak{u} = \mathfrak{v}$;

(4) $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \widetilde{\mathfrak{P}}(\mathfrak{v}, \mathfrak{u}, \tau)$;

(5) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) * \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{w},\tau') \precsim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{w},\tau+\tau')$;

(6) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(7) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(8) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u} = \mathfrak{v}$;

(9) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{u},\tau)$;

(10) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{w},\tau') \succsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{w},\tau+\tau')$;

(11) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(12) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(13) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u} = \mathfrak{v}$;

(14) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{u},\tau)$;

(15) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{w},\tau') \succsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{w},\tau+\tau')$;

(16) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous.

The Triplet $(\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}})$ is called a Complex Valued Neutrosophic Metric Space (CVNMS).

Definition 2.8. Let Ξ be a nonvoid set, $\theta \geq 1$ be a given real number, $*$, $\star$ are complex-valued continuous t-norm and t- conorm , $\widetilde{\mathfrak{P}}$, $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{Q}}$ are complex fuzzy sets on $\Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}}$ fulfilling the following assertions. Then $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\star,\theta)$ is called a Complex Valued Neutrosophic b-Metric Space (CVNbMS). For all $\mathfrak{u},\mathfrak{v},\mathfrak{w} \in \Xi$ and $\tau,\tau' \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$.

(1) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) + \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) + \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \precsim 3$;

(2) $\ddot{\mathfrak{o}} \prec \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)$;

(3) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \ell$ for every $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}} \Leftrightarrow \mathfrak{u} = \mathfrak{v}$;

(4) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{u},\tau)$;

(5) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) * \widetilde{\mathfrak{P}}(\mathfrak{v},\mathfrak{w},\tau') \precsim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{w},\theta(\tau+\tau'))$;

(6) $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(7) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(8) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u} = \mathfrak{v}$;

(9) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{u},\tau)$;

(10) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{L}}(\mathfrak{v},\mathfrak{w},\tau') \succsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{w},\theta(\tau+\tau'))$;

(11) $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous;

(12) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \prec \ell$;

(13) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \ddot{\mathfrak{o}}$, for all $\tau \in (0,\infty) \Leftrightarrow \mathfrak{u} = \mathfrak{v}$;

(14) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{u},\tau)$;

(15) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \star \widetilde{\mathfrak{Q}}(\mathfrak{v},\mathfrak{w},\tau') \succsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{w},\theta(\tau+\tau'))$;

(16) $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},.) : \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ is continuous.

Example 2.9 Let (Ξ,ρ,θ) be a b-Metric Space (bMS). Let $\tau_1 * \tau_2 = (\min\{\wp_1,\wp_2\},\min\{q_1,q_2\})$, $\tau_1 \star \tau_2 = (\max\{\wp_1,\wp_2\},\max\{q_1,q_2\})$ for all $\tau_1 = (\wp_1,q_1), \tau_2 = (\wp_2,q_2) \in \mathfrak{F}$. Let us consider the Complex

Fuzzy Sets[*CFS*] $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}} : \Xi^2 \times \mathfrak{H}_{\ddot{o}} \to \mathfrak{F}$ such that $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \frac{pq}{pq+\rho(\mathfrak{u},\mathfrak{v})}\ell, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \frac{\rho(\mathfrak{u},\mathfrak{v})}{pq+\rho(\mathfrak{u},\mathfrak{v})}\ell,\ \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) =$ $\frac{\rho(\mathfrak{u},\mathfrak{v})}{pq}\ell$, where $\tau = (p,q) \in \mathfrak{H}_{\ddot{o}}$. Then, $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is a *CVNbMS*.

Lemma 2.10 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* and $\tau_1, \tau_2 \in \mathbb{C}$. If $\tau_1 \prec \tau_2$, then $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_1) \precsim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\theta\tau_2)$, $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_1) \succsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\theta\tau_2)$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_1) \succsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\theta\tau_2)$ for all $\mathfrak{u},\mathfrak{v} \in \Xi$.

Proof. Let $\tau_1, \tau_2 \in \mathfrak{H}_{\ddot{o}}$ be such that $\tau_1 \prec \tau_2$.

Therefore, $\tau_2 - \tau_1 \in \mathfrak{H}_{\ddot{o}}$ and so that for all $\mathfrak{u},\mathfrak{v} \in \Xi$, we get $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_1) = \ell * \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_1) = \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{u},\tau_2 - \tau_1) * \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_1) \precsim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\theta\tau_2)$

$\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\theta\tau_2) \precsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{u},\tau_2 - \tau_1) \star \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_1) \precsim 0 \star \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_1)$ and

$\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\theta\tau_2) \precsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{u},\tau_2 - \tau_1) \star \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_1) \precsim 0 \star \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_1)$.

Definition 2.11 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* and $\{\mathfrak{u}_\iota\}$ be a sequence in Ξ.

(i) $\{\mathfrak{u}_\iota\}$ converges to $\mathfrak{u} \in \Xi$ if for every $\gamma \in \mathfrak{F}_{\ddot{o}}$ and every $\tau \in \mathfrak{H}_{\ddot{o}}$, there exists $\iota_0 \in \mathbb{N}$ such that, for every $\iota > \iota_0$, $\ell - \gamma \prec \widetilde{\mathfrak{P}}(\mathfrak{u}_\iota,\mathfrak{u},\tau)$, $\widetilde{\mathfrak{L}}(\mathfrak{u}_\iota,\mathfrak{u},\tau) \prec \gamma$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u}_\iota,\mathfrak{u},\tau) \prec \gamma$. We denote this by $\lim_{\iota\to\infty} \mathfrak{u}_\iota = \mathfrak{u}$.

(ii) $\{\mathfrak{u}_\iota\}$ in Ξ is named to be a Cauchy sequence in $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ if for every $\tau \in \mathfrak{H}_{\ddot{o}}$, $\lim_{\iota\to\infty} inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ell$, $\lim_{\iota\to\infty} sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ddot{o}$ and $\lim_{\iota\to\infty} sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ddot{o}$.

(iii) $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is known to be a complete *CVNbMS* if for every Cauchy sequence $\{\mathfrak{u}_\iota\}$ in $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$, there exists an $\mathfrak{u} \in \Xi$ such that $\lim_{\iota\to\infty} \mathfrak{u}_\iota = \mathfrak{u}$.

Lemma 2.12 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS*. A sequence $\{\mathfrak{u}_\iota\}$ in Ξ converge to $\mathfrak{u} \in \Xi \Leftrightarrow \lim_{\iota\to\infty} \widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ell$, $\lim_{\iota\to\infty} \widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ddot{o}$ and $\lim_{\iota\to\infty} \widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) = \ddot{o}$ holds for all $\tau \in \mathfrak{H}_{\ddot{o}}$.

3. Main Results

Theorem 3.1 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS* such that, for every sequence $\{\tau_\iota\}$ in $\mathfrak{H}_{\ddot{o}}$ with $\lim_{\iota\to\infty} \tau_\iota = \infty$, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_\iota) = \ell$, $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_\iota) = \ddot{o}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_\iota) = \ddot{o}$ for all $\mathfrak{u} \in \Xi$. Let $\mathfrak{k} : \Xi \to \Xi$ be a mapping satisfying

$$\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau),\ \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{ and } \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \quad (3.1.1)$$

For all $\mathfrak{u},\mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{o}}$ where $\delta \in (0,1)$. Then $\mathfrak{k}$ has a unique fixed point in Ξ.

Proof:

Let $\mathfrak{u}_0$ be a random element of Ξ and define the sequence $\{\mathfrak{u}_\iota\}$ in Ξ by the iterative method $\mathfrak{u}_\iota = \mathfrak{k}\mathfrak{u}_{\iota-1}$ for every $\iota \in$ N. If $\mathfrak{u}_\iota = \mathfrak{u}_{\iota-1}$ for some $\iota \in$ N, then $\mathfrak{u}_\iota$ is a fixed point of $\mathfrak{k}$.

So $\mathfrak{u}_\iota \neq \mathfrak{u}_{\iota-1}$ for every $\iota \in$ N. We claim that $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Define $\mathfrak{W}_\iota = \{\widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) : m > \iota\}$, $\mathfrak{N}_\iota = \{\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) : m > \iota\}$ and $\mathfrak{O}_\iota = \{\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) : m > \iota\}$ for all $\iota \in$ N and $\tau \in \mathfrak{H}_{\ddot{o}}$.

Since $\theta \prec \widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \precsim \ell$, $\theta \prec \widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \precsim \ell$ and $\theta \prec \widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau) \precsim \ell$ for every $m \in$ N with $m > \iota$ and from Remark (2.1)(ii), inf $\mathfrak{W}_\iota = \alpha_\iota$, sup$\mathfrak{N}_\iota = \beta_\iota$ and sup$\mathfrak{O}_\iota = \varrho_\iota$ exists for all $\iota \in$ N.

Using Lemma (2.10) and (3.1.1), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \precsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \tau) = \widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \quad (3.1.2)$$

$$\widetilde{\mathfrak{L}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \succsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \tau) = \widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \quad (3.1.3)$$

and $$\widetilde{\mathfrak{Q}}(\mathfrak{u}_m, \mathfrak{u}_\iota, \tau) \succsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}, \mathfrak{k}\mathfrak{v}, \tau) = \widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \quad (3.1.4)$$

for $\tau \in \mathfrak{H}_{\ddot{o}}$ and $m, \iota \in \mathrm{N}$ with $m > \iota$.

Since $\ddot{o} \precsim \alpha_\iota \precsim \alpha_{\iota+1} \precsim \ell$, $\ell \succsim \beta_\iota \succsim \beta_{\iota+1} \succsim \ddot{o}$ and $\ell \succsim \varrho_\iota \succsim \varrho_{\iota+1} \succsim \ddot{o}$ for all $\iota \in \mathrm{N}$ it follows that $\{\alpha_\iota\}$,$\{\beta_\iota\}$ and $\{\varrho_\iota\}$ are monotonic sequences in $\mathfrak{H}$.

Utilizing Remark (2.1)(i), there exists ℓ_0, ℓ' and $\overline{\ell} \in \mathfrak{H}$ such that

$$\lim_{\iota\to\infty} \alpha_\iota = \ell_0, \ \lim_{\iota\to\infty} \beta_\iota = \ell' \quad \text{and} \quad \lim_{\iota\to\infty} \varrho_\iota = \overline{\ell}. \quad (3.1.5)$$

Now, by repeatedly using the contractive condition (3.1.1), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-1}, \mathfrak{k}\mathfrak{u}_{\iota-1}, \frac{\delta\tau}{\theta}\right) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m-1}, \mathfrak{u}_{\iota-1}, \frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-2}, \mathfrak{k}\mathfrak{u}_{\iota-2}, \frac{\delta^2\tau}{\theta^2}\right) \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m-2}, \mathfrak{u}_{\iota-2}, \frac{\delta^3\tau}{\theta^3}\right) \succsim \cdots \succsim \widetilde{\mathfrak{P}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-1}, \mathfrak{k}\mathfrak{u}_{\iota-1}, \frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m-1}, \mathfrak{u}_{\iota-1}, \frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-2}, \mathfrak{k}\mathfrak{u}_{\iota-2}, \frac{\delta^2\tau}{\theta^2}\right) \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m-2}, \mathfrak{u}_{\iota-2}, \frac{\delta^3\tau}{\theta^3}\right) \precsim \cdots \precsim \widetilde{\mathfrak{L}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m, \mathfrak{u}_\iota, \frac{\delta\tau}{\theta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-1}, \mathfrak{k}\mathfrak{u}_{\iota-1}, \frac{\delta\tau}{\theta}\right) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m-1}, \mathfrak{u}_{\iota-1}, \frac{\delta^2\tau}{\theta^2}\right)$$

$$= \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-2}, \mathfrak{k}\mathfrak{u}_{\iota-2}, \frac{\delta^2\tau}{\theta^2}\right) \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m-2}, \mathfrak{u}_{\iota-2}, \frac{\delta^3\tau}{\theta^3}\right) \precsim \cdots \precsim \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right).$$

for $\tau \in \mathfrak{H}_{\ddot{o}}$ and $m, \iota \in \mathrm{N}$ with $m > \iota$.

Thus, $\alpha_{\iota+1} = inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \succsim inf_{m>\iota}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \succsim inf_{\mathfrak{v}\in\Xi.}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$,

$\beta_{\iota+1} = sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \precsim sup_{m>\iota}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$ and

$\varrho_{\iota+1} = sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{u}_{m+1}, \mathfrak{u}_{\iota+1}, \tau) \precsim sup_{m>\iota}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0, \mathfrak{u}_{m-\iota}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) \precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right)$.

Since $\lim_{\iota\to\infty} \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}} = \infty$, by using the hypothesis along with (3.1.5), we obtain

$\ell_0 \succsim \lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ell$, $\ell' \precsim \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ddot{o}$ and

$\overline{\ell} \precsim \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0, \mathfrak{v}, \frac{\delta^{\iota+1}\tau}{\theta^{\iota+1,}}\right) = \ddot{o}$.

This indicates that $\ell_0 = \ell$, $\ell' = \ddot{o}$ and $\overline{\ell} = \ddot{o}$. Thus, $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Since $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ is a *CVNbMS,* by Lemma (2.12), there exists a $\mathfrak{d} \in \Xi$ such that for all $\tau \in \mathfrak{H}_{\ddot{o}}$,

$\lim_{\iota\to\infty}\widetilde{\mathfrak{P}}(\mathfrak{u}_m,\mathfrak{d},\tau)=\ell,\ \lim_{\iota\to\infty}\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{d},\tau)=\ddot{\mathfrak{o}}$ and $\lim_{\iota\to\infty}\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{d},\tau)=\ddot{\mathfrak{o}}.$ (3.1.6)

We will demonstrate that $\mathfrak{d}$ is the fixed point of $\mathfrak{k}$. As a result of (5), (10) and (15) of definition (2.8), the contractive condition (3.1.1) we get,

$$\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)\succsim\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)*\widetilde{\mathfrak{P}}\left(\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)=\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)*\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)$$

$$\succsim\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)*\widetilde{\mathfrak{P}}\left(\mathfrak{u}_m,\mathfrak{d},\frac{\tau}{2\delta}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)\precsim\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{L}}\left(\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)=\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)$$

$$\precsim\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{L}}\left(\mathfrak{u}_m,\mathfrak{d},\frac{\tau}{2\delta}\right)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)\precsim\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)=\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right)$$

$$\precsim\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right)\star\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_m,\mathfrak{d},\frac{\tau}{2\delta}\right)$$

for any $\tau\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$. Taking the limit as $\iota\to\infty$, by (3.1.6) and Remark (2.2)(ii), we obtain $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)=\ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)=\ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau)=\ddot{\mathfrak{o}}$ and for all $\tau\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$, which gives $\mathfrak{d}=\mathfrak{k}\mathfrak{d}$.

To show that the fixed point $\mathfrak{d}$ is unique. Let $\mathfrak{z}$ be another fixed point of $\mathfrak{k}$, i.e., there is a $\tau\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$ with $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)\neq\ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)\neq\ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)\neq\ddot{\mathfrak{o}}$ from (3.1.1), we obtain that

$$\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau)\succsim\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right)=\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{y},\frac{\theta\tau}{\delta}\right)\succsim\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^2\tau}{\delta^2}\right)\ldots\succsim\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\succsim inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau)\precsim\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right)=\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\frac{\theta\tau}{\delta}\right)\precsim\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^2\tau}{\delta^2}\right)\ldots\precsim\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau)\precsim\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta\tau}{\delta}\right)=\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\frac{\theta\tau}{\delta}\right)\precsim\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^2\tau}{\delta^2}\right)\ldots\precsim\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$\precsim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right),\quad\text{for all }\iota\in\mathbb{N}.$$

Hence, since $\lim_{\iota\to\infty}\frac{\delta^{\iota}\tau}{\theta^{\iota}}=\infty$, the above inequality becomes $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)\succsim\ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)\precsim\ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)\precsim\ddot{\mathfrak{o}}$ which leads to a contradiction. Thus, we determine that the fixed point of $\mathfrak{k}$ is unique.

Example 3.2. Let $\Xi=[0,1]$ and let $\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}}:\Xi^2\times\mathfrak{H}_{\ddot{\mathfrak{o}}}\to\mathfrak{F}$ such that

$$\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)=\frac{pq}{pq+(\mathfrak{u}-\mathfrak{v})^2}\ell,\quad\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)=\frac{(\mathfrak{u}-\mathfrak{v})^2}{pq+(\mathfrak{u}-\mathfrak{v})^2}\ell\text{ and }\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)=\frac{(\mathfrak{u}-\mathfrak{v})^2}{pq}\ell\ ,$$

where $\tau=(p,q)\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$. Then, we can readily verify that $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\star,\theta)$ is a *CVNbMS* with $\theta=2$. We conclude that for any sequence $\{\mathfrak{u}_{\iota}\}$ in $\mathfrak{H}_{\ddot{\mathfrak{o}}}$ with $\lim_{\iota\to\infty}\tau_{\iota}=\infty$, we have

$\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)=\ell, \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)=\ddot{o}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)=\ddot{o}$ for all $\mathfrak{u}\in\Xi$. Let $\mathfrak{k}:\Xi\to\Xi$ be a mapping defined by $\mathfrak{k}\mathfrak{u}=\varsigma\,\mathfrak{u}^2$ where $0<\varsigma<\frac{1}{4}$. By a routine calculation, we see that

$\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)$, $\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)$ and $\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)$for every $\mathfrak{u},\mathfrak{v}\in\Xi$ and $\tau\in\mathfrak{H}_{\ddot{o}}$, where $\delta=4\varsigma$ and $0<\delta<1$. All the requirements of Theorem (3.1) are fulfilled and 0 is the unique fixed point of $\mathfrak{k}$.

Theorem 3.3. Let $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\ \star,\theta)$ be a *CVNbMS* such that, for every sequence $\{\tau_\iota\}$ in $\mathfrak{H}_{\ddot{o}}$ with $\lim_{\iota\to\infty}\tau_\iota=\infty$, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ell, \lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ddot{o}$ and $\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_\iota)=\ddot{o}$, for all $\mathfrak{u}\in\Xi$. Let $\mathfrak{k},\mathfrak{h}:\Xi\to\Xi$ be a mapping satisfying the following requirements:

(i) $\mathfrak{h}(\Xi)\subseteq\mathfrak{k}(\Xi)$,

(ii) $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ,

(iii) $\mathfrak{k}$ is continuous on Ξ,

(iv) $\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$, $\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$ and $\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u},\mathfrak{h}\,\mathfrak{v},\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau)$for all $\mathfrak{u},\mathfrak{v}\in\Xi$ and $\tau\in\mathfrak{H}_{\ddot{o}}$ where $0<\delta<1$. Then $\mathfrak{k}$ and $\mathfrak{h}$ have a unique common fixed point in Ξ.

Proof. Let $\mathfrak{u}_0\in\Xi$. Since $\mathfrak{h}(\Xi)\subseteq\mathfrak{k}(\Xi)$, we can choose an $\mathfrak{u}_1\in\Xi$ such that $\mathfrak{h}\mathfrak{u}_0=\mathfrak{k}\mathfrak{u}_1$. Repeating this procedure, we can choose $\mathfrak{u}_\iota\in\Xi$ such that $\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{u}_{\iota-1}$.

We claim that the sequence $\{\mathfrak{k}\mathfrak{u}_\iota\}$ is a Cauchy sequence. For every $\iota\in\mathrm{N}$ and $\tau\in\mathfrak{H}_{\ddot{o}}$, define
$\mathfrak{W}_\iota=\{\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau):m>\iota\}$, $\mathfrak{N}_\iota=\{\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau):m>\iota\}$ and $\mathfrak{O}_\iota=\{\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau):m>\iota\}$
for every $\iota\in\mathrm{N}$ and $\tau\in\mathfrak{H}_{\ddot{o}}$.

Since $\ddot{o}\prec\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\lesssim\ell$, $\ddot{o}\prec\widetilde{\mathfrak{L}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau)\lesssim\ell$ and $\ddot{o}\prec\widetilde{\mathfrak{Q}}(\mathfrak{u}_m,\mathfrak{u}_\iota,\tau)\lesssim\ell$, for every $m\in\mathrm{N}$ with $m>\iota$ and from Remark (2.1)(ii), inf $\mathfrak{k}\mathfrak{W}_\iota=\alpha_\iota$, sup $\mathfrak{k}\mathfrak{N}_\iota=\beta_\iota$ and sup$\mathfrak{k}\mathfrak{O}_\iota=\varrho_\iota$ exists for every $\iota\in\mathrm{N}$. Using Lemma(2.10) and (iv), we get

$$\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\lesssim\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)=\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau),$$

$$\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{v},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\delta\tau}{\theta}\right)\gtrsim\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{v},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau),$$

for $\tau\in\mathfrak{H}_{\ddot{o}}$ and $m,\iota\in\mathrm{N}$ with $m>\iota$.

Since $\ddot{o}\lesssim\alpha_\iota\lesssim\alpha_{\iota+1}\lesssim\ell,\ell\gtrsim\beta_\iota\gtrsim\beta_{\iota+1}\gtrsim\ddot{o}$ and $\ell\gtrsim\varrho_\iota\gtrsim\varrho_{\iota+1}\gtrsim\ddot{o}$, for all $\iota\in\mathrm{N}$ it follows that $\{\alpha_\iota\}$, $\{\beta_\iota\}$ and $\{\varrho_\iota\}$ are monotonic sequences in$\mathfrak{H}$.

So, utilizing Remark (2.1) (i), there exists an $\ell_0,\ell'$and $\tilde{\ell}\in\mathfrak{H}$ satisfying

$$\lim_{\iota\to\infty}\alpha_\iota=\ell_0,\ \lim_{\iota\to\infty}\beta_\iota=\ell'\text{ and }\lim_{\iota\to\infty}\varrho_\iota=\tilde{\ell} \tag{3.3.1}$$

By applying the condition (iv), we have

$$\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{P}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{P}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\gtrsim\cdots\gtrsim\widetilde{\mathfrak{P}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{L}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{L}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\lesssim\cdots\lesssim\widetilde{\mathfrak{L}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)=\widetilde{\mathfrak{Q}}(\mathfrak{h}\mathfrak{u}_m,\mathfrak{h}\mathfrak{u}_\iota,\tau)\lesssim\widetilde{\widetilde{\mathfrak{Q}}}\left(\mathfrak{k}\mathfrak{u}_m,\mathfrak{k}\mathfrak{u}_\iota,\frac{\theta\tau}{\delta}\right)\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u}_{m-1},\mathfrak{h}\mathfrak{u}_{\iota-1},\frac{\theta\tau}{\delta}\right)$$

$$\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-1},\mathfrak{k}\mathfrak{u}_{\iota-1},\frac{\theta^2\tau}{\delta^2}\right)=\widetilde{\mathfrak{Q}}\left(\mathfrak{h}\mathfrak{u}_{m-2},\mathfrak{h}\mathfrak{u}_{\iota-2},\frac{\theta^2\tau}{\delta^2}\right)$$

$$\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_{m-2},\mathfrak{k}\mathfrak{u}_{\iota-2},\frac{\theta^3\tau}{\delta^3}\right)\lesssim\cdots\lesssim\widetilde{\mathfrak{Q}}\left(\mathfrak{u}_0,\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right),$$

for $\tau\in\mathfrak{H}_{\ddot{\mathfrak{o}}}$ and $m,\iota\in\mathrm{N}$ with $m>\iota$. Thus,

$$\alpha_{\iota+1}=inf_{m>\iota}\widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\gtrsim inf_{m>\iota}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\gtrsim inf_{\mathfrak{v}\in\Xi.}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right).$$

$$\beta_{\iota+1}=sup_{m>\iota}\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\lesssim sup_{m>\iota}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\text{ and}$$

$$\varrho_{\iota+1}=sup_{m>\iota}\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u}_{m+1},\mathfrak{k}\mathfrak{u}_{\iota+1},\tau)\lesssim sup_{m>\iota}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{k}\mathfrak{u}_{m-\iota},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right)\lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1}}\right).$$

Since $\lim\limits_{\iota\to\infty}\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}=\infty$, by using the hypothesis along with (3.3.1), we obtain

$$\ell_0\gtrsim\lim_{\iota\to\infty}inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ell,\ \ell'\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}},$$

$$\ell'\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}}\text{ and }\bar{\ell}\lesssim\lim_{\iota\to\infty}sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_0,\mathfrak{v},\frac{\theta^{\iota+1}\tau}{\delta^{\iota+1,}}\right)=\ddot{\mathfrak{o}}$$

Which implies that $\ell_0=\ell$ and $\ell'=\ddot{\mathfrak{o}}$. Thus, $\{\mathfrak{k}\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ.

Using Lemma (2.12) and completeness of Ξ, there exists a $\mathfrak{d}\in\Xi$ such that $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{d}$.

Using (iv), we can check that the continuity of $\mathfrak{k}$ implies continuity of $\mathfrak{h}$. So, $\lim\limits_{\iota\to\infty}\mathfrak{h}\mathfrak{k}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{d}$.

Since $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ, we have $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{h}\mathfrak{u}_\iota=\mathfrak{h}\mathfrak{d}$.

Moreover, we know that $\lim\limits_{\iota\to\infty}\mathfrak{h}\mathfrak{u}_{\iota-1}=\mathfrak{d}$ so we get $\lim\limits_{\iota\to\infty}\mathfrak{k}\mathfrak{h}\mathfrak{u}_{\iota-1}=\mathfrak{k}\mathfrak{d}$.

Based on the uniqueness of limit, we get $\mathfrak{k}\mathfrak{d}=\mathfrak{h}\mathfrak{d}$ and therefore $\mathfrak{h}\mathfrak{h}\mathfrak{d}=\mathfrak{k}\mathfrak{h}\mathfrak{d}$.

Repeated use of the condition (iv) yields

$$\widetilde{\mathfrak{P}}(\mathfrak{hd},\mathfrak{hhd},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{kd},\mathfrak{khd},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta\tau}{\delta}\right) \gtrsim \cdots \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{khd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \gtrsim inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right)$$

$$\widetilde{\mathfrak{L}}(\mathfrak{hd},\mathfrak{hhd},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{kd},\mathfrak{khd},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta\tau}{\delta}\right) \lesssim \cdots \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{khd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{hd},\mathfrak{hhd},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{kd},\mathfrak{khd},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta\tau}{\delta}\right) \lesssim \cdots \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{hhd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right)$$

$$= \widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{khd},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota,}}\right).$$

Letting the limit as $\iota \to \infty$, and applying the hypothesis we get,
$\widetilde{\mathfrak{P}}(\mathfrak{hd},\mathfrak{hhd},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{hd},\mathfrak{hhd},\tau) = \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{hd},\mathfrak{hhd},\tau) = \ddot{\mathfrak{o}}$ which implies that $\mathfrak{hhd} = \mathfrak{khd} = \mathfrak{hd}$.
i.e., $\mathfrak{hd}$ is a common fixed point of $\mathfrak{k}$ and $\mathfrak{h}$.
We shall establish the uniqueness of the common fixed point $\mathfrak{hd}$.
Assume that $\mathfrak{hd}$ and $\mathfrak{z}$ are two distinct common fixed points of $\mathfrak{k}$ and $\mathfrak{h}$.
Utilizing (iv) with $\mathfrak{u} = \mathfrak{hd}$ and $\mathfrak{v} = \mathfrak{z}$, we find that,

$$\ell \gtrsim \widetilde{\mathfrak{P}}(\mathfrak{hd},\mathfrak{z},\tau){=}\widetilde{\mathfrak{P}}(\mathfrak{hhd},\mathfrak{hd},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{khd},\mathfrak{kz},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \gtrsim inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

$$\ddot{\mathfrak{o}} \lesssim \widetilde{\mathfrak{L}}(\mathfrak{hd},\mathfrak{z},\tau){=}\widetilde{\mathfrak{L}}(\mathfrak{hhd},\mathfrak{hd},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{khd},\mathfrak{kz},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \text{ and}$$

$$\ddot{\mathfrak{o}} \lesssim \widetilde{\mathfrak{Q}}(\mathfrak{hd},\mathfrak{z},\tau){=}\widetilde{\mathfrak{Q}}(\mathfrak{hhd},\mathfrak{hd},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{khd},\mathfrak{kz},\frac{\theta\tau}{\delta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta\tau}{\delta}\right) \ldots \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{z},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right) \lesssim sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}\left(\mathfrak{hd},\mathfrak{v},\frac{\theta^{\iota}\tau}{\delta^{\iota}}\right).$$

Since $\lim_{\iota\to\infty}\frac{\theta^{\iota}\tau}{\delta^{\iota}} = \infty$, we conclude that $\widetilde{\mathfrak{P}}(\mathfrak{hd},\mathfrak{z},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{hd},\mathfrak{z},\tau) = \ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{hd},\mathfrak{z},\tau) = \ddot{\mathfrak{o}}$

Thus, $\mathfrak{hd} = \mathfrak{z}$, this concludes the proof.

Example 3.4 Let $\Xi = [0,1]$ and let $\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ such that $\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = e^{-\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}}\ell$,

$\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = (1 - e^{-\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}})\ell$ and $\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) = (e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp+q}} - 1)\ell$ where $\tau = (\wp, q) \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$. Then, we can readily verify that $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*,\star,\theta)$ is a *CVNbMS* with $\theta = 4$. On the other hand, let $\lim_{\iota\to\infty}\tau_{\iota} = \infty$ for any sequence $\{\tau_{\iota}\}$ in $\mathfrak{H}_{\ddot{\mathfrak{o}}}$, where $\tau_{\iota} = (\wp_{\iota}, q_{\iota})$. Since $(\mathfrak{u}-\mathfrak{v})^2 \leq 1$ for every $\mathfrak{u},\mathfrak{v} \in \Xi$ it follows that

$$inf_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_{\iota}){=}inf_{\mathfrak{v}\in\Xi}\, e^{-(\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}})}\ell = e^{-(\frac{sup_{\mathfrak{v}\in\Xi}(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}})}\ell \gtrsim e^{-(\frac{1}{\wp_{\iota}+q_{\iota}})}\ell.$$

$$sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_{\iota}) = sup_{\mathfrak{v}\in\Xi}\left\{\ell - \frac{\ell}{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}}\right\} = \ell - \frac{\ell}{e^{\frac{sup(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}} \lesssim \ell - \frac{\ell}{e^{\frac{1}{\wp_{\iota}+q_{\iota}}}} \text{ and}$$

$$sup_{\mathfrak{v}\in\Xi}\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_{\iota}) = sup_{\mathfrak{v}\in\Xi}\left\{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}\ell - \ell\right\} = sup_{\mathfrak{v}\in\Xi}\left\{e^{\frac{(\mathfrak{u}-\mathfrak{v})^2}{\wp_{\iota}+q_{\iota}}}\ell - \ell\right\} \lesssim e^{\frac{1}{\wp_{\iota}+q_{\iota}}}\ell.$$

Therefore, we have $\lim_{\iota\to\infty} inf_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \gtrsim \lim_{\iota\to\infty} e^{-(\frac{1}{p_\iota+q_\iota})}\ell = \ell$,

$\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \lesssim \lim_{\iota\to\infty}(\ell - \frac{\ell}{e^{\frac{1}{p_\iota+q_\iota}}}) = \ddot{\mathrm{o}}$ and

$\lim_{\iota\to\infty} sup_{\mathfrak{v}\in\Xi}\, \widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau_\iota) \lesssim \lim_{\iota\to\infty}(e^{\frac{1}{p_\iota+q_\iota}}\ell) = \ddot{\mathrm{o}}$. Let $\mathfrak{k},\mathfrak{h} : \Xi \to \Xi$ be defined by $\mathfrak{k}\mathfrak{u} = \mathfrak{u}$ and, $\mathfrak{h}\mathfrak{u} = \frac{\mathfrak{u}}{4}$.

One can readily verify that $\mathfrak{h}(\Xi) \subseteq \mathfrak{k}(\Xi)$ and $\mathfrak{k}$ is continuous on Ξ. Furthermore, $\mathfrak{k}$ and $\mathfrak{h}$ commute on Ξ. Moreover, It is simple to demonstrate that condition (iv) true for every $\mathfrak{u},\mathfrak{v} \in [0,1]$ with $\delta = \frac{1}{4}$.

Definition.3.5 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a complete *CVNbMS.* The modified contraction condition for the mapping $\mathfrak{k} : \Xi \to \Xi$ as follows:

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau) \quad (1)$$

For all $\mathfrak{u},\mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{o}}$ where $\delta \in [0,1)$.

Theorem 3.6 Let $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a *CVNbMS,* and $\mathfrak{k} : \Xi \to \Xi$ be a mapping fulfilling the contraction condition (I). Then, $\mathfrak{k}$ has a unique common fixed point in Ξ.

Proof: Let $\mathfrak{u}_0$ be a random element of Ξ. Using induction, we can generate a sequence $\{\mathfrak{u}_\iota\}$ in Ξ such that $\mathfrak{u}_\iota = \mathfrak{k}\mathfrak{u}_{\iota-1}$ for every $\iota \in \mathbb{N}$. Continuing from the proof of Theorem (3.1) in [12], we examine that the sequence $\{\mathfrak{u}_\iota\}$ is a Cauchy sequence in Ξ and converges to some $\mathfrak{d} \in \Xi$.

We will demonstrate that $\mathfrak{d}$ is a fixed point of $\mathfrak{k}$. By the contractive condition (I), we have

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)$$

for all $\iota \in \mathbb{N}$ and $\tau \in \mathfrak{H}_{\ddot{o}}$. The above inequality demonstrates that

$$\ell(1-\delta) + \delta\widetilde{\mathfrak{P}}(\mathfrak{u}_\iota,\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\tau), \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) \text{and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{u},\mathfrak{k}\mathfrak{v},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau). \quad (3.6.1)$$

for all $\iota \in \mathbb{N}$ and $\tau \in \mathfrak{H}_{\ddot{o}}$.

Therefore,

$$\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \gtrsim \widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{u}_{\iota+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{P}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) * \widetilde{\mathfrak{P}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right).$$

$$\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{u}_{\iota+1},\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{L}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{L}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) \text{ and}$$

$$\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) \lesssim \widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{u}_{\iota+1},\ \mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) = \widetilde{\mathfrak{Q}}\left(\mathfrak{d},\mathfrak{u}_{\iota+1},\frac{\tau}{2\theta}\right) \star \widetilde{\mathfrak{Q}}\left(\mathfrak{k}\mathfrak{u}_\iota,\mathfrak{k}\mathfrak{d},\frac{\tau}{2\theta}\right) \text{ for any } \tau \in \mathfrak{H}_{\ddot{o}}.$$

Taking the limit as $\iota \to \infty$, from (3.6.1) and Remark (2.2) (ii), we determine that $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{k}\mathfrak{d},\tau) = \ddot{\mathrm{o}}$ for all $\tau \in \mathfrak{H}_{\ddot{o}}$, which yields $\mathfrak{k}\mathfrak{d} = \mathfrak{d}$.

To prove that the fixed point of $\mathfrak{k}$ is unique, assume that there exists another $\mathfrak{z} \in \Xi$ such that $\mathfrak{k}(\mathfrak{z}) = \mathfrak{z}$. Then, there is a $\tau \in \mathfrak{H}_{\ddot{o}}$ fulfilling $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$.

As a result of (I), we have

$$\ell - \widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) = \ell - \widetilde{\mathfrak{P}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)],\ \widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \text{ and } \widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \lesssim \delta\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau).$$

Since $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau) \neq \ddot{\mathrm{o}}$, we obtain

$Re(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$ or $Im(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$, $Re(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ or $Im(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ and $Re(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ or $Im(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$. Let $Re(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 1$, $Re(\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)) \neq 0$ and $Re\left(\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)\right) \neq 0$.

Therefore, we get

$1 - Re\left(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)\right) \precsim \delta[1 - Re(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau))] \precsim 1 - Re\left(\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau)\right)$,

$Re(\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \precsim \delta Re(\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)) \precsim Re(\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)) = Re(\widetilde{\mathfrak{L}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau))$ and
$Re(\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau) \precsim \delta Re(\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)) \precsim Re(\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)) = Re(\widetilde{\mathfrak{Q}}(\mathfrak{k}\mathfrak{d},\mathfrak{k}\mathfrak{z},\tau))$ which is a contradiction.
We can omit the details of the other since the other case is identical to this one.
Thus, $\widetilde{\mathfrak{P}}(\mathfrak{d},\mathfrak{z},\tau) = \ell$, $\widetilde{\mathfrak{L}}(\mathfrak{d},\mathfrak{z},\tau)=\ddot{\mathfrak{o}}$ and $\widetilde{\mathfrak{Q}}(\mathfrak{d},\mathfrak{z},\tau)=\ddot{\mathfrak{o}}$ for all $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$ and the proof is completed.
Example: 3.7 Let $\Xi = [0,1]$ and let $\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{\mathfrak{o}}} \to \mathfrak{F}$ such that

$\widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau) = \ell - \frac{(\mathfrak{u}-\mathfrak{v})^2}{1+\wp q}\ell, \widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \frac{(\mathfrak{u}-\mathfrak{v})^2}{1+\wp q}\ell$ and $\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau) = \frac{(\mathfrak{u}-\mathfrak{v})^2}{1+\wp q-(\mathfrak{u}-\mathfrak{v})^2}\ell$ where $\tau = (\wp, q) \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$.

Define the mapping $\mathfrak{k}: \Xi \to \Xi$ by $\mathfrak{k}\mathfrak{u} = \frac{\mathfrak{u}^2}{4}$. Therefore, we have

$\frac{(\mathfrak{k}\mathfrak{u}-\mathfrak{k}\mathfrak{v})^2}{1+\wp q}\ell \precsim \delta\frac{(\mathfrak{u}-\mathfrak{v})^2}{1+\wp q}\ell$ and $\frac{(\mathfrak{k}\mathfrak{u}-\mathfrak{k}\mathfrak{v})^2}{1+\wp q-(\mathfrak{k}\mathfrak{u}-\mathfrak{k}\mathfrak{v})^2}\ell \precsim \delta\frac{(\mathfrak{u}-\mathfrak{v})^2}{1+\wp q-(\mathfrak{u}-\mathfrak{v})^2}\ell$ where $\delta \in [\frac{1}{4}, 1)$. Thus, we determine that (I) holds, all the necessary hypotheses of Theorem (3.6) are fulfilled and thus we establish the existence and uniqueness of the fixed point of $\mathfrak{k}$ and 0 is the unique fixed point of $\mathfrak{k}$.

Corollary 3.8 Let $(\Xi,\widetilde{\mathfrak{P}},\widetilde{\mathfrak{L}},\widetilde{\mathfrak{Q}},*, \star,\theta)$ be a *CVNbMS* and $\mathfrak{k} : \Xi \to \Xi$ be a mapping satisfying $\ell - \widetilde{\mathfrak{P}}(\mathfrak{k}^\iota\mathfrak{u},\mathfrak{k}^\iota\mathfrak{v},\tau) \precsim \delta[\ell - \widetilde{\mathfrak{P}}(\mathfrak{u},\mathfrak{v},\tau)]$, $\widetilde{\mathfrak{L}}(\mathfrak{k}^\iota\mathfrak{u},\mathfrak{k}^\iota\mathfrak{v},\tau) \precsim \delta\widetilde{\mathfrak{L}}(\mathfrak{u},\mathfrak{v},\tau)$ and $\widetilde{\mathfrak{Q}}(\mathfrak{k}^\iota\mathfrak{u},\mathfrak{k}^\iota\mathfrak{v},\tau) \precsim \delta\widetilde{\mathfrak{Q}}(\mathfrak{u},\mathfrak{v},\tau)$ for every $\mathfrak{u},\mathfrak{v} \in \Xi$ and $\tau \in \mathfrak{H}_{\ddot{\mathfrak{o}}}$, where $0 \le \delta < 1$. Then, $\mathfrak{k}$ has a unique common fixed point in Ξ.

Proof: By Theorem (3.6), we get a unique $\mathfrak{u} \in \Xi$ such that $\mathfrak{k}^\iota\mathfrak{u} = \mathfrak{u}$. Since $\mathfrak{k}^\iota\mathfrak{k}\mathfrak{u} = \mathfrak{k}\mathfrak{k}^\iota\mathfrak{u} = \mathfrak{k}\mathfrak{u}$ and from uniqueness, we get $\mathfrak{k}\mathfrak{u} = \mathfrak{u}$. This demonstrates that $\mathfrak{k}$ has a unique fixed point in Ξ.

4. Application

Applying our main results from the previous part, we analyze the existence theorem for a solution to the following integral equation in this section:

$$\mathfrak{u}(\tilde{\mathfrak{s}})= \kappa(\tilde{\mathfrak{s}}) + \sigma\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}},\mathfrak{u}(\bar{\bar{\theta}})\right)d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1], \quad (2)$$

where

(i) κ is a continuous real-valued function on[0,1]; $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\psi(\tilde{\mathfrak{s}},\mathfrak{u}) \ge 0$ and there exists a $\delta \in [0,1)$ such that $|\psi(\tilde{\mathfrak{s}},\mathfrak{u}) - \psi(\tilde{\mathfrak{s}},\mathfrak{v})| \le \delta|\mathfrak{u} - \mathfrak{v}|$, for every $\mathfrak{u},\mathfrak{v} \in \mathbb{R}$;

(ii) $\beth : [0,1] \times [0,1]\,] \to \mathbb{R}$ is a continuous at $\tilde{\mathfrak{s}} \in [0,1]$ for every $\bar{\bar{\theta}} \in [0,1]$ and measurable at $\bar{\bar{\theta}} \in [0,1]$ for every $\tilde{\mathfrak{s}} \in [0,1]$. Moreover, $\beth(\tilde{\mathfrak{s}},\bar{\bar{\theta}}) \ge 0$ and $\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})d\bar{\bar{\theta}} \le \mathcal{L}$;

(iii) $\delta^2\mathcal{L}^2\sigma^2 \le \frac{1}{2}$.

Theorem 4.1. If the condition (i)-(iv) fulfilled. Then, the integral Eq. (2) has unique solution in (C[0,1], ℝ), where (C[0,1], ℝ) is the set of all continuous real valued functions on [0,1].

Proof: Let Ξ =(C[0,1], ℝ) and define a mapping $\mathfrak{k} : \Xi \to \Xi$ by

$\mathfrak{k}\mathfrak{u}(\tilde{\mathfrak{s}})= \kappa(\tilde{\mathfrak{s}}) + \sigma\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}},\mathfrak{u}(\bar{\bar{\theta}})\right)d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1]$, for all $\mathfrak{u} \in \Xi$ and for every $\tilde{\mathfrak{s}} \in [0,1]$.

We need to prove that the mapping $\mathfrak{k}$ fulfils all requirements of Theorem (3.6).

Define $\widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}} : \Xi^2 \times \mathfrak{H}_{\ddot{o}} \to \mathfrak{F}$ by $\widetilde{\mathfrak{P}}(\mathfrak{u}, \mathfrak{v}, \tau) = \ell - \sup_{\tilde{\mathfrak{s}}\in[0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\mathcal{p}\mathcal{q}}}\ell$,

$\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \sup_{\tilde{\mathfrak{s}}\in[0,1]} \frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\mathcal{p}\mathcal{q}}}\ \ell$ and $\widetilde{\mathfrak{L}}(\mathfrak{u}, \mathfrak{v}, \tau) = \left(\frac{\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\mathcal{p}\mathcal{q}}}}{1-\sup_{\tilde{\mathfrak{t}}\in[0,1]}\frac{(\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}}))^2}{e^{\mathcal{p}\mathcal{q}}}}\right)\ell$

where $\tau = (\mathcal{p}, \mathcal{q}) \in \mathfrak{H}_{\ddot{o}}$. Clearly, $(\Xi, \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{L}}, \widetilde{\mathfrak{Q}}, *, \star, \theta)$ be a complete CVNbMS.
Moreover, for every $\mathfrak{u}, \mathfrak{v} \in \Xi$ and $\tilde{\mathfrak{s}} \in [0,1]$, we get

$$|\mathfrak{f}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{f}\mathfrak{v}(\tilde{\mathfrak{s}})| = \sigma\left|\int_0^1 \beth(\tilde{\mathfrak{s}},\bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}},\mathfrak{u}(\bar{\bar{\theta}})\right) - \beth(\tilde{\mathfrak{s}},\bar{\bar{\theta}})\psi\left(\bar{\bar{\theta}},\mathfrak{v}(\bar{\bar{\theta}})\right)d\bar{\bar{\theta}}\right|$$

$$\leq \sigma\int_0^1 \beth(\tilde{\mathfrak{s}},\bar{\bar{\theta}})\left|\psi\left(\bar{\bar{\theta}},\mathfrak{u}(\bar{\bar{\theta}})\right) - \psi\left(\bar{\bar{\theta}},\mathfrak{v}(\bar{\bar{\theta}})\right)\right|d\theta \leq \sigma\int_0^1 \beth(\tilde{\mathfrak{s}},\bar{\bar{\theta}})\delta\left|\mathfrak{u}(\bar{\bar{\theta}}) - \mathfrak{v}(\bar{\bar{\theta}})\right|d\bar{\bar{\theta}}$$

$$\leq \sigma\mathcal{L}\ \delta\sup_{\tilde{\mathfrak{s}}\in[0,1]}|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|$$

Since, $\sup_{\tilde{\mathfrak{s}}\in[0,1]}|\mathfrak{f}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{f}\mathfrak{v}(\tilde{\mathfrak{s}})| \leq \sigma\mathcal{L}\ \delta\sup_{\tilde{\mathfrak{s}}\in[0,1]}|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|$

We get, $\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{f}\mathfrak{u}(\tilde{\mathfrak{s}})-\mathfrak{f}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}} \leq \sigma^2\mathcal{L}^2\delta^2\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}})-\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}} \leq \frac{1}{2}\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}})-\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}$ and

$$\left(\frac{\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{f}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{f}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}{1-\sup_{\tilde{\mathfrak{t}}\in[0,1]}\frac{|\mathfrak{f}\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{f}\mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}\right) \leq \sigma^2\mathcal{L}^2\delta^2\left(\frac{\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}{1-\sup_{\tilde{\mathfrak{t}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}\right) \leq \frac{1}{2}\frac{\sup_{\tilde{\mathfrak{s}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}{1-\sup_{\tilde{\mathfrak{t}}\in[0,1]}\frac{|\mathfrak{u}(\tilde{\mathfrak{s}}) - \mathfrak{v}(\tilde{\mathfrak{s}})|^2}{e^{\mathcal{p}\mathcal{q}}}}.$$

This establishes that the mapping $\mathfrak{f}$ fulfilling the contractive condition (1) in Theorem (3.6), and $\mathfrak{f}$ has a unique solution in (C [0,1], ℝ), i.e., the integral Eq. (2) has a unique solution in (C [0,1], ℝ).

Example 4.2 Take the integral equation

$$\mathfrak{u}(\tilde{\mathfrak{s}})=\frac{1}{1+\tilde{\mathfrak{s}}} + 2\int_0^1 \frac{\bar{\bar{\theta}}^2}{\tilde{\mathfrak{s}}^2+2}.\frac{|\cos\mathfrak{u}(\tilde{\mathfrak{s}})|}{5e^{\bar{\bar{\theta}}}}d\bar{\bar{\theta}}, \tilde{\mathfrak{s}} \in [0,1], \tag{4.2.1}$$

It can observed that the above equation is of the form (II), for $\sigma = 2$, $\kappa(\tilde{\mathfrak{s}}) = \frac{1}{1+\tilde{\mathfrak{s}}}$, $\xi(\tilde{\mathfrak{s}}, \bar{\bar{\theta}}) = \frac{\theta^2}{\tilde{\mathfrak{s}}+2}$, $\psi(\tilde{\mathfrak{s}}, \mathfrak{u}) = \frac{|\cos\tilde{\mathfrak{s}}|}{5e^{\tilde{\mathfrak{s}}}}$.
Clearly, ψ is continuous on $[0,1] \times \mathbb{R}$ and we get
$\left|\psi(\bar{\bar{\theta}}, \mathfrak{u}) - \psi(\bar{\bar{\theta}}, \mathfrak{v})\right| = \frac{1}{5e^{\tilde{\mathfrak{s}}}}\left||\cos\mathfrak{u}| - |\cos\mathfrak{v}|\right| \leq \frac{1}{5e^{\tilde{\mathfrak{s}}}}|\cos\mathfrak{u} - \cos\mathfrak{v}| \leq \frac{1}{5}|\cos\mathfrak{u} - \cos\mathfrak{v}| \leq \frac{1}{5}|\mathfrak{u} - \mathfrak{v}|$
for every $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$. Thus, ψ fulfills the condition (ii) of the integral equation (II) with $= \frac{1}{5}$. It is easy to verify that the mapping κ is continuous and $\int_0^1 \beth(\tilde{\mathfrak{s}}, \bar{\bar{\theta}})d\bar{\bar{\theta}} = \int_0^1 \frac{\bar{\bar{\theta}}^2}{\tilde{\mathfrak{s}}^2+2}\,d\bar{\bar{\theta}} = \frac{1}{\tilde{\mathfrak{s}}^2+2}\frac{1}{3} \leq \frac{1}{6} = \mathcal{L}$, the mapping ξ meets the condition (iii). We get $\sigma^2\mathcal{L}^2\delta^2 \leq \frac{1}{2}$.. Thus, the hypotheses (i), (ii), (iii), and (iv) are true. Using the Theorem (3. 6) leads us to the conclusion that the integral equation (II) has a unique solution in (C [0, 1], ℝ).

5. Conclusion

In this paper, we have defined complex valued neutrosophic metric like space and we have proved fixed point theorems for mappings on complex valued neutrosophic metric like space. We hope that the results proved in this paper will form new connections for those who are working in complex valued neutrosophic metric-like spaces.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

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Domination on Bipolar Fuzzy Graph Operations: Principles, Proofs, and Examples

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Abstract: Bipolar fuzzy graphs, capable of capturing situations with both positive and negative memberships, have found diverse applications in various disciplines, including decision-making, computer science, and social network analysis. This study investigates the domain of domination and global domination numbers within bipolar fuzzy graphs, owing to their relevance in these aforementioned practical fields. In this study, we introduce certain operations on bipolar fuzzy graphs, such as intersection$(\Gamma_1 \cap \Gamma_2)$, join $(\Gamma_1 + \Gamma_2)$, and union $(\Gamma_1 \cup \Gamma_2)$ of two graphs. Furthermore, we analyze the domination number $\gamma(\Gamma)$ and the global domination number $\gamma_g(\Gamma)$ for various operations on bipolar fuzzy graphs, including intersection, join, and union of fuzzy graphs and their complements.

Keywords: Bipolar Fuzzy Graph; Global Domination; Domination Number; Operations Fuzzy Graphs.

1. Introduction

L.A. Zadeh, who created the fuzzy set theory and fuzzy logic, originally suggested and studied the idea of "fuzzy sets" in 1965 [1]. By giving each element in a subset of a universal set a specific value in the closed interval [0, 1], this theory suggests a graded membership for each of such elements. Many scientific disciplines, including the fields of computer science, machine learning, analysis of decisions, the science of information, system sciences, controlling engineering, expertise systems, recognition of patterns, management science, and operation research, as well as a number of mathematical disciplines, including topology, algebra, geometry, graph theory, and analysis, have used Zadeh's ideas.

Rosenfeld 1975 [2] studied the notion of fuzzy graphs and numerous fuzzy analogs of graph-theoretic notions, such as the path, cycles, and connectedness. Zadeh 1987 [3] investigated the fuzzy relationship as well. Ore studied the mathematical definition of dominance in the graph in 1962 [4], while A. Somasundaram and S. Somasundaram examined various concepts of domination in fuzzy graphs [5].

Sampat-Kumar presented the first concept of global dominant sets in graphs in 1989 [6]. The notions of domination and global domination of some operations in fuzzy product graphs were presented by Haifa A. and Mahioub S. in [6], while the concepts of global domination number, domatic number, and global domatic number were introduced by Mahioub in [7]. Mordeson, J.N., and Peng C-S introduced and analyzed operations on the fuzzy graph in 1994 [8], and also in 2017 [9] Somasundaram presented more notions on domination in fuzzy graphs.

The study of domination and global domination numbers in bipolar fuzzy graphs has implications in fields such as operations research, game theory, and graph theory. By studying these important parameters, researchers can gain insight into the properties and behavior of complex systems modeled by bipolar fuzzy graphs.

Bipolar fuzzy graphs are a type of fuzzy graph where each vertex is assigned a pair of values that represent its positive and negative degrees. This paper studies the domination and global domination properties of these graphs, which are important concepts in network analysis. Domination refers to the minimum number of vertices needed to control or influence the entire graph, while global domination refers to the minimum number of vertices needed to control or influence any vertex in the graph. Additionally, Crisp graphs, being a fundamental mathematical construct, exhibit a plethora of operations that allow for their manipulation and analysis. These operations include but are not limited to, union, intersection, join, tensor product, Cartesian product, composition, strong product, disjunction, and symmetric difference of graphs. A comprehensive treatment of these operations is provided in [10-16]. Tobaili et al. [17] investigated hub number properties within the context of fuzzy graph structures. Further exploration into domination parameters on product fuzzy graphs was conducted by Ahmed and Alsharafi [18], with a specific focus on the semi-global domination number. In this study, we focus our attention on some of these operations, namely union, intersection, and join on bipolar fuzzy graphs, and discuss theorems and bounds of domination and global domination in such operations of the bipolar fuzzy graph.

Bipolar fuzzy graphs (BFG) are an extension of fuzzy graphs that can effectively capture uncertain or imprecise information in various applications. BFGs are used to define concepts such as covering, matching, and domination in graph theory when the vertices and edges are uncertain or imprecise. BFGs have been used in various domains, including disaster management, location selection, and medical diagnosis. The energy of a directed bipolar fuzzy graph is calculated as the sum of the absolute values of the eigenvalues of its adjacency matrix, and it can be used in solving multi-criteria decision-making problems [19-22].

Inverse domination in bipolar fuzzy graphs refers to the idea of an inverse dominating set (IDS) in which a set I is a dominating set of the complement of the dominating set D. The least IDS is called the inverse domination number. In addition, inverse domination has also been defined and studied in interval-valued fuzzy graphs, with bounds on the inverse domination number provided for different types of interval-valued fuzzy graphs. Furthermore, a new definition of inverse domination number has been introduced in the graphs, with bounds and results established for this parameter [23]. The cardinality, dominating set, independent set, total dominating number, independent dominating number, and redundancy number of bipolar fuzzy graphs have been introduced and investigated in [24]. The concept of domination in fuzzy graphs has been extended to bipolar frameworks, and various expanded concepts of bipolar fuzzy graphs have been obtained in [25].

This study suggests exploring the concepts of domination and global domination in some bipolar fuzzy graph operations. There are a few points that we want to highlight about the motivation and applications;

Bipolar fuzzy graphs offer a comprehensive approach to representing complex systems in which relationships can have both positive and negative aspects, unlike graphs that only consider positive membership.

Domination and global domination are concepts in graph theory that have applications in decision-making, computer science, and social network analysis. By studying these properties in graphs, we can gain fresh perspectives.

Investigating domination numbers helps us to understand how efficiently a set of vertices can control a graph. This has implications for modeling influence and control in world systems.

Analyzing operations like union, intersection, and join on graphs provides us with a mathematical framework to examine and manipulate these graphical models. This can be useful for algorithm development in data processing.

The insights obtained from this research, such as establishing bounds on domination numbers after operations, could reveal connections within fuzzy graph models of complex data.

The findings could have implications for fields such as machine learning, data mining, pattern recognition, and other disciplines dealing with data sets that require representation using bipolar fuzzy graphs.

2. Preliminaries

In this section, we review some definitions of graphs, fuzzy graphs, bipolar fuzzy graphs, and domination numbers in a bipolar fuzzy graph [7-10].

Definition 2.1: A crisp graph Γ is defined as an ordered pair $\Gamma = (V, E)$, where V is a set of vertices E is a set of edges, and each edge is a two-element subset of V. The edges of a crisp graph are present or absent, and there is no ambiguity or uncertainty about their existence. A fuzzy graph $\Gamma = (\lambda, \tau)$ is defined as:

Definition 2.2: A set V of vertices, where each vertex s is associated with a membership function $\lambda_v(s)$ that assigns a degree of membership to each element s in V. The membership function maps each element to a value between 0 and 1, where 0 indicates no membership, and 1 indicates full membership.

Definition 2.3: A set E of edges, where each edge e is associated with a membership function $\tau_e(s, t)$ that assigns a degree of membership to each pair of vertices (s, t) in E. The membership function maps each pair of vertices to a value between 0 and 1, where 0 indicates no membership and 1 indicates full membership.

Definition 2.4: The order and size of a fuzzy graph $\Gamma = (\lambda, \tau)$ are defined as follows:

The order of Γ is the sum of the degrees of membership of all vertices in Γ, that is, $p = \sum_{s\in V} \lambda(s)$. The size of Γ is the sum of the degrees of membership of all edges in Γ, that is, $q = \sum_{(s,t)\in E} \tau(s, t)$.

Definition 2.5: The complement of a fuzzy graph $\Gamma = (\lambda, \tau)$ is another fuzzy graph $\overline{\Gamma} = (\overline{\lambda}, \overline{\tau})$, defined as follows:

The vertex set of $\overline{\Gamma}$ is the same as the vertex set of Γ, i.e., $V(\overline{\Gamma}) = V(\Gamma)$.

The degree of membership of each vertex in $\overline{\Gamma}$ is the same as in Γ, that is, $\overline{\lambda}(t) = \lambda(t)$ for all $t \in V(\Gamma)$.

Definition 2.6: The degree of membership of each edge in $\overline{\Gamma}$ is the complement of the degree of membership of the corresponding edge in Γ, i.e., $\overline{\tau}(s, t) = \lambda(s) \wedge \lambda(t) - \tau(s, t)$ for all $(s, t) \in E(\Gamma)$.

Definition 2.7: A dominating set D of a fuzzy graph $\Gamma = (\lambda, \tau)$ is a subset of vertices such that every vertex $t \in V(\Gamma) - D$ is dominated by at least one vertex $s \in D$. In other words, for every vertex $t \in V(\Gamma) - D$, there exists a vertex $s \in D$ such that $\tau(s, t) \geq \lambda(s)$. So, a dominating set in a fuzzy graph is a subset of vertices that "control" the graph, in the sense that every non-dominated vertex is within a certain distance from a vertex in the dominating set.

Definition 2.8: A dominating set D of a fuzzy graph $\Gamma = (\lambda, \tau)$ is called a minimal dominating set if no proper subset of D is a dominating set of Γ. In other words, for every $t \in D$, the set $D - t$ is not a dominant set of Γ. Thus, a minimal dominating set is a dominating set that cannot be reduced in size while still maintaining the property of domination. It is the "smallest" dominating set possible for the given fuzzy graph.

Definition 2.9: The domination number of a fuzzy graph $\Gamma = (\lambda, \tau)$, denoted by $\gamma(\Gamma)$, is defined as the minimum fuzzy cardinality of all minimal dominating sets in Γ. In other words, $\gamma(\Gamma)$ is the smallest possible value of $\sum_{t\in D} \lambda(t)$ on all minimal dominating sets D of Γ. Intuitively, the domination number of a fuzzy graph measures the "influence" of the graph in the sense that it represents the minimum number of vertices needed to control the graph. A smaller number of dominations indicates a more efficient control structure, where a smaller number of vertices can dominate the entire graph. A vertex subset D of V in a fuzzy graph $\Gamma = (\lambda, \tau)$ is said to be a global dominating set of Γ if it is a dominating set of both Γ and its complement $\overline{\Gamma}$. In other words, every vertex in $V(\Gamma) - D$ is dominated by at least one vertex in D, and every vertex in $V(\overline{\Gamma}) - D$ is dominated by at least one vertex in D. So, a global dominating set in a fuzzy graph is a subset of vertices that control both the presence and absence of edges in the graph. It is a more stringent condition than a dominating set or an independent set, as it requires that the set dominates both the original graph and its complement.

Definition 2.10: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ denote two fuzzy graphs. We consider their join $\Gamma = \Gamma_1 + \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs, where E' is defined as the set of all edges joining the nodes of V_1 and V_2, under the assumption that $V_1 \cap V_2 \neq \phi$.

Furthermore, let us assume that Γ_1 and Γ_2 are fuzzy graphs. In this context, we define the joining of the two product fuzzy graphs denoted by $\Gamma = \Gamma_1 + \Gamma_2 : (\lambda_1 + \lambda_2, \tau_1 + \tau_2)$, as follows:

$$(\lambda_1 + \lambda_2)(s) = \begin{cases} (\lambda_1 \cup \lambda_2) & \text{if} \quad s \in V_1 \cap V_2 \\ \lambda_1(s); s \in V_1 - V_2 \\ \lambda_2(s); s \in V_2 - V_1 \end{cases} \tag{1}$$

and

$$(\tau_1 + \tau_2)(st) = \begin{cases} (\tau_1 \cup \tau_2) & \text{if} \quad st \in E_1 \cap E_2 \\ \tau_1(st); st \in E_1 - E_2 \\ \tau_2(st); st \in E_2 - E_1 \end{cases} \tag{2}$$

Definition 2.11: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ denote two fuzzy graphs. We consider their intersection $\Gamma^* = \Gamma_1^* \cap \Gamma_2^* = (V_1 \cap V_2, E_1 \cap E_2)$ of graphs, under the assumption that $V_1 \cap V_2 \neq \phi$.

Moreover, let us consider Γ_1 and Γ_2 as fuzzy graphs and define their intersection, denoted by $\Gamma = \Gamma_1 \cap \Gamma_2 : (\lambda_1 \cap \lambda_2, \tau_1 \cap \tau_2)$, as a product fuzzy graph. The intersection is defined as follows:

$$\lambda_1 \cap \lambda_2 = \{\min(\lambda_1, \lambda_2) \quad \text{ifs} \in V_1 \cap V_2. \tag{3}$$

and

$$\tau_1 \cap \tau_2 = \{\min(\tau_1, \tau_2) \quad \text{ifs} \in E_1 \cap E_2 \tag{4}$$

Definition 2.12: Let $\Gamma_1 = (\lambda_1, \tau_1)$ and $\Gamma_2 = (\lambda_2, \tau_2)$ be two fuzzy graphs considering the union $\Gamma = \Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2)$ of graphs, where $V_1 \cap V_2 \neq \phi$. Then the union of two fuzzy graphs Γ_1 and Γ_2 is a fuzzy graph $\Gamma = \Gamma_1 \cup \Gamma_2 : (\lambda_1 \cup \lambda_2\ ,\ \tau_1 \cup \tau_2)$ defined as follows:

$$(\lambda_1 \cup \lambda_2)(s) = \begin{cases} \max(\lambda_1, \lambda_2) & \text{ifs} \in V_1 \cap V_2 \\ \lambda_1(s); s \in V_1 - V_2 \\ \lambda_2(s); s \in V_2 - V_1 \end{cases} \tag{5}$$

and

$$(\tau_1 \cup \tau_2)(st) = \begin{cases} \max(\tau_1, \tau_2) & \text{ifst} \in E_1 \cap E_2 \\ \tau_1(st); st \in E_1 - E2 \\ \tau_2(st); st \in E_2 - E_1 \end{cases} \tag{6}$$

3. Results

This section studies some bipolar fuzzy graph operations and domination and global domination numbers on bipolar fuzzy graph operations.

3.1 Some Bipolar Fuzzy Graph Operations

The study of operations on bipolar fuzzy graphs can yield several potential benefits. Firstly, these operations can facilitate the analysis and interpretation of complex data sets that are difficult to model using traditional graphs. Second, by providing a mathematical framework for the manipulation of bipolar fuzzy graphs, these operations can aid in the development of algorithms for processing and analyzing large amounts of data. Finally, the study of operations on bipolar fuzzy graphs can lead to the discovery of new insights and relationships within data sets, which can have practical applications in fields such as machine learning, data mining, and pattern recognition. Within this section, we shall commence an exploration of certain operations on bipolar fuzzy graphs, namely, the intersection, the join, and the union.

Definition 3.1.1. Let $\Gamma_1 = (A_1, B_1)$ and $\Gamma_2 = (A_2, B_2)$ be two bipolar fuzzy graphs, where $A_1 = (\lambda_1^+, \lambda_1^-), B_1 = (\tau_1^+, \tau_1^-), A_2 = ((\lambda_2^+, \lambda_2^-)$ and $B_2 = (\tau_2^+, \tau_2^-)$ consider the intersection $\Gamma = \Gamma_1 \cap \Gamma_2 = (A_1 \cap A_2, B_1 \cap B_2)$ of graphs. Suppose that $V_1 \cap V_2 \neq \phi$, then the intersection of two bipolar fuzzy graphs Γ_1 & Γ_2 is a bipolar fuzzy graph $\Gamma = \Gamma_1 \cap \Gamma_2 = (A_1 \cap A_2, B_1 \cap B_2)$ defined as follows:

$$A_1 \cap A_2 = \begin{cases} (\lambda_1^+ \cap \lambda_2^+)(s) = \min(\lambda_1^+, \lambda_2^+)(s) & \text{ifs} \in V_1 \cap V_2 \\ (\lambda_1^- \cap \lambda_2^-)(s) = \max(\lambda_1^-, \lambda_2^-)(s) & \text{ifs} \in V_1 \cap V_2. \end{cases} \tag{7}$$

and

$$B_1 \cap B_2 = \begin{cases} (\tau_1^+ \cap \tau_2^+)(st) = \min(\tau_1^+, \tau_2^+)(st) & \text{ifst} \in E_1 \cap E_2 \\ (\tau_1^- \cap \tau_2^-)(st) = \max(\tau_1^-, \tau_2^-)(st)) & \text{ifst} \in E_1 \cap E_2 \end{cases} \tag{8}$$

Example 1. Let Γ_1 and Γ_2 be two bipolar fuzzy graphs such that $(\tau_1^+ \cap \tau_2^+)(st) = \min(\tau_1^+, \tau_2^+)(st)$ and $(\tau_1^- \cap \tau_2^-)(st) = \max(\tau_1^-, \tau_2^-)(st)$ given in Figure 1 and their intersection.

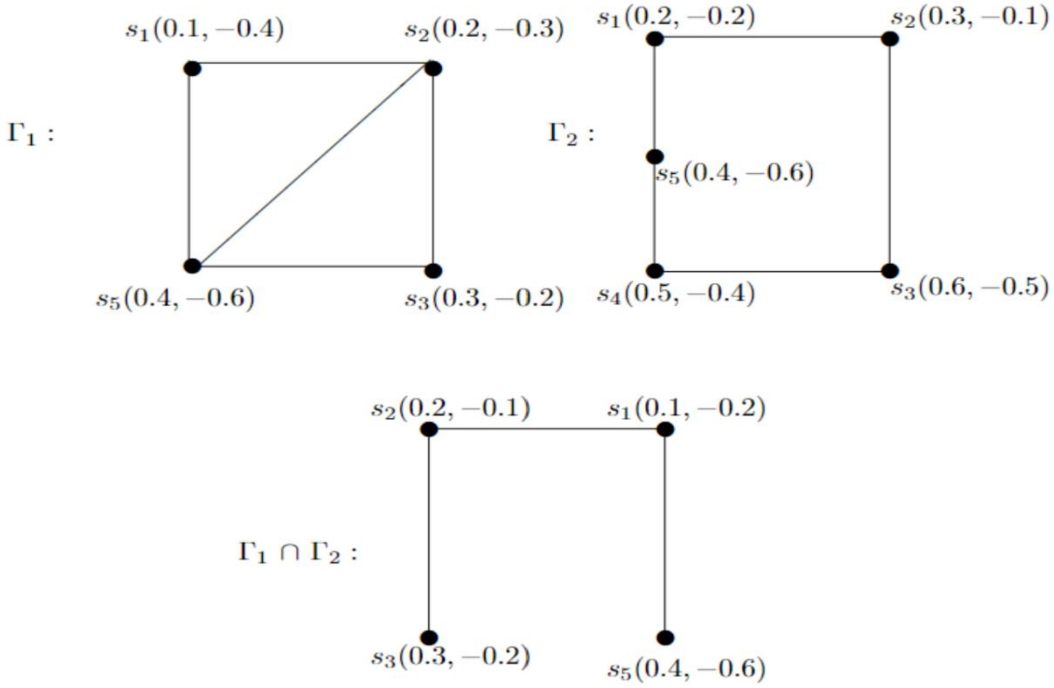

Figure 1. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 \cap \Gamma_2$.

Definition 3.1.2. Let $A_1 = (\lambda_1^+, \lambda_1^-)$ and $A_2 = (\lambda_2^+, \lambda_2^-)$ be bipolar fuzzy graphs subset of V_1 and V_2 and $B_1 = (\tau_1^+, \tau_1^-), B_2 = (\tau_2^+, \tau_2^-)$ be bipolar fuzzy graphs subset of $V_1 \times V_2$, and assume that $V_1 \cap V_2 \neq \phi$, then the join $\Gamma = (\Gamma_1 + \Gamma_2) = (A_1 + A_2, B_1 + B_2)$ is defined as follows:

$$(A_1 + A_2)(s) = \begin{cases} (\lambda_1^+ + \lambda_2^+)(s) = \max(\lambda_1^+, \lambda_2^+)(s) & \text{ifs} \in V_1 \cap V_2 \\ (\lambda_1^- + \lambda_2^-)(s) = \min(\lambda_1^-, \lambda_2^-)(s) & \text{ifs} \in V_1 \cap V_2 \\ (\lambda_1^+, \lambda_1^-)(s) & \text{ifs} \in V_1 - V_2 \\ (\lambda_2^+, \lambda_2^-)(s) & \text{ifs} \in V_2 - V_1 \end{cases} \tag{9}$$

and

$$(B_1 + B_2)(st) = \begin{cases} (\tau_1^+ + \tau_2^+)(st) = \max(\tau_1^+, \tau_2^+)(st) & \text{ifst} \in E_1 \cap E_2 \\ (\tau_1^- + \tau_2^-(st)) = \min(\tau_1^-, \tau_2^-)(st) & \text{ifst} \in E_1 \cap E_2 \\ (\tau_1^+, \tau_1^-)(st) & \text{ifst} \in E_1 - E2 \\ (\tau_2^+, \tau_2^-)(st) & \text{ifst} \in E_2 - E_1 \\ \max(\tau_1^+, \tau_2^+)(st) & \text{ifst} \in E' \\ \min(\tau_1^-, \tau_2^-)(st) & \text{ifst} \in E' \end{cases} \tag{10}$$

Example 2. For two bipolar fuzzy graphs Γ_1 and Γ_2, the joint $\Gamma_1 + \Gamma_2$ is given in Figure 2.

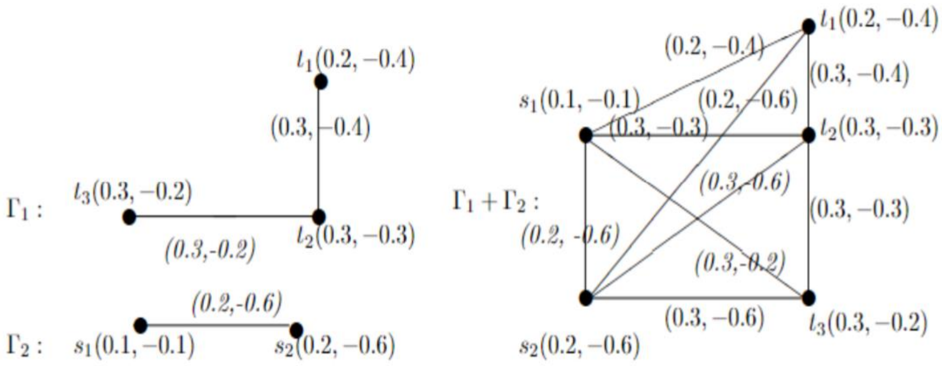

Figure 2. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 + \Gamma_2$.

Theorem 3.1.1. Let Γ_1 and Γ_2 be two bipolar fuzzy graphs, then, $\overline{\Gamma_1 + \Gamma_2} \neq \overline{\Gamma_1} + \overline{\Gamma_2}$.

The theorem states that the complement of the sum of two bipolar fuzzy graphs Γ_1 and Γ_2 is not equal to the sum of the complements of Γ_1 and Γ_2. In other words, De Morgan's laws of complementation do not hold for bipolar fuzzy graphs. Since the complement of the sum is a complete graph while the sum of the complements is a disjoint graph, we can conclude that De Morgan's laws of complementation do not hold for bipolar fuzzy graphs. This can be shown by considering a counterexample:

Example 3. Consider the bipolar fuzzy graphs $\Gamma_1 = (V_1, A_1, B_1)$, and $\Gamma_2 = (V_2, A_2, B_2)$, then $\Gamma_1 + \Gamma_2 = (V, A_1 + A_2, B_1 + B_1)$, and $\overline{\Gamma_1 + \Gamma_2} = (V, \overline{A_1 + A_2}, \overline{B_1 + B_1})$. Note that $\overline{\Gamma_1} = (V, \overline{A_1}, \overline{B_1})$, and $\overline{\Gamma_2} = (V, \overline{A_2}, \overline{B_2})$, where $A_1 = (\lambda_1^+, \lambda_1^-), B_1 = (\tau_1^+, \tau_1^-), A_2 = (\lambda_2^+, \lambda_2^-)$ and $B_2 = (\tau_2^+, \tau_2^-)$, such that $\tau_1^+(s,t) = \max(\lambda_1^+(s), \lambda_1^+(t))$, $\tau_1^-(s,t) = \min(\lambda_1^-(s), \lambda_1^-(t))$ $\forall$ $(s,t) \in E_1$, $\tau_2^+(s,t) = \max(\lambda_2^+(s), \lambda_2^+(t))$, $\tau_2^-(s,t) = \min(\lambda_2^-(s), \lambda_2^-(t))$, $\forall$ $(s,t) \in E_2$ which are respectively given in Figure 3, with $\overline{\Gamma_1} + \overline{\Gamma_2} = (V, \overline{A_1} + \overline{A_2}, \overline{B_1} + \overline{B_2})$.

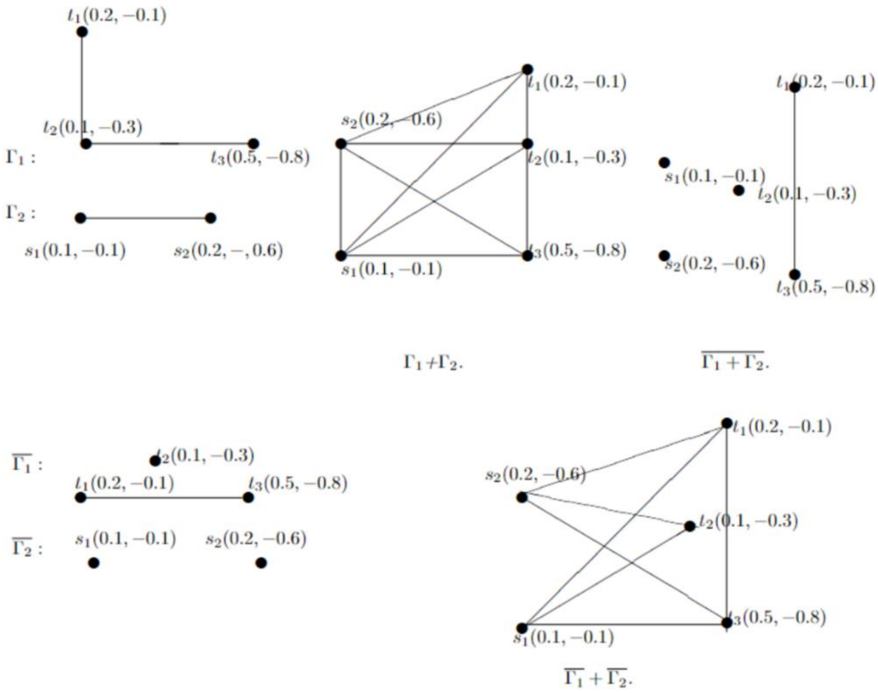

Figure 3. Graphs of Γ_1, Γ_2, $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and their $\Gamma_1 + \Gamma_2$, $\overline{\Gamma_1 + \Gamma_2}$, and $\overline{\Gamma}_1 + \overline{\Gamma}_2$.

Definition 3.1.3. The union of two bipolar fuzzy graphs Γ_1 and Γ_2 is a new bipolar fuzzy graph $\Gamma = \Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2)$: $(\lambda_1^+ \cup \lambda_2^+)(s), (\lambda_1^- \cup \lambda_2^-)(s), (\tau_1^+ \cup \tau_2^+)(st), (\tau^- \cup \tau_2^-)(st)$ such that $V_1 \cap V_2 \neq \phi$. These membership and relation grades are defined in the following way: For each vertex s in the union of V_1 and V_2, the positive and negative membership grades of s in Γ are defined as $(\lambda_1^+ \cup \lambda_2^+)(s)$ and $(\lambda_1^- \cup \lambda_2^-)(s)$, respectively. These grades are determined based on whether s is present in both Γ_1 and Γ_2 or in only one of the two graphs:

$$\begin{cases} (\lambda_1^+ \cup \lambda_2^+)(s) = \max(\lambda_1^+, \lambda_2^+)(s) & \text{if} s \in V_1 \cap V_2 \\ (\lambda_1^- \cup \lambda_2^-)(s) = \min(\lambda_1^-, \lambda_2^-)(s) & \text{if} s \in V_1 \cap V_2 \\ (\lambda_1^+, \lambda_1^-)(s) & \text{if} s \in V_1 - V_2 \\ (\lambda_2^+, \lambda_2^-)(s) & \text{if} s \in V_2 - V_1 \end{cases} \tag{11}$$

Similarly, for each edge st in the union of E_1 and E_2, the positive and negative relation grades of st in Γ are defined as $(\tau_1^+ \cup \tau_2^+)(st)$ and $(\tau_1^- \cup \tau_2^-)(st)$, respectively. These grades are also determined based on whether st is present in both Γ_1 and Γ_2 or in only one of the two graphs.

$$\begin{cases} (\tau_1^+ \cup \tau_2^+)(st) = \max(\tau_1^+, \tau_2^+)(st) & \text{if} st \in E_1 \cap E_2 \\ (\tau_1^- \cup \tau_2^-)(st) = \min(\tau_1^-, \tau_2^-)(st) & \text{if} st \in E_1 \cap E_2 \\ (\tau_1^+, \tau_2^+)(st) & \text{if} st \in E_1 - E_2 \\ (\tau_2^+, \tau_2^-)(st) & \text{if} st \in E_2 - E_1 \end{cases} \tag{12}$$

Example 4. Consider the two bipolar fuzzy graphs $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ be two bipolar fuzzy graphs such that $\tau_1^+(s,t) = \max(\lambda_1^+(s), \lambda_1^+(t)$, $\tau_1^-(s,t) = \min(\lambda_1^-(s), \lambda_1^-(t))$ $\forall$ $(s,t) \in E_1$, $\tau_2^+(s,t) = \max(\lambda_2^+(s), \lambda_2^+(t))$, $\tau_2^-(s,t) = \min(\lambda_2^-(s), \lambda_2^-(t))$, $\forall$ $(s,t) \in E_2$ and $\Gamma_1 \cup \Gamma_2$ are given in Figure 4.

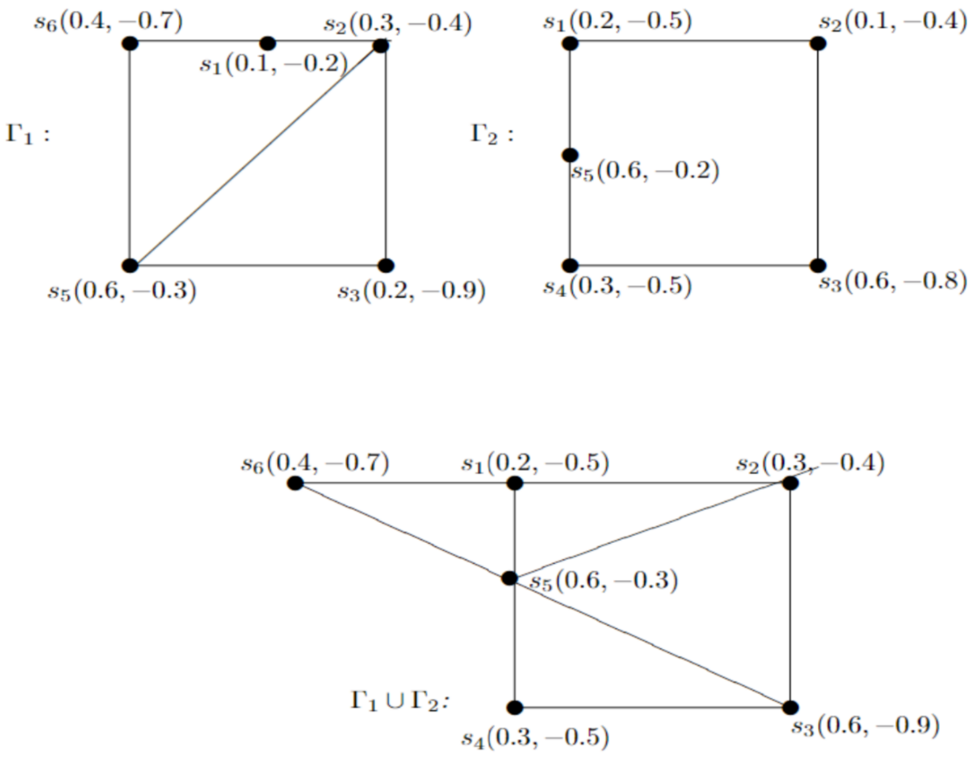

Figure 4. Graphs of Γ_1 , Γ_2 and their $\Gamma_1 \cup \Gamma_2$.

Theorem 3.1.2. Let $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ be two bipolar fuzzy graphs, then,

(i) $(\overline{\Gamma_1 + \Gamma_2}) = \overline{\Gamma_1} \cup \overline{\Gamma_2}$

(ii) $(\overline{\Gamma_1 \cup \Gamma_2}) = \overline{\Gamma_1} + \overline{\Gamma_2}$

Proof. Consider the identity map $I: V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove (i) it is enough to prove that

$A)(i)$ $(\overline{\lambda_1^+ + \lambda_2^+})(t_i) = (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i)$ and $(\overline{\lambda_1^- + \lambda_2^-})(t_i)$ $=(\overline{\lambda_1^-} \cup \overline{\lambda_2^-})(t_i)$

$A)(ii)$ $(\overline{\tau_1^+ + \tau_2^+})(t_i, t_j) = \overline{\tau_1^+} \cup \overline{\tau_2^+}(t_i, t_j)$and $(\overline{\tau_1^- + \tau_2^-})(t_i, t_j) = (\overline{\tau_1^-} \cup \overline{\tau_2^-})(t_i, t_j)$,

$A)(i)(\overline{\lambda_1^+ + \lambda_2^+})(t_i) = (\lambda_1^+ + \lambda_2^+)(t_i)$

$$= \begin{cases} \lambda_1^+(t_i); & t_i \in V_1 \\ \lambda_2^+(t_i); & t_i \in V_2 \end{cases} \tag{13}$$

$$= \begin{cases} \overline{\lambda_1^+}(t_i); & t_i \in V_1 \\ \overline{\lambda_2^+}(t_i); & t_i \in V_2 \end{cases} \tag{14}$$

$$= (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i).$$

Similarly $(\overline{\lambda_1^- + \lambda_2^-})(t_i)$ $=(\overline{\lambda_1^-} \cup \overline{\lambda_2^-})(t_i)$.

$A)(ii)$ $(\overline{\tau_1^+ + \tau_1^+})(t_i, t_j) = (\lambda_1^+ + \lambda_2^+)(t_i) \wedge (\lambda_1^+ + \lambda_2^+(t_j) - (\tau_1^+ + \tau_2^+)(t_i, t_j)$

$$= \begin{cases} \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_1^+(t_i t_j) & if \quad (t_i, t_j) \in E_1 \\ \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_2^+(t_i t_j) & if \quad (t_i, t_j) \in E_2 \\ \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) - \lambda_1^+(t_i) \wedge \lambda_2(t_j) & if \quad (t_i, t_j) \in E' \end{cases} \tag{15}$$

$$= \begin{cases} \overline{\tau_1^+}(t_i, t_i) & if \quad (t_i, t_j) \in E_1 \\ \overline{\tau_2^+}(t_i, t_i)) & if \quad (t_i, t_j) \in E_2 \\ 0 \quad if \quad (t_i, t_j) \in E' \end{cases} \tag{16}$$

$$= (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i).$$

Similarly $(\overline{\tau_1^- + \tau_2^-})(t_i, t_j) = \overline{(\tau_1^-} \cup \overline{\tau_2^-})(t_i, t_j)$.

Consider the identity map $I: V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove (ii), it is enough to prove

$A)(i)$ $(\overline{\lambda_1^+ \cup \lambda_2^+})(t_i) = \overline{(\lambda_1^+} + \overline{\lambda_2^+})(t_i)$and $(\overline{\lambda_1^- \cup \lambda_2^-})(t_i)$ =$(\overline{\lambda_1^-}) + (\overline{\lambda_2^-})(t_i)A)(ii)$ $(\overline{\tau_1^+ \cup \tau_2^+})(t_i, t_j) = (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i, t_j)$, and $(\overline{\tau_1^- \cup \tau_2^-})(t_i, t_j) = \overline{\tau_1^-} + \overline{\tau_2^-}(t_i, t_j)$

$A)(i)$ $(\overline{\lambda_1^+ \cup \lambda_2^+})(t_i) = (\lambda_1^+ \cup \lambda_2^+)(t_i)$

$$= \begin{cases} \lambda_1^+(t_i); & t_i \in V_1 \\ \lambda_2^+(t_i); & t_i \in V_2 \end{cases} \tag{17}$$

$$= \begin{cases} \overline{\lambda_1^+}(t_i); & t_i \in V_1 \\ \overline{\lambda_2^+}(t_i); & t_i \in V_2 \end{cases} \tag{18}$$

$$= (\overline{\lambda_1^+} \cup \overline{\lambda_2^+})(t_i) = (\overline{\lambda_1^+} + \overline{\lambda_2^+})(t_i).$$

Similarly $(\overline{\lambda_1^- \cup \lambda_2^-})(t_i)$ =$(\overline{\lambda_1^-}) + (\overline{\lambda_2^-})(t_i)$.

$A)(ii)$ $(\overline{\tau_1^+ \cup \tau_1^+})(t_i, t_j)$ = $(\lambda_1^+ \cup \lambda_2^+)(t_i) \wedge (\lambda_1 \cup \lambda_2(t_j)) - (\tau_1 \cup \tau_2)$

$$= \begin{cases} \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_1^+(t_i t_j) & if \quad (t_i, t_j) \in E_1 \\ \lambda_1^+(t_i) \wedge \lambda_1^+(t_j) - \tau_2^+(t_i t_j) & if \quad (t_i, t_j) \in E_2 \\ \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) - \lambda_1^+(t_i) \wedge \lambda_2^+(t_j) & if \quad (t_i, t_j) \in E' \end{cases} \tag{19}$$

$$= \begin{cases} \overline{\tau_1^+}(t_i, t_i) & if \quad (t_i, t_j) \in E_1 \\ \overline{\tau_2^+}(t_i, t_i)) & if \quad (t_i, t_j) \in E_2 \\ 0 \quad if \quad (t_i, t_j) \in E' \end{cases} \tag{20}$$

$$= (\overline{\tau_1^+} \cup \overline{\tau_2^+})(t_i, t_j) = (\overline{\tau_1^+} + \overline{\tau_2^+})(t_i, t_j).$$

Similarly $(\overline{\tau_1^- \cup \tau_2^-})(t_i, t_j) = \overline{(\tau_1^-} + \overline{\tau_2^-})(t_i, t_j)$.

3.2 Domination and Global Domination Number on Bipolar Fuzzy Graph Operations

Theorem 3.2.1. Let Γ_1 and Γ_2 be two disjoint bipolar fuzzy graphs. Then

$$\gamma(\Gamma_1 \cap \Gamma_2) = 0.$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a separate bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 = \phi$. Consequently, we can deduce that $\gamma(\Gamma_1 \cap \Gamma_2) = |D_1 \cap D_2| = |\phi| = 0$, where γ denotes the cardinality of a set.

Theorem 3.2.2. Let $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ be two bipolar fuzzy graphs such that $\tau_1^+(s,t) = max(\lambda_1^+(s), \lambda_1^+(t))$, $\tau_1^-(s,t) = min(\lambda_1^-(s), \lambda_1^-(t))$ for all $(s,t) \in E_1$, $\tau_2^+(s,t) = max(\lambda_2^+(s), \lambda_2^+(t))$, $\tau_2^-(s,t) = min(\lambda_2^-(s), \lambda_2^-(t)$, for all $(s,t) \in E_2$. Then,

$$\gamma(\Gamma_1 \cup \Gamma_2) = \gamma(\Gamma_1) + \gamma(\Gamma_1).$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 \neq \phi$. Then $D_1 \cup D_2$ is a dominating set of $\Gamma_1 \cup \Gamma_2$. Consequently, we can deduce that $\gamma(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = \gamma(\Gamma_1) + \gamma(\Gamma_2)$, where γ denotes the cardinality of a set.

Theorem 3.2.3. If Γ_1 and Γ_2 be any two not disjoint bipolar fuzzy graphs, then

$$\gamma(\Gamma_1 \cup \Gamma_2) = max(\gamma(\Gamma_1), \gamma(\Gamma_2)).$$

Proof. Let D_1 be a γ_1-set of a bipolar fuzzy graph Γ_1 and let D_2 be a γ_2-set of a bipolar fuzzy graph Γ_2. Then $D_1 \cup D_2$ is a dominating set of $\Gamma_1 \cup \Gamma_2$. Since Γ_1 and Γ_2 are not disjoint, then $D_1 \cap D_2 \neq \phi$ Hence $\gamma(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = max(\gamma(\Gamma_1), \gamma(\Gamma_1))$.

Theorem 3.2.4. If $\Gamma = \Gamma_1 + \Gamma_2$ is a complete bipolar fuzzy graph, then,

$$\gamma_g(\Gamma_1 + \Gamma_2) = p.$$

Proof. Consider a complete bipolar fuzzy graph $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two disjoint bipolar fuzzy graphs. In such a graph, every vertex has $(p-1)$ neighbors, where p is the number of vertices in the graph.
Since the complement of Γ is the null graph, the set of vertices V is the only global dominating set of both Γ and its complement $\overline{\Gamma}$.
Therefore, we can conclude that the global domination number $\gamma_g(\Gamma)$ of Γ is equal to p, where p is the number of vertices in V.

Theorem 3.2.5. If $\Gamma = \Gamma_1 + \Gamma_2 = (A_1 + A_2, B_1 + B_2)$ is a complete bipolar fuzzy graph, then

$$\gamma_g(\Gamma_1 + \Gamma_2) = \gamma_g(\overline{\Gamma_1 + \Gamma_2}).$$

Proof. Consider a bipolar fuzzy graph $\Gamma = \Gamma_1 + \Gamma_2 = (A_1 + A_2, B_1 + B_2)$, where Γ_1 and Γ_2 are two disjoint bipolar fuzzy graphs. Let D be a minimal global dominating set of Γ.

It can be observed that D is a dominating set of both Γ and its complement $\overline{\Gamma}$. This is because every vertex in $V(\Gamma)\backslash D$ is adjacent to at least one vertex in D, since D is a global dominating set. Therefore, D dominates all vertices in Γ, and its complement $V(\Gamma)\backslash D$ dominates all vertices in $\overline{\Gamma}$.

Furthermore, since D is a minimal global dominating set of Γ, it is also a minimal global dominating set of $\overline{\Gamma}$. This is because any global dominating set D' of $\overline{\Gamma}$ must also be a dominating set of Γ, since $\overline{\overline{\Gamma}} = \Gamma$. Therefore, $|D'| \geq |D|$.
From the above observations, we can conclude that the global domination number of Γ is equal to the global domination number of $\overline{\Gamma}$, i.e., $\gamma_g(\Gamma) = \gamma_g(\overline{\Gamma})$.

Theorem 3.2.6. Assume that Γ_1 and Γ_2 are two dis-joint bipolar fuzzy graphs. Then

$$\gamma_g(\Gamma_1 \cap \Gamma_2) = 0.$$

Proof. Consider a bipolar fuzzy graph Γ_1 with a global domination number γ_{g1} and a γ_{g1}-set D_1, as well as a bipolar fuzzy graph Γ_2 with a global domination number γ_{g2} and a γ_{g2}-set D_2.

Since Γ_1 and Γ_2 are disjoint, the intersection of D_1 and D_2 is non-empty. Hence, the size of the intersection, denoted by $|D_1 \cap D_2|$, is equal to zero since there are no common vertices in Γ_1 and Γ_2.

Therefore, the global domination number of the intersection of Γ_1 and Γ_2, denoted by $\Gamma_1 \cap \Gamma_2$, is also equal to zero, since the size of any minimal global dominating set of $\Gamma_1 \cap \Gamma_2$ is zero. Thus, $\gamma_g(\Gamma_1 \cap \Gamma_2) = |D_1 \cap D_2| = |\phi| = 0$.

Theorem 3.2.7. Consider two bipolar fuzzy graphs $\Gamma_1 = ((\lambda_1^+, \lambda_1^-), (\tau_1^+, \tau_1^-))$ and $\Gamma_2 = ((\lambda_2^+, \lambda_2^-), (\tau_2^+, \tau_2^-))$ such that

$\tau_1^+(s,t) = max(\lambda_1^+(s), \lambda_1^+(t)$, $\tau_1^-(s,t) = min(\lambda_1^-(s), \lambda_1^-(t)$ for all $(s,t) \in E_1$, $\tau_2^+(s,t) = \max(\lambda_2^+(s), \lambda_2^+(t)$, $\tau_2^-(s,t) = \min(\lambda_2^-(s), \lambda_2^-(t)$, for all $(s,t) \in E_2$, we claim that in this case, the global domination number of the union of the two graphs, denoted by $\Gamma_1 \cup \Gamma_2$, is equal to the sum of the global domination numbers of Γ_1 and Γ_2, i.e.,

$$\gamma_g(\Gamma_1 \cup \Gamma_2) = \gamma_g(\Gamma_1) + \gamma_g(\Gamma_2).$$

Proof. Let D_1 represent a γ_1-set of a bipolar fuzzy graph Γ_1, and let D_2 denote a γ_2-set of a bipolar fuzzy graph Γ_2. Given that Γ_1 and Γ_2 are disjoint, it follows that $D_1 \cap D_2 = \phi$. Then $D_1 \cup D_2$ is a global dominating set of $\Gamma_1 \cup \Gamma_2$. Consequently, we can deduce that $\gamma_g(\Gamma_1 \cup \Gamma_2) = |D_1 \cup D_2| = \gamma_g(\Gamma_1) + \gamma_g(\Gamma_2)$.

4. Conclusions

This study has explored the domain of domination and global domination numbers within the context of bipolar fuzzy graphs. We introduced and analyzed various operations on these graphs, including intersection, join, and union. Furthermore, we investigated the behavior of the domination number $\gamma(\Gamma)$ and the global domination number $\gamma_g(\Gamma)$ under these operations, encompassing not only the original graphs but also their complements. Much work still needs to be done, and here we mention some directions for future research, such as the relationship between domination and global domination numbers in bipolar fuzzy graphs under more complex operations, such as tensor product, Cartesian product, composition, strong product, disjunction, and symmetric difference of graphs. Other graph concepts like connectivity and independence numbers may also be investigated in bipolar fuzzy graphs, along with their relationships to domination measures.

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All authors contributed equally to this research.
Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.
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The authors declare that there is no conflict of interest in the research.

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Software Reliability Model Estimation for an Indeterministic Crime Cluster through Reinforcement Learning

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Abstract: The software reliability model estimates the probability of data failure in a specific environment, significantly impacting reliability and trustworthiness. The paper study focuses on cluster crime data, i.e., indeterministic in Neutrosophic Logic, using a software reliability model. The study utilizes reinforcement learning, Neutrosophic logic, and non-homogeneous Poisson process crime data to estimate indeterministic cluster data in crime. The "Non-homogeneous Poisson Process with Neutrosophic Logic" technique performs well in evaluating and deterring crime based on crime data analysis. The crime cluster involving offenders correctly classified as failure to accomplish does better than uncertain cluster reliability estimation with least squares and logistic regression analysis. The method enables crime prediction and prevention by using concave growth models to create an uncertain crime cluster, penalizing the correct person.

Keywords: Non-homogenous Poison Process; Neutrosophic Logic; Reinforcement Learning; Uncertain Crime Reliability Estimation.

1. Introduction

Crime clusters" are the tendency for crimes to congregate along the time, place, and other dimensions used to quantify them by Aparna [1]. Strategically, the ability to anticipate any crime based on timing, location, and other characteristics can help law enforcement by providing crucial information. Individuals with good self-discipline are more likely to commit crimes, while those with poor self-discipline are more likely to engage in illegal activities. A person has committed a crime when they blatantly violate the law through action, omission, or carelessness for which they may face punishment. A crime is an illegal act that violates a law or social standard, is punishable by law, and is approved by the government. Reliability refers to the consistency of measurement, ensuring results can be reproduced under the same circumstances [2]. While cluster integrity looks at the internal cohesion and separation of the clusters, cluster veracity assesses the external consistency and crime application of the clusters. The clustering analysis results can be accurate and beneficial when both variables are considered by J.A. Adeyiga [3]. Conducting a thorough investigation is crucial to determining if you are a party to the specific crime committed, as determining fault is challenging. Insufficient, uncertain data collection methods and poor-quality or malfunctioning data collection tools can produce unreliable crime data inquiries. Some traits are also more difficult to accurately quantify. To avoid this complexity, reliability estimation is used. It can measure how consistently a person is involved in the crime as a sort of average of the correlations between committing and silence, ranging from 0.0 to 1.0. Supervised machine learning is necessary for unlabeled clustering. When a crime is identified, clustering changes the classification [2]. Reliability is the application of crime data analytics, including AI machine learning, to predict when a committed crime investigation will fail or otherwise deteriorate so that it can be an inquiry or replaced before failing [4]. The software reliability growth model, divided into concave and S-shaped types shown in Figure 1, exhibits similar behavior, with the fault detection rate decreasing as faults are detected.

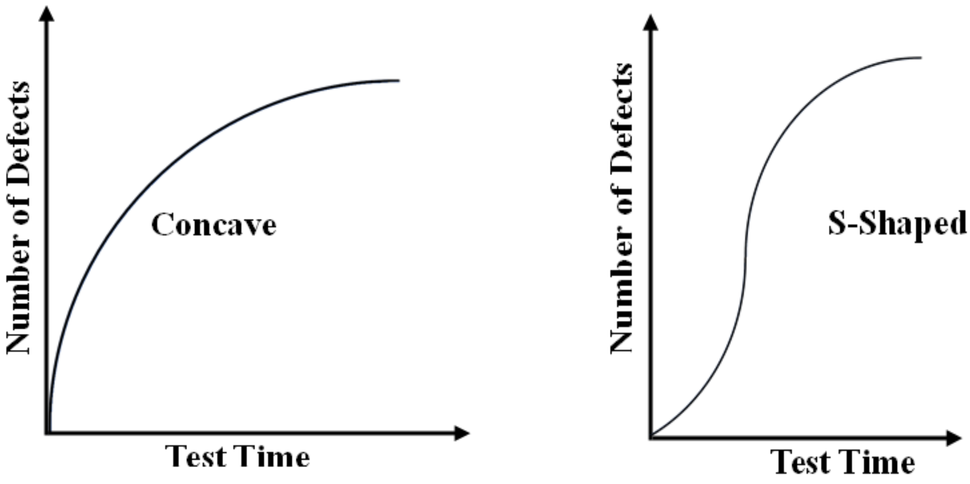


Figure 1. Concave and S-shaped models.

Defect density is the process of detecting defects in a crime application system during testing. It helps determine if a software system is ready for release, as proposed by Pushpa in 2019 [5]. However, identifying complete defects is challenging, especially for high-reliability software. To estimate defects, exponential software test coverage is used to measure thoroughness and estimate residual defect density. This method is easier to understand and visually observe. Reliability models are then used to evaluate the results.

The following six sections make up the correlation in this essay: Section 1's introduction and the proposed work in Section 2 using neutrosophic logic, a non-homogeneous poisoning process for crime clusters are covered in Section 2.1. Reinforcement learning is used for crime data analysis in Section 2.2. Uncertain cluster crime using least squares estimation in Section 2.3. A discussion of the experimental result is included in Section 2.4. The summary and projections for the future are found in Section 3 of the paper. References make mention of Section 4.

2. Proposed Work

The study utilizes neutrosophic logic and the non-homogenous Poisson process to analyze an uncertain crime cluster, focusing on the impact of software reliability on system reliability. The Contributions of this work are:

- To use hyperparameter control in the machine learning process using reinforcement learning on the uncertain crime cluster for a concave shape.
- To improve the uncertain cluster of crime using software reliability growth models.

The crime department utilizes a machine learning-based method called neutrosophic logic and a non-homogeneous Poisson process for crime investigation, which has a time limit for clusters.

2.1 Non-homogeneous Poisson Process-based Neutrosophic Logic for Crime Clusters

Neutrosophic logic is being utilized to create a non-homogeneous Poisson process for crime clusters. Veeraraghavan [6] introduced the Poisson process in stochastic processes $\{N(t)|t>=0\}$, counting actions and time t, for analyzing the non-homogeneous Poisson process on neutrosophic logic cluster criminals. N(t) is a random variable influenced by $N(t_n)$, which represents the number of crime cases identified at a specific time t and the number of criminals at time t_n.

$$P[N(t) = j \mid N(t_n) = i] = P[N(t) - N(t_n)] = j - i \quad (1)$$

where P[N(t) =j | process ending time], P[N(t) =i | Process Starting Time], j-i represents the process execution time. The neutrosophic logic rule can be used for continuous time-based Poisson processes, where criminals are involved in every crime detection system by Miguel Melgarejo [7].

$$\int_0^t N(t)dt = \int_0^1 (\mathrm{T} + \mathrm{I} + \mathrm{F})dt \quad (2)$$

The neutrosophic logic variable values in the same function N (t, s) vary between 0 to 1, as shown by evaluating the stochastic process $\int N(t)dt = \int (0 \leq \mathrm{T} + \mathrm{I} + \mathrm{F} \leq 1)$dt. The three-time interval crime data clusters in Neutrosophic logic, containing Certainty (T), Uncertainty (I), and False (F) which is not a criminal, is chosen and taken in the Non-homogeneous Poisson process. The rate parameter may change over time, and the general rate purpose function is given as λ(t). Here, T, I, and F are standard or non-standard real subsets of]-0, 1+[with not certainly any fitting together between them by Florentine [8].

$$\lambda_{a,b} = = \int_a^b \lambda(t)dt \quad (3)$$

The number of $\lambda_{a,b}$ on sets in the time interval (a, b], represented as N(b) - N(a), follows a poison process with associated parameters.

$$\mathrm{P}[\mathrm{N}(b) - \mathrm{N}(a) = \mathrm{K}] = \frac{e^{-\lambda_{a,b}}(\lambda_{a,b})^K}{K!}, K = 0.1, .., n \quad (4)$$

where K is the no. of events in the time interval between (a, b).
A time reason purpose in a Non-homogeneous Poisson process can be deterministic or autonomous, similar to a Cox procedure when λ(t) equals a constant rate proposed by Prasad [4].

2.2 Reinforcement Learning Used for Crime Cluster Data Analysis

Reinforcement learning (RL) is a method for customizing hyperparameters in crime data, transforming it into a supervised learning problem for model training, starting with a crime state and predicting an inquiry or investigation action introduced by Jagan Mohan [4, 5]. To anticipate future crime incentives, the model uses a discretized grid of hyperparameters, an uncertainty of crime loss function, policy curves, and qualitative learning techniques H: r = M (H). If a Reinforcement Learning model R is used to predict a value q with H and r, then q = R (H, r). The following R square error is minimized by the model (where g represents the discount rate for future rewards): (q' - (r + g*max q)) ^2. The network uses a linear layer output to predict q, simplifying policy gradient management and functioning as a classifier.

Next reward (Agreed/Silent) =M (next H). The crime type model is optimal for Hyperparameters with high crime rewards and silent low reward Hyperparameters, addressing the multi-label classification problem by Zhu et al. [9]. Cross entropy can be utilized to enhance the probability of a model producing certain Hyperparameters to 1, indicating our preference for them. L= (next H | current H, current r) * log e^{-P} accomplishes precisely that, but also balances the sample and reward value: L is equal to (next reward) *log e^{-P} (next H | current H, current r), where 0<-P<1.

2.3 Uncertain Cluster Crime using Least Square Estimation

Non-homogeneous Poisson process-based neutrosophic logic is utilized in crime case investigation to estimate the uncertainty of criminal cluster data using small sample sizes by Farrell [10]. It estimates Hyperparameters using failure intensity and best-possible mean values, obtaining coefficients for the equation Y= a + bX. The text discusses the use of Least Square estimation to estimate the probability of an uncertain cluster crime by Tsao Min [11].

Regression equation of x on y:

$$\sum x = b\sum y + Na \tag{5}$$

$$\sum xy = b\sum y^2 + a\sum y \tag{6}$$

Regression equation of y on x:

$$\sum y = b\sum x + Na \tag{7}$$

$$\sum xy = b\sum x^2 + a\sum x \tag{8}$$

The values of a and b can be easily determined by calculating the normal formula, allowing for easy determination of y and x.

The analysis of regression equations requires determining the appropriate criminal for the study. Establishing the relationship between dependent and independent criminals is crucial. Correlation, the linear relationship between two crime victims, is essential for this study, measured between observed variables by Win Bernic [12].

$$r = \frac{\sum_{i=1}^{n} y_i(x_i - \bar{x})}{\sqrt{[\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2]}} \tag{9}$$

The regression model uses a coefficient r to represent the mean of observed criminals, with values ranging from -1 to 1. A positive relationship indicates an increase or decrease in both criminals simultaneously, while a zero result indicates no or small linear relationship. A good fit includes a highly correlated dependent variable and independent criminals by Prasanth Sharma [13].

Independent criminals in regression can cause non-generalized, overfit models, leading to multicollinearity and conditioned X^TX. Perfect linear dependence can cause singular X^TX and infinite least squares estimates. The validity of a regression model is ensured by studying the residual standard error.

$$RSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \bar{y}_i)^2}{n-k}} \tag{10}$$

The equation estimates the difference between fitted and observed values, aiding in model cross-validation to prevent overfitting, and is explained in a separate section by Prasanth Sharma [14].

2.4 Uncertain Crime Cluster Using Logistic Regression

Logistic regression is a popular machine learning algorithm used to predict categorical dependent variables using independent variables. It uses a "Concave" shaped logistic function to predict probabilistic values between 0 and 1, similar to Linear Regression. This technique is used for classification problems, rather than regression uncertainty problem, and is similar to Linear Regression in its approach. Logistic Regression is a crucial machine learning algorithm that provides probabilities and classifies data using continuous and discrete datasets. It helps identify the most effective variables for classification in criminal investigations by Prasanth Sharma [14].

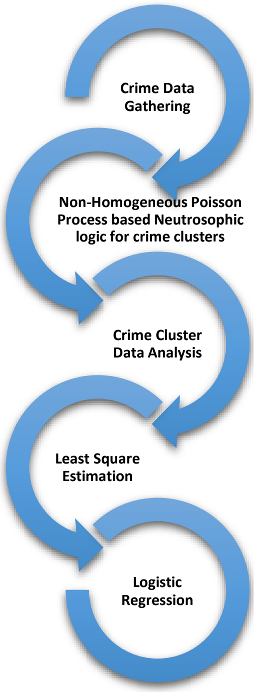


Figure 2. Working process to analyze an uncertain crime.

The categories of uncertain criminal data used include Murder, Rape, Robbery, and Auto-Theft as shown in Figure 3.

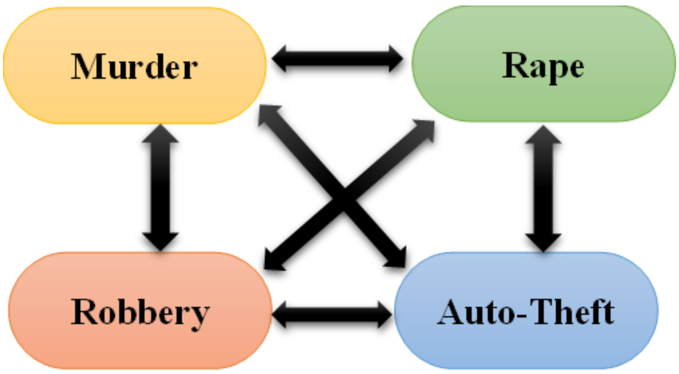
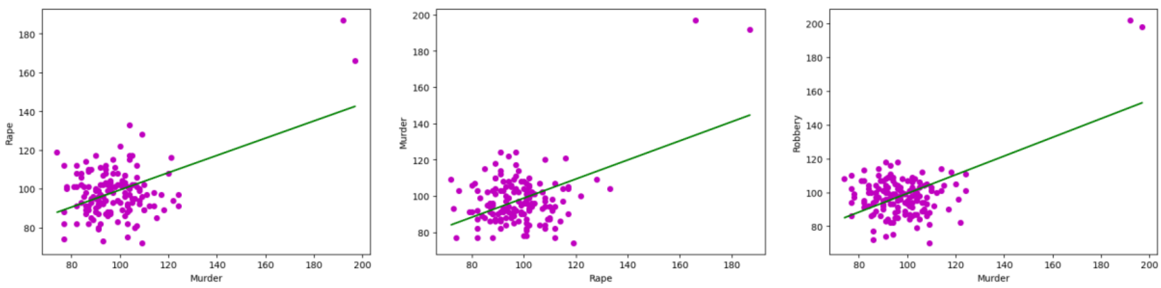


Figure 3. Regression statistics of uncertain criminals.

It will produce 12 combinations of regression analysis for each one that will be shown below in Figure 4 (a to k):



(a) (b) (c)

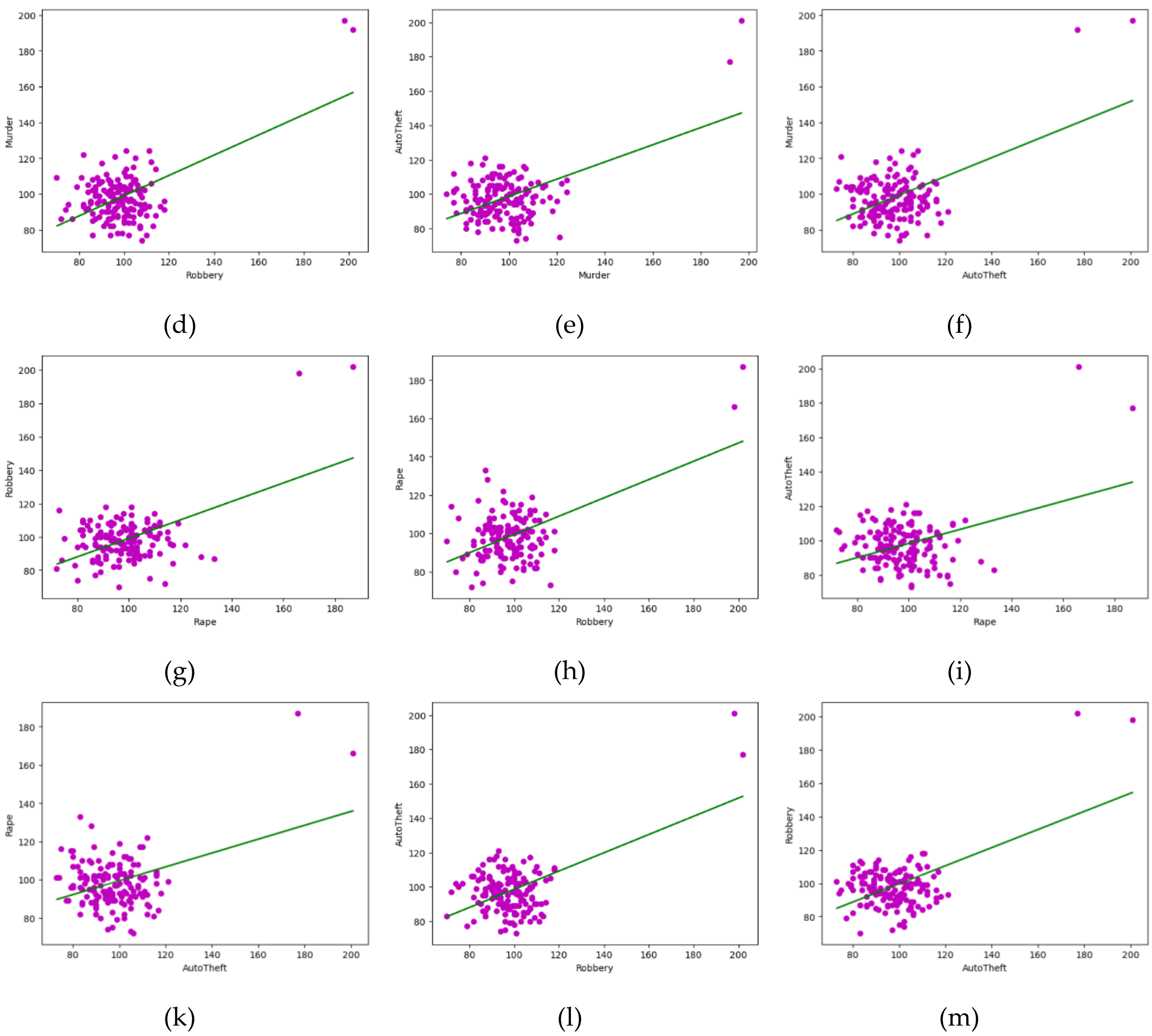


Figure 4. Regression analysis on uncertain criminals data.

3. Experimental Results

Experimental results in Tables (A and 1) show reliability estimation of criminal cases using neutrosophic logic and logistic regression on uncertain crime clusters, using indeterministic punishment data in a Concave-shape figure as shown in Figures 5-7.

Table 1. Crime punishment of uncertain criminals.

Regression Statistics	
Multiple R	0.668981671
R Square	0.447536476
Adjusted R Square	0.443561918
Standard Error	0.082019713
Observations	141

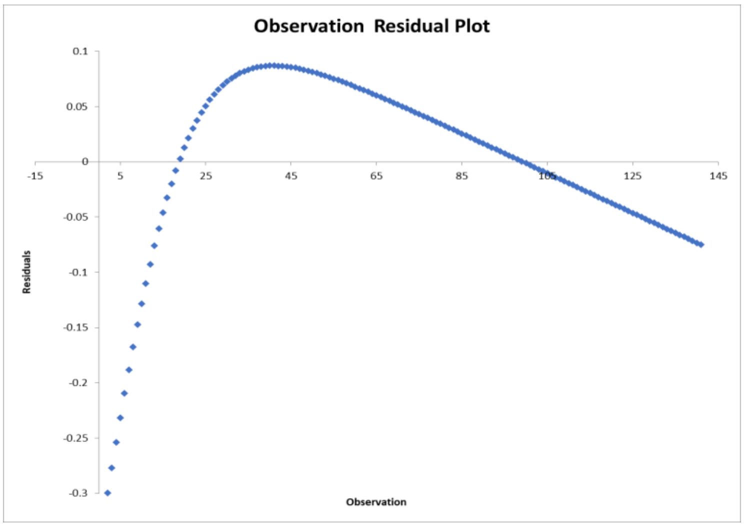


Figure 5. Observation residuals for uncertain criminal's punishment of crime cases.

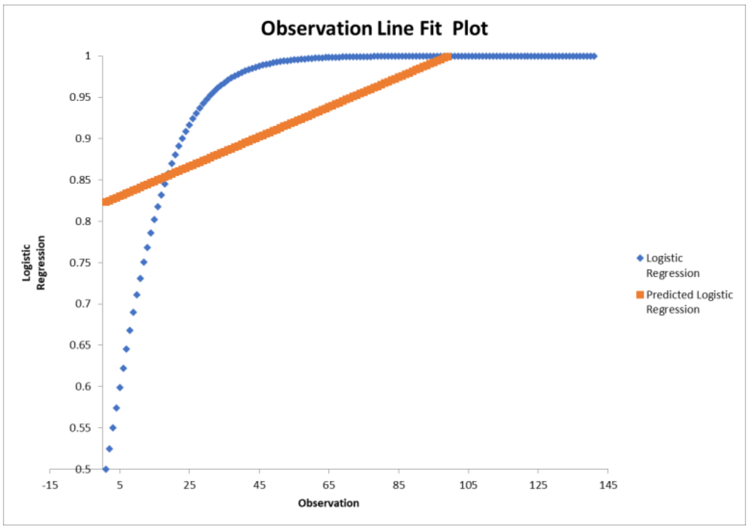


Figure 6. Observation line fit for uncertain criminal's punishment of crime cases.

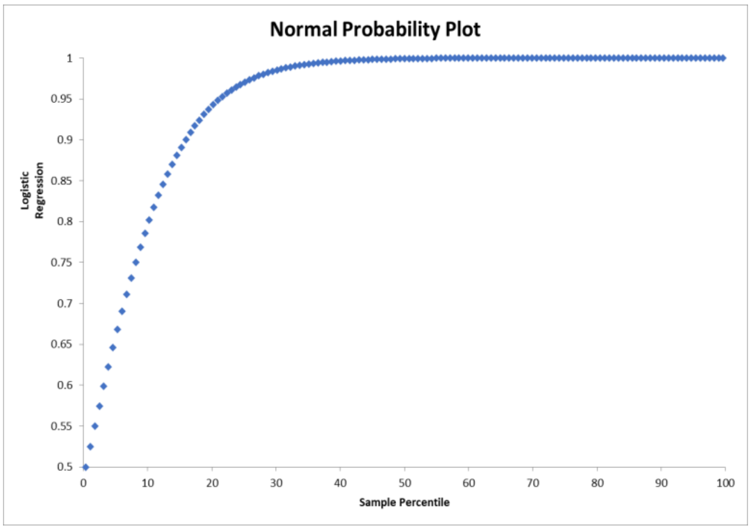


Figure 7. Concave-Shape of uncertain criminal's punishment of crime cases.

The software reliability growth model concave indicates a decrease in detection rate as faults are identified in crimes.

4. Conclusion

The likelihood that criminal data will work even if an investigation fails in a certain context has a big impact on cluster reliability. The study's main objective was to estimate software reliability models for a hazy crime cluster. In this respect, the criminal cluster predicts the non-homogeneous Poisson process, neutrosophic logic, and reinforcement learning technique. Using non-homogeneous Poisson process crime cluster data, logistic and least squares regression estimation, and neutrosophic logic-based crime cluster data, reinforcement learning classifies crimes, making it easier to anticipate crime probability based on crime data studied.

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Author Contributions
All authors contributed equally to this research.

Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest
The authors declare that there is no conflict of interest in the research.

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Appendix

Table A. Regression statistics of uncertain criminals.

RESIDUAL OUTPUT				PROBABILITY OUTPUT	
Observation	Predicted Logistic Regression	Residuals	Standard Residuals	Percentile	Logistic Regression
1	0.822998174	-0.322998174	-3.95219566	0.354609929	0.5
2	0.824798955	-0.299819768	-3.668585398	1.063829787	0.524979187
3	0.826599736	-0.276765739	-3.386497016	1.773049645	0.549833997
4	0.828400517	-0.253958001	-3.107422236	2.482269504	0.574442517
5	0.830201299	-0.231513638	-2.832793715	3.191489362	0.59868766
6	0.83200208	-0.209542749	-2.563958586	3.90070922	0.622459331
7	0.833802861	-0.188146555	-2.30215542	4.609929078	0.645656306
8	0.835603642	-0.16741587	-2.048495403	5.319148936	0.668187772
9	0.837404423	-0.147429942	-1.803948209	6.028368794	0.689974481
10	0.839205204	-0.128255702	-1.569332797	6.737588652	0.710949503
11	0.841005986	-0.109947407	-1.345313068	7.446808511	0.731058579
12	0.842806767	-0.092546661	-1.132398081	8.156028369	0.750260106
13	0.844607548	-0.076082764	-0.93094635	8.865248227	0.768524783
14	0.846408329	-0.060573346	-0.741173587	9.574468085	0.785834983
15	0.84820911	-0.046025222	-0.563163187	10.28368794	0.802183889
16	0.850009891	-0.032435415	-0.396878736	10.9929078	0.817574476
17	0.851810673	-0.019792287	-0.242177816	11.70212766	0.832018385
18	0.853611454	-0.008076719	-0.098826481	12.41134752	0.845534735
19	0.855412235	0.0027367	0.03348618	13.12056738	0.858148935
20	0.857213016	0.01267851	0.155133852	13.82978723	0.869891526
21	0.859013797	0.021783281	0.26653955	14.53900709	0.880797078
22	0.860814578	0.0300886	0.368163184	15.24822695	0.890903179
23	0.862615359	0.037634151	0.460490312	15.95744681	0.900249511
24	0.864416141	0.044460898	0.544022176	16.66666667	0.908877039
25	0.866216922	0.050610382	0.619267064	17.37588652	0.916827304
26	0.868017703	0.056124117	0.686732959	18.08510638	0.92414182
27	0.869818484	0.061043096	0.746921428	18.79432624	0.93086158
28	0.871619265	0.065407379	0.800322661	19.5035461	0.937026644
29	0.873420046	0.069255778	0.847411552	20.21276596	0.942675824

30	0.875220828	0.072625609	0.888644708	20.92198582	0.947846437
31	0.877021609	0.075552518	0.924458273	21.63120567	0.952574127
32	0.87882239	0.078070355	0.955266449	22.34042553	0.956892745
33	0.880623171	0.080211106	0.981460612	23.04964539	0.960834277
34	0.882423952	0.082004858	1.00340891	23.75886525	0.964428811
35	0.884224733	0.083479802	1.021456272	24.46808511	0.967704535
36	0.886025515	0.084662255	1.035924728	25.17730496	0.970687769
37	0.887826296	0.085576711	1.047113983	25.88652482	0.973403006
38	0.889627077	0.086245902	1.055302184	26.59574468	0.975872979
39	0.891427858	0.086690871	1.060746815	27.30496454	0.978118729
40	0.893228639	0.086931055	1.063685699	28.0141844	0.980159694
41	0.89502942	0.08698437	1.064338055	28.72340426	0.98201379
42	0.896830202	0.086867299	1.062905583	29.43262411	0.983697501
43	0.898630983	0.086594986	1.059573564	30.14184397	0.985225968
44	0.900431764	0.086181318	1.054511943	30.85106383	0.986613082
45	0.902232545	0.08563902	1.047876398	31.56028369	0.987871565
46	0.904033326	0.084979731	1.039809361	32.26950355	0.989013057
47	0.905834107	0.084214091	1.030441008	32.9787234	0.990048198
48	0.907634889	0.083351813	1.019890202	33.68794326	0.990986701
49	0.90943567	0.082401759	1.008265375	34.39716312	0.991837429
50	0.911236451	0.081372008	0.995665369	35.10638298	0.992608459
51	0.913037232	0.080269917	0.982180221	35.81560284	0.993307149
52	0.914838013	0.079102185	0.967891892	36.5248227	0.993940199
53	0.916638794	0.077874907	0.952874949	37.23404255	0.994513701
54	0.918439576	0.076593623	0.937197199	37.94326241	0.995033198
55	0.920240357	0.07526337	0.920920268	38.65248227	0.995503727
56	0.922041138	0.073888724	0.904100146	39.36170213	0.995929862
57	0.923841919	0.072473841	0.886787677	40.07092199	0.99631576
58	0.9256427	0.071022493	0.869029021	40.78014184	0.996665193
59	0.927443481	0.069538102	0.850866068	41.4893617	0.996981584
60	0.929244262	0.068023777	0.832336827	42.19858156	0.997268039
61	0.931045044	0.066482333	0.813475771	42.90780142	0.997527377
62	0.932845825	0.064916327	0.794314164	43.61702128	0.997762151
63	0.934646606	0.063328074	0.774880349	44.32624113	0.99797468
64	0.936447387	0.061719674	0.75520002	45.03546099	0.998167061
65	0.938248168	0.060093031	0.735296463	45.74468085	0.998341199
66	0.940048949	0.058449868	0.715190779	46.45390071	0.998498818
67	0.941849731	0.056791749	0.694902088	47.16312057	0.99864148
68	0.943650512	0.05512009	0.67444771	47.87234043	0.998770601
69	0.945451293	0.053436171	0.653843335	48.58156028	0.998887464
70	0.947252074	0.051741155	0.633103172	49.29078014	0.998993229
71	0.949052855	0.050036094	0.612240092	50	0.999088949
72	0.950853636	0.048321939	0.591265749	50.70921986	0.999175575
73	0.952654418	0.046599554	0.570190695	51.41843972	0.999253971

74	0.954455199	0.044869719	0.549024487	52.12765957	0.999324917
75	0.95625598	0.043133141	0.527775775	52.83687943	0.999389121
76	0.958056761	0.04139046	0.506452391	53.54609929	0.999447221
77	0.959857542	0.039642257	0.485061425	54.25531915	0.999499799
78	0.961658323	0.037889054	0.463609297	54.96453901	0.999547378
79	0.963459105	0.036131328	0.442101814	55.67375887	0.999590433
80	0.965259886	0.034369508	0.420544238	56.38297872	0.999629394
81	0.967060667	0.032603983	0.398941327	57.09219858	0.99966465
82	0.968861448	0.030835105	0.377297389	57.80141844	0.999696553
83	0.970662229	0.029063193	0.355616325	58.5106383	0.999725422
84	0.97246301	0.027288535	0.333901664	59.21985816	0.999751545
85	0.974263792	0.025511392	0.312156599	59.92907801	0.999775183
86	0.976064573	0.023732	0.290384021	60.63829787	0.999796573
87	0.977865354	0.021950574	0.268586547	61.34751773	0.999815928
88	0.979666135	0.020167307	0.246766543	62.05673759	0.999833442
89	0.981466916	0.018382373	0.224926153	62.76595745	0.99984929
90	0.983267697	0.016595932	0.203067314	63.4751773	0.99986363
91	0.985068479	0.014808127	0.181191782	64.18439716	0.999876605
92	0.98686926	0.013019087	0.159301145	64.89361702	0.999888347
93	0.988670041	0.01122893	0.137396839	65.60283688	0.999898971
94	0.990470822	0.009437762	0.115480165	66.31205674	0.999908584
95	0.992271603	0.00764568	0.093552299	67.0212766	0.999917283
96	0.994072384	0.005852769	0.071614306	67.73049645	0.999925154
97	0.995873165	0.00405911	0.049667149	68.43971631	0.999932276
98	0.997673947	0.002264774	0.0277117	69.14893617	0.99993872
99	0.999474728	0.000469824	0.005748748	69.85815603	0.999944551
100	1.001275509	-0.001325681	-0.016220994	70.56737589	0.999949828
101	1.00307629	-0.003121688	-0.038196878	71.27659574	0.999954602
102	1.004877071	-0.004918149	-0.060178321	71.9858156	0.999958922
103	1.006677852	-0.006715021	-0.082164794	72.69503546	0.999962831
104	1.008478634	-0.008512266	-0.104155818	73.40425532	0.999966368
105	1.010279415	-0.010309846	-0.126150961	74.11347518	0.999969568
106	1.012080196	-0.012107732	-0.14814983	74.82269504	0.999972464
107	1.013880977	-0.013905893	-0.17015207	75.53191489	0.999975085
108	1.015681758	-0.015704303	-0.192157362	76.24113475	0.999977456
109	1.017482539	-0.017502939	-0.214165414	76.95035461	0.999979601
110	1.019283321	-0.019301779	-0.236175964	77.65957447	0.999981542
111	1.021084102	-0.021100803	-0.258188775	78.36879433	0.999983299
112	1.022884883	-0.022899995	-0.280203631	79.07801418	0.999984888
113	1.024685664	-0.024699338	-0.302220337	79.78723404	0.999986326
114	1.026486445	-0.026498818	-0.324238717	80.4964539	0.999987627
115	1.028287226	-0.028298422	-0.346258613	81.20567376	0.999988805
116	1.030088008	-0.030098138	-0.36827988	81.91489362	0.99998987
117	1.031888789	-0.031897955	-0.390302387	82.62411348	0.999990834

118	1.03368957	-0.033697864	-0.412326017	83.33333333	0.999991706
119	1.035490351	-0.035497856	-0.434350662	84.04255319	0.999992495
120	1.037291132	-0.037297923	-0.456376227	84.75177305	0.99999321
121	1.039091913	-0.039098058	-0.478402622	85.46099291	0.999993856
122	1.040892695	-0.040898254	-0.500429771	86.17021277	0.999994441
123	1.042693476	-0.042698506	-0.5224576	86.87943262	0.99999497
124	1.044494257	-0.044498809	-0.544486045	87.58865248	0.999995448
125	1.046295038	-0.046299157	-0.566515048	88.29787234	0.999995881
126	1.048095819	-0.048099546	-0.588544554	89.0070922	0.999996273
127	1.0498966	-0.049899972	-0.610574518	89.71631206	0.999996628
128	1.051697382	-0.051700433	-0.632604894	90.42553191	0.999996949
129	1.053498163	-0.053500923	-0.654635644	91.13475177	0.999997239
130	1.055298944	-0.055301442	-0.676666732	91.84397163	0.999997502
131	1.057099725	-0.057101985	-0.698698126	92.55319149	0.99999774
132	1.058900506	-0.058902551	-0.720729796	93.26241135	0.999997955
133	1.060701287	-0.060703138	-0.742761717	93.97163121	0.999998149
134	1.062502069	-0.062503743	-0.764793865	94.68085106	0.999998326
135	1.06430285	-0.064304365	-0.786826218	95.39007092	0.999998485
136	1.066103631	-0.066105002	-0.808858756	96.09929078	0.999998629
137	1.067904412	-0.067905652	-0.830891462	96.80851064	0.99999876
138	1.069705193	-0.069706316	-0.852924321	97.5177305	0.999998878
139	1.071505974	-0.07150699	-0.874957316	98.22695035	0.999998984
140	1.073306755	-0.073307674	-0.896990436	98.93617021	0.999999081
141	1.075107537	-0.075108368	-0.919023669	99.64539007	0.999999168

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On Heptagonal Neutrosophic Semi-open Sets in Heptagonal Neutrosophic Topological Spaces: Testing Proofs by Examples

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Abstract: In terms of heptagonal neutrosophic topological spaces, the purpose of this paper is to present the idea of heptagonal neutrosophic semi-open sets. Additionally, we examine a few of its characterizations and heptagonal neutrosophic semi-interior and heptagonal neutrosophic semi-closure operators.

Keywords: Heptagonal Neutrosophic Topology; Heptagonal Neutrosophic Semi-open Set; Heptagonal Neutrosophic Semi-Interior and Heptagonal Neutrosophic Semi-Closure.

1. Introduction

In the year 1965, Zadeh [1] introduced and investigated fuzzy sets. An intuitionistic fuzzy set was first presented in 1986 by Atanassov [2]. Later, Coker [3] discovered intuitionistic fuzzy topological spaces in 1997. Florentin Smarandache [4] developed concepts such as neutrosophic logic and neutrosophic set in 1999. The truth, falsehood, and indeterminacy membership values are the three components on which he defined the neutrosophic set. The neutrosophic set was created in 2010 by Florentin Smarandache [5] as a generalization of intuitionistic fuzzy sets. In 2012, A.A. Salama and S.A. Albowi [6] introduced and developed the generalized neutrosophic set and generalized Neutrosophic topological spaces.

In 2014, Salama et al. [7] developed the concepts of neutrosophic closed sets and neutrosophic continuous functions. Salama [8] investigated the Basic Structure of Some Classes of Neutrosophic Crisp Nearly Open Sets & Possible Application to GIS Topology. In 2020, AL-Nafee et al. [9] explored New Types of Neutrosophic Crisp Closed Sets. In Neutrosophic Topological Spaces, Neutrosophic Semi-open sets were first introduced in 2016 by Iswarya P and K. Bageerathi [10].

Many scientists have constructed neutrosophic topological spaces on bipartitioned, quadripartitioned, and pentapartitioned neutrosophic sets. Kungumaraj et al. recently created heptagonal neutrosophic topological spaces [11]. The idea of heptagonal neutrosophic semi-open sets is introduced and its characterizations are studied in this study. Additionally, we present and investigate the heptagonal neutrosophic semi-interior and semi-closure operators.

The idea of heptagonal neutrosophic semi-open sets in heptagonal neutrosophic topological spaces is presented in this paper. The remaining part of the document is structured as follows: The preliminary information for a better comprehension of the study is contained in Section 2. In Section 3, the notion of the heptagonal neutrosophic semi-open set as well as the fundamental characteristics of these sets are introduced. The fundamental features of the heptagonal neutrosophic semi-interior operator are examined and the classical definition is presented in Section 4. The heptagonal neutrosophic semi-closure operator is defined classically and its fundamental features are examined in Section 5. The concluding Section 6 of the study contains the final results as well as some recommendations for additional research.

2. Preliminaries

Definition 2.1. [4] Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form A = {⟨x, α_A(x), β_A(x), γ_A(x)⟩: x ∈ X} where α_A(x), β_A(x), γ_A(x) represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element x ∈ X to the set A.

A Neutrosophic set A = {⟨x, α_A(x), β_A(x), γ_A(x)⟩: x ∈ X} can be identified as an ordered triple ⟨ α_A(x), β_A(x), γ_A(x)⟩in] −0, 1 +[on X.

Definition 2.2. [5] A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X that satisfies the following axioms:

(NT1) 0_N, 1_N∈ τ

(NT2) $G_1 \cap G_2$∈ τ for any G_1, G_2∈ τ

(NT3)∪G_i∈ τ ∀{G_i : i∈ J} ⊆ τ

The pair (X, τ) is used to represent a neutrosophic topological space τ over X.

Definition 2.3. [11] A heptagonal neutrosophic number S is defined and described as

S = < [(p, q, r, s, t, u, v); μ], [(p ′ , q′ , r′ , s′ , t′ , u′ , v′); ℰ] , [(p ″, q″, r″, s″, t″, u″, v″); η] > where μ, ℰ, η ∈ [0, 1].The truth membership function α : R ⇒ [0, μ], the indeterminacy membership function β : R⇒ [ℰ, 1], and the falsity membership function γ : R ⇒ [η, 1].

Using the ranking technique of heptagonal neutrosophic number is changed as,

$$\lambda = \frac{(p+q+r+s+t+u+v)}{7}$$

$$\mu = \frac{(p'+q'+r'+s'+t'+u'+v')}{7}$$

$$\delta = \frac{(p''+q''+r''+s''+t''+u''+v'')}{7}$$

Definition 2.4.[11] Let X be a non-empty set and A_{HN} and B_{HN} are HNS of the form A_{HN} = <x; $\lambda A_{HN}(x)$, $\mu A_{HN}(x)$, $\delta A_{HN}(x)$ >, B_{HN} = <x; $\lambda B_{HN}(x)$, $\mu B_{HN}(x)$, $\delta B_{HN}(x)$ >, then their heptagonal neutrosophic number operations may be defined as

- **Inclusive:**
  - (i) $A_{HN} \subseteq B_{HN} \Rightarrow \lambda A_{HN}(x) \le \lambda B_{HN}(x), \mu A_{HN}(x) \ge \mu B_{HN}(x), \delta A_{HN}(x) \ge \delta B_{HN}(x)$, for all x∈X.
  - (ii) $B_{HN} \subseteq A_{HN} \Rightarrow \lambda B_{HN}(x) \le \lambda A_{HN}(x), \mu B_{HN}(x) \ge \mu A_{HN}(x), \delta B_{HN}(x) \ge \delta A_{HN}(x)$, for all x∈X.
- **Union and Intersection:**
  - (iii) $A_{HN} \cup B_{HN}$ = {<x; $(\lambda A_{HN}(x) \vee \lambda B_{HN}(x), \mu A_{HN}(x) \wedge \mu B_{HN}(x), \delta A_{HN}(x) \wedge \delta B_{HN}(x))$ >}
  - (iv) $A_{HN} \cap B_{HN}$ = {<x; $(\lambda A_{HN}(x) \wedge \lambda B_{HN}(x), \mu A_{HN}(x) \vee \mu B_{HN}(x), \delta A_{HN}(x) \vee \delta B_{HN}(x))$ >}
- **Complement:**

  Let X be a non-empty set and A_{HN} be the HNS, A_{HN} = <x; $\lambda A_{HN}(x)$, $\mu A_{HN}(x)$, $\delta A_{HN}(x)$ >, then its complement is denoted by A'_{HN} and is defined by

  A'_{HN} = <x; $\delta A_{HN}(x)$, $1-\mu A_{HN}(x)$, $\lambda A_{HN}(x)$ > for all x∈X.
- **Universal and Empty set:**

  Let A_{HN} = <x; $\lambda A_{HN}(x)$, $\mu A_{HN}(x)$, $\delta A_{HN}(x)$ > be a HNS and the universal set I_A and O_A of A_{HN} is defined by
  - (v) I_{HN} = <x: (1,0,0)> for all x∈X.
  - (vi) O_{HN} = <x: (0,1,1)> for all x∈X.

Definition 2.5. [11] A Heptagonal neutrosophic topology (HNT) on a non-empty set X is a family τ of heptagonal neutrosophic subsets in X satisfies the following axioms:

(HNT1) $I_{HN}(x)$, $O_{HN}(x) \in \tau$
(HNT2)$\bigcup A_i \in \tau$,$\forall\{A_i : i \in J\} \subseteq \tau$
(HNT3) $A_1 \cap A_2 \in \tau$ for any A_1 , $A_2 \in \tau$
The pair (X, τ) is used to represent a heptagonal neutrosophic topological space τ over X. The sets in τ are called a heptagonal neutrosophic open set of X. The complement of heptagonal neutrosophic open sets are called heptagonal neutrosophic closed set of X.

Throughout this paper, we denote
HNS for heptagonal neutrosophic set
HNOS for heptagonal neutrosophic open set
HNCS for heptagonal neutrosophic closed set
HNTS for heptagonal neutrosophic topological space

Definition 2.6. [11] Let A be a HNS in HNTS (X, τ). Then,

- HNint(A_{HN}) = ⋃{G_{HN}: G_{HN} is a HNOS in X and $G_{HN} \subseteq A_{HN}$} is called a heptagonal neutrosophic interior of A. It is the largest HN-open subset contained in A_{HN}.
- HNcl(A_{HN}) = ⋂ {K_{HN}: K_{HN} is a HNCS in X and $A_{HN} \subseteq K_{HN}$} is called a heptagonal neutrosophic closure of A. It is the smallest HN-closed subset containing A_{HN}.

3. HN-Semi Open Sets

Definition 3.1: Let A_{HN} be a HNS of a HNTS X. Then A_{HN} is said to be a Heptagonal Neutrosophic Semi-open [written HN-SO] set of X if there exists a heptagonal neutrosophic open set HNO such that HNO $\subseteq A_{HN} \subseteq$ HNCl (HNO).

Example 3.2: Let X = {x,y} and A_{HN}, $B_{HN} \in$ HN(X).

A_{HN} = { <x; (λ:0.85,0.65,0.55,0.78,0.92,0.63,0.38), (μ: 0.75,0.95,0.63,0.48,0.56,0.88,0.78), (δ: 0.25,0.36,0.45,0.58,0.69,0.72,0.90)>, <y; (λ:0.83,0.65,0.72,0.98,0.66,0.53,0.92), (μ:0.73,0.53,0.45,0.38,0.92,0.75,0.63), (δ:0.45,0.35,0.25,0.95,0.85,0.65,0.15)>} and

B_{HN} = { <x; (λ:0.86,0.73,0.62,0.52,0.93,0.45,1), (μ:0.43,0.39,0.26,0.75,0.58,0.93,0.88), (δ:0.55,0.73,0.62,0.52,0.95,0.89,0.44)>, <y; (λ:0.73,0.62,0.51,0.42,0.33,0.29,0.19), (μ:0.82,0.92,1,0.61,0.54,0.76,0.46), (δ:0.19,0.23,0.63,0.52,0.95,0.82,1)>}

By Ranking Technique, (Definition 2.5)

A_{HN} = { <x; (λ:0.68), (μ:0.72), (δ:0.56)>, <y; (λ:0.76), (μ:0.63), (δ:0.52)>} and
B_{HN} = { <x; (λ:0.73), (μ:0.60), (δ:0.67)>, <y; (λ:0.44), (μ:0.73), (δ:0.62)>}

For simplicity, we write the Heptagonal Neutrosophic sets after ranking technique as

A_{HN} = { <x; (0.68, 0.72, 0.56)>, <y; (0.76, 0.63, 0.52)>} and
B_{HN} = { <x; (0.73, 0.60, 0.67)>, < y; (0.44, 0.73, 0.62)>}

Let X = {x,y} and HNTS τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN} } where
A_{HN} = { <x; (0.68, 0.72, 0.56)>, <y; (0.76, 0.63, 0.52)>}
B_{HN} = { <x; (0.73, 0.60, 0.67)>, <y; (0.44, 0.73, 0.62)>}
C_{HN} = { <x; (0.73, 0.60, 0.56)>, <y; (0.76, 0.63, 0.52)>}
D_{HN} = { <x; (0.68, 0.72, 0.67)>, <y; (0.44, 0.73, 0.62)>}

Consider the HNS after ranking technique
E_{HN} = { <x; (0.75, 0.52, 0.48)>, <y; (0.82, 0.59, 0.39)>}
F_{HN} = { <x; (0.58, 0.62, 0.75)>, <y; (0.25, 0.85, 0.75)>}
Then the HN-semi open sets of HN(X) are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, E_{HN}, F'_{HN}}

The following theorems are the characterization of the HN-SO set in HNTS.

Theorem 3.3: A subset A_{HN} in a HNTS X is a HN-Semi open set iff $A_{HN} \subseteq$ HNCl (HNInt (A_{HN})).
Proof:
Necessity: Let A_{HN} be a HN-semi open set in X. Then HNO $\subseteq$ A_{HN} $\subseteq$HNCl (HNO) for some heptagonal neutrosophic open set HNO. But HNO $\subseteq$HNInt (A_{HN}) and thus HNCl (HNO) $\subseteq$HNCl (HNInt (A_{HN})). Hence $A_{HN} \subseteq$HNCl (HNO) $\subseteq$HNCl (HNInt (A_{HN})).
Sufficiency: Let A_{HN} $\subseteq$HNCl (HNInt (A_{HN})). Since HNO = HNInt (A_{HN}), we have HNO $\subseteq$ $A_{HN} \subseteq$HNCl (HNO). Hence A_{HN} is a HN-Semi open set.

Theorem 3.4: Let (X, τ) be a HNTS. Then union of two HN-semi-open sets is again a HN-semi-open set in the HNTS X.
Proof: Let A_{HN} and B_{HN} are HN-semi open sets in X. Then A_{HN} $\subseteq$HNCl (HNInt (A_{HN})) and $B_{HN} \subseteq$HNCl (HNInt (B_{HN})). Therefore $A_{HN} \cup B_{HN} \subseteq$HNCl (HNInt ($A_{HN}$)) $\cup$HNCl (HNInt (B_{HN})) = HNCl (HNInt (A_{HN}) $\cup$HNInt (B_{HN})) $\subseteq$HNCl (HNInt ($A_{HN} \cup B_{HN}$)) [By Theorem 3.3]. Hence $A_{HN} \cup B_{HN}$ is a HN-semi open set in X.

Theorem 3.5: Let (X, τ) be a HNTS. Then union of a finite collection of HN-semi open sets is again a HN- semi open set in the HNTS X.
Proof: For each i∈Δ, $(A_{HN})_i$ is a HN-semi open sets in X. Then by theorem 3.3, $(A_{HN})_i \subseteq$HNCl (HNInt($(A_{HN})_i$)). Thus, $\cup_{i\in\Delta}$ $(A_{HN})_i \subseteq \cup_{i\in\Delta}$HNCl (HNInt($(A_{HN})_i$)) $\subseteq$HNCl ($\cup_{i\in\Delta}$HNInt($(A_{HN})_i$)). Hence $\cup_{i\in\Delta}$ $(A_{HN})_i \subseteq$HNCl (HNInt($\cup_{i\in\Delta}(A_{HN})_i$)). Therefore, the union of a finite collection of HN-semi open sets is again a HN- semi-open set in the HNTS X.

Remark 3.6: The intersection of any two HN-semi open sets need not be a HN- semi-open set as shown in the following example.

Example 3.7: Let X = {x,y} and τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN} } where
A_{HN} = { <x; (0.45,0.45,0.45,0.45,0.45,0.45,0.45)>, <y; (0.75,0.75,0.75,0.75,0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95,0.95,0.95,0.95,0.95)>, <y; (0.55,0.55,0.55,0.55,0.55,0.55,0.55)>}
By ranking technique,
A_{HN} = { <x; (0.45,0.45,0.45)>, <y; (0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95)>, <y; (0.55,0.55,0.55)>}
C_{HN} =$A_{HN} \cup B_{HN}$={<x; (0.95,0.45,0.45)>, <y; (0.75,0.55,0.55)>}
D_{HN} =$A_{HN} \cap B_{HN}$={<x; (0.45,0.95,0.95)>, <y; (0.55,0.75,0.75)>}
τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN} }is a HNTS.
Then the HN-semi open sets of HN(X) are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, B'_{HN}, C'_{HN}, D'_{HN}}.
Here $A_{HN} \cap B'_{HN}$ is not a HN-semi open set, since HNCl(HNInt($A_{HN} \cap B'_{HN}$))= C'_{HN} and $A_{HN} \cap B'_{HN} \nsubseteq C'_{HN}$.

Theorem 3.8: Let A_{HN} be a HNSO set in the HNTS X and suppose $A_{HN} \subseteq B_{HN} \subseteq HNCl(A_{HN})$. Then B_{HN} is HNSO set in X.

Proof: There exists a heptagonal neutrosophic open set HNO such that $HNO \subseteq A_{HN} \subseteq HNCl(HNO)$. Since, $A_{HN} \subseteq B_{HN}$, $HNO \subseteq B_{HN}$. But $HNCl(A_{HN}) \subseteq HNCl(HNO)$ and thus $B_{HN} \subseteq HNCl(HNO)$. Hence $HNO \subseteq B_{HN} \subseteq HNCl(HNO)$ and B_{HN} is HNSO set in X.

Theorem 3.9: Every heptagonal neutrosophic open set in the HNTS X is a HNSO set in X.

Proof: Let A be a heptagonal neutrosophic open set in HNTS X. Then $A_{HN} = HNInt(A_{HN})$. Also $HNInt(A_{HN}) \subseteq HNCl(HNInt(A_{HN}))$. This implies that $A_{HN} \subseteq HNCl(HNInt(A_{HN}))$. Hence by Theorem 3.3, A_{HN} is a HNSO set in X.

Remark 3.10: The converse of the above theorem need not be true as shown in the following example.

Example 3.11: From Example 3.7, B'_{HN}, C'_{HN}, D'_{HN} are HN-semi open sets, but not HN-open sets.

4. Heptagonal Neutrosophic Semi-Interior In Heptagonal Neutrosophic Topological Spaces

In this section, we introduce the heptagonal neutrosophic semi-interior operator and their properties in the heptagonal neutrosophic topological space.

Definition 4.1: Let (X, τ) be a HNTS. Then for a heptagonal neutrosophic subset A_{HN} of X, the heptagonal neutrosophic semi-interior of A_{HN} [HN-SInt (A_{HN}) for short] is the union of all heptagonal neutrosophic semi-open sets of X contained in A_{HN}.

$HN\text{-}SInt(A_{HN}) = \cup\{ S_{HN} : S_{HN}$ is a HNSO set in X and $S_{HN} \subseteq A_{HN}\}$

Proposition 4.2: Let (X, τ) be a HNTS. Then for any heptagonal neutrosophic subsets A_{HN} and B_{HN} of a HNTS X we have

(i) $HN\text{-}SInt(A_{HN}) \subseteq A_{HN}$
(ii) A_{HN} is HNSO set in X $\Leftrightarrow HN\text{-}SInt(A_{HN}) = A_{HN}$
(iii) $HN\text{-}SInt(HN\text{-}SInt(A_{HN})) = HN\text{-}SInt(A_{HN})$
(iv) If $A_{HN} \subseteq B_{HN}$ then $HN\text{-}SInt(A_{HN}) \subseteq HN\text{-}SInt(B_{HN})$
(v) $HN\text{-}SInt(A_{HN} \cap B_{HN}) = HN\text{-}SInt(A_{HN}) \cap HN\text{-}SInt(B_{HN})$
(vi) $HN\text{-}SInt(A_{HN}) \cup HN\text{-}SInt(B_{HN}) \subseteq HN\text{-}SInt(A_{HN} \cup B_{HN})$

Proof:

(i) Follows from Definition 4.1.

(ii) Let A_{HN} be a HNSO set in X. Then $A_{HN} \subseteq HN\text{-}SInt(A_{HN})$. By using (i) we get $A_{HN} = HN\text{-}SInt(A_{HN})$. Conversely assume that $A_{HN} = HN\text{-}SInt(A_{HN})$. By using Definition 4.1, A_{HN} is NSO set in X. Thus (ii) is proved.

(iii) By using (ii), $HN\text{-}SInt(HN\text{-}SInt(A_{HN})) = HN\text{-}SInt(A_{HN})$. This proves (iii). Since $A_{HN} \subseteq B_{HN}$, by using (i), $HN\text{-}SInt(A_{HN}) \subseteq A_{HN} \subseteq B_{HN}$. That is $HN\text{-}SInt(A_{HN}) \subseteq B_{HN}$. Thus (iii) is proved

(iv) By (iii), $HN\text{-}SInt(HN\text{-}SInt(A_{HN})) \subseteq HN\text{-}SInt(B_{HN})$. Thus $HN\text{-}SInt(A_{HN}) \subseteq HN\text{-}SInt(B_{HN})$. Thus (iv) is proved.

(v) Since $A_{HN} \cap B_{HN} \subseteq A_{HN}$ and $A_{HN} \cap B_{HN} \subseteq B_{HN}$, by using (iv), $HN\text{-}SInt(A_{HN} \cap B_{HN}) \subseteq HN\text{-}SInt(A_{HN})$ and $HN\text{-}SInt(A_{HN} \cap B_{HN}) \subseteq HN\text{-}SInt(B_{HN})$. This implies that
$HN\text{-}SInt(A_{HN} \cap B_{HN}) \subseteq HN\text{-}SInt(A_{HN}) \cap HN\text{-}SInt(B_{HN})$ ---(1).

By(i), HN-SInt(A_{HN})⊆A_{HN} and HN-SInt(B_{HN})⊆B_{HN}. This implies that HN-SInt(A_{HN})⋂HN-SInt(B_{HN}) ⊆ A_{HN}⋂B_{HN}.
Now by (iv), HN-SInt ((HN-SInt(A_{HN})⋂HN-SInt(B_{HN})) ⊆ HN-SInt(A_{HN}⋂ B_{HN}).
By (1), HN-SInt(HN-SInt (A_{HN}))⋂HN-SInt(HN-SInt(B_{HN}))⊆HN-SInt(A_{HN}⋂ B_{HN}).
By (iii), HN-SInt(A_{HN})⋂HN-SInt (B_{HN})⊆HN-SInt(A_{HN}⋂ B_{HN}) -----(2).
From (1) and (2), HN-SInt (A_{HN}⋂ B_{HN}) = HN-SInt(A_{HN})⋂HN-SInt(B_{HN}). Thus (v) is proved.

(vi) Since A_{HN}⊆ A_{HN}⋃B_{HN} and B_{HN}⊆ A_{HN}⋃ B_{HN}, by (iv), HN-SInt (A_{HN}) ⊆ HN-SInt (A_{HN}⋃ B_{HN}) and HN-SInt (B_{HN}) ⊆ HN-SInt (A_{HN}⋃ B_{HN}). This implies that,
HN-SInt (A_{HN}) ⋃ HN-SInt (B_{HN}) ⊆ HN-SInt (A_{HN}⋃ B_{HN}). Thus (vi) is proved.

The following example shows that the equality need not be held in Theorem 4.2 (vi).

Example 4.3: Let X = {x,y} and
A_{HN} = { <x; (0.45,0.45,0.45,0.45,0.45,0.45,0.45)>, <y; (0.75,0.75,0.75,0.75,0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95,0.95,0.95,0.95,0.95)>, <y; (0.55,0.55,0.55,0.55,0.55,0.55,0.55)>}
By ranking technique,
A_{HN} = { <x; (0.45,0.45,0.45)>, <y; (0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95)>, <y; (0.55,0.55,0.55)>}
C_{HN} =A_{HN}⋃B_{HN}={<x; (0.95,0.45,0.45)>, <y; (0.75,0.55,0.55)>}
D_{HN} =A_{HN}⋂B_{HN}={<x; (0.45,0.95,0.95)>, <y; (0.55,0.75,0.75)>}
Then, τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN} }is a HNTS
Consider the HNS after the ranking technique,
E_{HN} = { <x; (0.75,0.52,0.48)>, <y; (0.82,0.59,0.39)>}
Then the HN-semi open sets of HN(X) are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, B'_{HN}, C'_{HN}, D'_{HN}}.
Here, HN-SInt (A'_{HN}) ⋃ HN-SInt (E_{HN}) = C'_{HN}⋃ D_{HN} = C'_{HN}
HN-SInt (A'_{HN}⋃ E_{HN}) = D_{HN}
Hence, HN-SInt (A'_{HN}) ⋃ HN-SInt (E_{HN}) ≠ HN-SInt (A'_{HN}⋃ E_{HN}).

5. Heptagonal Neutrosophic Semi-Closure In Heptagonal Neutrosophic Topological Spaces

In this section, we introduce the heptagonal neutrosophic semi-closure operator and its properties in the heptagonal neutrosophic topological space.

Definition 5.1: Let (X,τ) be a HNTS. Then for a heptagonal neutrosophic subset A_{HN} of X, the heptagonal neutrosophic semi-closure of A_{HN} [HN-SCl (A_{HN}) for short] is the intersection of all heptagonal neutrosophic semi-closed sets of X contained in A_{HN}.
HN-SCl (A_{HN}) = ⋃{ K_{HN} : K_{HN} is a HNSC set in X and A_{HN}⊆ K_{HN}}.

Proposition 5.2: Let (X, τ) be a HNTS. Then for any heptagonal neutrosophic subsets A_{HN} and B_{HN} of a HNTS X we have

(i) A_{HN}⊆ HN-SCl (A_{HN})
(ii) A_{HN} is HNSC set in X ⇔ HN-SCl (A_{HN}) = A_{HN}
(iii) HN-SCl (HN-SCl (A_{HN})) = HN-SCl (A_{HN})
(iv) If A_{HN}⊆ B_{HN} then HN-SCl (A_{HN}) ⊆ HN-SCl (B_{HN})
(v) HN-SCl (A_{HN}⋂ B_{HN}) ⊆ HN-SCl (A_{HN}) ⋂ HN-SCl (B_{HN})
(vi) HN-SCl (A_{HN}) ⋃ HN-SCl (B_{HN}) = HN-SCl (A_{HN}⋃ B_{HN})

Proof:

(i) Follows from Definition 5.1.

(ii) Let A_{HN} be a HNSC set in X. Then A_{HN} contains HN-SCl(A_{HN}). Now by using (i), we get A_{HN} = HN-SCl(A_{HN}). Conversely assume that A_{HN} = HN-SCl(A_{HN}). By using Definition 5.1, A_{HN} is a HNSC set in X. Thus (ii) is proved.

(iii) By using (ii), HN-SCl(HN-SCl(A_{HN})) = HN-SCl(A_{HN}). This (iii) is proved.

(iv) Since $A_{HN} \subseteq B_{HN}$, by using (i), $B_{HN} \subseteq$ HN-SCl(B_{HN}) implies $A_{HN} \subseteq$ HN-SCl(B_{HN}) . But HN-SCl(A_{HN}) is the smallest closed set containing A_{HN}, hence HN-SCl(A_{HN}) $\subseteq$ HN-SCl(B_{HN}). Thus (iv) is proved.

(v) Since $A_{HN} \cap B_{HN} \subseteq A_{HN}$ and $A_{HN} \cap B_{HN} \subseteq B_{HN}$, by using (iv), HN-SCl ($A_{HN} \cap B_{HN}$) $\subseteq$ HN-SCl (A_{HN}) and HN-SCl($A_{HN} \cap B_{HN}$)$\subseteq$HN-SCl(B_{HN}). This implies that
HN-SCl($A_{HN} \cap B_{HN}$) $\subseteq$ HN-SCl(A_{HN})$\cap$HN-SCl(B_{HN}). Thus (v) is proved.

(vi) Since $A_{HN} \subseteq A_{HN} \cup B_{HN}$ and $B_{HN} \subseteq A_{HN} \cup B_{HN}$, by (iv), HN-SCl ($A_{HN}$) $\subseteq$ HN-SCl ($A_{HN} \cup B_{HN}$) and HN-SCl (B_{HN}) $\subseteq$ HN-SCl ($A_{HN} \cup B_{HN}$). This implies that,
HN-SCl (A_{HN}) $\cup$ HN-SCl (B_{HN}) $\subseteq$ HN-SCl ($A_{HN} \cup B_{HN}$) ------(1)
By(i), $A_{HN} \subseteq$ HN-SCl(A_{HN}) and $B_{HN} \subseteq$ HN-SCl(B_{HN}). This implies that
$A_{HN} \cup B_{HN} \subseteq$HN-SCl($A_{HN}$) $\cup$ HN-SCl(B_{HN}).
Now by (iv), HN-SCl($A_{HN} \cup B_{HN}$)$\subseteq$HN-SCl ((HN-SCl(A_{HN})$\cup$HN-SCl(B_{HN})).
By (1), HN-SCl($A_{HN} \cup B_{HN}$)$\subseteq$HN-SCl(HN-SCl (A_{HN}))$\cup$HN-SCl(HN-SCl(B_{HN})).
By (iii), HN-SCl($A_{HN} \cup B_{HN}$)$\subseteq$HN-SCl(A_{HN})$\cup$HN-SCl (B_{HN})-----(2).
From (1) and (2), HN-SCl ($A_{HN} \cup B_{HN}$) = HN-SCl(A_{HN})$\cup$HN-SCl(B_{HN}).
Thus (vi) is proved.

The following example shows that equality need not be held in Theorem 5.2 (vi).

Example 5.3: Let X = {x,y} and
A_{HN} = { <x; (0.45,0.45,0.45,0.45,0.45,0.45,0.45)>, <y; (0.75,0.75,0.75,0.75,0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95,0.95,0.95,0.95,0.95)>, <y; (0.55,0.55,0.55,0.55,0.55,0.55,0.55)>}
By ranking technique,
A_{HN} = { <x; (0.45,0.45,0.45)>, <y; (0.75,0.75,0.75)>}
B_{HN} = { <x; (0.95,0.95,0.95)>, <y; (0.55,0.55,0.55)>}
C_{HN} =$A_{HN} \cup B_{HN}$={<x; (0.95,0.45,0.45)>, <y; (0.75,0.55,0.55)>}
D_{HN} =$A_{HN} \cap B_{HN}$={<x; (0.45,0.95,0.95)>, <y; (0.55,0.75,0.75)>}
Then, τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}} is a HNTS.
Consider the HNS after the ranking technique,
E_{HN} = { <x; (0.75,0.52,0.48)>, <y; (0.82,0.59,0.39)>}
Then the HN-semi open sets of HN(X) are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, B'_{HN}, C'_{HN}, D'_{HN}}.
Here, HN-SCl (A'_{HN}) $\cup$ HN-SCl (E_{HN}) = $C'_{HN} \cup D_{HN}$ = C'_{HN}
HN-SCl ($A'_{HN} \cup E_{HN}$) = D_{HN}
Hence, HN-SCl (A'_{HN}) $\cup$ HN-SCl (E_{HN}) ≠ HN-SCl ($A'_{HN} \cup E_{HN}$)

Proposition 5.4: Let (X, τ) be a HNTS. Then for any heptagonal neutrosophic subsets $A_{HN}$of a HNTS X, we have

(i) (HN-SInt(A_{HN}))′ =HN-SCl(A'_{HN})

(ii) (HN-SCl(A_{HN}))′ =HN-SInt(A'_{HN})

Proof:

(i) By definition 4.1, HN-SInt (A_{HN}) = ⋃ { S_{HN} : S_{HN} is a HNSO set in X and $S_{HN} \subseteq A_{HN}$}

Taking the complement on both sides,

(HN-SInt(A_{HN}))′ = ⋂ { S'_{HN} : S'_{HN} is a HNSC set in X and $A'_{HN} \subseteq S'_{HN}$}

Now, replace S'_{HN} with K_{HN}, we get

(HN-SInt(A_{HN}))′ = ⋂ {K_{HN} : K_{HN} is a HNSC set in X and $A'_{HN} \subseteq K_{HN}$}

By definition 5.1, (HN-SInt(A_{HN}))′ =HN-SCl(A'_{HN}). Thus (i) is proved.

(ii) From (i) for the HNS A'_{HN}

We write, (HN-SInt(A'_{HN}))′ =HN-SCl(A_{HN})

Taking the complement on both sides we get

HN-SInt(A'_{HN}) = (HN-SCl(A_{HN}))′. Thus (ii) is proved.

6. Conclusion

The notion of heptagonal neutrosophic semi-open sets and their characterization were presented and examined in this paper. It can also be expanded upon in the areas of quotient, continuous, and contra-continuous mappings. It is possible to investigate the set's homeomorphism, connectedness, and compactness in further detail.

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Author Contributions

All authors contributed equally to this research.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

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