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Uncertain Labeling Graphs and Uncertain Graph Classes (with Survey for Various Uncertain Sets)

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Abstract

Graph theory, a branch of mathematics, studies the relationships between entities using vertices and edges. Uncertain Graph Theory has emerged within this field to model the uncertainties present in real-world networks. Graph labeling involves assigning labels, typically integers, to the vertices or edges of a graph according to specific rules or constraints.

This paper introduces the concept of the Turiyam Neutrosophic Labeling Graph, which extends the traditional graph framework by incorporating four membership values—truth, indeterminacy, falsity, and a liberal state—at each vertex and edge. This approach enables a more nuanced representation of complex relationships. Additionally, we discuss the Single-Valued Pentapartitioned Neutrosophic Labeling Graph.The paper also examines the relationships between these novel graph concepts and other established types of graphs. In the Future Directions section, we propose several new classes of Uncertain Graphs and Labeling Graphs.

And the appendix of this paper details the findings from an investigation into set concepts within Uncertain Theory. These set concepts have inspired numerous proposals and studies by various researchers, driven by their applications, mathematical properties, and research interests.

Keywords: Neutrosophic graph, Fuzzy graph, Plithogenic graph, Labeling Graph, Fuzzy Set

1 | Introduction

1.1 | Graph Theory

Graph theory is a fundamental branch of mathematics that uses vertices (nodes) and edges (connections) to represent relationships within networks [142, 796, 233, 148]. A graph is a useful concept that can represent relationships or state transitions between concepts (sets) using edges (relations). Given its importance, graph theory has been extensively researched in various fields [315, 275], including real-world applications [130, 404],

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the development of graph neural networks [628, 740], circuits design [735, 116], chemical graph theory[700, 119], Bayesian network theory [527], bioinformatics [14, 696], protein structures[312, 699], Project Management[562, 691], and graph databases [99, 483]. Additionally, extensive research is conducted on graph algorithms [131, 690], the complexity of graph and related problems [533, 106], graph classes[148], and graph parameters[627, 329].

1.2 | Graph Coloring and Graph Labeling

Graph Coloring is one of the most widely studied topics in this field, involving the assignment of colors to the vertices or edges of a graph such that no two adjacent elements share the same color [371, 418]. Various extensions and specialized forms of Graph Coloring have been developed, including Circular Coloring [508], Defective Coloring [191, 192], Fractional Coloring [450], Hamiltonian Coloring [177], Complete Coloring [781, 768], Total Coloring [403, 564], and Radio Coloring [270]. For further information, readers can refer to introductory and survey literature on graph coloring [513, 372].

The Graph Labeling problem discussed in this paper is a variation of Graph Coloring. It involves assigning labels, typically integers, to the vertices or edges of a graph according to specific rules or constraints [286, 287]. This problem has been further extended to Uncertain Graphs, which have also been a significant area of research [290]. Several related concepts include Graceful Labeling [488], Edge-Graceful Labeling [218, 582], Harmonious Labeling [268, 416], and Lucky Labeling [198, 484].

Both Graph Labeling and Graph Coloring have been extensively studied with respect to computational complexity, algorithms, and their underlying mathematical properties [708, 558, 409]. For further reading, introductory and survey literature on Graph Labeling is available [285, 149, 284].

1.3 | Uncertain Graph Theory

To handle uncertainty in the world, various uncertain concepts such as Fuzzy Set [775], Neutrosophic Set [655], Fuzzy Logic [112, 688], Neutrosophic Logic [658, 670], Fuzzy number[632, 173], Neutrosophic number[3], and Fuzzy Matroid [311] have been proposed and are continuously being studied, including their applications.

This paper delves into various models of uncertain graphs, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam, and Plithogenic Graphs. These models enhance classical graph theory by introducing different forms of uncertainty, providing a more flexible framework for analyzing complex and ambiguous relationships. Such models have practical applications across diverse fields and have led to the development of several related graph classes [279, 274].

Derived concepts in graph theory include Bipolar Graphs [35, 611], Picture Graphs [636, 103], Interval-valued Graphs [639, 155], and Single-valued Uncertain Graphs [156, 39]. In addition, other types of graphs such as Vague Graphs [30], N-Graphs [20], Rough Graphs [228, 9], Four-Valued Fuzzy Graphs [282], Hesitant Fuzzy Graphs [282], Intuitionistic Hesitant Fuzzy Graphs [282], Quadripartitioned Neutrosophic Graphs [574, 355], Pentapartitioned Neutrosophic Graphs [462], and Spherical Fuzzy Graphs are also recognized.

Fundamental concepts like Fuzzy Sets and Neutrosophic Sets are well-documented in the literature, providing the foundation for these uncertain graph models [655, 543, 240, 214]. For further details on each Uncertain Set, please refer to the Appendix of this paper as needed.

Given the vast body of literature and the numerous applications, the study of uncertain graphs is of considerable importance. For a more detailed overview, readers are encouraged to refer to existing surveys [282, 279].

1.4 | Our Contribution in This Paper

Here, we outline Our Contribution in This Paper. As with traditional graph theory, uncertain graph theory has also given rise to a wide variety of graph classes [40, 34].

Within this framework, our paper introduces the Turiyam Neutrosophic Labeling Graph, an extension of traditional graph theory that assigns four membership values—truth, indeterminacy, falsity, and liberal state—to each vertex and edge. This approach enables a more nuanced representation of complex relationships [279, 282]. Additionally, related concepts like Turiyam Neutrosophic Sets and Turiyam Neutrosophic Rings have been explored in prior studies [643, 644]. The Turiyam Neutrosophic Labeling Graph is, essentially, an extension of the classic Labeling Graph into the realm of Turiyam Neutrosophic Graphs. And the Pentapartitioned Neutrosophic Graph assigns five values—truth, contradiction, ignorance, unknown, and falsity—to each vertex and edge, effectively addressing complex uncertainties [208, 572, 353]. Consequently, the development of a Single-Valued Pentapartitioned Neutrosophic Labeling Graph is considered.

Furthermore, this paper examines the relationships between these graph concepts and other graph types. The Future Directions section proposes several new classes of Uncertain Graphs and Labeling Graphs, aiming to inspire further research into these advanced graph concepts.

The appendix presents the findings from an investigation into set concepts within Uncertain Theory. These set concepts have been the subject of numerous proposals and studies by various researchers, driven by their research themes, applications, and mathematical properties. It is hoped that this work will contribute to the acceleration of research in Uncertain Theory.

2 | Preliminaries and Definitions

This section provides an overview of the fundamental definitions and notations used throughout the paper. Some basic concepts from set theory are applied in parts of this paper. Please refer to the relevant papers or surveys as needed [338, 419, 369].

2.1 | Basic Graph Concepts

Below are some of the foundational concepts in graph theory. For more comprehensive information on graph theory and its notations, refer to [233, 445, 715].

Definition 1 (Graph)**.** [233] A *graph* G is a mathematical structure that represents relationships between objects. It consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge connects a pair of vertices. Formally, a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges.

Definition 2 (Degree)**.** [233] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is defined as the number of edges connected to v. For undirected graphs, the degree is given by:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

For directed graphs, the *in-degree* $\deg^-(v)$ refers to the number of edges directed towards v, while the *out-degree* $\deg^+(v)$ represents the number of edges directed away from v.

2.2 | Uncertain Graph

This paper addresses Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam Neutrosophic, Vague, Single-Valued Pentapartitioned Neutrosophic, and Plithogenic concepts. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing ”Contradiction” with ”Liberal” [654].

Definition 3 (Unified Uncertain Graphs Framework)**.** (cf.[279]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

(1) *Fuzzy Graph* [528, 604, 734]*:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$.

(2) *Intuitionistic Fuzzy Graph (IFG)* [497, 207, 132]*:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ and $\nu_B(u, v) \in [0, 1]$, with $\mu_B(u, v) + \nu_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [662, 663, 155]*:*

- Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0, 1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

(4) *Turiyam Neutrosophic Graph* [289, 288]*:*

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where each component is in $[0, 1]$ and $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple.

(5) *Vague Graph* [30, 29, 591]*:*

- Each vertex $v \in V$ is assigned a pair $(\tau(v), \phi(v))$, where $\tau(v) \in [0, 1]$ is the degree of truth-membership and $\phi(v) \in [0, 1]$ is the degree of false-membership, with $\tau(v) + \phi(v) \leq 1$.
- The grade of membership is characterized by the interval $[\tau(v), 1 - \phi(v)]$.
- Each edge $e = (u, v) \in E$ is assigned a pair $(\tau(e), \phi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\}, \quad \phi(e) \geq \max\{\phi(u), \phi(v)\}.$$

(6) *Hesitant Fuzzy Graph* [747, 314]*:*

- Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0, 1]$, denoted $\sigma(v) \subseteq [0, 1]$.
- Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0, 1]$.
- Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.

(7) *Single-Valued Pentapartitioned Neutrosophic Graph* [208, 572]*:*

- Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
  - $T(v) \in [0, 1]$ is the truth-membership degree.
  - $C(v) \in [0, 1]$ is the contradiction-membership degree.
  - $R(v) \in [0, 1]$ is the ignorance-membership degree.
  - $U(v) \in [0, 1]$ is the unknown-membership degree.
  - $F(v) \in [0, 1]$ is the false-membership degree.
  - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
- Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

Recently, Plithogenic Graphs have been introduced as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, serving as a graphical representation of Plithogenic Sets [661]. These graphs have gained attention and are currently being studied extensively [682, 680]. Below, we provide a formal definition.

Definition 4. [661, 660] Let S be a universal set and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following properties for all $a, b \in Pv$:

(1) *Reflexivity of the Contradiction Function*: $pCF(a, a) = 0$.

(2) *Symmetry of the Contradiction Function*: $pCF(a, b) = pCF(b, a)$.

Example 5. Consider $s, t \in \{1, 2, 3, 4, 5\}$.

- For $s = t = 1$, PS is called a *Plithogenic Fuzzy Set* and denoted by PFS.
- For $s = 2, t = 1$, PS is called a *Plithogenic Intuitionistic Fuzzy Set* and denoted by $PIFS$.
- For $s = 3, t = 1$, PS is called a *Plithogenic Neutrosophic Set* and denoted by PNS.
- For $s = 4, t = 1$, PS is called a *Plithogenic Turiyam Neutrosophic Set* and denoted by $PTuS$.

Definition 6. [682] Let $G = (V, E)$ be a crisp graph with vertex set V and edge set $E \subseteq V \times V$. A *Plithogenic Graph* PG is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with vertices.
  - Ml is the range of possible values for the attribute l.
  - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
  - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with edges.
  - Nm is the range of possible values for the attribute m.
  - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
  - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between x and y.

(2) *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 7. Examples of Plithogenic Graphs:

- For $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph.*
- For $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph.*
- For $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph.*
- For $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph.*

Theorem 8. [282] *In each graph class, the following relationships hold:*

- *An empty graph and a null graph can be represented as 2-valued and 3-valued graphs.*
- *Any edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Any fuzzy graph can be converted into a 3-valued graph by mapping fuzzy membership values of vertices and edges to* $\{-1, 0, 1\}$.
- *An Intuitionistic Fuzzy Graph can be simplified to a Fuzzy Graph by setting the non-membership function to zero.*
- *A Neutrosophic Graph can be simplified to an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*
- *Every Extended Turiyam Neutrosophic Graph generalizes the Turiyam Neutrosophic Graph.*
- *Plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Extended Turiyam Neutrosophic Graphs.*
- *Any general Plithogenic Graph can be transformed into other types such as General Turiyam Neutrosophic Graph, General Fuzzy Graph, General Intuitionistic Fuzzy Graph, Four-Valued Fuzzy graph, Ambiguous graph, Picture Fuzzy Graph, Hesitant Fuzzy Graph, Intuitionistic Hesitant Fuzzy Graph, Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Quadripartitioned Neutrosophic graph, Pentapartitioned Neutrosophic graph, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs, and Spherical Fuzzy Graphs.*

For reference, the relationships between the graphs are illustrated in Figure 1.

The following is an unresolved question presented in this paper:

Question 9. Is it mathematically feasible to define a Plithogenic Labeling Graph correctly? Furthermore, can it serve as a generalized concept encompassing Fuzzy Labeling Graphs and Neutrosophic Labeling Graphs?

2.3 | Fuzzy Labeling Graph and Neutrosophic Labeling Graph

As mentioned in the introduction, research on Labeling Graphs has also been conducted within the framework of Uncertain Graphs. The definitions of Fuzzy Labeling Graph [593, 145, 134], Intuitionistic Fuzzy Labeling Graph[540, 616], and Neutrosophic Labeling Graph[679, 313] are provided below.

Definition 10 (Labeling Graph)**.** Let $G = (V, E)$ be a classical graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *labeling graph*, denoted as $G_L = (V, \sigma, \mu)$, is defined by the following components:

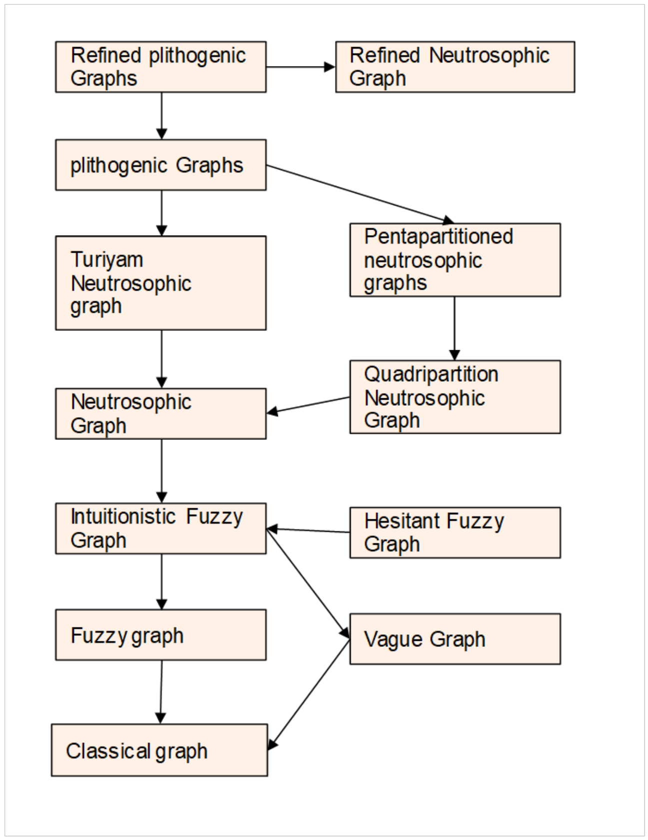

FIGURE 1. Some Uncertain graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

(1) *Vertex Labeling:*

$$\sigma : V \to L_V,$$

where L_V is the set of possible labels for vertices. The function σ assigns a label $\sigma(v) \in L_V$ to each vertex $v \in V$.

(2) *Edge Labeling:*

$$\mu : E \to L_E,$$

where L_E is the set of possible labels for edges. The function μ assigns a label $\mu(e) \in L_E$ to each edge $e \in E$.

(3) *Labeling Rules:* The labeling functions σ and μ must satisfy specific rules or constraints that depend on the type of labeling graph being considered. These rules can include, but are not limited to:
   - *Distinct Labels:* All labels assigned to vertices and edges may need to be distinct, depending on the context of the labeling.
   - *Consistency Rules:* There may be consistency rules between vertex labels and edge labels. For example, an edge $e = (u, v)$ may have a label that depends on the labels of the vertices u and v.
   - *Constraints on Label Values:* The labels may need to satisfy additional constraints, such as numerical relationships (e.g., sum, product, minimum, maximum) or logical conditions.

Example 11. Consider a simple labeling graph where $G = (V, E)$ is an undirected graph with:

- $V = \{v_1, v_2, v_3\}$,
- $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$.

Let $L_V = \{1, 2, 3\}$ and $L_E = \{a, b, c\}$. A possible labeling for this graph can be:

- Vertex labeling: $\sigma(v_1) = 1,\ \sigma(v_2) = 2,\ \sigma(v_3) = 3$.
- Edge labeling: $\mu((v_1, v_2)) = a,\ \mu((v_2, v_3)) = b,\ \mu((v_3, v_1)) = c$.

This satisfies the definition of a labeling graph.

Definition 12 (Fuzzy Labeling Graph)**.** Let $G = (V, E)$ be a graph with a set of vertices V and a set of edges E. A *fuzzy labeling graph*, denoted by $G_\omega = (\sigma_\omega, \mu_\omega)$, is defined as follows:

- $\sigma_\omega : V \to [0, 1]$ is a fuzzy membership function that assigns each vertex $u \in V$ a membership value $\sigma_\omega(u)$.
- $\mu_\omega : V \times V \to [0, 1]$ is a fuzzy membership function that assigns each edge $(u, v) \in E$ a membership value $\mu_\omega(u, v)$.

The function ω is a bijection from the set of all nodes and edges of G to the interval $[0, 1]$, satisfying the following condition:

$$\mu_\omega(u, v) < \min\{\sigma_\omega(u), \sigma_\omega(v)\}, \quad \text{for all } u, v \in V.$$

A graph G_ω is called a *fuzzy labeling graph* if it has a fuzzy labeling as described above.

Definition 13 (Intuitionistic Fuzzy Labeling Graph)**.** Let $G = (V, E)$ be a graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. An *intuitionistic fuzzy labeling graph*, denoted by $G = (V, \sigma, \mu)$, is defined as follows:

- The functions $\sigma_1 : V \to [0, 1]$ and $\sigma_2 : V \to [0, 1]$ assign to each vertex $v \in V$ a membership degree $\sigma_1(v)$ and a non-membership degree $\sigma_2(v)$, respectively, such that $0 \leq \sigma_1(v) + \sigma_2(v) \leq 1$.
- The functions $\mu_1 : E \to [0, 1]$ and $\mu_2 : E \to [0, 1]$ assign to each edge $e = (u, v) \in E$ a membership degree $\mu_1(e)$ and a non-membership degree $\mu_2(e)$, respectively, such that $\mu_1(e) + \mu_2(e) \leq 1$.
- The functions $\sigma_1, \sigma_2, \mu_1, \mu_2$ are bijective, meaning that all membership and non-membership values are distinct.
- For every edge $e = (u, v) \in E$:
$$\mu_1(e) \leq \min\{\sigma_1(u), \sigma_1(v)\}, \quad \mu_2(e) \geq \max\{\sigma_2(u), \sigma_2(v)\}.$$

Definition 14 (Neutrosophic Labeling Graph)**.** Let $G^* = (V, \sigma, \mu)$ be a neutrosophic graph, where:

- V is a set of vertices, denoted as $V = \{v_1, v_2, \ldots, v_n\}$.
- $\sigma = (T_1, I_1, F_1)$, where:
$$T_1 : V \to [0, 1], \quad I_1 : V \to [0, 1], \quad F_1 : V \to [0, 1]$$
represent the truth-membership, indeterminacy-membership, and falsity-membership functions of vertices, respectively. For each vertex $v_i \in V$, the sum satisfies:
$$0 \leq T_1(v_i) + I_1(v_i) + F_1(v_i) \leq 3.$$
- $\mu = (T_2, I_2, F_2)$, where:
$$T_2 : V \times V \to [0, 1], \quad I_2 : V \times V \to [0, 1], \quad F_2 : V \times V \to [0, 1]$$
represent the truth-membership, indeterminacy-membership, and falsity-membership functions of edges, respectively. For each edge $(v_i, v_j) \in V \times V$, the conditions hold:
$$T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\}, \quad I_2(v_i, v_j) \leq \min\{I_1(v_i), I_1(v_j)\}, \quad F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\},$$
and the sum satisfies:
$$0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3.$$

A neutrosophic graph $G^* = (V, \sigma, \mu)$ is called a *neutrosophic labeling graph* if:

(1) The functions T_1, I_1, F_1, T_2, I_2, and F_2 are bijective, meaning each vertex and edge has distinct values for the truth, indeterminacy, and falsity memberships.

(2) For every edge $(v_i, v_j) \in V \times V$:

$$T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\}, \quad I_2(v_i, v_j) \leq \min\{I_1(v_i), I_1(v_j)\}, \quad F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\}.$$

(3) The sum of the membership values for each edge satisfies:

$$0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3.$$

3 | Result: Labeling Graph

The results of this paper are described below.

3.1 | Turiyam Neutrosophic Labeling Graph

In this paper, we examine the definition and basic mathematical structure of the Turiyam Neutrosophic Labeling Graph. The definition is provided below.

Definition 15 (Turiyam Neutrosophic Labeling Graph)**.** Let $G = (V, E)$ be a graph. A *Turiyam Neutrosophic labeling graph*, denoted by $G = (V, \sigma, \mu)$, is defined as follows:

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (T(v), I(v), F(v), L(v))$, where:
  - $T(v), I(v), F(v), L(v) \in [0, 1]$ represent the truth-membership, indeterminacy-membership, falsity-membership, and latent-membership degrees, respectively.
  - The sum satisfies $0 \leq T(v) + I(v) + F(v) + L(v) \leq 1$.
  - The functions T, I, F, L are bijective; that is, each vertex has distinct membership values.
- Each edge $e = (u, v) \in E$ is assigned a quadruple $\mu(e) = (T(e), I(e), F(e), L(e))$, satisfying:
  - $T(e) \leq \min\{T(u), T(v)\}, \quad I(e) \leq \min\{I(u), I(v)\}, \quad F(e) \geq \max\{F(u), F(v)\}, \quad L(e) \leq \min\{L(u), L(v)\}$.
  - The sum satisfies $0 \leq T(e) + I(e) + F(e) + L(e) \leq 1$.
  - The functions T, I, F, L for edges are bijective.

Theorem 16. *A Turiyam Neutrosophic Labeling Graph can be transformed into:*

(1) *A Fuzzy Labeling Graph by considering only the truth-membership degree $T(v)$ for vertices and $T(e)$ for edges.*

(2) *An Intuitionistic Fuzzy Labeling Graph by mapping $T(v)$ to the membership degree and $F(v)$ to the non-membership degree, ensuring $T(v) + F(v) \leq 1$.*

(3) *A Neutrosophic Labeling Graph by using $T(v), I(v), F(v)$ and ignoring $L(v)$ or appropriately mapping $L(v)$ into the indeterminacy component.*

Proof: (1) *Turiyam Neutrosophic to Fuzzy:* Define the fuzzy membership function $\sigma_\omega(v) = T(v)$ for vertices and $\mu_\omega(e) = T(e)$ for edges. Since $T(v) \in [0, 1]$ and the functions are bijective, G becomes a fuzzy labeling graph.

(2) *Turiyam Neutrosophic to Intuitionistic Fuzzy:* Set the membership degree $\sigma_1(v) = T(v)$ and the non-membership degree $\sigma_2(v) = F(v)$ for vertices, ensuring $T(v)+F(v) \leq 1$. Similarly for edges, $\mu_1(e) = T(e)$, $\mu_2(e) = F(e)$.

(3) *Turiyam Neutrosophic to Neutrosophic:* Use $\sigma(v) = (T(v), I(v), F(v))$ for vertices, disregarding $L(v)$ or incorporating $L(v)$ into $I(v)$ by setting $I'(v) = I(v) + L(v)$ while ensuring $T(v) + I'(v) + F(v) \leq 3$.

□

Theorem 17. *In a Turiyam Neutrosophic Labeling Graph $G = (V, \sigma, \mu)$, the sum of the membership degrees for any vertex $v \in V$ satisfies $0 \leq T(v) + I(v) + F(v) + L(v) \leq 1$.*

Proof: By the definition of a Turiyam Neutrosophic Labeling Graph, each vertex $v \in V$ is assigned a quadruple of membership degrees $\sigma(v) = (T(v), I(v), F(v), L(v))$, where $T(v), I(v), F(v), L(v) \in [0, 1]$, and the sum satisfies:

$$0 \leq T(v) + I(v) + F(v) + L(v) \leq 1.$$

This condition is inherent to the structure of a Turiyam Neutrosophic Labeling Graph to ensure that the combined degrees represent valid probabilities or degrees of membership. Therefore, for any vertex $v \in V$, the sum of its membership degrees adheres to the specified inequality. □

Theorem 18. *In a Turiyam Neutrosophic Labeling Graph, if all vertices have $L(v) = 0$, then the graph reduces to a Neutrosophic Labeling Graph.*

Proof: In a Turiyam Neutrosophic Labeling Graph, each vertex $v \in V$ has a membership quadruple $\sigma(v) = (T(v), I(v), F(v), L(v))$. If $L(v) = 0$ for all $v \in V$, the quadruple simplifies to $\sigma(v) = (T(v), I(v), F(v), 0)$.

Since $L(v) = 0$, we can disregard the latent membership component. The remaining components $(T(v), I(v), F(v))$ correspond exactly to the membership degrees in a Neutrosophic Labeling Graph. Therefore, the Turiyam Neutrosophic Labeling Graph effectively becomes a Neutrosophic Labeling Graph when $L(v) = 0$ for all vertices. □

Theorem 19. *The union of two Turiyam Neutrosophic Labeling Graphs G_1 and G_2 is a Turiyam Neutrosophic Labeling Graph if the membership values of overlapping vertices and edges are distinct.*

Proof: Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two Turiyam Neutrosophic Labeling Graphs. Define the union graph $G = (V, \sigma, \mu)$, where:

$$V = V_1 \cup V_2, \quad E = E_1 \cup E_2.$$

For each vertex $v \in V$:

$$\sigma(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V_1 - V_2, \\ \sigma_2(v), & \text{if } v \in V_2 - V_1, \\ \text{distinct values}, & \text{if } v \in V_1 \cap V_2. \end{cases}$$

Similarly, define μ for edges. Since the membership values of overlapping vertices and edges are distinct, the bijectivity of the labeling functions σ and μ is preserved. Additionally, the membership degrees satisfy the Turiyam Neutrosophic conditions for all vertices and edges in G. Therefore, G is a Turiyam Neutrosophic Labeling Graph. □

Theorem 20. *Every Turiyam Neutrosophic Labeling Subgraph of a Turiyam Neutrosophic Labeling Graph is itself a Turiyam Neutrosophic Labeling Graph.*

Proof: Let $G = (V, \sigma, \mu)$ be a Turiyam Neutrosophic Labeling Graph, and let $H = (V', \sigma', \mu')$ be a subgraph of G, where $V' \subseteq V$ and $E' \subseteq E$.

Define σ' and μ' as the restrictions of σ and μ to V' and E', respectively. Since σ and μ satisfy the Turiyam Neutrosophic conditions (membership degrees in $[0, 1]$, sums not exceeding 1, bijectivity), their restrictions σ' and μ' also satisfy these conditions.

Therefore, H inherits all the properties of a Turiyam Neutrosophic Labeling Graph from G and is itself a Turiyam Neutrosophic Labeling Graph. □

Theorem 21. *A Turiyam Neutrosophic Labeling Graph is connected if there exists a path between any two vertices where the minimum of the truth-membership degrees along the path is non-zero.*

Proof: In a Turiyam Neutrosophic Labeling Graph, the truth-membership degree $T(e)$ of an edge e represents the degree to which that edge truly exists. If, for any two vertices $u, v \in V$, there exists a path P such that $\min_{e\in P} T(e) > 0$, then there is a non-zero degree of truth that each edge along the path exists.

This implies that there is a connection between u and v through edges that are not entirely false or indeterminate. Therefore, the graph is connected in the sense that there are paths with non-zero truth-membership degrees between any pair of vertices. □

Theorem 22. *In a Turiyam Neutrosophic Labeling Graph, the strength of a path P between vertices u and v is given by:*

$$S(P) = \left(\min_{e\in P} T(e), \max_{e\in P} I(e), \max_{e\in P} F(e), \max_{e\in P} L(e)\right).$$

Proof: The strength of a path P in a Turiyam Neutrosophic Labeling Graph is determined by aggregating the membership degrees of its constituent edges:

- $T(P) = \min_{e\in P} T(e)$: The truth-membership degree of the path is limited by the weakest link (edge) in terms of truth.
- $I(P) = \max_{e\in P} I(e)$: The indeterminacy of the path is governed by the edge with the highest indeterminacy.
- $F(P) = \max_{e\in P} F(e)$: The falsity of the path is influenced by the edge with the highest falsity degree.
- $L(P) = \max_{e\in P} L(e)$: The latent membership degree of the path is determined by the edge with the highest latent degree.

This method of calculating the path's strength ensures that the overall assessment reflects the most restrictive (for truth) and the most significant (for indeterminacy, falsity, and latent degrees) characteristics of the path. □

Theorem 23. *If a Turiyam Neutrosophic Labeling Graph has all vertices with $T(v) = 1$ and $I(v) = F(v) = L(v) = 0$, it reduces to a crisp graph.*

Proof: When $T(v) = 1$ and $I(v) = F(v) = L(v) = 0$ for all vertices $v \in V$, each vertex is fully included in the graph without any uncertainty, falsity, or latent characteristics.

Similarly, if all edges $e \in E$ have $T(e) = 1$ and $I(e) = F(e) = L(e) = 0$, then every edge definitively exists.

Under these conditions, the Turiyam Neutrosophic Labeling Graph behaves exactly like a classical crisp graph, where the presence of vertices and edges is certain and unambiguous. Thus, the graph reduces to a crisp graph. □

Theorem 24. *The complement of a Turiyam Neutrosophic Labeling Graph is also a Turiyam Neutrosophic Labeling Graph if the complement operation adjusts the membership degrees appropriately.*

Proof: The complement $\overline{G}$ of a Turiyam Neutrosophic Labeling Graph $G = (V, \sigma, \mu)$ involves:

- Replacing each edge $e \in E$ with its non-existent counterpart $e \notin E$.
- Adjusting the membership degrees for vertices and edges to reflect the complementarity.

For vertices, since they exist in both G and $\overline{G}$, their membership degrees remain the same or are adjusted in a way that maintains the Turiyam Neutrosophic conditions.

For edges, the membership degrees are adjusted appropriately, for example:

$$\overline{T}(e) = 1 - T(e), \quad \overline{I}(e) = I(e), \quad \overline{F}(e) = F(e), \quad \overline{L}(e) = L(e).$$

This ensures that the sum of membership degrees for edges in $\overline{G}$ still satisfies $0 \leq \overline{T}(e) + \overline{I}(e) + \overline{F}(e) + \overline{L}(e) \leq 1$.

Provided these adjustments maintain the Turiyam Neutrosophic conditions, $\overline{G}$ is also a Turiyam Neutrosophic Labeling Graph. □

Theorem 25. *In a Turiyam Neutrosophic Labeling Graph, the sum of the membership degrees for any edge* $e = (u, v)$ *satisfies* $0 \leq T(e) + I(e) + F(e) + L(e) \leq 1$.

Proof: By the definition of a Turiyam Neutrosophic Labeling Graph, each edge $e \in E$ is assigned a quadruple $\mu(e) = (T(e), I(e), F(e), L(e))$, where each component lies in the interval $[0, 1]$, and the sum of these components satisfies:

$$0 \leq T(e) + I(e) + F(e) + L(e) \leq 1.$$

This condition ensures that the membership degrees for edges are valid and collectively represent a coherent state of existence, uncertainty, falsity, and latency for each edge in the graph. □

3.2 | Single-Valued Pentapartitioned Neutrosophic Labeling Graph

In this paper, we examine the definition and basic mathematical structure of the Single-Valued Pentapartitioned Neutrosophic Labeling Graph. The definition is provided below.

Definition 26 (Single-Valued Pentapartitioned Neutrosophic Labeling Graph)**.** Let $G = (V, E)$ be a graph.

- *Vertex Labeling:*
  - Each vertex $v \in V$ is assigned a pentuple:
    $$\sigma(v) = (T(v), C(v), R(v), U(v), F(v)),$$
    where $T(v), C(v), R(v), U(v), F(v) \in [0, 1]$ and
    $$T(v) + C(v) + R(v) + U(v) + F(v) \leq 5.$$
  - The functions T, C, R, U, F are bijective.
- *Edge Labeling:*
  - Each edge $e = (u, v) \in E$ is assigned a pentuple:
    $$\mu(e) = (T(e), C(e), R(e), U(e), F(e)),$$
    where:
    $$\begin{aligned} T(e) &\leq \min\{T(u), T(v)\}, \\ C(e) &\leq \min\{C(u), C(v)\}, \\ R(e) &\geq \max\{R(u), R(v)\}, \\ U(e) &\geq \max\{U(u), U(v)\}, \\ F(e) &\geq \max\{F(u), F(v)\}, \end{aligned}$$
    and
    $$T(e) + C(e) + R(e) + U(e) + F(e) \leq 5.$$
  - The functions T, C, R, U, F for edges are bijective.

Theorem 27. *Every Single-Valued Pentapartitioned Neutrosophic Labeling Graph can be transformed into a Turiyam Neutrosophic Labeling Graph and a Neutrosophic Labeling Graph.*

Proof: *Transformation to Turiyam Neutrosophic Labeling Graph*

Define the mappings for each vertex $v \in V$:

$$\begin{aligned} t(v) &= T(v), \\ iv(v) &= C(v) + R(v), \\ fv(v) &= F(v), \\ lv(v) &= U(v). \end{aligned}$$

Verification of Membership Degrees:

The components $t(v), iv(v), fv(v), lv(v) \in [0, 1]$ since they are sums or values of components in $[0, 1]$.

The sum:

$$t(v) + iv(v) + fv(v) + lv(v) = T(v) + C(v) + R(v) + F(v) + U(v) \leq 5.$$

Normalization:

Since the total sum should be ≤ 4 in a Turiyam Neutrosophic Labeling Graph, we normalize:

$$\begin{aligned} t'(v) &= \frac{t(v)}{5} \times 4, \\ iv'(v) &= \frac{iv(v)}{5} \times 4, \\ fv'(v) &= \frac{fv(v)}{5} \times 4, \\ lv'(v) &= \frac{lv(v)}{5} \times 4. \end{aligned}$$

Thus, the sum becomes:

$$t'(v) + iv'(v) + fv'(v) + lv'(v) = \frac{4}{5} \left(T(v) + C(v) + R(v) + F(v) + U(v) \right) \leq 4.$$

Transformation to Neutrosophic Labeling Graph

Define the mappings for each vertex $v \in V$:

$$\begin{aligned} T_N(v) &= T(v), \\ I_N(v) &= C(v) + R(v) + U(v), \\ F_N(v) &= F(v). \end{aligned}$$

Verification of Membership Degrees:

The components $T_N(v), I_N(v), F_N(v) \in [0, 1]$.

The sum:

$$T_N(v) + I_N(v) + F_N(v) = T(v) + C(v) + R(v) + U(v) + F(v) \leq 5.$$

Normalization:

Since the total sum should be ≤ 3 in a Neutrosophic Labeling Graph, we normalize:

$$\begin{aligned} T'_N(v) &= \frac{T_N(v)}{5} \times 3, \\ I'_N(v) &= \frac{I_N(v)}{5} \times 3, \\ F'_N(v) &= \frac{F_N(v)}{5} \times 3. \end{aligned}$$

Thus, the sum becomes:

$$T'_N(v) + I'_N(v) + F'_N(v) = \frac{3}{5}(T(v) + C(v) + R(v) + U(v) + F(v)) \leq 3.$$

Therefore, we obtain a Turiyam Neutrosophic Labeling Graph and a Neutrosophic Labeling Graph from the given Single-Valued Pentapartitioned Neutrosophic Labeling Graph. □

For reference, the relationships between the labeling graphs are illustrated in Figure 3. The author hopes that the exploration of these classes of Uncertain Labeling Graphs will continue to advance in the future.

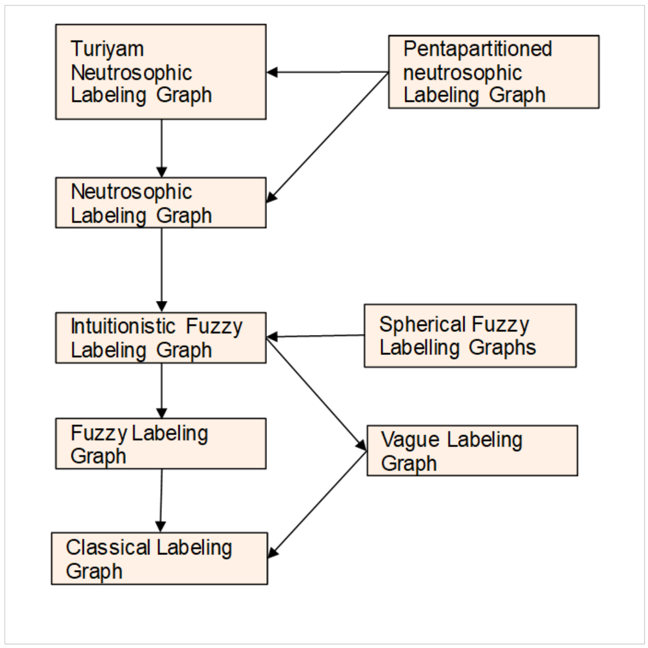

FIGURE 2. Some Uncertain labeling graphs Hierarchy. The labeling graph class at the origin of an arrow contains the graph class at the destination of the arrow.

4 | Future tasks and Discussion

The future directions of this research are outlined as follows.

4.1 | Future tasks: Neutrosophic Magic labeling Graph and other graphs

Initially, we plan to extend labeling concepts such as magic labeling [681, 519, 679], anti-magic labeling [432, 637, 725, 227], and bi-magic labeling [135] to uncertain graphs, along with exploring related parameters. Additionally, we aim to investigate the adaptation of labeling in digraphs [114, 469, 143] to uncertain graphs and to develop a version of HyperFuzzy Set [379, 511] in the context of Labeling Graphs.

Additionally, we aim to study the extension of Constant Intuitionistic Fuzzy Graphs and Fuzzy Tolerance Graphs[615, 206] to Single-Valued Pentapartitioned Neutrosophic Graphs, as well as labeling problems associated with them.

4.2 | Future tasks: Fuzzy L(h, k)-Labeling Graph

Looking ahead, we plan to investigate the L(h, k)-Labeling Graph, a well-established topic in general graph theory [165, 164, 166]. Building upon this, we aim to extend these labeling problems to fuzzy graphs and explore their mathematical characteristics. Although still in the conceptual phase, the fuzzy L(h, k)-Labeling Graph has been defined, and we plan to examine its mathematical structure further as needed.

Definition 28 (L(h, k)-Labeling Graph)**.** Let $G = (V, E)$ be a simple, undirected graph where V is the set of vertices and E is the set of edges. An *L(h, k)-labeling* of G is a function $f : V \to \mathbb{N}$ that assigns non-negative integer labels to the vertices of G such that the following conditions hold:

(1) For any two adjacent vertices $u, v \in V$ (i.e., $(u, v) \in E$), the absolute difference between their labels is at least h:

$$|f(u) - f(v)| \geq h.$$

(2) For any two vertices $u, v \in V$ that have a common neighbor (i.e., there exists a vertex w such that $(u, w) \in E$ and $(v, w) \in E$), the absolute difference between their labels is at least k:

$$|f(u) - f(v)| \geq k.$$

Remark 29.

- *Special Cases:*
  - *If $h = 0$ and $k = 1$, the L(h, k)-labeling problem becomes the classical vertex coloring problem.*
  - *If $h = 1$ and $k = 1$, the problem becomes the distance-2 coloring or D2-vertex coloring problem.*
  - *If $h = 2$ and $k = 1$, it corresponds to the radio coloring problem.*
- *Generalizations: The L(h, k)-labeling problem can model several applications in communication networks, such as frequency assignment, where h represents interference constraints between directly connected nodes, and k represents constraints between nodes with a common neighbor.*

Definition 30 (Fuzzy L(h, k)-Labeling Graph)**.** Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, where:

- $\sigma : V \to [0, 1]$ is the vertex membership function.
- $\mu : E \to [0, 1]$ is the edge membership function, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $(u, v) \in E$.

An *L(h, k)-labeling* of G is defined via the membership functions σ and μ, satisfying the following conditions:

(1) For any two adjacent vertices $u, v \in V$ (i.e., $\mu(u, v) > 0$):

$$|\sigma(u) - \sigma(v)| \geq h \cdot \mu(u, v).$$

(2) For any two vertices $u, v \in V$ that have a common neighbor $w \in V$ with $\mu(u, w) > 0$ and $\mu(v, w) > 0$:

$$|\sigma(u) - \sigma(v)| \geq k \cdot \min\{\mu(u, w), \mu(v, w)\}.$$

A fuzzy graph satisfying these conditions is called a *Fuzzy L(h, k)-Labeling Graph*.

4.3 | Discussion: Other Uncertain Graph Classes

Research on uncertain graph classes is continually evolving, extending beyond concepts like labeling graphs. Although still in the conceptual stage, several new graph classes have been proposed as follows.

4.3.1 | Meta Graph Class

The Meta Set is a set concept studied as a derivative of the Fuzzy Set. We are considering extending this concept to graph theory [677, 676, 678]. Although still in the conceptual stage, its definition is presented below.

Definition 31 (Infinite Binary Tree)**.** [167, 67, 168] An *infinite binary tree* $T = (V, E)$ is a tree structure defined as follows:

- The set V of *vertices* consists of all finite binary sequences. Each vertex represents a unique sequence $v = (v_1, v_2, \ldots, v_n)$ where $v_i \in \{0, 1\}$ for all i and n is finite.
- The set E of *edges* consists of ordered pairs (u, v) where v is obtained from u by appending exactly one additional binary digit (either 0 or 1) to the end of u. Formally, if $u = (u_1, u_2, \ldots, u_n)$, then $v = (u_1, u_2, \ldots, u_n, b)$ with $b \in \{0, 1\}$.

Each vertex has exactly two children, corresponding to appending 0 and 1 respectively, and exactly one parent, obtained by removing the last binary digit. The root of T is the empty sequence (), denoted by ϵ.

Definition 32 (Meta Set)**.** [678] A *meta set* is a crisp set that is either the empty set $\emptyset$, or it has the form:

$$\tau = \{\langle \sigma, p \rangle : \sigma \text{ is a meta set}, \ p \in T\},$$

where:

- T is the full infinite binary tree,
- $\langle \cdot, \cdot \rangle$ denotes an ordered pair.

Elements of a meta set are ordered pairs, where the first element σ is a *potential element* (also a meta set), and the second element p is a condition from the binary tree T, representing the degree of membership. This definition is recursive and is founded by the empty set $\emptyset$, which itself is considered a meta set.

Definition 33 (Domain of a Meta Set)**.** [678] The *domain* of a meta set τ, denoted $\text{dom}(\tau)$, is the set of its potential elements:

$$\text{dom}(\tau) = \{\sigma : \langle \sigma, p \rangle \in \tau\}.$$

Definition 34 (Range of a Meta Set)**.** [678] The *range* of a meta set τ, denoted $\text{ran}(\tau)$, is defined as:

$$\text{ran}(\tau) = \{p : \langle \sigma, p \rangle \in \tau\}.$$

Definition 35. In a meta graph $G_m = (V_m, E_m, \tau)$:

- Each vertex $v \in V_m$ is a meta set, defined as:

  $$v = \{\langle \sigma, p \rangle : \sigma \text{ is a meta set}, \ p \in T\},$$

  where σ is a potential element, and p is a condition in T.
- Each edge $e \in E_m$, connecting vertices $v_i, v_j \in V_m$, is labeled by a meta set $\tau(e)$:

  $$\tau(e) = \{\langle \eta, q \rangle : \eta \text{ is a meta set}, \ q \in T\}.$$

- The domain and range of the meta sets in vertices and edge labels are defined as:
  - *Domain of a vertex:*

    $$\text{dom}(v) = \{\sigma : \langle \sigma, p \rangle \in v\}.$$

  - *Range of a vertex:*

    $$\text{ran}(v) = \{p : \langle \sigma, p \rangle \in v\}.$$

  - *Domain of an edge label:*

    $$\text{dom}(\tau(e)) = \{\eta : \langle \eta, q \rangle \in \tau(e)\}.$$

  - *Range of an edge label:*

    $$\text{ran}(\tau(e)) = \{q : \langle \eta, q \rangle \in \tau(e)\}.$$

4.3.2 | Extended Hesitant Fuzzy Graph and Dual Extended Hesitant Fuzzy Graph

In recent years, the concepts of the Extended Hesitant Fuzzy Set [815, 422, 626] and Dual Extended Hesitant Fuzzy Set [64] have been defined, receiving considerable attention in the same way as Hesitant Fuzzy Sets[697, 698, 747] and Dual Hesitant Fuzzy Sets [816, 254]. Likewise, the Hesitant Fuzzy Graph[314, 531] and Dual Hesitant Fuzzy Graph[115] are already established concepts in graph theory. Inspired by these developments, we propose the definitions for Extended Hesitant Fuzzy Graphs and Dual Extended Hesitant Fuzzy Graphs, with the aim of exploring their mathematical structures and potential applications in future studies.

Definition 36 (Hesitant Fuzzy Set (HFS))**.** [698] Let X be a non-empty set. A *Hesitant Fuzzy Set* (HFS) on X is a mapping $h_M : X \to P^*([0,1])$, where $P^*([0,1])$ denotes the set of all non-empty subsets of $[0,1]$. For each element $x \in X$, the corresponding set $h_M(x) \subseteq [0,1]$ represents the set of possible membership degrees assigned to x.

When the membership set $h_M(x)$ is finite, the HFS is called a *Typical Hesitant Fuzzy Set* (THFS). In this case, the typical hesitant fuzzy element (THFE) $h_M(x)$ can be expressed as:

$$h_M(x) = \{h_1, h_2, \dots, h_{l(x)}\},$$

where $h_1 < h_2 < \cdots < h_{l(x)}$, and $h_M(x)^- = h_1$ and $h_M(x)^+ = h_{l(x)}$ represent the minimum and maximum membership degrees, respectively.

In the general case, the minimum and maximum of an HFE $h_M(x)$ are defined as:

$$h_M(x)^- = \inf\{\gamma : \gamma \in h_M(x)\} \quad \text{and} \quad h_M(x)^+ = \sup\{\gamma : \gamma \in h_M(x)\}.$$

Thus, a hesitant fuzzy set M on X can be represented as:

$$M = \{(x, h_M(x)) : x \in X\}.$$

Definition 37 (Extended Hesitant Fuzzy Set (EHFS))**.** [815] Let X be a non-empty set. An *Extended Hesitant Fuzzy Set* (EHFS) of degree m on X is a set defined as:

$$H_X^m = \{\langle x, H(x)\rangle : x \in X\},$$

where each $H(x)$ is an *Extended Hesitant Fuzzy Element* (EHFE) of degree m, defined as the Cartesian product of m non-empty subsets of $[0,1]$:

$$H(x) = \prod_{i=1}^{m} H_i(x),$$

where $H_i(x) \subseteq [0,1]$ for each i and $H(x) \subseteq [0,1]^m$.

Definition 38 (Dual Hesitant Fuzzy Set (DHFS))**.** [64] Let X be a non-empty set. A *Dual Hesitant Fuzzy Set* (DHFS) on X is a set defined as:

$$D = \{\langle x, h(x), g(x)\rangle : x \in X\},$$

where each pair $(h(x), g(x))$ is a *Dual Hesitant Fuzzy Element* (DHFE) associated with x, where:

$$h(x), g(x) \subseteq [0,1],$$

and satisfy $\gamma + \eta \leq 1$ for all $\gamma \in h(x)$ and $\eta \in g(x)$. The supremum values $h^+ = \sup h(x)$ and $g^+ = \sup g(x)$ also satisfy $h^+ + g^+ \leq 1$.

Definition 39 (Hesitant Fuzzy Graph (HFG))**.** A *Hesitant Fuzzy Graph* (HFG) is defined as a four-tuple $G = (V, E, \sigma, \mu)$, where:

- V is a set of vertices,
- $E \subseteq V \times V$ is a set of edges,
- $\sigma : V \to S_f([0,1])$ is the vertex membership function, where $S_f([0,1])$ denotes the collection of all finite subsets of $[0,1]$. For each vertex $v \in V$, $\sigma(v) \subseteq [0,1]$ represents a hesitant fuzzy element (HFE) containing possible membership values of v in V,
- $\mu : E \to S_f([0,1])$ is the edge membership function. For each edge $e \in E$, $\mu(e) \subseteq [0,1]$ represents a hesitant fuzzy element containing possible membership values of e in E.

Thus, each vertex $v \in V$ and each edge $e \in E$ in the hesitant fuzzy graph G is associated with a finite subset of membership values in $[0,1]$, allowing for multiple degrees of membership to reflect hesitancy.

The hesitant fuzzy graph G can be represented as:

$$G = \{(v, \sigma(v)) : v \in V\} \cup \{(e, \mu(e)) : e \in E\}.$$

Definition 40 (Extended Hesitant Fuzzy Graph (EHFG))**.** An *Extended Hesitant Fuzzy Graph* (EHFG) of degree m is a five-tuple $G = (V, E, \sigma, \mu, m)$, where:

- V is a set of vertices,
- $E \subseteq V \times V$ is a set of edges,
- $\sigma : V \to S_f([0,1]^m)$ is the vertex membership function, where $S_f([0,1]^m)$ denotes the collection of all finite subsets of $[0,1]^m$. For each vertex $v \in V$, $\sigma(v) \subseteq [0,1]^m$ represents an *Extended Hesitant Fuzzy Element* (EHFE) containing multiple membership values of v with degree m,
- $\mu : E \to S_f([0,1]^m)$ is the edge membership function, where for each edge $e \in E$, $\mu(e) \subseteq [0,1]^m$ is an EHFE of degree m, representing multiple membership values for e.

Thus, each vertex $v \in V$ and each edge $e \in E$ in the EHFG G is associated with a finite subset of m-tuples from $[0,1]^m$, allowing for hesitancy across multiple dimensions of membership.

The extended hesitant fuzzy graph G can be expressed as:

$$G = \{(v, \sigma(v)) : v \in V\} \cup \{(e, \mu(e)) : e \in E\}.$$

Definition 41 (Dual Extended Hesitant Fuzzy Graph (DEHFG))**.** A *Dual Extended Hesitant Fuzzy Graph* (DEHFG) is defined as a six-tuple $G = (V, E, \sigma, \mu, h, g)$, where:

- V is a set of vertices,
- $E \subseteq V \times V$ is a set of edges,
- $h : V \to S_f([0,1])$ and $g : V \to S_f([0,1])$ are the vertex membership and non-membership functions, respectively, where each $h(v)$ and $g(v)$ for $v \in V$ are subsets of $[0,1]$,
- $\sigma : E \to S_f([0,1])$ and $\mu : E \to S_f([0,1])$ are the edge membership and non-membership functions, respectively, where each $\sigma(e)$ and $\mu(e)$ for $e \in E$ are subsets of $[0,1]$,
- The conditions $h^+(v) + g^+(v) \leq 1$ for vertices and $\sigma^+(e) + \mu^+(e) \leq 1$ for edges are satisfied, where $h^+(v) = \sup h(v)$, $g^+(v) = \sup g(v)$, $\sigma^+(e) = \sup \sigma(e)$, and $\mu^+(e) = \sup \mu(e)$.

In a DEHFG, each vertex $v \in V$ and each edge $e \in E$ are associated with dual hesitant fuzzy elements, allowing for both membership and non-membership values to reflect hesitancy.

The dual extended hesitant fuzzy graph G can be represented as:

$$G = \{(v, h(v), g(v)) : v \in V\} \cup \{(e, \sigma(e), \mu(e)) : e \in E\}.$$

4.3.3 | Cohesive Fuzzy Graph

The Cohesive Fuzzy Set is a generalized concept that extends both the Complex Fuzzy Set[704, 584, 232] and the Hesitant Fuzzy Set[617, 697]. This concept is extended to graphs to form a Cohesive Fuzzy Graph. Although it remains in the conceptual phase, we outline its definition along with related concepts as follows.

Definition 42 (Cohesive Fuzzy Set (CHFS))**.** [748] Let S be a fixed universe of discourse and $T \subset S$ a fuzzy set defined over S. A *Cohesive Fuzzy Set* on T is defined by a membership function h_T that, when applied to each $x \in S$, returns a subset of the unit circle in the complex plane, representing the possible membership degrees of x in T.

For each $x \in S$, the membership degree $h_T(x)$ is expressed as a set of complex numbers in the form:

$$h_T(x) = \{r_T(x) \exp(i w_T(x)) : r_T(x) \in [0,1],\ w_T(x) \in \mathbb{R}\},$$

where $r_T(x)$ represents the magnitude of membership, $w_T(x)$ represents the phase in radians, and $i = \sqrt{-1}$ denotes the imaginary unit.

The cohesive fuzzy set T is therefore represented as:

$$T = \{\langle x, h_T(x) \rangle : x \in S\}.$$

Definition 43 (Cohesive Fuzzy Graph (CHFG))**.** Let $G = (V, E)$ be a classical graph where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Cohesive Fuzzy Graph* (CHFG) $\tilde{G}$ on G is defined as a four-tuple $\tilde{G} = (V, E, h_V, h_E)$, where:

- V is the vertex set,
- $E \subseteq V \times V$ is the edge set,
- $h_V : V \to S_c([0,1])$ is the vertex cohesive membership function, where $S_c([0,1])$ denotes the collection of all subsets of the unit circle in the complex plane. For each vertex $v \in V$, $h_V(v) \subseteq \{r_V(v)\exp(iw_V(v)) : r_V(v) \in [0,1], w_V(v) \in \mathbb{R}\}$, representing the cohesive membership degrees of v,
- $h_E : E \to S_c([0,1])$ is the edge cohesive membership function, where for each edge $e \in E$, $h_E(e) \subseteq \{r_E(e)\exp(iw_E(e)) : r_E(e) \in [0,1], w_E(e) \in \mathbb{R}\}$, representing the cohesive membership degrees of e.

Each membership function returns a set of complex numbers within the unit circle, with $r_V(v)$ and $r_E(e)$ representing the magnitudes of membership for vertices and edges, respectively, and $w_V(v)$ and $w_E(e)$ representing the phase angles in radians.

The cohesive fuzzy graph $\tilde{G}$ can be expressed as:

$$\tilde{G} = \{\langle v, h_V(v)\rangle : v \in V\} \cup \{\langle e, h_E(e)\rangle : e \in E\}.$$

4.3.4 | Hesitant Fuzzy Linguistic Term Graph

We intend to further explore the extension of Hesitant Fuzzy Linguistic Term Sets [602, 732] to graphs. Numerous related concepts, such as Fuzzy Linguistic Term Sets, are already well-known. With this context in mind, we present preliminary definitions, including those still in the conceptual phase, along with relevant related concepts as outlined below.

Definition 44 (Hesitant Fuzzy Linguistic Term Set (HFLTS))**.** Let $S = \{s_0, s_1, \dots, s_g\}$ be an ordered linguistic term set.

A *Hesitant Fuzzy Linguistic Term Set* (HFLTS) H_S is an ordered finite subset of consecutive linguistic terms from S.

Formally, an HFLTS H_S is defined as:

$$H_S = \{s_i, s_{i+1}, \dots, s_{i+k}\}, \quad \text{for some } 0 \le i \le i + k \le g.$$

The empty HFLTS and the full HFLTS are special cases:

- *Empty HFLTS*: $H_S = \emptyset$.
- *Full HFLTS*: $H_S = S$.

Any other HFLTS contains at least one linguistic term from S.

Definition 45 (Hesitant Fuzzy Linguistic Term Graph (HFLTG))**.** Let $G = (V, E)$ be a classical graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Let $S = \{s_0, s_1, \dots, s_g\}$ be an ordered linguistic term set.

A *Hesitant Fuzzy Linguistic Term Graph* (HFLTG) is a quadruple $G = (V, E, \sigma, \mu)$, where:

- V is the vertex set.
- E is the edge set.
- $\sigma : V \to \mathcal{H}(S)$ is the vertex HFLTS function, assigning to each vertex $v \in V$ a hesitant fuzzy linguistic term set $\sigma(v) \subseteq S$.
- $\mu : E \to \mathcal{H}(S)$ is the edge HFLTS function, assigning to each edge $e \in E$ a hesitant fuzzy linguistic term set $\mu(e) \subseteq S$.

Here, $\mathcal{H}(S)$ denotes the set of all HFLTS over S. Thus, each vertex $v \in V$ and each edge $e \in E$ is associated with an HFLTS, representing the possible linguistic evaluations or degrees assigned to them.

4.3.5 | Triangular Dense Fuzzy Graph Class

The Dense Fuzzy Set is one of the related concepts of the Fuzzy Set, and it has been extensively studied [212, 211]. Although still in the conceptual stage, extending these concepts to graphs would result in the following.

Definition 46 (Dense Fuzzy Set (DFS))**.** Let $\tilde{A}$ be a fuzzy set whose components are defined by a sequence of functions $\{f_n\}$ generated from the mapping of natural numbers $\mathbb{N}$ to a crisp number $x \in \mathbb{R}$. If all components of $\{f_n\}$ converge to x as $n \to \infty$, then $\tilde{A}$ is called a *dense fuzzy set (DFS).*

Definition 47 (Triangular Dense Fuzzy Set (TDFS))**.** Let $\tilde{A}$ be a fuzzy number defined by:

$$\tilde{A} = (a_n, b_n, c_n),$$

where a_n, b_n, c_n are sequences of functions such that:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 1,$$

and $\lim_{n\to\infty} b_n = x$, where x is a crisp singleton. Then, $\tilde{A}$ is called a *Triangular Dense Fuzzy Set (TDFS).*

Definition 48 (Dense Fuzzy Graph (DFG))**.** Let $G = (V, E)$ be a classical graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. Let $\tilde{G}$ be a fuzzy graph defined as $\tilde{G} = (V, \tilde{E})$, where $\tilde{E}$ represents the fuzzy edges. The fuzzy membership function for edges, $\mu_{\tilde{E}} : E \to [0, 1]$, is given by a sequence of functions $\{f_n\}$ generated from the mapping of natural numbers $\mathbb{N}$ to a crisp number x. If all components $f_n(e)$ converge to x as $n \to \infty$ for each edge $e \in E$, then $\tilde{G}$ is called a *Dense Fuzzy Graph (DFG).*

Definition 49 (Triangular Dense Fuzzy Graph (TDFG))**.** Let $G = (V, E)$ be a classical graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A fuzzy graph $\tilde{G} = (V, \tilde{E})$ is called a *Triangular Dense Fuzzy Graph (TDFG)* if the fuzzy membership function for edges, $\mu_{\tilde{E}} : E \to [0, 1]$, is defined by a triangular fuzzy set represented by three sequences $\{a_n\}, \{b_n\}, \{c_n\}$ for each edge $e \in E$, where:

- $a_n(e), b_n(e), c_n(e)$ are sequences of functions such that:

  $$\lim_{n\to\infty} a_n(e) = \lim_{n\to\infty} c_n(e) = 1,$$

  and

  $$\lim_{n\to\infty} b_n(e) = x,$$

  where $x \in [0, 1]$ is a crisp singleton.
- The fuzzy set $\tilde{E}(e)$ converges to the crisp singleton x for each edge $e \in E$, thereby forming a TDFS for each edge.

Thus, $\tilde{G}$ is called a Triangular Dense Fuzzy Graph if the fuzzy edge set $\tilde{E}$ is a collection of TDFSs over the edges of G.

4.3.6 | (3, 2)-Fuzzy Graph Class

The (3, 2)-Fuzzy Set is one of the related concepts of the Fuzzy Set, and it has been extensively studied[709, 356]. Although still in the conceptual stage, extending these concepts to graphs would result in the following.

Definition 50 (Universal Set)**.** [332, 269] A *universal set* U is a set that contains all the elements under consideration for a particular discussion or problem. Every other set in that context is a subset of the universal set. Formally, if A is any set in the context, then:

$$A \subseteq U.$$

The universal set U is often assumed to be large enough to include all relevant elements, and its specific definition may vary depending on the scope of the problem.

Remark 51. *The universal set U is typically represented differently based on the context:*

- *In the context of natural numbers, $U = \mathbb{N}$ (the set of all natural numbers).*

- *In the context of real numbers, $U = \mathbb{R}$ (the set of all real numbers).*
- *For finite problems, U may be a finite set encompassing all relevant objects.*

Definition 52 ((3, 2)-Fuzzy Set)**.** [709, 356, 360] Let X be a universal set. A *(3, 2)-fuzzy set D*, denoted briefly as (3, 2)-FS, is defined as follows:

$$D = \{\langle r, \alpha_D(r), \beta_D(r)\rangle : r \in X\},$$

where:

- $\alpha_D(r) : X \to [0, 1]$ is the degree of membership of $r \in X$ to the set D,
- $\beta_D(r) : X \to [0, 1]$ is the degree of non-membership of $r \in X$ to the set D,
- The following condition holds:

$$0 \leq (\alpha_D(r))^3 + (\beta_D(r))^2 \leq 1.$$

The *degree of indeterminacy* of $r \in X$ with respect to D is defined by:

$$\pi_D(r) = \sqrt[5]{1 - \left((\alpha_D(r))^3 + (\beta_D(r))^2\right)}.$$

Remark 53. *It is clear that the following inequality holds for each $r \in X$:*

$$(\alpha_D(r))^3 + (\beta_D(r))^2 + (\pi_D(r))^5 = 1.$$

Additionally, $\pi_D(r) = 0$ whenever $(\alpha_D(r))^3 + (\beta_D(r))^2 = 1$.

Definition 54 ((3, 2)-Fuzzy Graph)**.** Let $G = (V, E)$ be a classical graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *(3, 2)-fuzzy graph $\tilde{G} = (V, \sigma, \mu)$*, denoted briefly as (3, 2)-FG, is defined as follows:

- The vertex set V is associated with a (3, 2)-fuzzy set, where each vertex $v \in V$ has:
  - A membership degree $\sigma_\alpha(v) : V \to [0, 1]$,
  - A non-membership degree $\sigma_\beta(v) : V \to [0, 1]$,
  - A degree of indeterminacy $\pi_\sigma(v)$ defined as:

$$\pi_\sigma(v) = \sqrt[5]{1 - \left((\sigma_\alpha(v))^3 + (\sigma_\beta(v))^2\right)},$$

    where the condition $0 \leq (\sigma_\alpha(v))^3 + (\sigma_\beta(v))^2 \leq 1$ must hold for all $v \in V$.

- The edge set E is also associated with a (3, 2)-fuzzy set, where each edge $e = (u, v) \in E$ has:
  - A membership degree $\mu_\alpha(e) : E \to [0, 1]$,
  - A non-membership degree $\mu_\beta(e) : E \to [0, 1]$,
  - A degree of indeterminacy $\pi_\mu(e)$ defined as:

$$\pi_\mu(e) = \sqrt[5]{1 - \left((\mu_\alpha(e))^3 + (\mu_\beta(e))^2\right)},$$

    where the condition $0 \leq (\mu_\alpha(e))^3 + (\mu_\beta(e))^2 \leq 1$ must hold for all $e \in E$.

Thus, a (3, 2)-fuzzy graph $\tilde{G} = (V, \sigma, \mu)$ is characterized by the triplet:

$$\tilde{G} = \{\langle v, \sigma_\alpha(v), \sigma_\beta(v), \pi_\sigma(v)\rangle : v \in V\} \cup \{\langle e, \mu_\alpha(e), \mu_\beta(e), \pi_\mu(e)\rangle : e \in E\}.$$

Remark 55. *In a (3, 2)-fuzzy graph, it is evident that for each vertex $v \in V$ and each edge $e \in E$, the following conditions hold:*

$$(\sigma_\alpha(v))^3 + \left(\sigma_\beta(v)\right)^2 + (\pi_\sigma(v))^5 = 1,$$

and

$$(\mu_\alpha(e))^3 + \left(\mu_\beta(e)\right)^2 + \left(\pi_\mu(e)\right)^5 = 1.$$

The degree of indeterminacy $\pi_\sigma(v)$ or $\pi_\mu(e)$ becomes zero when the sum of the membership and non-membership degrees equals 1.

4.3.7 | (2,1)-Fuzzy Graph

The (2,1)-Fuzzy Set is one of the related concepts of the Fuzzy Set, and it has been extensively studied[55]. Although still in the conceptual stage, extending these concepts to graphs would result in the following(cf.[391]).

Definition 56 ((2,1)-Fuzzy Set)**.** [55] Let B be a universal set. A *(2,1)-Fuzzy Set* Ω, defined over B, is represented as:

$$\Omega = \{\langle \nu, \delta_\Omega(\nu), \lambda_\Omega(\nu)\rangle : \nu \in B\},$$

where:

- $\delta_\Omega : B \to [0,1]$ is the membership function,
- $\lambda_\Omega : B \to [0,1]$ is the non-membership function,
- The constraint $0 \le (\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) \le 1$ holds for all $\nu \in B$.

The indeterminacy degree with respect to a (2,1)-FS Ω is defined as:

$$\zeta_\Omega(\nu) = \left(1 - \left((\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu)\right)\right)^{\frac{2}{3}}, \quad \forall\, \nu \in B.$$

It follows that:

$$(\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) + (\zeta_\Omega(\nu))^{\frac{3}{2}} = 1.$$

The indeterminacy degree $\zeta_\Omega(\nu)$ becomes 0 whenever $(\delta_\Omega(\nu))^2 + \lambda_\Omega(\nu) = 1$.

For simplicity, we denote the (2,1)-Fuzzy Set $\Omega = (\delta_\Omega, \lambda_\Omega)$.

Remark 57. *The (2,1)-Fuzzy Set can be viewed as an intermediate concept between Intuitionistic Fuzzy Sets (IFS) and Pythagorean Fuzzy Sets (PFS), where:*

- *Every IFS is a (2,1)-FS.*
- *Every (2,1)-FS is a PFS.*

Definition 58 ((2,1)-Fuzzy Graph)**.** Let $G = (V, E)$ be a classical graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *(2,1)-Fuzzy Graph* $\tilde{G} = (V, \sigma, \mu)$, denoted briefly as (2,1)-FG, is defined as follows:

- The vertex set V is associated with a (2,1)-fuzzy set, where each vertex $v \in V$ has:
  - A membership degree $\delta_\sigma(v) : V \to [0,1]$,
  - A non-membership degree $\lambda_\sigma(v) : V \to [0,1]$,
  - An indeterminacy degree $\zeta_\sigma(v)$ defined as:

    $$\zeta_\sigma(v) = \left(1 - \left((\delta_\sigma(v))^2 + \lambda_\sigma(v)\right)\right)^{\frac{2}{3}}, \quad \forall\, v \in V,$$

    with the condition $0 \le (\delta_\sigma(v))^2 + \lambda_\sigma(v) \le 1$ holding for all $v \in V$.
- The edge set E is also associated with a (2,1)-fuzzy set, where each edge $e = (u, v) \in E$ has:
  - A membership degree $\delta_\mu(e) : E \to [0,1]$,

– A non-membership degree $\lambda_\mu(e) : E \to [0,1]$,

– An indeterminacy degree $\zeta_\mu(e)$ defined as:

$$\zeta_\mu(e) = \left(1 - \left(\left(\delta_\mu(e)\right)^2 + \lambda_\mu(e)\right)\right)^{\frac{2}{3}}, \quad \forall\, e \in E,$$

with the condition $0 \leq \left(\delta_\mu(e)\right)^2 + \lambda_\mu(e) \leq 1$ holding for all $e \in E$.

Thus, a (2,1)-fuzzy graph $\tilde{G} = (V, \sigma, \mu)$ can be represented as:

$$\tilde{G} = \{\langle v, \delta_\sigma(v), \lambda_\sigma(v), \zeta_\sigma(v)\rangle : v \in V\} \cup \left\{\langle e, \delta_\mu(e), \lambda_\mu(e), \zeta_\mu(e)\rangle : e \in E\right\}.$$

Remark 59. *In a (2,1)-Fuzzy Graph, it is evident that for each vertex $v \in V$ and each edge $e \in E$, the following conditions hold:*

$$(\delta_\sigma(v))^2 + \lambda_\sigma(v) + (\zeta_\sigma(v))^{\frac{3}{2}} = 1,$$

and

$$\left(\delta_\mu(e)\right)^2 + \lambda_\mu(e) + \left(\zeta_\mu(e)\right)^{\frac{3}{2}} = 1.$$

The indeterminacy degree becomes zero when the sum of the membership and non-membership degrees equals 1.

The generalized (m, n)-Fuzzy Set [58, 692] and the further generalized (m, a, n)-Fuzzy Neutrosophic Set[693] are defined. Referring to these, we also define the (m, a, b, n)-Fuzzy Turiyam Neutrosophic Set.

Definition 60 ((m, n)-Fuzzy Set)**.** [58, 692] Let m, n be positive real numbers. The *(m, n)-Fuzzy set* (abbreviated as (m, n)-FS) E over the universal set U is defined for each $m, n \geq 1$ as follows:

$$E = \{\langle a, \beta_E(a), \lambda_E(a)\rangle : a \in U\},$$

where $\beta_E, \lambda_E : U \to [0,1]$ are functions that represent the degrees of membership and non-membership, respectively, for every $a \in U$ under the constraint:

$$0 \leq (\beta_E(a))^m + (\lambda_E(a))^n \leq 1.$$

The degree of indeterminacy with respect to an (m, n)-FS E is a function $\alpha_E : U \to [0,1]$ defined by:

$$\alpha_E(a) = (1 - ((\beta_E(a))^m + (\lambda_E(a))^n))^{\frac{1}{mn}}$$

for each $a \in U$.

It follows that:

$$(\beta_E(a))^m + (\lambda_E(a))^n + (\alpha_E(a))^{mn} = 1.$$

Additionally, note that $\alpha_E(a) = 0$ whenever $(\beta_E(a))^m + (\lambda_E(a))^n = 1$.

For simplicity, we denote the (m, n)-FS $E = \{\langle a, \beta_E(a), \lambda_E(a)\rangle : a \in U\}$ by $E = (\beta_E, \lambda_E)$. The family of all (m, n)-FSs defined over U is symbolized by $\mathcal{I}_{(m,n)\text{-FS}}$.

Definition 61 ((m,a,n)-Fuzzy Neutrosophic Set)**.** [693] Let V be a non-empty set. A *(m, a, n)-Fuzzy Neutrosophic Set* (abbreviated as (m, a, n)-FNS) H in V is an object of the form:

$$H = \{\langle v, \Upsilon_H(v), \varpi_H(v), \Omega_H(v)\rangle : v \in V\},$$

where $\Upsilon_H : V \to [0,1]$, $\varpi_H : V \to [0,1]$, and $\Omega_H : V \to [0,1]$ represent the membership, indeterminacy, and non-membership functions, respectively. This structure satisfies the following conditions for all $v \in V$:

$$0 \leq (\Upsilon_H(v))^m + (\Omega_H(v))^n \leq 1, \quad 0 \leq (\varpi_H(v))^a \leq 1,$$

where $m, a, n \in \mathbb{N}$. Here, $\Upsilon_H(v)$ and $\Omega_H(v)$ are dependent components, while $\varpi_H(v)$ is an independent component.

Remark 62. *The (m, a, n)-Fuzzy Neutrosophic Set $H = \{\langle v, \Upsilon_H(v), \varpi_H(v), \Omega_H(v)\rangle : v \in V\}$ satisfies:*

$$0 \leq (\Upsilon_H(v))^m + (\varpi_H(v))^a + (\Omega_H(v))^n \leq 2.$$

For simplicity, an (m, a, n)-FNS over V is denoted by $(\Upsilon_H, \varpi_H, \Omega_H)$.

Definition 63 ((m,a,b,n)-Fuzzy Turiyam Neutrosophic Set)**.** Let V be a non-empty set. A (m, a, b, n)*-Fuzzy Turiyam Neutrosophic Set* (abbreviated as (m, a, b, n)-FTS) T in V is an object of the form:

$$T = \{\langle v, \Upsilon_T(v), \varpi_T(v), \Omega_T(v), \Lambda_T(v)\rangle : v \in V\},$$

where $\Upsilon_T : V \to [0, 1]$, $\varpi_T : V \to [0, 1]$, $\Omega_T : V \to [0, 1]$, and $\Lambda_T : V \to [0, 1]$ represent the truth-membership, indeterminacy, falsity-membership, and liberal-state functions, respectively. This structure satisfies the following conditions for all $v \in V$:

$$0 \le (\Upsilon_T(v))^m + (\Omega_T(v))^n \le 1, \quad 0 \le (\varpi_T(v))^a \le 1, \quad 0 \le (\Lambda_T(v))^b \le 1,$$

where $m, a, b, n \in \mathbb{N}$.

Remark 64. *For simplicity, an (m, a, b, n)-Fuzzy Turiyam Neutrosophic Set $T = \{\langle v, \Upsilon_T(v), \varpi_T(v), \Omega_T(v), \Lambda_T(v)\rangle : v \in V\}$ over V is denoted by $(\Upsilon_T, \varpi_T, \Omega_T, \Lambda_T)$.*

The concepts extended to graph theory are as follows. Moving forward, we aim to examine the relationships between these graphs and other types of graphs.

Definition 65 ((m, n)-Fuzzy Graph)**.** Let $G = (V, E)$ be a simple undirected graph. An *(m, n)-Fuzzy Graph* $\tilde{G} = (V, E, \beta_V, \lambda_V, \beta_E, \lambda_E)$ is defined by assigning to each vertex $v \in V$ membership and non-membership degrees $\beta_V(v), \lambda_V(v) \in [0, 1]$, and to each edge $e \in E$ membership and non-membership degrees $\beta_E(e), \lambda_E(e) \in [0, 1]$, satisfying:

For all $v \in V$:

$$0 \le (\beta_V(v))^m + (\lambda_V(v))^n \le 1,$$
$$\alpha_V(v) = (1 - ((\beta_V(v))^m + (\lambda_V(v))^n))^{\frac{1}{mn}}.$$

Similarly, for all $e \in E$:

$$0 \le (\beta_E(e))^m + (\lambda_E(e))^n \le 1,$$
$$\alpha_E(e) = (1 - ((\beta_E(e))^m + (\lambda_E(e))^n))^{\frac{1}{mn}}.$$

It follows that:

$$(\beta_V(v))^m + (\lambda_V(v))^n + (\alpha_V(v))^{mn} = 1,$$
$$(\beta_E(e))^m + (\lambda_E(e))^n + (\alpha_E(e))^{mn} = 1.$$

Definition 66 ((m, a, n)-Fuzzy Neutrosophic Graph)**.** Let $G = (V, E)$ be a simple undirected graph. An *(m, a, n)-Fuzzy Neutrosophic Graph* $\tilde{G} = (V, E, \beta_V, \alpha_V, \lambda_V, \beta_E, \alpha_E, \lambda_E)$ is defined by assigning to each vertex $v \in V$:

- Membership degree $\beta_V(v) \in [0, 1]$,
- Indeterminacy degree $\alpha_V(v) \in [0, 1]$,
- Non-membership degree $\lambda_V(v) \in [0, 1]$,

and to each edge $e \in E$:

- Membership degree $\beta_E(e) \in [0, 1]$,
- Indeterminacy degree $\alpha_E(e) \in [0, 1]$,
- Non-membership degree $\lambda_E(e) \in [0, 1]$.

These functions satisfy:

For all $v \in V$:

$$0 \le (\beta_V(v))^m + (\lambda_V(v))^n \le 1, \quad 0 \le (\alpha_V(v))^a \le 1.$$

Similarly, for all $e \in E$:

$$0 \le (\beta_E(e))^m + (\lambda_E(e))^n \le 1, \quad 0 \le (\alpha_E(e))^a \le 1.$$

Here, $\beta_V(v)$ and $\lambda_V(v)$ are dependent components, while $\alpha_V(v)$ is an independent component.

It follows that:

$$0 \leq (\beta_V(v))^m + (\alpha_V(v))^a + (\lambda_V(v))^n \leq 2.$$

Definition 67 ((m, a, b, n)-Fuzzy Turiyam Neutrosophic Graph)**.** Let $G = (V, E)$ be a simple undirected graph. An *(m, a, b, n)-Fuzzy Turiyam Neutrosophic Graph* $\tilde{G} = (V, E, \beta_V, \alpha_V, \gamma_V, \lambda_V, \beta_E, \alpha_E, \gamma_E, \lambda_E)$ is defined by assigning to each vertex $v \in V$:

- Truth-membership degree $\beta_V(v) \in [0, 1]$,
- Indeterminacy degree $\alpha_V(v) \in [0, 1]$,
- Falsity-membership degree $\lambda_V(v) \in [0, 1]$,
- Liberal-state degree $\gamma_V(v) \in [0, 1]$,

and to each edge $e \in E$:

- Truth-membership degree $\beta_E(e) \in [0, 1]$,
- Indeterminacy degree $\alpha_E(e) \in [0, 1]$,
- Falsity-membership degree $\lambda_E(e) \in [0, 1]$,
- Liberal-state degree $\gamma_E(e) \in [0, 1]$,

These functions satisfy:

For all $v \in V$:

$$0 \leq (\beta_V(v))^m + (\lambda_V(v))^n \leq 1, \quad 0 \leq (\alpha_V(v))^a \leq 1, \quad 0 \leq (\gamma_V(v))^b \leq 1.$$

Similarly, for all $e \in E$:

$$0 \leq (\beta_E(e))^m + (\lambda_E(e))^n \leq 1, \quad 0 \leq (\alpha_E(e))^a \leq 1, \quad 0 \leq (\gamma_E(e))^b \leq 1.$$

Theorem 68. *An (m, n)-Fuzzy Graph reduces to a standard fuzzy graph when $m = n = 1$.*

Proof: When $m = n = 1$, the conditions for an (m, n)-Fuzzy Graph become:

$$0 \leq \beta_V(v) + \lambda_V(v) \leq 1,$$

$$\alpha_V(v) = (1 - (\beta_V(v) + \lambda_V(v)))\,.$$

Since $\alpha_V(v) \geq 0$, it follows that:

$$\beta_V(v) + \lambda_V(v) \leq 1.$$

This implies that the membership degree $\beta_V(v)$ alone satisfies $0 \leq \beta_V(v) \leq 1$. Thus, the graph reduces to a standard fuzzy graph where only the membership function $\beta_V(v)$ is considered. □

Theorem 69. *Any subgraph of an (m, n)-Fuzzy Graph is also an (m, n)-Fuzzy Graph.*

Proof: Let $\tilde{G} = (V, E, \beta_V, \lambda_V, \beta_E, \lambda_E)$ be an (m, n)-Fuzzy Graph. Consider a subgraph $\tilde{G}' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. Define the functions $\beta'_V, \lambda'_V, \beta'_E, \lambda'_E$ as the restrictions of $\beta_V, \lambda_V, \beta_E, \lambda_E$ to V' and E'. Since the original functions satisfy the conditions of an (m, n)-Fuzzy Graph, their restrictions will also satisfy:

$$0 \leq (\beta'_V(v))^m + (\lambda'_V(v))^n \leq 1, \quad \forall v \in V',$$

and similarly for edges. Therefore, $\tilde{G}'$ is an (m, n)-Fuzzy Graph. □

Theorem 70. *Under the transformation $m \leftrightarrow n$, the complement of an (m, n)-Fuzzy Graph is an (n, m)-Fuzzy Graph.*

Proof: Let $\tilde{G} = (V, E, \beta_V, \lambda_V, \beta_E, \lambda_E)$ be an (m, n)-Fuzzy Graph. Define the complement $\tilde{G}^c = (V, E, \beta_V^c, \lambda_V^c, \beta_E^c, \lambda_E^c)$ by:

$$\beta_V^c(v) = \lambda_V(v), \quad \lambda_V^c(v) = \beta_V(v), \quad \forall v \in V,$$

and similarly for edges. Then,

$$(\beta_V^c(v))^n + (\lambda_V^c(v))^m = (\lambda_V(v))^n + (\beta_V(v))^m \leq 1,$$

since $(\beta_V(v))^m + (\lambda_V(v))^n \leq 1$. Thus, $\tilde{G}^c$ satisfies the conditions of an (n, m)-Fuzzy Graph. □

Theorem 71. *The union of two (m, n)-Fuzzy Graphs is an (m, n)-Fuzzy Graph.*

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two (m, n)-Fuzzy Graphs with membership functions β_V^1, λ_V^1 and β_V^2, λ_V^2, respectively. Define the union $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ by:

$$\beta_V(v) = \max\{\beta_V^1(v), \beta_V^2(v)\}, \quad \lambda_V(v) = \min\{\lambda_V^1(v), \lambda_V^2(v)\}, \quad \forall v \in V.$$

Since $\beta_V^i(v), \lambda_V^i(v) \in [0, 1]$, their maxima and minima also lie in $[0, 1]$. We need to show that:

$$0 \leq (\beta_V(v))^m + (\lambda_V(v))^n \leq 1.$$

Since $\beta_V(v) \geq \beta_V^i(v)$ and $\lambda_V(v) \leq \lambda_V^i(v)$, we have:

$$(\beta_V(v))^m + (\lambda_V(v))^n \leq (\beta_V^i(v))^m + (\lambda_V^i(v))^n \leq 1.$$

Thus, the union $\tilde{G}$ is an (m, n)-Fuzzy Graph. □

Theorem 72. *The intersection of two (m, n)-Fuzzy Graphs is an (m, n)-Fuzzy Graph.*

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two (m, n)-Fuzzy Graphs. Define the intersection $\tilde{G} = \tilde{G}_1 \cap \tilde{G}_2$ by:

$$\beta_V(v) = \min\{\beta_V^1(v), \beta_V^2(v)\}, \quad \lambda_V(v) = \max\{\lambda_V^1(v), \lambda_V^2(v)\}, \quad \forall v \in V.$$

Since $\beta_V(v) \leq \beta_V^i(v)$ and $\lambda_V(v) \geq \lambda_V^i(v)$, we have:

$$(\beta_V(v))^m + (\lambda_V(v))^n \leq (\beta_V^i(v))^m + (\lambda_V^i(v))^n \leq 1.$$

Therefore, the intersection $\tilde{G}$ is an (m, n)-Fuzzy Graph. □

Theorem 73. *An (m, a, n)-Fuzzy Neutrosophic Graph reduces to an (m, n)-Fuzzy Graph when the indeterminacy degree $\alpha_V(v) = 0$ for all $v \in V$.*

Proof: If $\alpha_V(v) = 0$ for all $v \in V$, the conditions of an (m, a, n)-Fuzzy Neutrosophic Graph reduce to:

$$0 \leq (\beta_V(v))^m + (\lambda_V(v))^n \leq 1,$$

which are exactly the conditions for an (m, n)-Fuzzy Graph. Therefore, the graph reduces to an (m, n)-Fuzzy Graph. □

Theorem 74. *An (m, a, b, n)-Fuzzy Turiyam Neutrosophic Graph reduces to an (m, a, n)-Fuzzy Neutrosophic Graph when the liberal-state degree $\gamma_V(v) = 0$ for all $v \in V$.*

Proof: If $\gamma_V(v) = 0$ for all $v \in V$, the conditions of an (m, a, b, n)-Fuzzy Turiyam Neutrosophic Graph reduce to those of an (m, a, n)-Fuzzy Neutrosophic Graph:

$$0 \leq (\beta_V(v))^m + (\lambda_V(v))^n \leq 1, \quad 0 \leq (\alpha_V(v))^a \leq 1.$$

Thus, the graph reduces to an (m, a, n)-Fuzzy Neutrosophic Graph. □

Theorem 75. *The complement of an (m, a, n)-Fuzzy Neutrosophic Graph is also an (m, a, n)-Fuzzy Neutrosophic Graph.*

Proof: Let $\tilde{G} = (V, E, \beta_V, \alpha_V, \lambda_V, \beta_E, \alpha_E, \lambda_E)$ be an (m, a, n)-Fuzzy Neutrosophic Graph. Define the complement $\tilde{G}^c$ by:

$$\beta_V^c(v) = \lambda_V(v), \quad \lambda_V^c(v) = \beta_V(v), \quad \alpha_V^c(v) = \alpha_V(v), \quad \forall v \in V.$$

Since $\beta_V(v), \lambda_V(v) \in [0, 1]$, their roles are swapped in the complement. The indeterminacy degree $\alpha_V(v)$ remains unchanged. The conditions for $\tilde{G}^c$ become:

$$0 \leq (\beta_V^c(v))^m + (\lambda_V^c(v))^n = (\lambda_V(v))^m + (\beta_V(v))^n \leq 1,$$

which holds because $(\beta_V(v))^n + (\lambda_V(v))^m \leq 1$. Therefore, $\tilde{G}^c$ is an (m, a, n)-Fuzzy Neutrosophic Graph. □

Theorem 76. *The union of two (m, a, n)-Fuzzy Neutrosophic Graphs is an (m, a, n)-Fuzzy Neutrosophic Graph.*

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two (m, a, n)-Fuzzy Neutrosophic Graphs. Define the union $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ by:

$$\beta_V(v) = \max\{\beta_V^1(v), \beta_V^2(v)\}, \quad \lambda_V(v) = \min\{\lambda_V^1(v), \lambda_V^2(v)\}, \quad \alpha_V(v) = \max\{\alpha_V^1(v), \alpha_V^2(v)\}, \quad \forall v \in V.$$

Since $\beta_V(v), \lambda_V(v), \alpha_V(v) \in [0, 1]$, and the maxima and minima preserve these bounds, the conditions of an (m, a, n)-Fuzzy Neutrosophic Graph are satisfied for $\tilde{G}$. □

4.3.8 | Omega-Soft Graph

The Ω-Soft Set is a set studied using concepts from Pythagorean Fuzzy Sets and Soft Sets [406, 408, 407]. We aim to extend these ideas to graphs and examine their mathematical structure. Although still in the conceptual stage, we define this extension as follows.

Definition 77 (Ω-Soft Set)**.** [406] Let E be a Pythagorean Fuzzy Set (PFS) over a parameter set P. The function $F_E(f)$ is called a *Pythagorean fuzzy parametrized Ω-soft set* if F_E is a Pythagorean fuzzy soft set (PFSS) over the universal set U and $f : E \to L$ is a PFS on E.

In this definition, $F_E(f)$ can be represented as:

$$F_E(f) = \{(x, m_F(x), n_F(x), f_F(x)) : x \in P,\ f_F(x) \in L,\ m_F(x) \in [0, 1],\ n_F(x) \in [0, 1]\}.$$

Here, the functions m_F and n_F are called the *membership function* and *non-membership function* of the Ω-soft set, respectively. The values $m_F(x)$ and $n_F(x)$ represent the degree of importance and unimportance of the parameter x. The elements of the parameter f_F are denoted by (m_f, n_f).

The set of all Ω-soft sets on U is denoted by $\Omega(U)$.

Definition 78 (Ω-Soft Graph)**.** Let $G = (V, E)$ be a simple undirected graph, where V is the set of vertices and E is the set of edges. An *Ω-Soft Graph* $\tilde{G} = (V, E, m_V, n_V, m_E, n_E, f_V, f_E)$ is defined by assigning to each vertex $v \in V$ and each edge $e \in E$:

- A membership degree $m_V(v) \in [0, 1]$ and non-membership degree $n_V(v) \in [0, 1]$ for each vertex $v \in V$,
- A membership degree $m_E(e) \in [0, 1]$ and non-membership degree $n_E(e) \in [0, 1]$ for each edge $e \in E$,
- A function $f_V : V \to L$ assigning each vertex a parameter $f_V(v) \in L$,
- A function $f_E : E \to L$ assigning each edge a parameter $f_E(e) \in L$,

where L is a set of labels or attributes associated with vertices and edges. The functions m_V and n_V are called the *membership function* and *non-membership function* for the vertices of the Ω-soft graph, while m_E and n_E are the *membership function* and *non-membership function* for the edges of the Ω-soft graph.

The conditions for an Ω-soft graph are given by:

$$0 \leq m_V(v) + n_V(v) \leq 1, \quad 0 \leq m_E(e) + n_E(e) \leq 1$$

for all $v \in V$ and $e \in E$. The set of all Ω-soft graphs defined over a graph G is denoted by $\Omega(G)$.

4.3.9 | Fuzzy Quadrigeminal Graph

Consider the graph version of a Fuzzy Quadrigeminal Set. The concept of a Fuzzy Quadrigeminal Set[61] is highly similar to concepts such as the Turiyam Neutrosophic Set[118, 640] and the Ambiguous Set (Ambiguous Graph)[645, 279, 645]. Moreover, it can be generalized using the Single-Valued Pentapartitioned Neutrosophic Set.

Definition 79. Let X be a non-empty set of objects. A *Fuzzy Quadrigeminal Set* (abbreviated as FQS) Q in X is defined as:

$$Q = \left\{ \langle q, \Upsilon_Q(q), \Omega_Q(q), \mathbb{V}_Q(q), \partial_Q(q) \rangle : q \in X \right\},$$

where each element $q \in X$ is associated with four degree values:

- $\Upsilon_Q : X \to [0,1]$ is called the *degree of extreme-belongingness* to Q,
- $\Omega_Q : X \to [0,1]$ is called the *degree of very-belongingness* to Q,
- $\mathbb{V}_Q : X \to [0,1]$ is called the *degree of moderate-belongingness* to Q,
- $\partial_Q : X \to [0,1]$ is called the *degree of weak-belongingness* to Q.

These degree values represent the membership of q in Q under four distinct levels of belongingness. For each element $q \in X$, the sum of the four membership degrees satisfies the following condition:

$$\Upsilon_Q(q) + \Omega_Q(q) + \mathbb{V}_Q(q) + \partial_Q(q) \leq 1.$$

Definition 80 (Fuzzy Quadrigeminal Graph)**.** Let $G = (V, E)$ be a simple undirected graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Fuzzy Quadrigeminal Graph* $\tilde{G} = (V, E, \Upsilon, \Omega, \mathbb{V}, \partial)$ is defined by assigning to each vertex $v \in V$ and each edge $e \in E$ four membership degrees representing varying levels of belongingness:

- $\Upsilon_V : V \to [0,1]$, the *degree of extreme-belongingness* for vertices,
- $\Omega_V : V \to [0,1]$, the *degree of very-belongingness* for vertices,
- $\mathbb{V}_V : V \to [0,1]$, the *degree of moderate-belongingness* for vertices,
- $\partial_V : V \to [0,1]$, the *degree of weak-belongingness* for vertices.

Similarly, each edge $e \in E$ is assigned the following membership functions:

- $\Upsilon_E : E \to [0,1]$, the *degree of extreme-belongingness* for edges,
- $\Omega_E : E \to [0,1]$, the *degree of very-belongingness* for edges,
- $\mathbb{V}_E : E \to [0,1]$, the *degree of moderate-belongingness* for edges,
- $\partial_E : E \to [0,1]$, the *degree of weak-belongingness* for edges.

For all $v \in V$ and $e \in E$, the membership degrees satisfy the following conditions:

$$\Upsilon_V(v) + \Omega_V(v) + \mathbb{V}_V(v) + \partial_V(v) \leq 1,$$
$$\Upsilon_E(e) + \Omega_E(e) + \mathbb{V}_E(e) + \partial_E(e) \leq 1.$$

These conditions ensure that the sum of degrees of belongingness does not exceed unity, allowing the Fuzzy Quadrigeminal Graph $\tilde{G}$ to represent different levels of uncertainty and belongingness across vertices and edges.

Theorem 81. *A Turiyam Neutrosophic Graph and a Single-Valued Pentapartitioned Neutrosophic Graph can both be transformed into a Fuzzy Quadrigeminal Graph.*

Proof: To show that both a Turiyam Neutrosophic Graph and a Single-Valued Pentapartitioned Neutrosophic Graph (SVPN Graph) can be transformed into a Fuzzy Quadrigeminal Graph, we will construct mappings from the membership degrees of each graph type to the four membership degrees of a Fuzzy Quadrigeminal Graph.

A Turiyam Neutrosophic Graph assigns each vertex $v \in V$ a quadruple:

$$\sigma(v) = (t(v), iv(v), fv(v), lv(v)),$$

such that $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.

To transform this into a Fuzzy Quadrigeminal Graph $\tilde{G}$, we define four degrees for each vertex v in V:

$$\Upsilon(v) = t(v), \quad \Omega(v) = iv(v), \quad \mathbb{V}(v) = fv(v), \quad \partial(v) = lv(v).$$

These four values now correspond to the degrees of extreme, very, moderate, and weak belongingness in the Fuzzy Quadrigeminal Graph, satisfying:

$$\Upsilon(v) + \Omega(v) + \mathbb{V}(v) + \partial(v) = t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$

Thus, the transformation preserves the structure of the Fuzzy Quadrigeminal Graph.

In an SVPN Graph, each vertex $v \in V$ is assigned a quintuple:

$$\sigma(v) = (T(v), C(v), R(v), U(v), F(v)),$$

with the constraint $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.

To map this to a Fuzzy Quadrigeminal Graph, we can combine some membership degrees, as follows:

$$\Upsilon(v) = T(v), \quad \Omega(v) = C(v) + R(v), \quad \mathbb{V}(v) = U(v), \quad \partial(v) = F(v).$$

Then, the total sum for each vertex satisfies:

$$\Upsilon(v) + \Omega(v) + \mathbb{V}(v) + \partial(v) = T(v) + (C(v) + R(v)) + U(v) + F(v) \leq 5.$$

By adjusting the upper bound to ensure that the sum does not exceed 1, we normalize the degrees if necessary, allowing an SVPN Graph to be represented within the Fuzzy Quadrigeminal structure. □

4.3.10 | Support-Intuitionistic Fuzzy Graph

A Support-Intuitionistic Fuzzy Set extends Intuitionistic Fuzzy Sets by adding a "support-membership" function, enabling richer representation of membership uncertainties (cf.[515, 102]). A Support-Neutrosophic Set, a generalization of the Support-Intuitionistic Fuzzy Set, is also recognized[694]. I intend to extend these concepts to graph theory and explore their mathematical characteristics(cf.[534]).

Definition 82 (Intuitionistic Fuzzy Set)**.** [772, 414] An intuitionistic fuzzy set (IF set) A on the universe X is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is the *degree of membership* of x in A and $\nu_A(x) \in [0,1]$ is the *degree of non-membership* of x in A, with the condition that:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Definition 83 (Support-Intuitionistic Fuzzy Set)**.** A support-intuitionistic fuzzy set (SIF set) A on the universe X is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x), \sigma_A(x)\rangle : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is the *degree of membership* of x in A, $\nu_A(x) \in [0,1]$ is the *degree of non-membership* of x in A, and $\sigma_A(x) \in [0,1]$ is the *degree of support-membership* of x in A, with the condition that:

$$0 \leq \mu_A(x) + \nu_A(x) + \sigma_A(x) \leq 1.$$

Remark 84.

- *An element $x \in X$ is called the worst element in A if:*

$$\mu_A(x) = 0, \quad \nu_A(x) = 1, \quad \sigma_A(x) = 0.$$

- *An element $x \in X$ is called the best element in A if:*

$$\mu_A(x) = 1, \quad \nu_A(x) = 0, \quad \sigma_A(x) = 0.$$

- *A support-intuitionistic fuzzy set reduces to an intuitionistic fuzzy set when $\sigma_A(x) = 0$ for all $x \in X$.*
- *A support-intuitionistic fuzzy set reduces to a fuzzy set when $\nu_A(x) = 0$ and $\sigma_A(x) = 0$ for all $x \in X$.*

Definition 85 (Neutrosophic Set)**.** [656, 655, 657, 718] A neutrosophic set A on the universe U is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\},$$

where $T_A : U \to [0, 1]$ is the *truth-membership function*, $I_A : U \to [0, 1]$ is the *indeterminacy-membership function*, and $F_A : U \to [0, 1]$ is the *falsity-membership function*.

Definition 86 (Support-Neutrosophic Set)**.** A support-neutrosophic set (SNS) A on the universe U is characterized by four functions: the truth-membership function T_A, the indeterminacy-membership function I_A, the falsity-membership function F_A, and an additional support-membership function s_A, defined as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x), s_A(x)\rangle : x \in U\},$$

where $T_A(x), I_A(x), F_A(x), s_A(x) \in [0, 1]$ for each $x \in U$. In general, there is no restriction on the sum of $T_A(x), I_A(x)$, and $F_A(x)$, so:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3,$$

and $0 \leq s_A(x) \leq 1$.

Remark 87.

- *An element $x \in U$ is called the worst element in A if:*

$$T_A(x) = 0, \quad I_A(x) = 0, \quad F_A(x) = 1, \quad s_A(x) = 0.$$

- *An element $x \in U$ is called the best element in A if:*

$$T_A(x) = 1, \quad I_A(x) = 0, \quad F_A(x) = 0, \quad s_A(x) = 1.$$

- *A support-neutrosophic set A reduces to a neutrosophic set if $s_A(x) = c \in [0, 1]$ for all $x \in U$.*
- *A support-neutrosophic set A is called a support-standard neutrosophic set if:*

$$T_A(x) + I_A(x) + F_A(x) \leq 1,$$

for all $x \in U$.

Definition 88 (Support-Intuitionistic Fuzzy Graph)**.** Let $G = (V, E)$ be a simple undirected graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

A *Support-Intuitionistic Fuzzy Graph* (SIFG) $\tilde{G} = (V, E, \mu_V, \nu_V, \sigma_V, \mu_E, \nu_E, \sigma_E)$ is defined by associating with each vertex $v \in V$:

- $\mu_V(v) \in [0, 1]$: the degree of membership of v in $\tilde{G}$,
- $\nu_V(v) \in [0, 1]$: the degree of non-membership of v in $\tilde{G}$,
- $\sigma_V(v) \in [0, 1]$: the degree of support-membership of v in $\tilde{G}$,

satisfying the condition:

$$0 \leq \mu_V(v) + \nu_V(v) + \sigma_V(v) \leq 1.$$

Similarly, for each edge $e \in E$:

- $\mu_E(e) \in [0, 1]$: the degree of membership of e in $\tilde{G}$,
- $\nu_E(e) \in [0, 1]$: the degree of non-membership of e in $\tilde{G}$,
- $\sigma_E(e) \in [0, 1]$: the degree of support-membership of e in $\tilde{G}$,

satisfying the condition:

$$0 \leq \mu_E(e) + \nu_E(e) + \sigma_E(e) \leq 1.$$

Definition 89 (Support-Neutrosophic Graph)**.** Let $G = (V, E)$ be a simple undirected graph.

A *Support-Neutrosophic Graph* (SNG) $\tilde{G} = (V, E, T_V, I_V, F_V, s_V, T_E, I_E, F_E, s_E)$ is defined by associating with each vertex $v \in V$:

- $T_V(v) \in [0, 1]$: the truth-membership degree of v,
- $I_V(v) \in [0, 1]$: the indeterminacy-membership degree of v,
- $F_V(v) \in [0, 1]$: the falsity-membership degree of v,
- $s_V(v) \in [0, 1]$: the support-membership degree of v.

Similarly, for each edge $e \in E$:

- $T_E(e) \in [0, 1]$: the truth-membership degree of e,
- $I_E(e) \in [0, 1]$: the indeterminacy-membership degree of e,
- $F_E(e) \in [0, 1]$: the falsity-membership degree of e,
- $s_E(e) \in [0, 1]$: the support-membership degree of e.

In general, there is no restriction on the sum $T_V(v) + I_V(v) + F_V(v)$, so:

$$0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3,$$

and $0 \leq s_V(v) \leq 1$.

Theorem 90 (Complement of a Support-Intuitionistic Fuzzy Graph)**.** *Let $\tilde{G} = (V, E, \mu_V, \nu_V, \sigma_V, \mu_E, \nu_E, \sigma_E)$ be a Support-Intuitionistic Fuzzy Graph. The complement graph $\tilde{G}^c = (V, E, \mu_V^c, \nu_V^c, \sigma_V^c, \mu_E^c, \nu_E^c, \sigma_E^c)$ is defined by:*

For all $v \in V$:

$$\mu_V^c(v) = \nu_V(v), \quad \nu_V^c(v) = \mu_V(v), \quad \sigma_V^c(v) = 1 - \mu_V(v) - \nu_V(v) - \sigma_V(v).$$

For all $e \in E$:

$$\mu_E^c(e) = \nu_E(e), \quad \nu_E^c(e) = \mu_E(e), \quad \sigma_E^c(e) = 1 - \mu_E(e) - \nu_E(e) - \sigma_E(e).$$

Then $\tilde{G}^c$ is also a Support-Intuitionistic Fuzzy Graph.

Proof: We need to show that for all $v \in V$:

(1) $\mu_V^c(v) \in [0, 1]$,

(2) $\nu_V^c(v) \in [0, 1]$,

(3) $\sigma_V^c(v) \in [0, 1]$,

(4) $0 \leq \mu_V^c(v) + \nu_V^c(v) + \sigma_V^c(v) \leq 1$.

Given that $\mu_V(v), \nu_V(v), \sigma_V(v) \in [0, 1]$ and $0 \leq \mu_V(v) + \nu_V(v) + \sigma_V(v) \leq 1$, we have that $\tilde{G}^c$ satisfies the conditions for a Support-Intuitionistic Fuzzy Graph. Similarly, we can show that for all $e \in E$, the conditions hold, thus proving the theorem. □

Theorem 91 (Union of Two Support-Intuitionistic Fuzzy Graphs)**.** *Let $\tilde{G}_1 = (V, E, \mu_V^1, \nu_V^1, \sigma_V^1, \mu_E^1, \nu_E^1, \sigma_E^1)$ and $\tilde{G}_2 = (V, E, \mu_V^2, \nu_V^2, \sigma_V^2, \mu_E^2, \nu_E^2, \sigma_E^2)$ be two Support-Intuitionistic Fuzzy Graphs. Define the union graph $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ by:*

For all $v \in V$:

$$\mu_V(v) = \max\{\mu_V^1(v), \mu_V^2(v)\}, \quad \nu_V(v) = \min\{\nu_V^1(v), \nu_V^2(v)\}, \quad \sigma_V(v) = \max\{\sigma_V^1(v), \sigma_V^2(v)\}.$$

For all $e \in E$:

$$\mu_E(e) = \max\{\mu_E^1(e), \mu_E^2(e)\}, \quad \nu_E(e) = \min\{\nu_E^1(e), \nu_E^2(e)\}, \quad \sigma_E(e) = \max\{\sigma_E^1(e), \sigma_E^2(e)\}.$$

Then $\tilde{G}$ is a Support-Intuitionistic Fuzzy Graph.

Proof: Since the maximal and minimal of values in $[0, 1]$ remain within $[0, 1]$, and since the sum of membership values is bounded as required, $\tilde{G}$ satisfies the conditions for a Support-Intuitionistic Fuzzy Graph. □

Theorem 92 (Complement of a Support-Neutrosophic Graph)**.** *Let $\tilde{G} = (V, E, T_V, I_V, F_V, s_V, T_E, I_E, F_E, s_E)$ be a Support-Neutrosophic Graph. The complement graph $\tilde{G}^c = (V, E, T_V^c, I_V^c, F_V^c, s_V^c, T_E^c, I_E^c, F_E^c, s_E^c)$ is defined by:*

For all $v \in V$:

$$T_V^c(v) = F_V(v), \quad I_V^c(v) = I_V(v), \quad F_V^c(v) = T_V(v), \quad s_V^c(v) = 1 - s_V(v).$$

Similarly for all $e \in E$.

Then $\tilde{G}^c$ is also a Support-Neutrosophic Graph.

Proof: Since $T_V(v), F_V(v), s_V(v) \in [0, 1]$ and the sum $T_V(v) + I_V(v) + F_V(v)$ has no further restrictions, $\tilde{G}^c$ is verified to satisfy the conditions for a Support-Neutrosophic Graph. □

Theorem 93. *Every Support-Intuitionistic Fuzzy Graph reduces to an Intuitionistic Fuzzy Graph when the support-membership functions are zero.*

Proof: Let $\tilde{G} = (V, E, \mu_V, \nu_V, \sigma_V, \mu_E, \nu_E, \sigma_E)$ be a Support-Intuitionistic Fuzzy Graph. If $\sigma_V(v) = 0$ and $\sigma_E(e) = 0$ for all $v \in V$ and $e \in E$, then the conditions for each vertex $v \in V$ and each edge $e \in E$ reduce to:

$$0 \leq \mu_V(v) + \nu_V(v) \leq 1, \quad 0 \leq \mu_E(e) + \nu_E(e) \leq 1,$$

which are precisely the conditions defining an Intuitionistic Fuzzy Graph. Thus, under these conditions, the Support-Intuitionistic Fuzzy Graph $\tilde{G}$ is equivalent to an Intuitionistic Fuzzy Graph. □

Theorem 94. *If in a Support-Neutrosophic Graph, the support-membership functions are constants, the graph reduces to a Neutrosophic Graph.*

Proof: Let $\tilde{G} = (V, E, T_V, I_V, F_V, s_V, T_E, I_E, F_E, s_E)$ be a Support-Neutrosophic Graph. If $s_V(v) = c \in [0, 1]$ and $s_E(e) = c$ for all $v \in V$ and $e \in E$, then the support-membership functions s_V and s_E are constants and do not vary with v or e. This means that the support-membership functions do not influence the variability of the graph, effectively making the graph equivalent to a standard Neutrosophic Graph, where each element is only described by T, I, and F functions. □

4.3.11 | p, q, r-spherical fuzzy Graph

A p, q, r-spherical fuzzy set is a type of fuzzy set defined by three membership degrees (positive, neutral, negative) with flexibility through parameters p and q. These parameters allow varied conditions for uncertainty representation[580, 579]. I would like to consider these as graph concepts and examine future concepts based on them.

Definition 95. [580] Let S be a non-empty finite set. A p, q, r*-spherical fuzzy set* (p, q, r-SFS) over an element $s \in S$ is defined as follows:

$$S = \{ (s, (\zeta_S(s), \eta_S(s), \xi_S(s))) \mid s \in S\}$$

where $\zeta_S(s), \eta_S(s), \xi_S(s) : S \to [0, 1]$ represent the positive, neutral, and negative membership degrees, respectively, of an element $s \in S$. These values satisfy the following conditions:

- $0 \leq \zeta_S(s), \eta_S(s), \xi_S(s) \leq 1$

- $\zeta_S(s)^p + \eta_S(s)^r + \xi_S(s)^q \leq 1$

where p and q are positive integers, and r is the least common multiple (LCM) of p and q, i.e., $r = \text{LCM}(p, q)$. The degree of negation (uncertainty) is given by

$$\tau(s) = 1 - \zeta_S(s)^p - \eta_S(s)^r - \xi_S(s)^q$$

A triplet (ζ, η, ξ) that satisfies the above condition is called a *p, q, r-spherical fuzzy number* (p, q-SFN).

Definition 96. Let $G = (V, E)$ be a graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *p, q, r-spherical fuzzy graph* is defined by assigning a p, q, r-spherical fuzzy number to each vertex $v \in V$ and each edge $e \in E$, denoted as $S_v = (\zeta_v, \eta_v, \xi_v)$ and $S_e = (\zeta_e, \eta_e, \xi_e)$, respectively. These assignments satisfy the following conditions:

(1) For all $v \in V$ and $e \in E$,
$$0 \leq \zeta_v, \eta_v, \xi_v, \zeta_e, \eta_e, \xi_e \leq 1$$

(2) The membership degrees satisfy
$$\zeta_v^p + \eta_v^r + \xi_v^q \leq 1, \quad \zeta_e^p + \eta_e^r + \xi_e^q \leq 1$$
where $r = \text{LCM}(p, q)$.

The *degree of hesitation* (or non-membership) for a vertex v and an edge e are given by:

$$\tau_v = 1 - \zeta_v^p - \eta_v^r - \xi_v^q, \quad \tau_e = 1 - \zeta_e^p - \eta_e^r - \xi_e^q$$

The p, q, r-spherical fuzzy graph structure provides flexibility in modeling complex systems by adjusting the parameters p and q to suit specific applications and requirements.

Note: p and q are positive integers that can satisfy:

- $p = q$, $p < q$, or $p > q$
- $r = \text{LCM}(p, q)$ is used to ensure consistency in the calculation of neutral membership degrees.

By defining p, q, r-spherical fuzzy graphs in this way, we extend traditional fuzzy graph concepts to accommodate a wider range of uncertainty and vagueness, useful in advanced decision-making and modeling scenarios.

4.3.12 | Controlled Graph

The Controlled Set is one of the related concepts of the Fuzzy Set, and it has been extensively studied [196, 197]. Although still in the conceptual stage, extending these concepts to graphs would result in the following.

Definition 97 (α-Set)**.** [196] Let E be a universe, and let $\alpha : E \to [0, 1]$ be a function. We call E an *α-set*.

Definition 98 (α-Controlled Set)**.** Let E be an *α-set*. The set E is called an *α-controlled set* if, for every element $x \in E$, there exists an element $y \in E$ such that:
$$1 - \alpha(x) = \alpha(y).$$
The family of α-controlled sets on a universe E is denoted by $E \in \text{CS}(\alpha)$.

Definition 99 (Control Set)**.** [196] Let $E \in \text{CS}(\alpha)$ and $a \in E$. The *control set* of a, denoted by a, is defined as:
$$a = \{b \in E \mid 1 - \alpha(a) = \alpha(b)\}.$$

Definition 100 ((α, α^*)-Mapping)**.** [196] Let E be an *α-set*. We define a mapping $\alpha^* : E \to [0, 1]$ as follows:

$$\alpha^*(x) = \begin{cases} 1 - \alpha(x), & \text{if } x \in E_\alpha, \\ \sup_y \alpha(y), & \text{if } y \in E \text{ and } 3\alpha(x) < 1 - \alpha(y), \\ 0, & \text{otherwise,} \end{cases}$$

where $E_\alpha = \bigcup_{a \in E} a$.

Definition 101 ((α, α^*)-Controlled Set)**.** [196] Let E be an *α-set*. The set $A = \{\langle x, \alpha(x), \alpha^*(x)\rangle \mid x \in E\}$ is called an *(α, α^*)-controlled set*.

Remark 102. *Every (α, α^*)-controlled set is an intuitionistic fuzzy set, but the converse is not necessarily true.*

Definition 103 (α-Controlled Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. Let $\alpha : V \to [0, 1]$ be a function that assigns a membership degree to each vertex in V. The graph G is called an *α-controlled graph* if it satisfies the following conditions:

(1) *α-Controlled Vertices:* For each vertex $v \in V$, there exists a vertex $u \in V$ such that:

$$1 - \alpha(v) = \alpha(u).$$

(2) *α-Controlled Edges:* For each edge $e = (v_1, v_2) \in E$, there exists an edge $e' = (u_1, u_2) \in E$ such that:

$$1 - \min\{\alpha(v_1), \alpha(v_2)\} = \max\{\alpha(u_1), \alpha(u_2)\}.$$

The set

$$G = \{\langle v, \alpha(v)\rangle \mid v \in V\} \cup \{\langle e, \alpha(e)\rangle \mid e \in E\}$$

is called the *α-controlled graph*.

Definition 104 (Controlled Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. The graph G is called a *controlled graph* if there exists a function $\alpha : V \to [0, 1]$ such that G becomes an α-controlled graph as defined above.

In other words, a controlled graph is characterized by the existence of a function α that assigns membership degrees to the vertices of V, ensuring that for each vertex $v \in V$, there is a corresponding vertex $u \in V$ with:

$$1 - \alpha(v) = \alpha(u),$$

and for each edge $e = (v_1, v_2) \in E$, there is a corresponding edge $e' = (u_1, u_2) \in E$ with:

$$1 - \min\{\alpha(v_1), \alpha(v_2)\} = \max\{\alpha(u_1), \alpha(u_2)\}.$$

Definition 105 ((α, α^*)-Controlled Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. Let $\alpha : V \to [0, 1]$ be a function that assigns a membership degree to each vertex in V. The graph G is called an *(α, α^*)-controlled graph* if it satisfies the following conditions:

(1) *(α, α^*)-Controlled Vertices:* For each vertex $v \in V$, there exists a vertex $u \in V$ such that:

$$1 - \alpha(v) = \alpha(u).$$

Define $\alpha^* : V \to [0, 1]$ as:

$$\alpha^*(v) = \begin{cases} 1 - \alpha(v), & \text{if } v \in V_\alpha, \\ \sup_{u \in V} \alpha(u), & \text{if } 3\alpha(v) < 1 - \alpha(u) \text{ for some } u \in V, \\ 0, & \text{otherwise,} \end{cases}$$

where $V_\alpha = \bigcup_{v \in V} \{v\}$.

(2) *(α, α^*)-Controlled Edges:* For each edge $e = (v_1, v_2) \in E$, there exists an edge $e' = (u_1, u_2) \in E$ such that:

$$1 - \min\{\alpha(v_1), \alpha(v_2)\} = \max\{\alpha(u_1), \alpha(u_2)\}.$$

Define the edge mapping $\alpha^* : E \to [0, 1]$ as:

$$\alpha^*(e) = \begin{cases} 1 - \min\{\alpha(v_1), \alpha(v_2)\}, & \text{if } e \in E_\alpha, \\ \sup_{e' \in E} \alpha(e'), & \text{if } 3\min\{\alpha(v_1), \alpha(v_2)\} < 1 - \max\{\alpha(u_1), \alpha(u_2)\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $E_\alpha = \bigcup_{e \in E} \{e\}$.

The set

$$G = \{\langle v, \alpha(v), \alpha^*(v)\rangle \mid v \in V\} \cup \{\langle e, \alpha(e), \alpha^*(e)\rangle \mid e \in E\}$$

is called the *(α, α^*)-controlled graph.*

4.3.13 | Extending Other Sets to Graph Theory

In set theory, many uncertain sets and related concepts are known (ex.[775, 381, 494, 779, 776]). In the future, we plan to explore the mathematical characteristics of these extended graph concepts. For example, we would like to consider the following concepts. Furthermore, we plan to explore applications in areas such as Decision Making(cf.[377, 139]) and Neural Networks(cf.[340, 689]).

- Extend Fuzzy Superior Mandelbrot Set [451] to graph theory
- Extend Fuzzy Mandelbric Set[556] to graph theory
- Extend Time-sequential hesitant fuzzy set [479, 480] to graph theory
- Extend Eigen Fuzzy Set [620, 517] to graph theory
- Extend Eigen spherical fuzzy set [364] to graph theory
- Extend Fuzzy Julia Set[498] to graph theory
- Extend Affine Fuzzy Set[188, 621] to graph theory
- Extend Fuzzy Open Sets[520, 133] and Related Concepts [204, 741] to graph theory.
- Extend Doubt Fuzzy Set[605, 213] to graph theory
- Extend (a, b)-Fuzzy soft sets [56] to graph theory
- Extend Quaternion Set[642] to graph theory
- Extend Spectral Fuzzy Set[529] to graph theory
- Extend Decomposed Fuzzy Set [171, 172, 382] to graph theory
- Extend Genuine Sets [331, 223] to Genuine Graph
- Extend Tolerance Rough Fuzzy Sets [784, 96] to Tolerance Rough Fuzzy Graph
- Extend Twofold fuzzy sets [473, 239] to graph theory
- Extend Hybrid Fuzzy Sets [176, 545] to Hybrid Fuzzy Graph
- Extend Linguistic intuitionistic fuzzy sets [795, 521] to graph theory.
- Extend Level Fuzzy Sets [573, 438] to Level Fuzzy Graph
- Extend the Bell-Shaped Fuzzy Set [184, 186, 185] to Bell-Shaped Fuzzy Graph
- Extend the Hyperbolic Fuzzy Set [242, 234, 243] to Hyperbolic Fuzzy Graph
- Extend Power Root Fuzzy Set [57, 606, 361] to graph theory
- Extend the Probabilistic Fuzzy Set [325, 174] to Probabilistic Fuzzy Graph
- Extend Conditional Fuzzy Set [722] to graph theory
- Extend the Hexagonal Fuzzy Set [635, 492] to Hexagonal Fuzzy Graph
- Extend the Sigmoid Fuzzy Set [231] to Sigmoid Fuzzy Graph
- Extend the Convex Fuzzy Set [444, 417] to Convex Fuzzy Graph
- Extend Fuzzy connected sets [180, 701] to graph theory
- Extend discrete fuzzy sets [127, 390] to graph theory.
- Extend Atanassov Intuitionistic Fuzzy Sets [129, 298] to graph theory

- Extend the Gray Fuzzy Set [683, 348] to Gray Fuzzy Graph
- Extend the Granular Fuzzy Set [761, 435] to Granular Fuzzy Graph
- Extend the Continuous Fuzzy Set [769, 425, 583] to Continuous Fuzzy Graph
- Extend Symmetric Fuzzy Set [570, 120] to graph theory
- Extend Shadowed Fuzzy Set [544, 170] to graph theory
- Extend Stochastic Fuzzy Set [303] to graph theory
- Extend Fuzzy Power Set [122, 737] to Fuzzy Power Graph
- Extend Hyperfuzzy Sets [379, 510] to graph theory
- Extend Hesitant Bifuzzy Set[179, 328, 327] to graph theory.
- Extend Boolean fuzzy sets[235, 467] to graph theory.
- Extend Paraconsistent Set [266, 731] to Paraconsistent graph[279].

Additionally, we plan to explore mathematical properties and applications by combining the above concepts with Rough Sets (Rough Graphs) [537, 542, 541], Soft Sets (Soft Graphs) [460, 74, 63], Thick set[238, 225], quasi set[411, 161], and Soft Expert Sets (Soft Expert Graphs)[87, 88] as needed.

5 | Appendix: Various Uncertain Sets

In the realm of concepts that address uncertainty, Uncertain Sets, akin to Uncertain Graph Theory, are actively researched by numerous scholars. This section provides an overview of various types of Uncertain Sets. Due to their wide-ranging applications and mathematical properties, numerous sets are continually proposed and analyzed in various academic papers and by researchers. This appendix presents the findings from an investigation into these concepts, intended to support future advancements in the field of Uncertain Theory by researchers.

First, the definition of a Fuzzy Set is provided below [775]. Various extended concepts have been proposed based on the following Fuzzy Set.

Definition 106. [775] Let U be a universe of discourse. A *Fuzzy Set* F in U is defined as:

$$F = \{(x, \mu_F(x)) : x \in U\},$$

where:

- $\mu_F(x) : U \to [0, 1]$ is the membership function, representing the degree of membership of each element $x \in U$.

The function $\mu_F(x)$ assigns a membership grade to each element x, indicating how strongly x belongs to the fuzzy set F.

Example 107. Consider the universe of discourse $U = \{1, 2, 3, 4, 5\}$. Define a fuzzy set F over U as:

$$F = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 0.4), (5, 1.0)\}.$$

In this example, the membership function $\mu_F(x)$ is defined as:

$$\mu_F(1) = 0.2,$$
$$\mu_F(2) = 0.5,$$
$$\mu_F(3) = 0.8,$$
$$\mu_F(4) = 0.4,$$
$$\mu_F(5) = 1.0.$$

This indicates that element 1 has a membership degree of 0.2, element 2 has a membership degree of 0.5, and so on.

The diagram below illustrates the relationships among Uncertain Sets. As the diagram alone may not provide a comprehensive explanation, please refer to the relevant papers as needed.

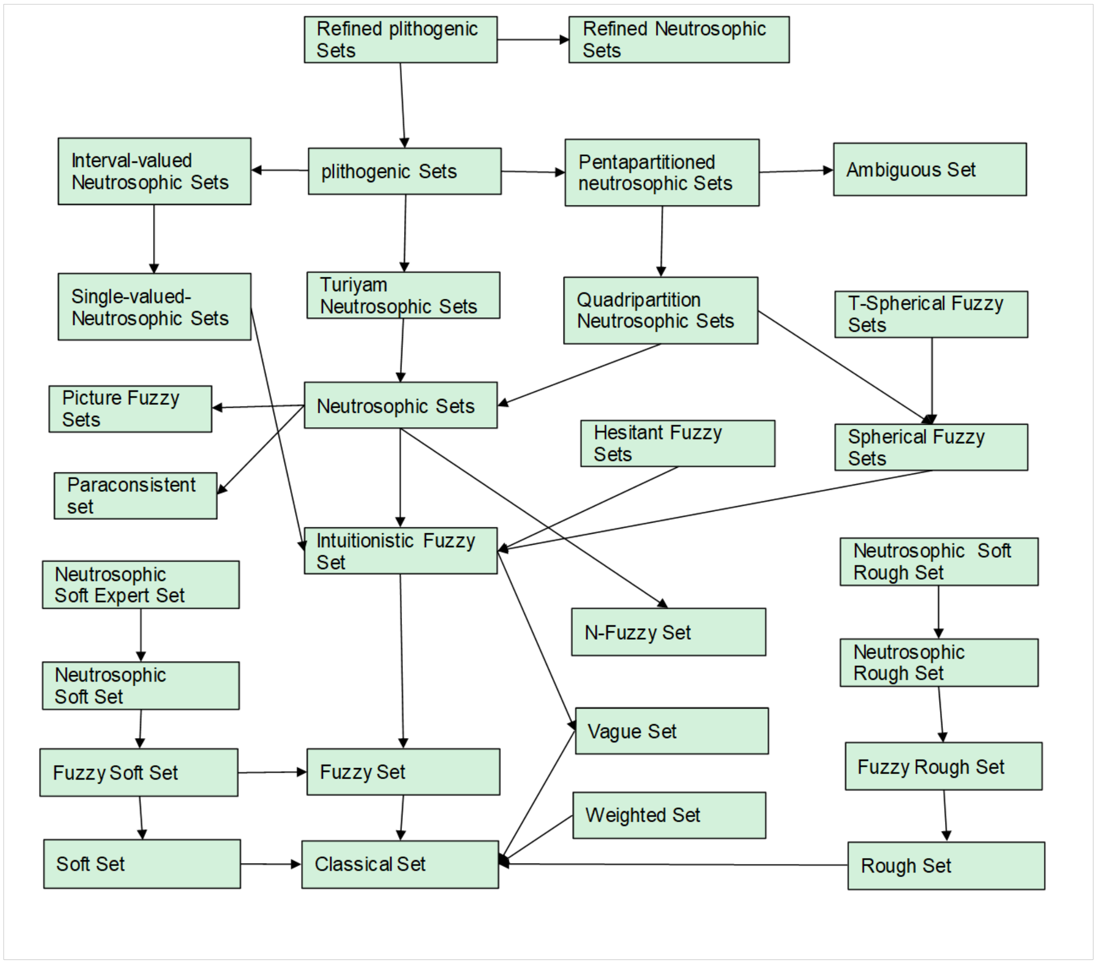

FIGURE 3. Some Uncertain Sets Hierarchy. The Set class at the origin of an arrow contains the set class at the destination of the arrow.

If you wish to examine the relationships among graphs instead of sets, please refer to the surveys as needed [280, 283, 282, 274].

5.1 | Intuitionistic fuzzy set

Intuitionistic Fuzzy Sets (IFS) extend classical fuzzy sets by incorporating both membership and non-membership degrees for each element. This approach includes a hesitation margin, reflecting uncertainty in membership assessment[111, 109]. The definition and related concepts of a Intuitionistic Fuzzy Set are outlined below.

Definition 108. [111] Let U be a universe of discourse. An *Intuitionistic Fuzzy Set* (IFS) A in U is defined as:

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\},$$

where:

- $\mu_A(x) : U \to [0, 1]$ is the membership function, representing the degree of membership of each element $x \in U$.
- $\nu_A(x) : U \to [0, 1]$ is the non-membership function, representing the degree of non-membership of each element $x \in U$.

These functions satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in U.$$

The value $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree, reflecting the uncertainty regarding the membership of x in A.

Notation 109. In this paper, we define the term "Related sets" as a set that either extends or restricts a corresponding set in some way.

Theorem 110. *The following are examples of related sets, including but not limited to:*

- *Interval Valued Intuitionistic Fuzzy Sets* [117, 618, 110]*: Interval-valued intuitionistic fuzzy sets (IVIFSs) extend intuitionistic fuzzy sets by associating each element with intervals for membership and non-membership degrees, enhancing uncertainty representation.*
- *Circular intuitionistic fuzzy sets* [400, 743]*: Circular intuitionistic fuzzy sets extend intuitionistic fuzzy sets by adding a radius parameter to represent uncertainty geometrically, enhancing complex decision-making with more comprehensive information.*
- *Triangular Intuitionistic Fuzzy Sets* [506, 774, 507]*: Triangular intuitionistic fuzzy sets (TIFSs) extend intuitionistic fuzzy sets by using triangular-shaped membership and non-membership functions to better handle uncertainty.*
- *Type-2 Intuitionistic Fuzzy Sets* [210, 101, 650]*: Type-2 intuitionistic fuzzy sets (T2IFSs) enhance standard intuitionistic fuzzy sets by incorporating uncertainty within membership and non-membership degrees, allowing more robust representation of complex uncertainties. Related Concepts include Type-3 Intuitionistic Fuzzy Sets*[169]*, triangular interval type-2 intuitionistic fuzzy sets*[301] *,and Symmetric triangular interval type-2 intuitionistic fuzzy sets* [651]*.*

5.2 | Bipolar Fuzzy Sets

Bipolar fuzzy sets extend traditional fuzzy sets by representing both positive and negative membership degrees, capturing both support and opposition [41, 800]. The definition and related concepts of a Bipolar Fuzzy Set are outlined below.

Definition 111. [800] Let U be a universe of discourse. A extitBipolar Fuzzy Set (BFS) B over U is defined as:

$$B = \{(m, \mu_B^+(m), \vartheta_B^-(m)) : m \in U\},$$

where:

- $\mu_B^+(m) : U \to [0, 1]$ is the positive membership function, representing the grade of positive membership of each element $m \in U$.
- $\vartheta_B^-(m) : U \to [-1, 0]$ is the negative membership function, representing the grade of negative membership of each element $m \in U$.

These functions satisfy the condition:

$$-1 \leq \mu_B^+(m) + \vartheta_B^-(m) \leq 1, \quad \forall m \in U.$$

The positive and negative membership functions ensure that each element has a degree of positive and negative association with the set, reflecting the bipolar nature of the set.

Theorem 112. *The following are examples of related sets, including but not limited to:*

- *Complex bipolar fuzzy sets*[320]*: Complex bipolar fuzzy sets extend fuzzy sets by integrating complex-valued positive and negative memberships, effectively representing nuanced uncertainty and oppositional characteristics within problems. Related concepts include Complex Bipolar Fuzzy N-Soft Sets*[257]*, Bipolar Complex Fuzzy Soft Sets*[92, 457]*, Bipolar complex picture fuzzy soft Sets*[365]*, Bipolar Complex Intuitionistic Fuzzy N-Soft Sets*[458]*, and Complex Bipolar Multi-Fuzzy Sets*[89]*.*

- *m-Polar Fuzzy Sets*[21, 309]*: m-Polar fuzzy sets generalize bipolar fuzzy sets, mapping elements to* $[0, 1]^m$*, facilitating representation of multi-agent, multi-attribute uncertainty and multipolar information. Related concepts include Pythagorean m-polar fuzzy sets*[413]*, q-rung orthopair m-polar fuzzy sets*[596]*, Cubic m-Polar Fuzzy Sets*[300]*, Doubt m-Polar Fuzzy Sets*[62]*, m-polar Q-hesitant anti-fuzzy set*[95]*, and Soft rough Pythagorean m-polar fuzzy sets* [597]*.*
- *Cubic bipolar fuzzy set*[599]*: Cubic bipolar fuzzy sets integrate bipolar fuzzy and interval-valued fuzzy information, representing dual-sided membership and interval-based uncertainty simultaneously for complex data modeling.*
- *Neutrosophic Bipolar Fuzzy Sets*[335]*: Neutrosophic Bipolar Fuzzy Set integrates neutrosophic sets with bipolar fuzzy sets to model uncertainty, indeterminacy, and dual-sided opinions simultaneously.*
- *Spatial Bipolar Fuzzy Sets*[140]*: Spatial Bipolar Fuzzy Sets model positive and negative spatial information, handling imprecision and incomplete data, crucial for spatial reasoning.*
- *Bipolar Pythagorean Fuzzy Sets*[463]*: Bipolar Pythagorean Fuzzy Sets (BPFSs) generalize fuzzy, bipolar fuzzy, and intuitionistic fuzzy sets, enabling flexible handling of uncertain decision-making data with positive and negative membership degrees.*
- *Spherical Bipolar Fuzzy Sets*[561]*: Bipolar Spherical Fuzzy Sets (BSFSs) combine bipolar fuzzy and spherical fuzzy sets, incorporating truth, abstinence, and non-membership grades for enhanced multi-criteria decision-making.*
- *Tripolar Fuzzy Sets*[590, 589]*: These Sets are 3-polar Fuzzy Sets.*

The diagram below (Figure4) illustrates the relationships among Bipolar Sets. As the diagram alone may not provide a comprehensive explanation, please refer to the relevant papers as needed.

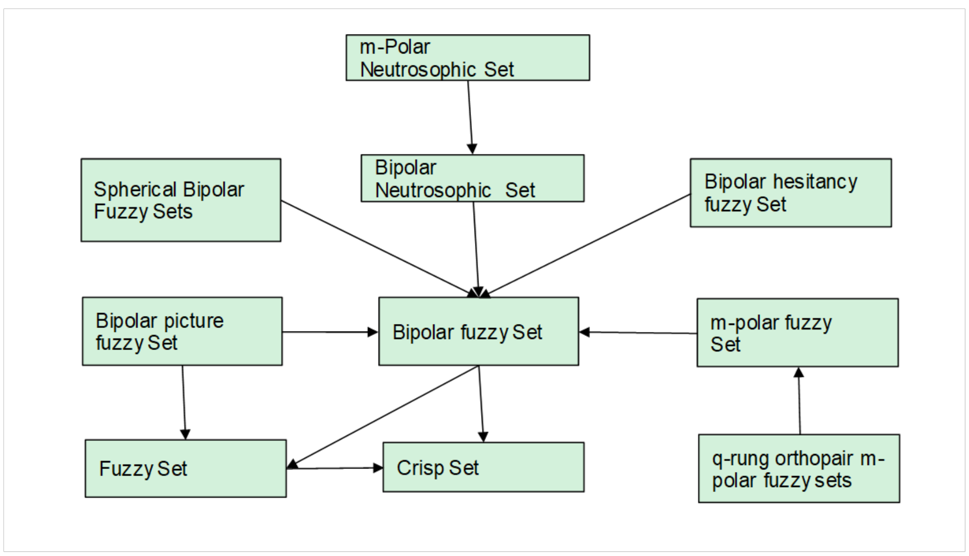

Figure 4. Some Bipolar Sets Hierarchy. The Set class at the origin of an arrow contains the set class at the destination of the arrow.

5.3 | Neutrosophic Set

Neutrosophic sets, introduced by Florentin Smarandache, extend fuzzy sets to handle uncertainty, inconsistency, and indeterminacy using truth, indeterminacy, and falsity values for better modeling [657, 656, 655]. The definition and related concepts of a Neutrosophic Set are outlined below.

Definition 113 (Neutrosophic Set)**.** [656, 655, 657, 718] A neutrosophic set A on the universe U is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\},$$

where $T_A : U \to [0,1]$ is the *truth-membership function*, $I_A : U \to [0,1]$ is the *indeterminacy-membership function*, and $F_A : U \to [0,1]$ is the *falsity-membership function*.

Theorem 114. *The following are examples of related sets, including but not limited to:*

- *Bipolar Neutrosophic Set* [221, 706, 222]*: Bipolar Neutrosophic Sets integrate positive/negative truth, indeterminacy, and falsity functions, enabling flexible handling of uncertain, complex decision-making problems.*
- *Interval Valued Neutrosophic Set* [773, 786, 767]*: Interval-valued neutrosophic sets use intervals for truth, indeterminacy, and falsity membership degrees, enhancing flexibility in uncertain decision-making scenarios.*
- *single-valued neutrosophic sets* [447, 812]*: Single Valued Neutrosophic Sets extend fuzzy and intuitionistic sets, representing truth, indeterminacy, and falsity independently for enhanced decision-making flexibility. Related concepts include Type-2 single-valued neutrosophic sets* [388]*.*
- *Single valued quadripartitioned neutrosophic sets* [144, 178]*: Single-Valued Quadripartitioned Neutrosophic Sets (SVQNS) extend neutrosophic sets by defining degrees of truth, contradiction, unknown, and falsity for each element, enabling refined analysis. Related concepts include Bipolar quadripartitioned single valued neutrosophic sets* [652]*.*
- *Hyperneutrosophic Set* [273, 281]*: A definition that extends the Neutrosophic Set within the framework of Hyperstructures* [276, 665]*.*
- *Neutrosophic Offset* [668, 162]*: Neutrosophic Offset is a concept that extends the range of the membership function in the Neutrosophic Set beyond* $[0,1]$*.*

5.4 | Spherical fuzzy Set

Spherical fuzzy sets (SFS) are an advanced extension of fuzzy sets, defining positive, neutral, and negative membership degrees for elements, constrained by $P^2 + I^2 + N^2 \leq 1$ [322, 454, 83]. The definition and related concepts of a spherical fuzzy set are outlined below.

Definition 115. [322] Let $R \neq \emptyset$ be a universe set. A *spherical fuzzy set* (SFS) J in R is defined as:

$$J = \left\{\langle r, P_j(r), I_j(r), N_j(r)\rangle \mid r \in R\right\}, \tag{1}$$

where $P_j : R \to [0,1]$, $I_j : R \to [0,1]$, and $N_j : R \to [0,1]$ are functions that represent the positive, neutral, and negative membership degrees of each $r \in R$, respectively. These functions satisfy the condition:

$$0 \leq P_j^2(r) + I_j^2(r) + N_j^2(r) \leq 1 \quad \text{for all } r \in R. \tag{2}$$

Theorem 116. *The following are examples of related sets, including but not limited to:*

- *T-spherical fuzzy sets* [703, 705, 739]*: A T-Spherical Fuzzy Set extends spherical fuzzy sets by allowing custom powers for membership, abstinence, and non-membership degrees, enhancing flexibility.*
- *Interval-Valued Spherical Fuzzy Set* [321, 317, 555]*: Interval-Valued Spherical Fuzzy Sets (IV-SFS) extend Spherical Fuzzy Sets by defining membership, non-membership, and indeterminacy with intervals, offering increased flexibility in uncertain scenarios.*
- *Complex Spherical Fuzzy Set* [31]*: A Complex Spherical Fuzzy Set (CSFS) integrates complex fuzzy sets with spherical fuzzy sets, using three parameters—truth, abstinence, and falsity—to manage uncertain information in decision-making, within unit-disc constraints.*
- *Complex Spherical Fuzzy N -Soft Sets* [37, 28, 38]*: Related Concepts include Complex T-Spherical Fuzzy N-Soft Sets*[36]*.*

- *Spherical linear Diophantine fuzzy sets* [108, 598]*: Spherical Linear Diophantine Fuzzy Sets integrate control parameters for modeling uncertainties in multi-criteria decision-making, enhancing flexibility and independence.*
- *Spherical Linear Diophantine Fuzzy Soft Rough Sets* [336]*: Spherical Linear Diophantine Fuzzy Soft Rough Sets integrate fuzzy, soft, and rough set theories for enhanced decision-making, modeling complex uncertainties with flexible parameterization.*
- *Eigen spherical fuzzy set* [318, 364]*: Eigen spherical fuzzy sets extend the concept of eigen fuzzy sets, applying them to spherical fuzzy relations, aiding complex decision-making with refined computational compositions. Related concepts include Eigen fuzzy sets*[471, 518].
- *spherical fuzzy soft expert sets* [526]*: Spherical fuzzy soft expert sets combine soft expert sets with spherical fuzzy sets to better model uncertainty and expert opinions for decision-making. Related concepts include T-spherical fuzzy hypersoft sets*[123] *and spherical fuzzy N-soft expert sets*[26].

5.5 | Hesitant Fuzzy Set

Hesitant Fuzzy Sets (HFSs) represent situations where the membership degree of an element to a set can have multiple possible values, reflecting hesitation or uncertainty in decision-making [617, 249, 378, 509]. The definition and related concepts of a hesitant fuzzy set are outlined below.

Definition 117. [697] Let X be a reference set. A hesitant fuzzy set H on X is defined as a function h that returns a subset of membership values in $[0,1]$:

$$h : X \to \mathcal{P}([0,1]), \tag{3}$$

where $\mathcal{P}([0,1])$ denotes the power set of $[0,1]$. For each $x \in X$, $h(x)$ represents the possible membership degrees of x to the set H.

Each subset $h(x)$ is called a hesitant fuzzy element (HFE). The set of all HFEs in H is denoted by:

$$H = \bigcup_{x \in X} h(x). \tag{4}$$

Theorem 118. *The following are examples of related sets, including but not limited to:*

- *T-spherical hesitant fuzzy sets*[52]*: T-Spherical Hesitant Fuzzy Sets combine T-spherical fuzzy and hesitant fuzzy sets to manage complex truth, indeterminacy, and falsity membership values effectively. Related concepts include T-spherical type-2 hesitant fuzzy sets*[525].
- *Generalized hesitant fuzzy sets* [138, 567, 814]*: Generalized hesitant fuzzy sets (G-HFSs) extend hesitant fuzzy sets to include both crisp and interval-valued memberships, enhancing decision-making under uncertainty. Related concepts include Weighted Generalized Hesitant Fuzzy Sets* [811].
- *Interval-valued hesitant fuzzy sets* [253, 255]*: Interval-valued hesitant fuzzy sets (IVHFSs) extend hesitant fuzzy sets by allowing membership degrees as interval values, accommodating greater uncertainty. Related concepts include probabilistic interval-valued hesitant fuzzy sets* [73]*, Complex Interval-Valued Hesitant Fuzzy Sets*[252]*, Interval-valued hesitant fuzzy linguistic sets*[719]*, Weighted Interval-Valued Hesitant Fuzzy Sets*[783]*, and Dual Interval-Valued Hesitant Fuzzy Sets*[255].
- *Probabilistic hesitant fuzzy set* [256, 333]*: Probabilistic Hesitant Fuzzy Sets (PHFSs) extend hesitant fuzzy sets by incorporating probability, representing membership degrees with associated likelihoods for more nuanced decision-making. Related concepts include probabilistic hesitant fuzzy rough set* [373] *continuous probabilistic hesitant fuzzy sets*[291]*, and Probabilistic dual hesitant fuzzy set*[333].
- *Single-valued neutrosophic hesitant fuzzy sets* [766, 478]*: Single-valued neutrosophic hesitant fuzzy sets integrate truth, indeterminacy, and falsity degrees with hesitation in membership, handling complex uncertainty and decision-making scenarios effectively. Related concepts include neutrosophic cubic hesitant fuzzy sets* [595] *and Single valued neutrosophic type-2 hesitant fuzzy sets*[524].

- *Pythagorean hesitant fuzzy sets* [402]*: Pythagorean hesitant fuzzy sets extend hesitant fuzzy sets by allowing the squared sum of membership and non-membership degrees. Related concepts include Fermatean hesitant fuzzy sets* [405, 487]*, Pythagorean probabilistic hesitant fuzzy sets*[566]*, and Fermatean probabilistic hesitant-fuzzy sets*[565]*.*
- *Complex hesitant fuzzy sets* [456, 299]*: Complex hesitant fuzzy sets (CHFS) extend hesitant fuzzy sets by incorporating complex-valued membership degrees, providing two-dimensional information to handle complex decision-making. As a graph concept, the complex hesitant fuzzy graph is known* [5, 90]*.*
- *Multi-hesitant fuzzy sets* [546]*: Hesitant fuzzy sets allow an element's membership degree to include multiple possible values between 0 and 1, representing decision uncertainty.*
- *intuitionistic hesitant fuzzy sets* [547, 128]*: Intuitionistic hesitant fuzzy sets incorporate multiple potential membership and non-membership values for elements, enhancing uncertainty handling in decision-making processes. Related concepts include D-intuitionistic hesitant fuzzy sets*[486, 427] *and probabilistic interval-valued intuitionistic hesitant fuzzy sets*[785]*.*
- *Cubic hesitant fuzzy sets* [452]*: Cubic Hesitant Fuzzy Sets (CHFSs) are an extended form combining interval-valued hesitant fuzzy elements and standard hesitant fuzzy elements to enhance decision-making. This allows handling more complex uncertainties.*
- *Hesitant fuzzy soft sets* [711, 516]*: Hesitant fuzzy soft sets combine soft set theory and hesitant fuzzy sets to manage uncertainties. They facilitate operations like union and intersection. Related concepts include Hesitant fuzzy N-soft set*[23]*, Interval-Valued Hesitant Fuzzy Soft Sets*[756, 794]*, Weighted hesitant fuzzy soft set*[736]*, Intertemporal Hesitant Fuzzy Soft Sets*[442]*, Hesitant linguistic expression soft sets*[443]*, Generalized hesitant fuzzy soft sets*[421, 136]*, hesitant multi-fuzzy soft set*[229]*, Hesitant Bipolar-Valued Fuzzy Soft Sets*[720]*, interval-valued intuitionistic hesitant fuzzy soft sets*[550]*, and dual hesitant fuzzy soft sets*[105]*.*
- *Picture hesitant fuzzy set* [723, 587, 80]*: Picture Hesitant Fuzzy Set (PHFS) combines Picture Fuzzy Sets and Hesitant Fuzzy Sets, representing multiple positive, neutral, negative, and refusal memberships for decision-making. Related concepts include picture type-2 hesitant fuzzy sets* [523]*, Interval-valued Picture Hesitant Fuzzy Sets*[11]*, and probabilistic picture-hesitant fuzzy set*[587]
- *Hesitant fuzzy rough set* [374, 755]*: Hesitant Fuzzy Rough Set integrates rough set theory with hesitant fuzzy sets, defining lower and upper approximations for uncertain decision-making scenarios. Related concepts include Generalized hesitant fuzzy rough sets* [631]*, Interval-valued dual hesitant fuzzy rough set*[420]*, hesitant neutrosophic rough set*[808]*, Single Valued Neutrosophic Hesitant Fuzzy Rough Set*[94]*, and Dual hesitant fuzzy rough set*[793]*.*

5.6 | Complex Fuzzy Set

A Complex Fuzzy Set assigns complex-valued membership to elements, combining an amplitude within $[0, 1]$ and a phase angle between $[0, 2\pi]$ [160, 585]. As a graph concept, the complex intuitionistic fuzzy graph[762] and Complex Pythagorean Dombi fuzzy graph[32] are known. The definition and related concepts of a Complex Fuzzy Set are outlined below.

Definition 119. [585] Let X be a universe of discourse. A *Complex Fuzzy Set* C in X is defined by a complex-valued membership function:

$$C = \{(x, \gamma_C(x)) : x \in X\},$$

where:

$$\gamma_C(x) = p_C(x) \cdot e^{j\phi_C(x)}, \quad p_C(x) \in [0, 1], \quad \phi_C(x) \in [0, 2\pi].$$

Here:

- $p_C(x)$ is the amplitude term representing the magnitude of membership.
- $\phi_C(x)$ is the phase term representing the phase angle.

The membership function $\gamma_C(x)$ assigns a complex value to each element x, which lies within the unit circle in the complex plane.

Theorem 120. *The following are examples of related sets, including but not limited to:*

- *Complex bipolar fuzzy sets*[455, 320]*: Complex bipolar fuzzy sets extend fuzzy sets by integrating complex-valued positive and negative memberships, effectively representing nuanced uncertainty and oppositional characteristics within problems.*
- *Interval-valued complex fuzzy sets* [199, 671]*: Interval-Valued Complex Fuzzy Sets (IVCFS) extend complex fuzzy sets, representing membership with interval-valued amplitudes and complex phase, enhancing modeling flexibility.*
- *complex fuzzy soft sets* [782, 92]*: Complex Intuitionistic Fuzzy Soft Sets combine complex-valued membership, non-membership functions with soft set parameters, handling uncertainty, periodicity, and decision-making. Related concepts include complex intuitionistic fuzzy soft sets* [571, 415]*, interval-complex neutrosophic soft sets*[60]*, bipolar complex intuitionistic fuzzy N-soft sets*[453]*, complex vague soft sets*[629]*, and Complex fermatean fuzzy N-soft sets*[27]*.*
- *Complex multi-fuzzy sets* [48, 89]*: Complex multi-fuzzy sets extend multi-fuzzy sets by incorporating complex-valued multi-membership functions, handling uncertainty and periodic data through amplitude and phase terms. Related concepts include Complex multi-fuzzy soft set*[49]*, complex multi-fuzzy hypersoft set*[607] *and Complex multi-fuzzy soft expert set*[50]*.*
- *Complex Neutrosophic Set* [77, 78]*: An extension of the Complex Fuzzy Set, derived from the Neutrosophic Set.*

5.7 | Picture Fuzzy set

A Picture Fuzzy Set (PFS) generalizes fuzzy sets by incorporating four membership degrees: positive, neutral, negative, and refusal. It captures more complex, nuanced decision information [649, 724]. The definition and related concepts of a Picture Fuzzy set are outlined below.

Definition 121. [193] Let X be a universe of discourse. A *Picture Fuzzy Set* (PFS) A in X is defined as:

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where:

- $\mu_A(x) : X \to [0, 1]$ is the degree of positive membership of x in A.
- $\eta_A(x) : X \to [0, 1]$ is the degree of neutral membership of x in A.
- $\nu_A(x) : X \to [0, 1]$ is the degree of negative membership of x in A.

These functions satisfy the condition:

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.$$

The value $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called the degree of refusal membership of x in A.

Theorem 122. *The following are examples of related sets, including but not limited to:*

- *picture hesitant fuzzy sets* [10]*: A Picture Hesitant Fuzzy Set (PHFS) combines Picture Fuzzy Sets and Hesitant Fuzzy Sets, allowing multiple membership values for positive, neutral, negative assessments.*
- *Picture Fuzzy Rough Set* [12, 12]*: A Picture Fuzzy Rough Set integrates picture fuzzy sets and rough set theory, enabling nuanced approximations with positive, neutral, negative, and refusal memberships.*
- *complex picture fuzzy sets: Complex Picture Fuzzy Sets extend picture fuzzy sets by representing membership, non-membership, and neutrality degrees in a complex plane, enhancing multidimensional problem modeling*[81, 397]*.*

- *Pythagorean picture fuzzy sets* [194, 195]*: Pythagorean Picture Fuzzy Sets combine Picture Fuzzy Sets and Pythagorean Fuzzy Sets, handling membership, non-membership, and neutrality degrees squared.*
- *Picture Fuzzy Soft Sets* [395, 757]*: Picture fuzzy soft sets (PFSS) combine picture fuzzy sets and soft sets, representing uncertainty with positive, neutral, and negative membership, supporting decision-making with multi-faceted attributes.*

  Related concepts include Generalized picture fuzzy soft sets[399] *and Multi-valued picture fuzzy soft sets*[366]*.*

5.8 | Type-n fuzzy sets

A *Type-n Fuzzy Set* is a fuzzy set whose membership function assigns a *Type-*$(n-1)$ *fuzzy set* to each element, allowing multi-level uncertainty representation [601, 780].

Theorem 123. *The following are examples of related sets, including but not limited to:*

- *Type-2 fuzzy sets* [475, 746, 474]*: Type-2 fuzzy sets extend traditional fuzzy sets by incorporating a range of membership degrees, addressing higher uncertainty in decision-making and modeling perceptions. Related concepts include interval type 2 fuzzy sets* [306, 226]*, Closed General Type-2 Fuzzy Sets*[738]*, Type-n Neutrosophic Set*[146]*, and triangular interval type-2 fuzzy sets*[505]*.*
- *Type-2 hesitant fuzzy sets* [346, 389]*: Type-2 hesitant fuzzy sets extend hesitant fuzzy sets by incorporating fuzzy membership levels, addressing repeated and uncertain membership degrees for better decision-making.*
- *Type-3 fuzzy sets* [169, 412, 307]*: Type-3 fuzzy sets extend Type-2 by adding a tertiary membership function, capturing more complex uncertainty in multi-layered decision-making.*

The diagram below (Figure5) illustrates the relationships among Bipolar Sets. As the diagram alone may not provide a comprehensive explanation, please refer to the relevant papers as needed.

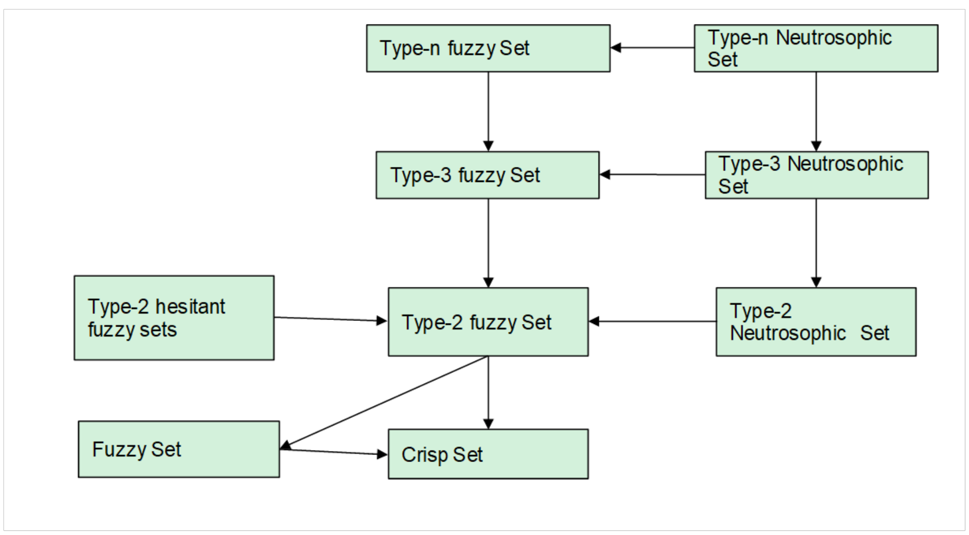

Figure 5. Some Type-2 Sets Hierarchy. The Set class at the origin of an arrow contains the set class at the destination of the arrow.

5.9 | q-rung orthopair fuzzy sets, Pythagorean fuzzy sets, and Fermatean fuzzy sets

The q-rung orthopair fuzzy sets offer a highly flexible type of fuzzy set [75, 549]. Known related (generalizable) concepts include Pythagorean fuzzy sets[551, 801, 733] and Fermatean fuzzy sets[393, 370]. The definition and related concepts of a q-rung orthopair fuzzy sets are outlined below.

Definition 124 (q-Rung Orthopair Fuzzy Set)**.** [75] Let X be a universal set. A *q-rung orthopair fuzzy set* (qROFS) R on X is defined as:

$$R = \{(t_i, \xi_R(t_i), \nu_R(t_i)) \mid t_i \in X\},$$

where $\xi_R : X \to [0,1]$ and $\nu_R : X \to [0,1]$ are the membership and non-membership functions, respectively, and they satisfy the following condition for a given $q \geq 1$:

$$(\xi_R(t_i))^q + (\nu_R(t_i))^q \leq 1, \quad \forall t_i \in X.$$

The *hesitancy degree* $\pi_R(t_i)$ of the element t_i is defined as:

$$\pi_R(t_i) = (1 - ((\xi_R(t_i))^q + (\nu_R(t_i))^q))^{1/q}.$$

For simplicity, a q-rung orthopair fuzzy value (qROFV) can be denoted as $(\xi_R(t_i), \nu_R(t_i))$.

Theorem 125. *The following are examples of related sets, including but not limited to:*

- *n-Pythagorean fuzzy sets* [158, 159]*: n-Pythagorean Fuzzy Sets are an extension of Pythagorean Fuzzy Sets where nth power of membership and non-membership sum* ≤ 1.
- *(m, n)-fuzzy set* [58, 692]*: (m, n)-Fuzzy sets are fuzzy sets where the mth power of membership and nth power of non-membership degrees sum up to* ≤ 1*. Related concepts include (p, q, r)−Fractional fuzzy sets*[319]*.*
- *Pythagorean fuzzy subsets* [749]*: Subset concepts of Pythagorean fuzzy set.*
- *Constrained Pythagorean fuzzy sets* [530]*: Constrained Pythagorean fuzzy sets integrate Pythagorean fuzzy sets with probabilistic reliability, enabling richer expression of fuzzy and stochastic information.*
- *Disc Pythagorean fuzzy sets* [398]*: Disc Pythagorean fuzzy sets (D-PFSs) are extensions of Pythagorean fuzzy sets, using circular representations with variable radii for better flexibility.*
- *Circular Pythagorean fuzzy sets* [147]*: Circular Pythagorean Fuzzy Sets (C-PFSs) model fuzziness using circles, allowing flexible membership and non-membership representation with improved decision-making sensitivity. Related concepts include Circular intuitionistic fuzzy sets* [401, 93]*.*
- *hesitant Pythagorean fuzzy sets* [431]*: Hesitant fuzzy sets version of Pythagorean fuzzy sets.*
- *Refined pythagorean fuzzy sets* [609]*: Refined Pythagorean Fuzzy Sets enhance traditional fuzzy sets by incorporating sub-grades for membership and non-membership, improving flexibility in complex scenarios.*
- *interval-valued Pythagorean fuzzy set* [548, 803, 292]*: An Interval-Valued Pythagorean Fuzzy Set (IVPFS) represents uncertain data with interval-valued membership and non-membership functions, supporting enhanced decision-making flexibility. Related concepts include Linguistic interval-valued Pythagorean fuzzy sets* [293]*.*
- *Bipolar pythagorean fuzzy sets* [463, 8]*: A Bipolar Pythagorean Fuzzy Set (BPFS) extends Pythagorean fuzzy sets by including positive and negative membership degrees for flexible decision-making. Related concepts include Bipolar Pythagorean Neutrosophic Sets* [7] *and bipolar n,m-rung orthopair fuzzy sets*[358, 359]*.*
- *Generalized orthopair fuzzy sets* [750, 237]*: Generalized orthopair fuzzy sets extend intuitionistic and Pythagorean fuzzy sets, allowing flexible membership and non-membership degrees, enhancing decision-making expressiveness. Related concepts include probabilistic generalized orthopair fuzzy sets*[264, 263]*.*
- *Quintic fuzzy sets*[244, 271]*: Quintic fuzzy sets are advanced fuzzy sets where the fifth power sum of membership and non-membership degrees is less than 1, enhancing flexibility. Related concepts include Q-Rung orthopair fuzzy sets*[588] *and quartic fuzzy set*[104, 271]*.*

The hierarchy of Some Uncertain q-rung Sets is illustrated below. It is hoped that future research will investigate whether these sets can be extended to Turiyam Neutrosophic Sets or other Uncertain Sets.

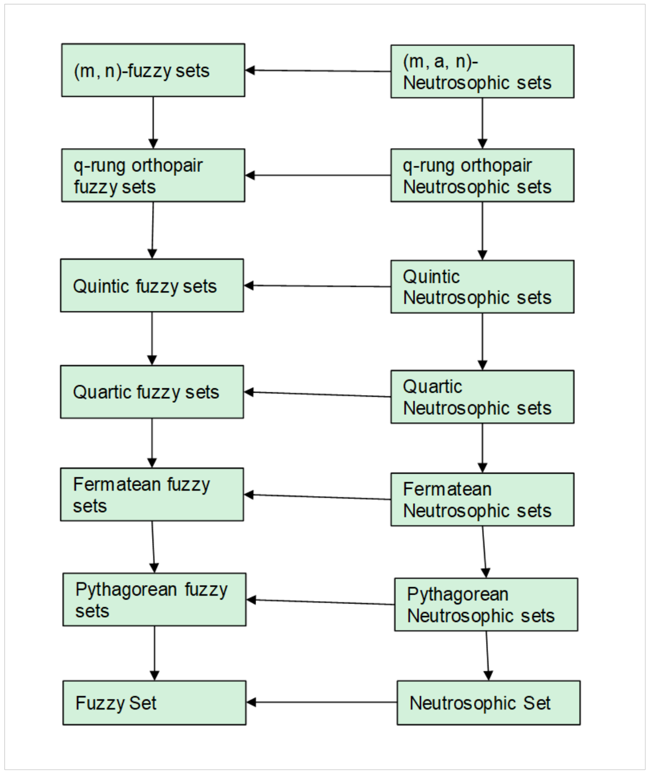

FIGURE 6. Some Uncertain q-rung Sets Hierarchy. The Set class at the origin of an arrow contains the set class at the destination of the arrow.

Question 126. Can these concepts be extended to other Uncertain Sets, such as Ambiguous Sets, Turiyam Neutrosophic Sets, and Plithogenic Sets?

5.10 | Linguistic Set

A linguistic set uses words or linguistic terms as elements to represent and analyze human qualitative judgments, enhancing decision-making. As an example, the definition of Hesitant fuzzy linguistic term sets is provided below [602, 732].

Definition 127. Let S be a linguistic term set defined as $S = \{s_0, s_1, \ldots, s_g\}$. A **Hesitant Fuzzy Linguistic Term Set (HFLTS)** H_S is defined as an ordered finite subset of consecutive linguistic terms from S, such that:

$$H_S(\vartheta) = \{s_i \mid s_i \in S\}$$

where $H_S(\vartheta)$ represents the HFLTS associated with the linguistic variable ϑ. Additionally, the empty and full HFLTSs are defined as:

- **Empty HFLTS**: $H_S(\vartheta) = \emptyset$
- **Full HFLTS**: $H_S(\vartheta) = S$

Any other HFLTS must contain at least one linguistic term from S.

Theorem 128. *The following are examples of related sets, including but not limited to:*

- *Hesitant fuzzy linguistic term set* [602, 732]*: A hesitant fuzzy linguistic term set (HFLTS) allows multiple linguistic terms to represent uncertainty or hesitation in decision-making, enhancing flexibility in evaluations. Related concepts include double hierarchy hesitant fuzzy linguistic term set*[729, 493], *Hesitant Intuitionistic Fuzzy Linguistic Sets*[477]*, Hesitant Probabilistic Fuzzy Linguistic Sets*[376, 17]*, and Interval-Valued Hesitant Fuzzy Linguistic Sets* [818, 568]*.*

- *fuzzy linguistic sets* [107]*: Fuzzy linguistic sets use linguistic variables to express imprecise or qualitative information, aiding in decision-making and human cognitive evaluations. Related concepts include Fermatean fuzzy linguistic set* [183, 436]*, picture fuzzy linguistic set*[182, 107]*, Pythagorean Fuzzy Linguistic Sets*[813]*, Quasirung orthopair fuzzy linguistic sets*[809]*, Hesitant Picture Fuzzy Linguistic Sets*[351]*, and Multiple-Valued Picture Fuzzy Linguistic Set*[754]*.*
- *Single-Valued Neutrosophic Linguistic Set* [569, 765]*: Single-valued neutrosophic linguistic sets represent qualitative, fuzzy information using linguistic terms with associated truth, falsity, and indeterminacy degrees. Related concepts include Interval-Valued Neutrosophic Linguistic Set* [181]*, Hesitant Neutrosophic Linguistic Sets*[532]*, and Multi-Valued Neutrosophic Linguistic Set*[384]*.*
- *Fuzzy Linguistic Soft Set* [15, 461]*: Related concepts include Intuitionistic Fuzzy Linguistic Soft Set* [316]*, Hesitant Fuzzy Linguistic Term Soft Sets*[439]*, and Multi-Valued Neutrosophic Linguistic Soft Set*[384]*.*

Question 129. Can these concepts be extended to other Uncertain Sets, such as Ambiguous Sets, Turiyam Neutrosophic Sets, and Plithogenic Sets?

5.11 | Soft Set

A soft set is a mathematical framework introduced by Molodtsov for handling uncertainty. It maps parameters to subsets of a universe, aiding decision-making [460, 673, 248]. The HyperSoft Set is a generalized concept of the soft set [608, 624, 659]. The definition of a Soft Set, along with an example and related concepts, is briefly provided below.

Definition 130. [491] Let U be a universe of discourse and E be a set of parameters. Denote $P(U)$ as the power set of U. A **soft set** over U is defined as a pair (F, A), where $A \subseteq E$ and F is a mapping given by:

$$F : A \to P(U).$$

This means that for each parameter $e \in A$, $F(e)$ is a subset of U. The set $F(e)$ is called the set of e-approximate elements of the soft set (F, A).

Example 131. [460] Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of houses and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters, where:

- e_1 stands for “expensive”,
- e_2 stands for “beautiful”,
- e_3 stands for “wooden”,
- e_4 stands for “cheap”,
- e_5 stands for “in the green surroundings”.

Suppose F is defined as:

$$F(e_1) = \{h_2, h_4\}, \quad F(e_2) = \{h_1, h_3\}, \quad F(e_3) = \{h_3, h_4, h_5\},$$
$$F(e_4) = \{h_1, h_3, h_5\}, \quad F(e_5) = \{h_1\}.$$

Then, the soft set (F, E) represents the following collection of descriptions:

- “Expensive houses” = $\{h_2, h_4\}$,
- “Beautiful houses” = $\{h_1, h_3\}$,
- “Wooden houses” = $\{h_3, h_4, h_5\}$,
- “Cheap houses” = $\{h_1, h_3, h_5\}$,

- "Houses in the green surroundings" = $\{h_1\}$.

Theorem 132. *The following are examples of related sets, including but not limited to:*

- *Fuzzy Soft Set*[410, 163]*: A Fuzzy Soft Set combines fuzzy sets and soft sets, mapping parameters to fuzzy subsets, aiding in handling uncertainty in data. Related concepts include multi-fuzzy soft set*[758]*, intuitionistic fuzzy soft set* [297, 295]*, Flexible Fuzzy Soft Set*[66]*, dual hesitant fuzzy soft* [296]*, trapezoidal fuzzy soft set*[742]*, hesitant multi-fuzzy soft set*[229]*, q-Rung Orthopair Fuzzy Soft Set*[819]*, Inverse fuzzy soft set*[394]*, Complex Multi-Fuzzy Soft Set*[49]*, and Interval-Valued Picture Fuzzy Soft Set*[395]*.*
- *Pythagorean Fuzzy Soft Sets* [504]*: A Pythagorean fuzzy soft set extends intuitionistic fuzzy soft sets by incorporating Pythagorean fuzzy parameters for enhanced decision-making applications. Related concepts include Quadripartitioned neutrosophic pythagorean soft set* [575]*, Neutrosophic pythagorean soft set*[576]*, Pythagorean fuzzy N-soft sets*[791]*, and Pentapartitioned neutrosophic pythagorean soft set* [577, 16]*.*
- *Z-Numbers Soft Set* [806, 433]*: Z-Numbers Soft Set combines Z-numbers' uncertainty* [777, 499, 778] *and reliability aspects with soft sets, enhancing decision-making by modeling judgment precision and reliability.*
- *FP-intuitionistic multi fuzzy N-soft set* [201, 202]*: FP-intuitionistic multi fuzzy N-soft set extends intuitionistic fuzzy sets with fuzzy-parameterized multi-fuzzy elements for enhanced group decision-making.*
- *Soft Expert Sets* [87, 88, 610]*: Soft expert sets combine opinions from multiple experts within a soft set framework, enabling comprehensive decision analysis without complex operations. Ideal for consensus studies. Related concepts include Fuzzy N-Soft Expert Sets*[72]*, Generalized vague soft expert set* [69, 70]*, Fuzzy Soft Expert Sets* [86, 88]*, Vague Soft Expert Sets* [71, 554]*, Complex Fuzzy Soft Expert Sets* [50, 51]*, Spherical Fuzzy Soft Expert Sets* [553, 552]*, Hypersoft Expert Sets* [362, 363]*, Generalized neutrosophic soft expert set*[707]*, Bipolar Neutrosophic Soft Expert Set*[613, 612]*, Interval-Valued Neutrosophic Soft Expert Set* [137, 59]*, Bipolar fuzzy soft expert set*[47]*, and spherical fuzzy N-soft expert sets*[26]*.*
- *Intersectional soft sets* [672, 500]*: Intersectional soft sets are mathematical structures used to model relationships by intersecting data sets, preserving common elements, and defining shared properties for decision-making contexts.*
- *Ranked soft sets* [623]*: Ranked soft sets extend soft sets by ordering elements based on qualitative preferences without numeric evaluation, enhancing decision-making.*
- *Cluster soft sets* [97]*: Cluster soft sets extend soft sets by incorporating cluster points, forming a structure that refines soft topology for deeper analysis. Related concepts include Baire category soft sets*[98]*.*
- *Generalized intuitionistic fuzzy soft set* [6, 294]*: Generalized intuitionistic fuzzy soft sets (GIFSS) extend intuitionistic fuzzy soft sets by incorporating a moderator's assessment, enhancing decision-making reliability through added validation. Related concepts include Generalized fuzzy soft sets* [187, 744, 745]*, Generalised multi-fuzzy bipolar soft sets*[396]*, and Generalized q-Rung Orthopair Fuzzy Soft Sets*[339]*.*
- *neutrosophic soft set* [386, 614, 459]*: Neutrosophic soft sets combine neutrosophic sets and soft sets to handle uncertainty, inconsistency, and vagueness by mapping parameters to neutrosophic sets. Related concepts include Intuitionistic neutrosophic soft set* [152]*, Generalized neutrosophic soft set*[150]*, Bipolar neutrosophic soft sets*[79]*, Q-neutrosophic soft set*[4]*, Bipolar Quadripartitioned Neutrosophic Soft Set*[586]*, Neutrosophic Bipolar Vague Soft Set*[503]*, Linguistic single-valued neutrosophic soft sets*[383]*, and Interval-valued neutrosophic soft sets* [501, 151, 220]*.*
- *Hypersoft Set* [625, 203]*: A concept that extends the soft set using hyperstructures. Related notions include the Treesoft set*[305, 539] *and the superhypersoft set* [666, 664]*.*

Question 133. Is it possible to extend various sets such as Fuzzy Sets, Neutrosophic Sets, and N-Fuzzy Sets using the concept of Z-Number Sets? Additionally, can these be applied as logical concepts or graph concepts?

5.12 | Rough Set

Rough sets represent uncertainty by defining lower and upper approximations using equivalence relations, identifying elements that certainly or possibly belong to a set, creating boundary regions for analysis [537, 542, 541]. The related concepts are introduced as follows.

Theorem 134. *The following are examples of related sets, including but not limited to:*

- *rough fuzzy sets*[495, 578]*: A Rough Fuzzy Set combines rough set and fuzzy set theories to manage uncertainty through both vagueness and coarseness in data analysis. Related concepts include Generalized fuzzy rough sets* [482, 261]*, Generalized intuitionistic fuzzy rough sets*[804]*, Generalized Interval-Valued Fuzzy Rough Set* [345, 792]*, Generalized hesitant fuzzy rough sets*[631]*, Covering-Based Generalized Rough Fuzzy Sets*[265]*, and partition-based fuzzy rough sets*[481]*.*
- *Soft rough sets* [262, 630, 430]*: Soft rough sets integrate rough set theory and soft sets to manage vagueness and uncertainty. They utilize soft approximation spaces for lower and upper approximations, enhancing applications in complex decision-making scenarios. Related concepts include Soft rough fuzzy sets*[476]*, generalized soft rough sets*[245, 76]*, intuitionistic fuzzy N-soft rough sets*[24]*, Interval-valued intuitionistic fuzzy soft rough sets*[502]*, multi-soft rough sets*[423]*, inverse soft rough sets*[224]*, Bipolar soft rough sets*[387]*, neutrosophic soft rough sets*[771]*, Dual Hesitant Fuzzy Soft Rough Sets*[1]*, T-Spherical Fuzzy Soft Rough Sets*[790]*, Interval-Valued Neutrosophic Soft Rough Sets*[153]*, m-polar fuzzy soft rough sets*[25]*, modified soft rough sets*[424]*, q-Rung Orthopair Fuzzy Soft Rough Sets*[728]*, and N-soft rough sets*[788]*.*

  The concept of Soft Rough Graph is also known in graph theory [33, 22]*.*
- *Pythagorean fuzzy soft rough sets* [352]*: A Pythagorean Fuzzy Soft Set is a generalized fuzzy set framework that incorporates Pythagorean membership and non-membership for decision-making applications. Related concepts include Soft rough Pythagorean m-polar fuzzy sets* [597]*.*
- *Double-Quantitative Rough Sets* [426, 323]*: Double-quantitative rough set incorporates both relative (precision) and absolute (grade) quantitative measures, enhancing rough set models' capability to handle complex, nuanced data.*
- *Granular rough sets* [464, 760, 712]*: Granular Rough Sets are a type of rough set that utilizes information granules, or clusters of indiscernible elements, for precise concept approximations. Related concepts include granular shadowed sets*[760]*, Multigranulation soft rough sets*[251]*, and Co-Granular Rough Sets*[465]*.*
- *Hybrid Rough Set* [603, 337, 538]*: Hybrid Rough Sets combine rough sets with other methods, like neural networks, to enhance decision-making and classification accuracy.*
- *Rough neutrosophic sets* [154, 751]*: Rough neutrosophic sets combine rough set theory and neutrosophic logic, using upper and lower approximations with truth, indeterminacy, and falsity memberships to handle incomplete information. Related concepts include interval rough neutrosophic sets* [667, 557] *and Rough neutrosophic multisets*[82]*.*
- *Multi-granulation neutrosophic rough sets* [141, 807]*: Multi-granulation neutrosophic rough sets (MGNRS) extend rough sets by leveraging multiple neutrosophic relations, enhancing approximation methods for handling complex uncertainties.*
- *Single valued neutrosophic refined rough set* [124]*:Single-valued neutrosophic refined rough sets combine neutrosophic refined sets and rough sets, using refined truth, indeterminacy, and falsity components to handle imprecise data effectively.*
- *Rough pentapartitioned neutrosophic set* [209]*: A Rough Pentapartitioned Neutrosophic Set combines rough set theory with pentapartitioned neutrosophic sets, providing detailed approximations and handling complex uncertainties across five membership components.*

For reference, part of the relationships surrounding Soft Sets and Rough Sets is illustrated in the diagram.

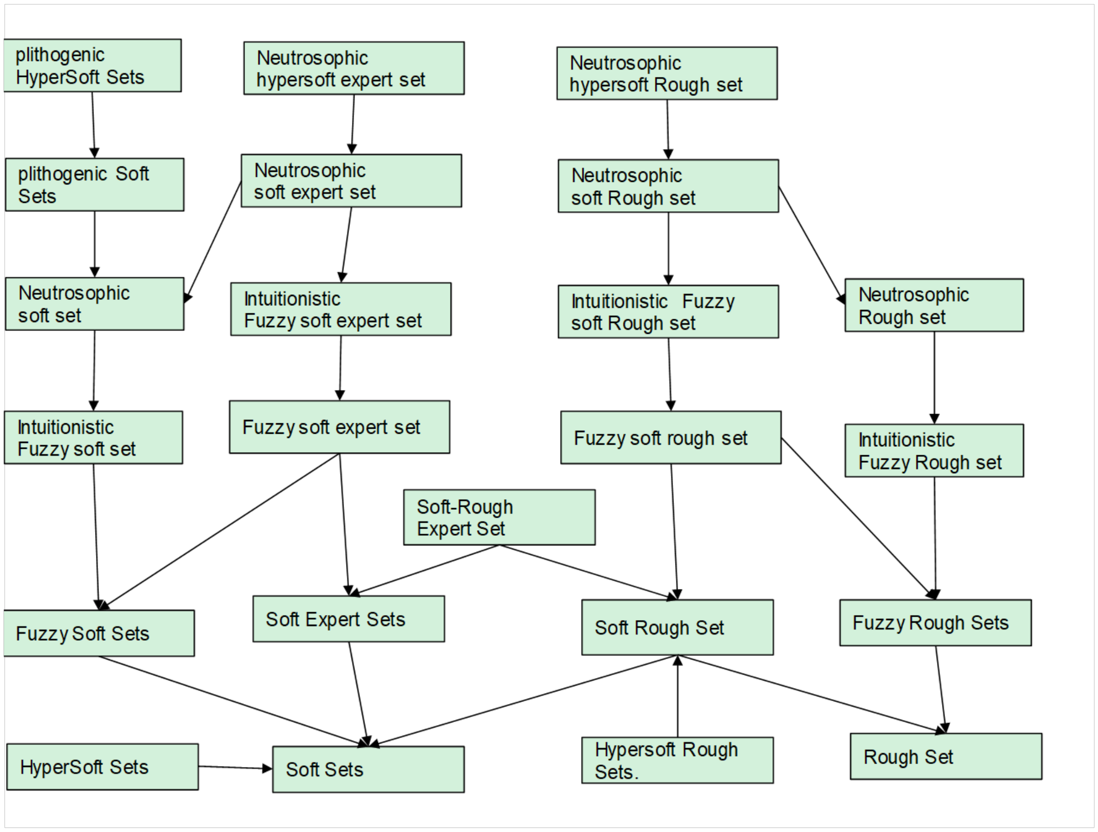

FIGURE 7. Some Soft Sets and Rough Sets Hierarchy. The Set class at the origin of an arrow contains the set class at the destination of the arrow.

5.13 | Weighted Set

A weighted set associates each element with a positive real number representing its weight, affecting selection or analysis processes [789, 753].

Theorem 135. *The following are examples of related sets, including but not limited to:*

- *Weighted Fuzzy Set*[535, 536] *:A weighted fuzzy subset assigns not only a membership degree to each element but also a weight representing the element's importance. Related concepts include Weighted fuzzy soft multiset*[200]*.*
- *Weighted Multiset* [216, 217]*: A weighted multiset extends a multiset by associating each element with a weight, reflecting its importance or frequency.*

Question 136. Is it possible to create a graph representation of the above concepts? Additionally, can these concepts be extended to other Uncertain Sets, such as Ambiguous Sets, Turiyam Neutrosophic Sets, and Plithogenic Sets?

5.14 | Probabilistic Set

A probabilistic set generalizes traditional sets by incorporating probability measures. Each element has an associated probability, representing uncertainty and allowing stochastic reasoning within a set-theoretic framework[343].

Theorem 137. *The following are examples of related sets, including but not limited to:*

- *Probabilistic fuzzy set* [350, 429, 175]*: A Probabilistic Fuzzy Set integrates probabilistic measures with fuzzy sets, allowing elements to have probabilistic membership degrees, enhancing uncertainty modeling. Related concepts include Hesitant probabilistic fuzzy set*[324]*, Probabilistic dual hesitant fuzzy set*[334]*, Multigranulation Pythagorean fuzzy probabilistic rough sets*[787]*, Asymmetry-Center Probabilistic Fuzzy Set*[730]*, general probabilistic fuzzy set*[349]*, probabilistic neutrosophic sets*[810]*, and Complex probabilistic fuzzy set*[380]*.*
- *Probabilistic Soft Set* [258, 817, 259]*: Probabilistic soft sets combine soft set theory with probability, mapping parameters to probability distributions to manage uncertainty in decision-making. Related concepts include probabilistic hesitant N-soft sets*[726] *and Probabilistic Hesitant Fuzzy Soft Set*[440]*.*
- *Probabilistic rough set* [764, 759, 448]*: Probabilistic rough sets extend classical rough set theory using conditional probabilities, enabling flexible approximations with thresholds for uncertain and variable precision analyses. Related concepts include Serial Probabilistic Rough Set*[485] *and Fuzzy probabilistic rough set*[752]*.*

Question 138. Is it possible to create a graph representation of the above concepts? Additionally, can these concepts be extended to other Uncertain Sets, such as Ambiguous Sets, Turiyam Neutrosophic Sets, and Plithogenic Sets?

5.15 | Other Fuzzy Set

Various related concepts to Fuzzy Sets are being proposed daily. An example is provided below. Additionally, please refer to the Discussion subsection for further set concepts.

Theorem 139. *The following are examples of related sets, including but not limited to:*

- *random fuzzy sets* [190, 308]*: Random fuzzy sets (RFS) extend fuzzy sets by integrating randomness, allowing modeling of data generation processes that yield fuzzy outcomes. They generalize concepts of probability to fuzzy data.*
- *Nonstationary Fuzzy Set* [600, 302]*: A* Nonstationary Fuzzy Set *is a fuzzy set with a time-dependent membership function, allowing membership degrees to vary over time.*
- *Gaussian Fuzzy Set* [368, 367]*: A Gaussian Fuzzy Set uses a Gaussian function for membership, typically defined by a mean and standard deviation, enabling smooth membership transitions.*
- *cosine fuzzy set* [267]*: A Cosine Fuzzy Set applies a cosine-based membership function over a closed interval, smoothly scaling element membership degrees.*
- *Multidimensional Fuzzy Sets* [215, 375]*: Multidimensional fuzzy sets generalize n-dimensional fuzzy sets, allowing elements with varying dimensions to address incomplete or unequal evaluations.*
- *Separable fuzzy soft sets*[65]*: A separable fuzzy soft set normalizes positive and negative attributes using fuzzy complements, facilitating better decision-making by modular aggregation of parameter classes.*
- *Axiomatic fuzzy set* [428, 441]*: An axiomatic fuzzy set is a framework in which fuzzy set theory is formalized using specific axioms for consistent decision-making and analysis. Related concepts include Axiomatic fuzzy rough sets*[496]*.*
- *N-fuzzy sets*[634]*: N-fuzzy sets extend traditional fuzzy sets, using the co-domain for membership functions, enabling representation of negative characteristics effectively. Related concepts include Intuitionistic N-fuzzy sets*[2]*.*
- *Convex fuzzy sets* [444, 446]*: Convex fuzzy sets are fuzzy sets where, for any two points x, y and a weight $\alpha \in [0, 1]$, the condition*
  $$\mu(\alpha x + (1 - \alpha)y) \geq \min(\mu(x), \mu(y))$$
  holds, ensuring that the membership function maintains convexity.
- *L-Fuzzy Sets* [449, 592]*: Related concepts include Rough L-fuzzy sets*[674, 304]*, Intuitionistic L-fuzzy sets*[2]*, and Hesitant L -Fuzzy Sets*[219]*.*

- *Three-way fuzzy sets* [721, 347]*: Three-Way Fuzzy Sets incorporate three decision perspectives—acceptance, rejection, and indeterminacy—enhancing decision-making under uncertainty with comprehensive evaluations.*
- *Neuro Fuzzy set* [326, 490, 19]*: A Neuro-fuzzy set combines neural networks and fuzzy logic to create systems capable of handling imprecision and learning complex patterns adaptively.*
- *Fuzzy Alpha-level set* [805, 702, 727]*: An alpha-level set of a fuzzy set is the crisp set containing all elements whose membership degree is at least alpha. Related concepts include alpha-level fuzzy rough sets* [695]*.*

Question 140. What mathematical structures can be observed when the above Fuzzy Set is extended to Neutrosophic Set, Ambiguous Set, Turiyam Neutrosophic Set, Plithogenic Set, and similar concepts?

5.16 | Other Uncertain Set

Various concepts beyond Fuzzy Sets and Neutrosophic Sets are also known. An example is provided below. Additionally, please refer to the Discussion subsection for further set concepts.

Theorem 141. *The following are examples of related sets, including but not limited to:*

- *Vague set* [30, 619, 594]*: A Vague Set is a fuzzy set extension with both truth- and false-membership functions to model uncertainty in memberships. Related concepts include step-vague set*[798, 797]*, Neutrosophic vague set*[84]*, Cubic vague set*[68, 330]*, Complex vague set*[638]*, rough vague sets*[799, 763]*, Complex vague soft set*[236, 802]*, and Neutrosophic Bipolar Vague Set*[354]*.*
- *Turiyam Neutrosophic set*[643, 644, 289, 279] *:A Turiyam Neutrosophic Set extends neutrosophic sets, incorporating four dimensions: truth, indeterminacy, falsity, and a unique "liberal" value for broader uncertainty modeling.*
- *Meta Set* [677, 676, 678]*: Meta sets are generalizations of fuzzy sets that describe imprecise data, structured within classical set theory and enabling hierarchical representation.*
- *Ambiguous Set* [647, 646, 648]*: An Ambiguous Set is characterized by four membership values (true, false, partially true, partially false) to model complex uncertainty and indeterminacy.*
- *Paraconsistent set* [266, 731] *:A paraconsistent set allows contradictory elements by treating membership and non-membership as independent properties, addressing logical inconsistencies in set theory. Related concepts include paraconsistent rough sets* [713, 714]*.*
- *toll sets* [113]*: Toll sets are mathematical constructs where membership is defined through a cost function. These functions map elements to non-negative values, representing "costs" instead of traditional binary inclusion.*
- *Plithogenic Sets* [675, 277]*: Plithogenic Sets are an extension of fuzzy and neutrosophic sets that represent objects with multiple attributes, each associated with varying appurtenance and contradiction degrees. Related concepts include plithogenic hypersoft sets* [470, 230]*, Complex plithogenic set*[641]*, Plithogenic Cubic Sets*[653, 100]*, Plithogenic Offset*[278, 272]*, and Plithogenic Soft Set*[85]*.*

Question 142. Is it possible to transform Ambiguous Sets, Turiyam Neutrosophic Sets, and Plithogenic Sets into Bipolar, Interval-valued, Cubic, and Type structures and apply them to decision-making? Additionally, can these sets be represented as graph concepts?

5.17 | Other Set (Non Uncertain Set)

For reference, we will also introduce the concept of Non-Uncertain Sets. There are numerous concepts of sets that do not handle uncertainty. By combining these concepts with Uncertain Sets(Fuzzy Set, Neutrosophic Set, etc..), we hope to explore whether any new mathematical characteristics or applications can emerge. Some concepts have indeed yielded interesting results when transformed into Uncertain Sets, so we believe there is considerable potential for further exploration.

Theorem 143. *Below are examples of related sets, including but not limited to:*

- *Crisp Set* [669]*: A Crisp Set is a classical set where elements have binary membership; each element either fully belongs (1) or doesn't belong (0).*
- *Hyperfinite Set* [342]*: Hyperfinite sets are large finite sets examined in nonstandard analysis, often employed to study functions or measures with unique properties through nonstandard techniques.*
- *Partition Set* [91, 250]*: A set divided into distinct subsets that collectively represent the entire original set.*
- *Topological Set* [434]*: A set characterized by topological properties.*
- *Stochastic Set* [560, 559, 560]*: A set defined by probabilistic properties, with specified distributions for its elements.*
- *Quantum Set* [522, 687, 189]*: A set in which each element possesses quantum states, incorporating concepts such as superposition and uncertainty. Related notions include quantum fuzzy sets* [466]*.*
- *Power Set* [310, 385]*: A set comprising all possible subsets of a given set. Related concepts include Fuzzy Power Set* [122, 737]*.*
- *Dense Set* [241]*: A set in which there is an element between any two distinct elements. Related concepts include Somewhere Dense Sets*[54, 246]*, Dense Fuzzy Set*[684]*, Triangular dense fuzzy lock sets*[212]*, and Triangular dense fuzzy sets*[211]*.*
- *Boolean Set* [18]*: A set represented using Boolean values. Related concepts include Boolean fuzzy sets*[235, 467]*.*
- *Infinite Set* [685]*: A set with infinitely many elements. While computer science often deals with finite sets, exploring unique characteristics of infinite sets is intriguing.*
- *Affine Set* [125, 126]*: A set that exists within an affine space in linear algebra. Related concepts include Affine Fuzzy Set*[188, 621]*.*
- *Directed Set* [157, 468]*: A set equipped with a direction or orientation. Related concepts include fuzzy Directed Set*[437]*.*
- *Conformal Set* [512, 121]*: A set where elements maintain geometric conformity.*
- *Asymmetric Set* [472]*: A set defined by the absence of symmetry in the relationships between its elements. Related concepts include symmetric set*[341]*.*
- *Nested Set* [514, 260]*: A set structured such that its elements are nested within one another, forming a hierarchical arrangement.*
- *Closed Set*[717, 46, 53]*: A closed set includes all its boundary points, meaning if a sequence within the set converges, its limit also belongs to the set. Related concepts include fuzzy closed sets*[247, 13]*, Fuzzy L-Closed sets*[686, 42]*, Fuzzy W-closed sets*[44]*, intuitionistic fuzzy closed sets*[581, 622]*, Generalized fuzzy closed sets*[205]*, and Fuzzy Neutrosophic Weakly-Generalized Closed Sets*[489]*.*
- *Open Set*[45, 710]*: An open set is one where every point within the set has a neighborhood entirely contained in the set, excluding boundary points. Related concepts include Fuzzy open sets*[633, 344]*, Fuzzy M-Open Sets*[43]*, and hesitant fuzzy open sets*[357]*.*
- *Alternative Set*[716, 563]*: An Alternative Set is a mathematical structure representing multiple options or choices, allowing for diverse potential outcomes or selections. Related concepts include alternative fuzzy set* [770, 392]*.*

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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