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Some New Interpretations of the Connectives in Classical Logic

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Abstract

In this paper, the connectives: conditional, conjunction and disjunction of the classical logic are given new interpretations without violating the laws of contradiction and excluded middle unlike fuzzy logic. It is shown that these connectives are equivalent to those of conditional probability, intersection and union of sets/events.

Keywords: Standard Notations of Symbolic Logic, Set Theory and Probability Theory, Union/Disjunction, Intersection/Conjunction, Complement/Negation, σ-algebra of Sets/Events, Coefficient of Uncertainty/Indeterminancy (μ/1-μ), Fuzzy Logic.

1 | Introduction

Consider the set Ω of adults of a certain locality, classified into (i) employed and (ii) unemployed. This classification is unsatisfactory if we consider the fact that there may be adults who are partially employed or employed with a meager income. It is thus necessary to introduce another class (iii) under-employed. As another example, let Ω be the set of adult males in a village, classified into two sets (i) A: those who shave themselves (ii) B: those who are shaved by the village barber, who is a member of Ω. Since the barber belongs to both A and B, the truth value of the proposition (iii) that the barber belongs to the set A/B may be taken as $\mu/1-\mu$. Due to insufficient information $\mu/1-\mu$ will be called a coefficient of uncertainty/indeterminacy. Similarly in the first example, we may assign the truth value $\mu/1-\mu\ (0 \leq \mu \leq 1)$ that the class (iii) is part of class (i)/(ii). A simple way of assigning values for $\mu/1-\mu$ is explained in Examples 1 and 2 of Section 9. To analyze conditional statements of the type "if A is B then C is D", the probabilistic concepts of Section 8 are useful.

2 | Connectives in Symbolic Logic and Lattices

In the propositional logic, the lattice operations $-/'$ for negation, ∧ for conjunction, ∨ for disjunction, and → for the conditional operation are applied to the propositions A and B according to Table 1.

It is further assumed that

(i) $(\mathrm{A} \to \mathrm{B}) \equiv [\mathrm{A} \to (\mathrm{A} \wedge \mathrm{B})]$. According to the column (4), (i) is equivalent to

(ii) $(\overline{\mathrm{A}} \vee \mathrm{B}) \equiv [\overline{\mathrm{A}} \vee (\mathrm{A} \wedge \mathrm{B})]$

By considering the truth values of (ii) we have

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(iii) $T(\overline{A} \vee B) \neq T[\overline{A} \vee (A \wedge B)]$ according to column (5) of the table. For if we suppose that $T(B) > T(A) > \frac{1}{2}$ then LHS = $\max\{T(\overline{A}), T(B)\} = T(B)$ and similarly RHS = T(A) so that LHS and RHS are different. This indicates that the max operator in column (5) is to be properly checked. Since (ii) and (iii) are in contradiction (ii) and hence (i) has to be checked; hence the meaning of $(A \rightarrow B)$ has to be revised. Again $T(A \vee \overline{A}) = \max\{T(A), 1 - T(A)\} \neq 1$, in general, and $T(A \wedge \overline{A}) = \min\{T(A), 1 - T(A)\} \neq 0$ in general i.e. the law of excluded middle and the law of contradiction are violated in classical logic and fuzzy logic. These violations occur solely because of the operations min/max for conjunction/disjunction according to column (5) of the above table. Thus all three operations: →, ∧ and ∨ have to be modified to keep the LOEM and LOC always true. This can be done by the new rule of conditional proposition, the new multiplication theorem/addition theorem for conjunction/disjunction.

Table 1. Some of lattice operations.

Sl. No.	Connective or Operator	Symbol	Notation	Truth Value
(1)	(2)	(3)	(4)	(5)
1	Negation	– / ′	$\overline{A}$ / A′	$T(\overline{A}) = 1 - T(A)$
2	Conjunction	∧	$A \wedge B$	$\min\{T(A), T(B)\}$
3	Disjunction	∨	$A \vee B$	$\max\{T(A), T(B)\}$
4	Conditional Operation	→	$A \rightarrow B \equiv \overline{A}$	$\max\{1 - T(A), T(B)\}$

3 | Rule for T(A→B)

Consider the modus ponens, modus tollens, and syllogism: A, (A→B) implies B, hence A, B or A and B

$$\therefore T(A)T(A \rightarrow B) = T(A, B) \text{ or } T(A \wedge B)$$

$$\therefore T(A \rightarrow B) = \frac{T(A, B)}{T(A)} \text{ or } \frac{T(A \wedge B)}{T(A)}$$

Similarly

$$T(A \wedge B \wedge C) = T(A)\, T(A \rightarrow B) T(\overline{A \wedge B} \rightarrow C)$$

since $T(\overline{A \wedge B} \rightarrow C) = \frac{T(A \wedge B \wedge C)}{T(A \wedge B)}$ where bar does not mean negation.

These are exactly analogous to the laws of conditional probability:

$$P(A/B) = \frac{P(AB)}{P(B)} \text{ and } P\,(B/A) = \frac{P(AB)}{P(A)}$$

It follows that we can redefine $P(A/B) = P(B \rightarrow A)$ and $P(B/A) = P(A \rightarrow B)$ analogous to the above formulae.

Hence $T(A \rightarrow B) = T(B/A)$ and $T(A/B) = T(B \rightarrow A)$.

4 | Rules for Conjunction and Disjunction

It can be proved that

(i) $T(A \vee A) = T(A \wedge A) = T(A)$

(ii) $T(A \wedge B) = \min\{T(A), T(B)\} - d$

(iii) $T(A \vee B) = \max\{T(A), T(B)\} + d$ where

(iv) $d = \min[\min\{T(A), T(\overline{A})\}, \min\{T(B), T(\overline{B})\}]$

(v) $T(A \vee B) = T(A) + T(B) - T(A \wedge B)$

Proof

$T(A \wedge B) \leq T(A)$ and $T(A \wedge B) \leq T(B)$ imply $T(A \wedge B) \leq \text{Min}\{T(A), T(B)\}$

Similarly

$$\max\{T(A), T(B)\} \leq T(A \vee B)$$

since

$$T(A) \leq T(A \wedge B) \text{ and } T(B) \leq T(A \wedge B)$$

These inequalities can be re-written as

$$T(A \wedge B) = \min\{T(A), T(B)\} - d_1 \text{ and } T(A \vee B) = \max\{T(A), T(B)\} + d_2.$$

Adding these

$$T(A \wedge B) + T(A \vee B) = \min\{T(A), T(B)\} + \max\{T(A), T(B)\} + d_2 - d_1$$

i.e. $T(A) + T(B) = T(A) + T(B) + d_2 - d_1 \Rightarrow d_2 - d_1 = 0 \therefore d_2 = d_1 = d$, say.

Thus we get (ii) $T(A \wedge B) = \min\{T(A), T(B)\} - d$

(iii) $T(A \vee B) = \max\{T(A), T(B)\} + d$

Case 1: (when B = A). The LHS of these equations coincide

$$\therefore T(A \wedge A) = T(A \vee A) \Rightarrow$$

$$\min\{T(A), T(A)\} - d = \max\{T(A), T(A)\} + d$$

i.e. $T(A) - d = T(A) + d$ or $d = 0$. Thus we get (i)

Case 2: (when B ≠ A)

Letting $B = \overline{A}$ implies

$$T(A \wedge \overline{A}) = \min\{T(A), T(\overline{A})\} - d \text{ and } T(A \vee \overline{A}) = \max\{T(A), T(\overline{A})\} + d$$

To make $T(A \wedge \overline{A}) \geq 0$ we must take $d \leq \min\{T(A), T(\overline{A})\}$

Similarly

$$d \leq \min\{TB), T(\overline{B}) \therefore d = \min[\min\{T(A), T(\overline{A})\}, \min\{T(B), T(\overline{B})\}]$$

Thus we get (iv). Adding (ii) and (iii) we get (v). It is readily shown that (i) to (iv) are in agreement with LOEM/LOC, since

$$T(A \wedge \overline{A}) = \min\{T(A), T(\overline{A})\} - \min\{T(A), T(\overline{A})\} = 0$$

implies $A \wedge \overline{A} = \phi$ so that $T(A \vee \overline{A}) = 1$ or $(A \vee \overline{A}) = \Omega$, the universal set.

5 | Truth Table involving Truth Values $\alpha/1-\alpha$ and $\alpha'/1-\alpha'$

The classical truth value table involving (0/1) can be upgraded by using the formulae of 4(i) to 4(iv). Thus we get Table 2.

Table 2

T(P)	T(Q)	T(P∧Q)	T(P∨Q)	T(P → Q)	Limiting value of T(P → Q)
(1)	(2)	(3)	(4)	(5)	(6)
α	$1-\alpha'$	$\min\{\alpha_1, 1-\alpha'\}-d$	$\max\{\alpha, 1-\alpha'\}+d$	$\min\left\{1, \frac{1-\alpha'}{\alpha}\right\}-\frac{d}{\alpha}$	1
α	α'	$\min\{\alpha_1, \alpha'\}-d$	$\max\{\alpha, \alpha'\}+d$	$\min\left\{1, \frac{\alpha'}{\alpha}\right\}-\frac{d}{\alpha}$	1
$1-\alpha$	$1-\alpha'$	$\min\{1-\alpha, 1-\alpha'\} -d$	$\max\{1-\alpha, 1-\alpha'\} +d$	$\min\left\{1, \frac{1-\alpha'}{1-\alpha}\right\}-\frac{d}{1-\alpha}$	1
$1-\alpha$	α'	$\min\{1-\alpha, \alpha'\}-d$	$\max\{1-\alpha, \alpha'\}+d$	$\min\left\{\frac{\alpha'}{1-\alpha} 1\right\}-\frac{d}{1-\alpha}$	0

where $d = \min[\min\{\alpha, 1-\alpha\}, \min\{\alpha', 1-\alpha'\}] \to 0$ as $\alpha \to 0, \alpha' \to 0$ and $\frac{\alpha'}{\alpha} \to 1$

The truth values in column (6) are identical to the classical truth value of (P→Q).

When $\alpha = \alpha' = \frac{1}{2}$ then all four rows are identical, columns (3) and (5) contain zeros only, and column (4) contains 1 only.

6 | Interpretations of the Lattice Operators

The lattice operators are

(i) $\mathrm{T}(\mathrm{A} \wedge \mathrm{B}) = \min\{(\mathrm{T}(\mathrm{A}), \mathrm{T}(\mathrm{B})\}$ and

(ii) $\mathrm{T}(\mathrm{A} \vee \mathrm{B}) = \max\{\mathrm{T}(\mathrm{A}), \mathrm{T}(\mathrm{B})\}$

Those formulae are equivalent to 4(ii) and 4(iii) with $d = 0$. We know that $d = 0$ iff $\mathrm{A}/\overline{\mathrm{A}}$ coincides with ϕ or $\mathrm{B}/\overline{\mathrm{B}}$ coincides with ϕ. But this will be true only in the trivial case.

From 6(i) and 6(ii), it follows that

$$\mathrm{T}(\mathrm{A} \wedge \overline{\mathrm{A}}) = \min\{\mathrm{T}(\mathrm{A}), \mathrm{T}(\overline{\mathrm{A}})\} \neq 0$$

$$\mathrm{T}(\mathrm{A} \vee \overline{\mathrm{A}}) = \max\{\mathrm{T}(\mathrm{A}), \mathrm{T}(\overline{\mathrm{A}})\} \neq 1$$

i.e. $\mathrm{A} \wedge \overline{\mathrm{A}} \neq \phi$ and $\mathrm{A} \vee \overline{\mathrm{A}} \neq \Omega$

i.e. 6(i) and 6(ii) lead to a rejection of LOEM/LOC

Again if we use the min/max rule for the product/sum of numbers in problems of the scalar product of two vectors or the product of two matrices, the answers will be different from their exact values. Hence in computations, it is desirable to use the theory of Sections 3, 4, and 5 rather than that of 6.

7 | Space of Propositional Logic and Probability Space

In this section, it is shown that the space of propositional logic is a probability space. Then it will follow that ∧≡∩ and ∨≡∪.

Consider the universal set $\Omega = \{\overline{0}, \overline{1}\}$ where $\overline{0}$ stands for 'No' to a question and $\overline{1}$ stands for 'Yes' to the same question with truth values $\mathrm{T}(\overline{0}) = \alpha$ and $\mathrm{T}(\overline{1}) = 1-\alpha$. Now it is possible to define a random variable X taking values 0 and 1 with $\mathrm{P}(\mathrm{X}=0) = \mathrm{T}(\overline{0}) = \alpha$ and $\mathrm{P}(\mathrm{X}=1) = \mathrm{T}(\overline{1}) = 1-\alpha$.

Clearly $\overline{0}$ and $\overline{1}$ are complementary propositions and {0} and {1} complementary events; either is a disguised form of the other and we have $\mathrm{P}(..) = \mathrm{T}\,(..)$. Let $\Sigma = \{\phi, \Omega, \mathrm{E}_0 \mathrm{E}_1\}$ where $\mathrm{E}_0 = \{\overline{0}\}$ or $\{\mathrm{X}=0\}$ and $\overline{\mathrm{E}}_1 =$

$\overline{\{1\}}$ or $\{X = 1\}$. Clearly $\bar{\phi} = \Omega$ and $\bar{E}_0 = E_1$ and vice-versa. It follows that Σ is sigma-algebra [1,4] of subsets of Ω satisfying the axioms (i) Ω ∈ Σ (ii) E ∈ Σ implies $\bar{E} \in \Sigma$ (iii) $\{E_n\} \subset \Sigma$ implies $(\vee E_n) \in \Sigma$

Further the set function T (..) defined by $T: \Sigma \to [0,1]$ is a probability function P (..) satisfying

(iv) (a) $T(E) \geq 0$ for any E ∈ Σ (b) $T(\Omega) = 1$(c) $T(\bar{E}) = 1 - T(E)$ (d) $T\left(\underset{n}{\vee} E_n\right) = \sum_n T(E_n)$

whenever $E_i \wedge E_j = \phi$ for $i \neq j$ and Σ on the RHS of (d) is used as a summation symbol.

Thus the propositional space (Ω, Σ, T( )) is the probability space (Ω, Σ, P( )) so that ∧ ≡ ∩ and ∨ = ∪ and T(E) ≡ P(E). Hence we can interchangeably use T and P in the remaining sections.

Now consider a probability space consisting of a random variable X on a sample space Ω with a cumulative probability function $F(x)$ [3,6,7]. The statement $x = x_\alpha$ is a simple proposition as well as a simple event and $a \leq x \leq b$ represents a compound proposition as well as a compound event.

We may define

$$T(a \leq x \leq b) = P(a \leq x \leq b) = F(b) - F(a).$$

If $A_\alpha = (-\infty, x_\alpha)$ then $T(A_\alpha) = P(A_\alpha) = F(x_\alpha)$

Thus it is possible to compute the truth value of propositions from the C.D.F., $F(x)$. If X is a discrete variable, then $T(X = x) = P(X = x) = F(x) - F(x-1)$.

8 | Conditional Statements Involving Linguistic Clusters

In a frequency distribution of a data set on the variables X, the usual class-width is uniform. But considering the heterogeneous nature of the data set, it is desirable to use clusters defined by linguistic terms such as Low, Medium, High, etc. Since the class limits of clusters have a vague nature, the principle of uncertainty/indeterminacy can be applied by making the class limits depend on $\mu_1/1-\mu_1$, etc. Thus a cluster is defined as an interval with vague endpoints depending on the coefficients of uncertainty or a set of points whose infimum/supremum depends on coefficients of uncertainty. The clusters stated above can be denoted by $L(x_1), M(x_2), H(x_3)$ etc where x_1, x_2, x_3 denote representative values in the clusters. The union of domains of these clusters will be the domain of X. If $f_1, f_2, f_3, \ldots, f_m$ denote frequencies of these clusters (which may depend on coefficients of uncertainty), then we may write $X = \left\{\frac{f_1}{x_1}, \frac{f_2}{x_2}, \ldots, \frac{f_m}{x_m}\right\}$ as a short notation of the distribution [2, 8].

If $N = \Sigma f_i$ and $f(x_i) = \frac{f_i}{N}$ or $M = \max\{f_1, f_2, \ldots, f_m\}$ and $\mu(x_i) = \frac{f_i}{M}$ then we may rewrite

$$X = N\left\{\frac{f(x_1)}{x_1}, \frac{f(x_2)}{x_2}, \ldots, \frac{f(x_m)}{x_m}\right\}$$

or

$$X = M\left\{\frac{\mu(x_1)}{x_1}, \frac{\mu(x_2)}{x_2}, \ldots, \frac{\mu(x_m)}{x_m}\right\}$$

Similarly, the bivariate frequency distribution of two random variables X, Y may be expressed in the form

$$(X, Y) = \left\{\frac{f_{ij}}{(x_i, y_j)} / i = 1 \text{ to } m;\ j = 1 \text{ to } n\right\}$$

By defining $N_0 = \Sigma\Sigma f_{ij}$ and $f_{ij} = N_0 f(x_i, y_j)$, we may redefine

$$(X, Y) = \left\{\frac{f(x_i, y_j)}{(x_i, y_j)} / i = 1 \text{ to } m, j = 1 \text{ to } m\right\}$$

as the bivariate truth value/probability distribution.

The marginal distributions are $f_x(x) = \sum_y f\,(x, y)$ and $f_y(y) = \sum_x f\,(x, y)$

The conditional distributions are

$$f_1(y/x) \text{ or } \mathrm{R}(x, y) = \mathrm{T}(\mathrm{X} \to \mathrm{Y}) = \frac{f(x, y)}{f_x(x)}$$

and

$$f_2(x/y) \text{ or } \mathrm{S}(y, x) = \mathrm{T}(\mathrm{Y} \to \mathrm{X}) = \frac{f(x, y)}{f_y(y)}$$

Let $\mathrm{P}(x) = \mathrm{N}_0(p_1, p_2, \dots, \mathrm{p}_m$ formed by the truth values $f_x(x_1) = p_1, f_x(x_2) = p_2$ of clusters of X and $\mathrm{Q}(y) = \mathrm{N}(p'_1, p'_2, \dots, p'_n)$ formed by the truth values of clusters of Y

$$f_y(y_1) = p'_1, f_y(y_2) = p'_2, \dots, f_y(y_n) = p'_n.$$

Now we have

(i) $\sum_x \mathrm{P}(x) \circ \mathrm{T}(\mathrm{X} \to \mathrm{Y}) = \sum_x \mathrm{P}(x) \circ \mathrm{R}\,(x, y) = \mathrm{N}_0 \sum_x f_x\,(x) \circ \mathrm{R}\,(x, y)$

$$= \mathrm{N}_0 \sum_x f_x\,(x) \frac{f(x,\ y)}{f_x(x)} = \mathrm{N}_0 \sum_x f(x, y)$$

$$= \{\mathrm{N}_0 f_y(y)\} = \mathrm{N}_0(p'_1, p'_2, \cdots, p'_n)$$

i.e. P∘R = Q (*y*)

(ii) $\sum_y \mathrm{Q}\,(y) \circ \mathrm{P}(\mathrm{Y} \to \mathrm{X}) = \sum_y \mathrm{Q}(y) \circ \mathrm{S}(y, x) = \sum_y \mathrm{N}_0\, f_y(y) \circ \mathrm{S}(y, x)$

$$= \mathrm{N}_0 \sum_y f_y(y) \circ \frac{f(x, y)}{f_y(y)} = \mathrm{N}_0 \sum_y f\,(x, y)$$

$$= \{\mathrm{N}_0 f_x(x)\} = \mathrm{N}_0(p_1, P_2, \dots, p_m)$$

i.e. $\mathrm{Q} \circ \mathrm{S} = \mathrm{P}(x)$

Clearly $\mathrm{P}(x)$ is $1 \times m$ matrix; $\mathrm{Q}(y)$ is $1 \times n$ matrix, $\mathrm{R}(x, y)$ is $m \times n$ matrix, and $\mathrm{S}(y, x)$ is $n \times m$ matrix. Also $\mathrm{Q} = \mathrm{P} \circ \mathrm{R} = (\mathrm{Q} \circ \mathrm{S}) \circ \mathrm{R} = \mathrm{Q} \circ (\mathrm{S} \circ \mathrm{R})$ does not imply $\mathrm{S} \circ \mathrm{R}$ is the identity matrix I_n. Similarly, $\mathrm{R} \circ \mathrm{S}$ is not the identity matrix I_m, since cancellation law is not permissible in a matrix equation. Hence $\mathrm{S} \neq \mathrm{R}^\dagger$ and $\mathrm{R} \neq \mathrm{S}^\dagger$ where † denotes generalized inverse. The conditional propositions 'if P then Q' and 'if Q then P' are equivalent to the equations (ii) $\mathrm{Q} = \mathrm{P} \circ \mathrm{R}$ and (ii) $\mathrm{P} = \mathrm{Q} \circ \mathrm{S}$ respectively. For single clusters of X, we have $\mathrm{P}_1(x) = \mathrm{N}_0(p_1, 0, \dots, 0)$, $\mathrm{P}_2(x) = \mathrm{N}_0(0, p_2, \dots, 0)$, $\mathrm{P}_3(x) = \mathrm{N}_0(0, 0, p_3, \dots, 0)$, etc there being m components for each $\mathrm{P}_i(x)$. Similarly for single clusters of Y, we have $\mathrm{Q}_1(y) = \mathrm{N}_0(p'_1, 0, \dots, 0)$, etc there being n components for $\mathrm{Q}_j(y)$ etc of the cluster of Y. Hence the conditional statement

'if '*x*' is $\mathrm{P}_i(x)$ then '*y*' is $\mathrm{Q}_j(y)$' is equivalent to $\mathrm{Q}_j(y) = \mathrm{P}_i(x) \circ \mathrm{R}\,(x, y)$ and can be represented as in the given Figure 1.

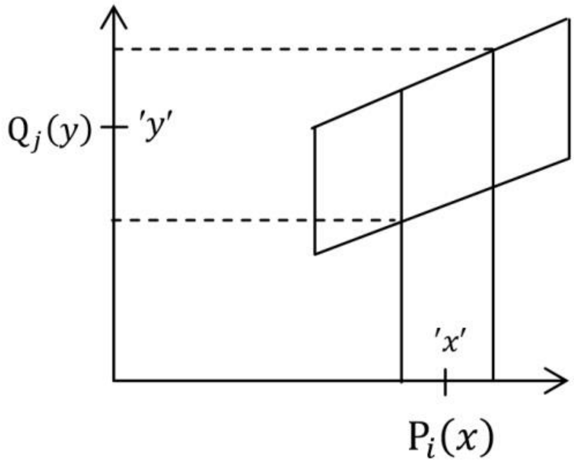


Figure 1. Representation of $['x' \text{ is } P_i(x)] \to ['y' \text{ is } Q_j(y)]$.

It may be noted that the endpoints of cluster intervals and cluster frequencies contain the coefficients $\mu_1/\overline{\mu}_1, \mu_2/\overline{\mu}_2$ etc. Conditional statements like (i) 'if speed is medium, then the mileage is moderate' (ii) 'if fruit is ripe, then the taste is good' etc. can be treated by the method of this section.

For the computations of $P(x) \circ R(x, y)$, using the approximate methods of max/min operations, different users select different operations and different expressions for $R(x, y)$. There are at least twelve such approximations. Some of them are known by the names: max-min, min-min, min-max, and max-product, etc are found in books on Fuzzy Logic and Applications [5]. However according to the theory of this section, the computation is unique.

9 | Worked Examples

9.1 | Example 1

Suppose a frequency distribution has clusters defined by linguistic terms: Low, Medium, and High, denoted by L, M, and H respectively such that (i) L contains those values in the interval [0, 20] with truth value $\mu_1 = 1$ and those values in [20, 35] with truth value $\mu_1 = \mu_{11} \in [0, 1]$, (ii) M contains those values in [20, 35] with truth value $\mu_2 = \mu_{21} \in [0, 1]$ and those values in [35, 45] with truth value μ_2 =1 and those values in [45, 60] with truth value $\mu_2 = \mu_{22} \in [0,1]$, (iii) H contains those values in [45, 60] with truth value $\mu_3 = \mu_{31} \in [0,1]$ and those values in [60, 100] with truth value $\mu_3 = 1$.

Suppose *x* is measured from the '0' point to the right at two x-axis at a distance of 1 unit from each other.

By definition of slope of oblique lines, we have Figure 2.

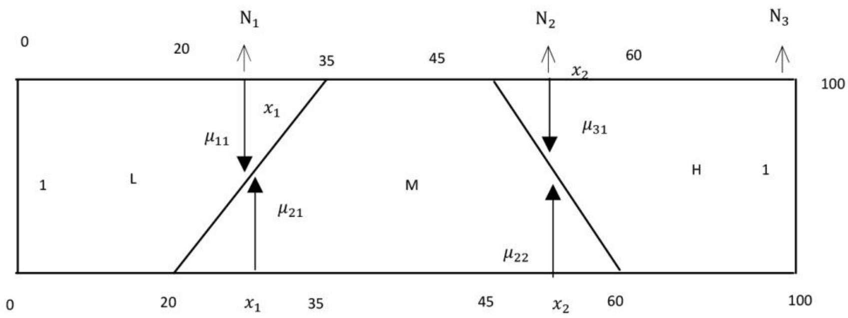


Figure 2

$$\mu_1 = \begin{cases} 1, x\epsilon\ [0,20] \\ \mu_{11} = \frac{35-x}{15}, x \in [20,35] \end{cases} \text{or } \mathrm{L}(x) = [0, 35 - 15\ \mu_{11}] = [0, 20 + 15\ \overline{\mu}_{11}]$$

$$\mu_2 = \begin{cases} \mu_{21} = \dfrac{x-20}{15}, x \in [20,35] \\ 1, x \in [35,45] \qquad \text{or } \mathrm{M}(x) \\ \mu_{22} = \dfrac{60-x}{15}, x \in [45,60] \end{cases}$$

$$= \left[20, 20 + 15\ \overline{\mu}_{11}\right] \cup [35,45] \cup \left[45, 45 + 15\ \overline{\mu}_{22}\right]$$

$$\mu_3 = \begin{cases} \mu_{31} = \dfrac{x-45}{15}, x \in [45{,}60] \\ 1, x \in [60,100] \end{cases} \text{or } \mathrm{H}(x) = [45 + 15\ \mu_{31}, 100$$

If $\mu_{21} + \mu_{11} = 1$ and $\mu_{22} + \mu_{31} = 1$ then the Venn diagrams of L, M, and H will not overlap.

Suppose $\mathrm{F}_{20}, \mathrm{F}_{35}, \mathrm{F}_{45}, \mathrm{F}_{60}$ and F_{100} denote cumulative frequencies corresponding to the values $x = 20, 35, 45, 60,$ and 100 and $\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3$ the cumulative frequencies are to be estimated at x_1, x_2 and x_3. From the figure 2, by considering the areas of the 3 trapezium, we have

$$N_1 = F_{20} + \frac{1}{2}(F_{35} - F_{20}) = \frac{1}{2}(F_{20} - F_{35})$$

$$N_1 + N_2 = F_{45} + \frac{1}{2}(F_{60} - F_{45}) = \frac{1}{2}(F_{45} - F_{60})$$

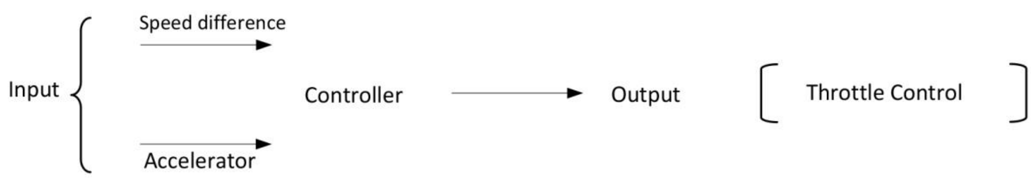
$$N_1 + N_2 + N_3 = F_{100}$$

Clearly, the cluster frequencies of L, M, and H can be determined from the last three conditions. It is clear that the cluster frequencies of L, M, H and the cluster intervals depend on the coefficients of uncertainty: μ_1, μ_2, μ_3.

Clearly $\mathrm{N}_1, \mathrm{N}_2 - \mathrm{N}_1$ and $\mathrm{N}_3 = \mathrm{N}$ are the cluster frequencies of L, M, H, and (i) the cluster intervals and (ii) cluster frequencies depend on the coefficients of uncertainty: μ_1, μ_2, μ_3.

9.2 | Example 2

Consider the problem of a certain control system consisting of two cluster inputs: (i) speed difference (ii) acceleration and one cluster output: throttle control.



All three types of clusters are [5] classified into: NL (negative large), NM (negative medium), NS (negative small), ZE (zero), PS (positive small), PM (positive medium), and PL (positive large) as described in Figure 3.

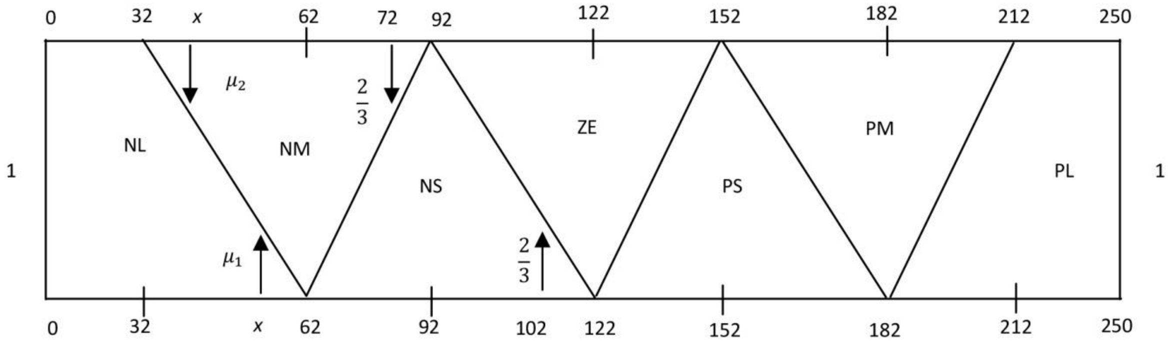


Figure 3

Suppose the following inference rules are given [5]

Rule 1. (If speed diff: is NL) and (acceleration is ZE) then (throttle control is PL)

Rule 2. (If speed diff: is ZE) and (acceleration is NL) then (throttle control is PL)

Rule 3. (If speed diff: is NM) and (acceleration is ZE) then (throttle control is PM)

Rule 4. (If speed diff: is ZE) and (acceleration is NM) then (throttle control is PM)

Rule 5. (If speed diff: is NS) and (acceleration is PS) then (throttle control is PS)

Rule 6. (If speed diff: is ZE) and (acceleration is NS) then (throttle control is PS)

Rule 7. (If speed diff: is PS) and (acceleration is NS) then (throttle control is NS)

Rule 8. (If speed diff: is PL) and (acceleration is ZE) then (throttle control is NL)

It is required to compute the output (throttle control) value when the speed difference = 102 and acceleration = 72 are the inputs.

Let x be the distances measured horizontally from the vertical line at the '0' point and $\mu_1, \mu_2, \ldots, \mu_7$ the coefficients of indeterminacy corresponding to the linguistic clusters NL, NM, NS, ZE, PS, PM, and PL. By using the formula for the slope of oblique lines

$$\mu_1 = \begin{cases} 1, x \in [0, 32] \\ \frac{62 - x}{30}, x \in [32,62] \end{cases} \mu_2 = \begin{cases} \frac{x - 32}{30}, x \in [32,62] \\ \frac{92 - x}{30}, x \in [62, 92] \end{cases} \mu_3 = \begin{cases} \frac{x - 62}{30}, x \in [62,92] \\ \frac{122 - x}{30}, x \in [92, 122] \end{cases}$$

$$\mu_4 = \begin{cases} \frac{x - 92}{30}, x \in [92, 122] \\ \frac{152 - x}{30}, x \in [122, 152] \end{cases} \mu_5 = \begin{cases} \frac{x - 122}{30}, x \in [122, 152] \\ \frac{182 - x}{30}, x \in [152, 182] \end{cases} \mu_6 = \begin{cases} \frac{x - 152}{30}, x \in [152,182] \\ \frac{212 - x}{30}, x \in [182, 212] \end{cases} \mu_7 = \begin{cases} \frac{x - 182}{30}, x \in [182, 212] \\ 1, x \in [212, 250] \end{cases}$$

For the speed difference = 102 and acceleration = 72, the details of the computation of estimated output (= 172) are given in Table 3.

Table 3

Speed Difference (102)	$\mu_{NS} = \mu_3(102) = \frac{2}{3}$	$\mu_{ZE} = \mu_4(102) = \frac{1}{3}$	Remaining $\mu's = 0$
Acceleration (72)	$\mu_{NM} = \mu_2(72) = \frac{2}{3}$	$\mu_{NS} = \mu_3(72) = \frac{1}{3}$	Remaining $\mu's = 0$

Next the truth values of combined inputs according to the given rules of inference are computed, by using the product rule of truth values (Table 4):

Alternatively, we can dispose of the inference rules 1 to 8, by using a frequency distribution of observed outputs, corresponding to a bivariate cluster containing (102, 72). The required consecutive pair of output clusters can be located by computing the average of the observed outputs. The simple average of the central coordinates of this pair will give the point estimate of the output.

Table 4

Rule	Truth Value of Combined Inputs	Computed Value	Output Cluster	Central Coordinate	Estimated Weight	Weighted Average (output)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	$\mu_{NL}\,\mu_{ZE}$	0	-			
2	$\mu_{ZE}\mu_{NL}$	0	-			
3	$\mu_{NM}\,\mu_{ZE}$	0	-			
4	$\mu_{ZE}\,\mu_{NM}$	$\frac{2}{9}$	PM	182		$182 \times \frac{2}{3} + 152 \times \frac{1}{3} = 172$
5	$\mu_{NS}\,\mu_{PS}$	0	-			
6	$\mu_{ZE}\,\mu_{NS}$	$\frac{1}{9}$	PS	152		
7	$\mu_{PS}\,\mu_{NS}$	0	-			
8	$\mu_{PL}\,\mu_{ZE}$	0	-			

9.3 | Example 3

Let $f(x,y) = \frac{1}{72}(2x + 3y)$; $x = 0, 1, 2$ and $y = 1, 2, 3$. The joint probability distribution can be exhibited as a (Table 5):

Table 5. The joint probability distribution.

Y/ X	1	2	3	$f_x(x)$
0	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{5}{72}$	$\frac{1}{9}$	$\frac{11}{72}$	$\frac{1}{3}$
2	$\frac{7}{72}$	$\frac{5}{36}$	$\frac{13}{72}$	$\frac{5}{12}$
$f_y(y)$	$\frac{5}{24}$	$\frac{1}{3}$	$\frac{11}{24}$	1

$$P(x) = N_0(p_1, p_2, p_3) = N_0\left(\frac{1}{4}, \frac{1}{3}, \frac{5}{12}\right)$$

$$Q(y) = N_0\left(\frac{5}{24}, \frac{1}{3}, \frac{11}{24}\right)$$

where N_0 is the total frequency of the bivariate frequency distribution.

$T(X \to Y) = R(x, y) = \frac{f(x,y)}{f_X(x)} = f_1(Y/X = x)$ is exhibited in Table 6.

Table 6

Y $f_1(y/x)$	1	2	3	Total
$f_1(y/x = 0)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	1
$f_1(y/x = 1)$	$\frac{5}{24}$	$\frac{1}{3}$	$\frac{11}{24}$	1
$f_1(y/x = 2)$	$\frac{7}{30}$	$\frac{1}{3}$	$\frac{13}{30}$	1

$T(Y \to X) = S(y, x) = \frac{f(x,y)}{f_Y(y)} = f_2(X/Y = y)$ is similarly found to be (Table 7):

Table 7

X $f_2(x/y)$	0	1	2	Total
$f_2(x/y = 1)$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{7}{15}$	1
$f_2(x/y = 2)$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	1
$f_2(x/y = 3)$	$\frac{3}{11}$	$\frac{1}{3}$	$\frac{13}{33}$	1

$$T(Y)_0 T(Y \to X) = f_y(Y)_0\, S(y, x) = \begin{bmatrix} \frac{5}{24} & \frac{1}{3} & \frac{1}{24} \end{bmatrix}_0 \begin{bmatrix} \frac{1}{5} & \frac{1}{3} & \frac{7}{15} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{3}{11} & \frac{1}{3} & \frac{13}{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \end{bmatrix} = [f_x(x)] = T(X)$$

Similarly, it is shown that

$$T(X)_0 T(X \to Y) = T(Y)$$

These are equivalent to the statements

$$Q(y)_0\, S(y, x) = P(x) \text{ and } P(x)_0 R(x, y) = Q(y)$$

Since $P(x) = N_0(p_1, p_2, p_3)$ and $Q(y) = N_0(p_1', p_2', p_3')$

Now let us approximate $P(x)$ by $P_x' = N_0\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$, say

then $Q(y)$ will be approximated by $Q'(y) = P_0' R$

i.e. $Q'(y) = N_0\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right) \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{5}{24} & \frac{1}{3} & \frac{11}{24} \\ \frac{7}{30} & \frac{1}{3} & \frac{13}{30} \end{bmatrix} = N_0\left(\frac{77}{360}, \frac{1}{3}, \frac{163}{360}\right)$

i.e. If $N_0\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$ is P_x', then $N_0\left(\frac{77}{360}, \frac{1}{3}, \frac{163}{360}\right)$ is $Q'(y)$ which is of the form:

If x is P' then y is Q'.

10 | Conclusion

The max/min operations for disjunction/conjunction of lattice theory or Fuzzy theories lead to the violation of LOC and LOEM; this violation does not arise according to the new interpretations of Sections 3 to 5.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] De Barra, G. (2013) Measure Theory and Integration, New Age International (P) Ltd, New Delhi, India.
[2] Gupta, C.B., Singh, S.R., & Sandeep Kumar (2010) Advanced Discrete Structures, IK International Publishing House Pvt. Ltd., New Delhi, India.
[3] Henry Stark & John F. Woods (2002) Probability and Random Processes with Applications, Pearson Education Asia, Indian Reprint.
[4] Klimov, G. (1987) Probability Theory and Mathematical Statistics, MIR Publishers, Moscow.
[5] Rajasekaran, S., & Vijayalakshmi Pai, G.A., (2003) Neural Networks, Fuzzy Logic and Genetic Algorithms, PHI Pvt. Ltd., New Delhi, India.
[6] Rizwan, U., (2002) Mathematical Foundations, Sci-Tech Publications (India) Pvt. Ltd., Chennai, India.
[7] Sundara Pandian, (2013) Probability, Statistics and Queuing Theory, PHI Learning Pvt. Ltd., Delhi, India.
[8] Swapan Kumar Sarkar, (2014) Discreet Mathematics, S. Chand & Co., Ltd., Ramnagar, New Delhi, India.