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Budget-Constrained Linear Utility Maximization and the Hurwicz (1951) Criterion

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Abstract

We show that for a trader displaying state-dependent risk-neutrality, budget constrained maximization based on the Hurwicz criterion for "non-probabilistic" uncertainty with the degree of pessimism being less than half, reduces to expected utility maximization under "the equal ignorance principle", with all wealth being invested in positive return being available in just one state of nature and nothing at all in other states of nature. Such a portfolio is an extreme point of the set of budget-constrained expected utility maximizing portfolios, with equiprobable states of nature. With more than two uncertain states of nature, the above result extends to the case where the degree of pessimism is equal to half. We also show that if for a degree of pessimism, greater than or equal to half, an optimal solution to budget constrained maximization based on the Hurwicz criterion is not of this type, then it must be a "budget-constrained max-min utility maximizing" portfolio.

Keywords: Budget, Risk-Neutrality, Uncertainty, Portfolio, Non-Probabilistic, Optimal Solution.

1 | Introduction

The context of this paper is the same as in [1], where we "consider a trader in a "bourse" who in the current period is initially endowed with the opportunity of recovering monetary wealth in a future period, the amount recovered in the future period being dependent on the state of nature that will be realized in the future period. The trader does not know which of a non-empty finite set of (mutually exclusive) states of nature is going to be realized in the future period. In the current period, the trader can trade its initial endowments of state-dependent future wealth with other traders in the bourse at current prices, where the price for each state of nature is the payment in the current period for one unit of wealth being recovered in this state of nature. In addition, or the trader can spend the "cash" that it owns, to buy state-dependent future wealth in the bourse. Thus, state-dependent prices at which state-dependent returns of monetary wealth may be traded in the bourse along with the monetary value of the initial wealth available with the trader define the budget constraint faced by the trader. The trader's problem is to choose a "portfolio of future state-dependent wealth" (briefly referred to as a portfolio) that satisfies the budget constraint faced by the trader. Such a portfolio is a "random variable". We do not allow "short selling" in our model. Thus, the budget constraint is equivalent to the assumption that expenditure by the trader on a chosen portfolio cannot exceed the value of its (investible) monetary wealth".

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The trader's utility of monetary wealth in each future state of nature is given by a possibly state-dependent Bernoulli "linear" utility function for wealth. A comprehensive exposition of the early stages of the analysis of decision-making under uncertainty with state-dependent preferences is available in [2]. However, the significance of state-dependent linear utility functions for money is that they fit comfortably with the concept of expected utility based on Ramsey-de Finetti probabilities, with brief discussions along with intuitive motivation of such probabilities being available in [3] and [4]. The Ramsey-de Finetti subjective probability of a "son" (say i) that is assessed by an agent is the price (say p_i) that the agent would be willing to pay for a simple bet that returns one unit of money if 'i' occurs and nothing otherwise so that the expected monetary value of the simple bet to the agent is zero. Thus, if the average utility of money in son 'i' is a constant, say $\mu_i > 0$, then for one unit of money in son i, the agent will be willing to forego $\mu_i p_i$ units of utility and for ξ units of money in son i the agent will be willing to forego $\mu_i p_i \xi$ units of utility, the latter being the utility the agent willingly forgoes for ξ simple bets of the type we have just discussed. ξ simple bets, each of which returns one unit of money if 'i' occurs and nothing otherwise, is identical to a bet that returns ξa unit of money if 'i' occurs and nothing otherwise. Thus, with state-dependent linear utility functions for money, Ramsey-de Finetti probabilities and expected utilities are "perfectly economically consistent" with one another.

In this paper, unlike in [1], we assume that the trader is "non-probabilistically" (as opposed to "probabilistically" or "quantifiably") uncertain about the future states of nature. Probabilistic or quantifiable uncertainty is known as "risk". The context of what is discussed here is not "risk" but "complete ambiguity", which is generally referred to as "uncertainty" (See for instance, [5-7]).

In this paper, we study the choice function which chooses from each budget set those portfolios that maximize the objective function determined by the "Hurwicz criterion" as defined in [8]. The criterion is applied to the profiles of Bernoulli utilities, corresponding to the portfolios that satisfy the budget constraint. The following paragraph paraphrases the one on the "Hurwicz criterion" appearing in the biography of Leonid Hurwicz in Wikipedia (https://en.wikipedia.org/wiki/Leonid_Hurwicz):

First presented in 1950, the Hurwicz criterion combines ideas in [9] with work done in 1812 by Pierre-Simon Laplace (see pages 58-59 of [10]). Hurwicz criterion gives each alternative a value which is "a weighted sum of its worst and best possible" utility or pay-off, across all states of nature. The weight assigned to the worst utility value in each utility profile is known as an index of pessimism. Variations have been proposed ever since and some corrections came very soon from Leonard Jimmie Savage in 1954 (see [5]). These four approaches– Laplace, Wald, Hurwicz, and Savage– have been studied, corrected, and applied for over several decades by many different people including John Milnor, G. L. S. Shackle, Daniel Ellsberg (see [11]), R. Duncan Luce and Howard Raiffa, in a field some date back to Jacob Bernoulli (see [12]).

In this paper, we show that for a trader displaying state-dependent risk-neutrality, budget-constrained maximization based on the Hurwicz criterion with the degree of pessimism being less than half, reduces to expected utility maximization under "the equal ignorance principle", with all wealth being invested in the positive return being available in just one state of nature and nothing at all in other states of nature. Such a portfolio is an extreme point of the set of budget-constrained expected utility maximizing portfolios, with equiprobable states of nature. With more than two uncertain states of nature, the above result extends to the case where the degree of pessimism is equal to half. Such a result seems to be "personally" (i.e., to the author of the paper) to be somewhat "counter-intuitive.

The conclusion that the chosen portfolios maximize expected utility assuming equiprobable states of nature, is undoubtedly very convincing and non-problematic. The counterintuitive aspect of the conclusion is that the trader dumps its entire wealth on a positive return being available in exactly "one" future state of nature, rather than dividing it equally among all states of nature that maximize the "bang per buck". In our context, is the Hurwicz criterion a balancing act between "optimism" and "pessimism" or is it the strategy of a "desperate gambler"?

In a final selection, we show, that if for a degree of pessimism, greater than or equal to half, an optimal solution to budget-constrained maximization based on the Hurwicz criterion is not of the type that maximizes expected utility under "the equal ignorance principle", with all wealth being invested in the positive return being available in just one state of nature, then it must be a "budget-constrained max-min utility-maximizing" portfolio.

2 | Analysis Framework

For some positive integer $L \geq 2$, let $\{1, 2, \ldots, L\}$ denote the finite set of states of nature.

Given x, $y \in \mathbb{R}^L$, let $y^T x$ denote $\sum_{j=1}^{L} y_j x_j$.

A **portfolio of future monetary returns** (briefly referred to as a **portfolio)** is an element $x \in \mathbb{R}^L_+$.

A price vector 'p' is an element of $\mathbb{R}^L_{++}$.

At price vector 'p', a portfolio x can be viewed as an insurance policy which for a total premium of $p^T x$, returns $x_j + p^T x$ units of money in state of nature 'j' for each $j \in \{1, 2, \ldots, L\}$.

The monetary value of the trader's initial endowment m is a strictly positive real number, i.e., m is an element of $\mathbb{R}_{++}$.

A pair $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ is called a **price-wealth pair**.

Notation: In what follows we will let $\Delta^{L-1} = \{p \in \mathbb{R}^L_+ \mid \sum_{j=1}^{L} p_j = 1\}$ denote the L-1 dimensional simplex.

Given a price vector p, **the competitive budget set at** p is the set $\{x \in \mathbb{R}^L_+ \mid p^T x \leq m\}$ denoted B(p, m).

Given a price-wealth pair (p, m) a portfolio $x \in B(p, m)$ is said to be a **feasible portfolio** at (p, m).

Given a state of nature j, an extended real-valued function on the set of non-negative real numbers u_j (i.e., a function u_j with domain being the set of non-negative real numbers and co-domain the set $[-\infty, +\infty)$) such that for all $a > 0$, $u_j(a) > -\infty$ is said to be a **Bernoulli utility function** (for monetary wealth) of the trader, where $u_j(a)$ is the utility of monetary wealth to the trader in state of nature j.

In the case of "non-probabilistic uncertainty", there is a solution procedure based on the Hurwicz "β-pessimism-optimism criterion" which for each $\beta \in [0,1]$, each L- tuple of Bernoulli-utility functions $(u_1, \ldots, u_L)$ and each price-wealth pair (p, m) requires the trader to solve the problem:

Maximize $\beta[\min_{k \in \{1,\ldots,L\}} u_k(x_k)] + (1-\beta)[\max_{k \in \{1,\ldots,L\}} u_k(x_k)]$ subject to $x \in B(p, m)$.

A **linear utility profile** is a vector $\alpha \in \mathbb{R}^L_{++}$ such that the trader's Bernoulli utility function for wealth in state of nature $j \in \{1, \ldots, L\}$ u_j satisfies $u_j(a) = \alpha_j a$ for all non-negative real numbers a.

In the case of "non-probabilistic uncertainty", the Hurwicz "β-pessimism-optimism criterion" for $\beta \in [0,1]$ for a linear utility profile $\alpha \in \mathbb{R}^L_{++}$ requires that at every price-wealth pair (p, m) the trader solves the following problem.

Maximize $\beta[\min_{k \in \{1,\ldots,L\}} \alpha_k x_k] + (1-\beta)[\max_{k \in \{1,\ldots,L\}} \alpha_k x_k]$, subject to $x \in B(p, m)$……**(Problem H-β)**

Note: H-β is meant to be an abbreviation for Hurwicz-β.

The purpose of this note is to show that for $\beta \in [0, \frac{1}{2})$, x solves Problem H-β <u>if and only if</u> there exists $j \in \underset{k \in \{1,\ldots,L\}}{\operatorname{argmax}} \frac{\alpha_k}{p_k}$ such that $x_j = \frac{m}{p_j}$ and $x_k = 0$ for all $k \neq j$. Further if $L > 2$, then the equivalence holds for $\beta = \frac{1}{2}$ as well.

Thus for $\beta \in [0, \frac{1}{2})$, x solves Problem H-β if and only if:

a) x solves

Maximize $\sum_{j=1}^{L} \alpha_j x_j$, subject to $x \in B(p, m)$

and

b) $\{j \mid x_j > 0\}$ is a singleton.

Further, if L > 2, then the equivalence holds for $\beta = \frac{1}{2}$ as well.

3 | Preliminaries

In this section, we provide some lemmas.

Lemma 1: Given $\beta \in [0, 1)$ and a linear utility profile α, consider Problem H-β.

If x solves the above problem and $\max_{k \in \{1,\ldots,L\}} \alpha_k x_k > \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, then $\operatorname{argmax}_{k \in \{1,\ldots,L\}} \alpha_k x_k$ is a singleton.

Proof: If $x \in B(p, m)$ with $\alpha_j x_j = \alpha_h x_h = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k > \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$ for some $j \neq h$, then $\beta[\min_{k \in \{1,\ldots,L\}} \alpha_k y_k] + (1-\beta)[\max_{k \in \{1,\ldots,L\}} \alpha_k y_k] > \beta[\min_{k \in \{1,\ldots,L\}} \alpha_k x_k] + (1-\beta)[\max_{k \in \{1,\ldots,L\}} \alpha_k x_k]$, where $y_i = x_i$ if $\alpha_i x_i \neq \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$, $y_i = \frac{\min_{k \in \{1,\ldots,L\}} \alpha_k y_k}{\alpha_i}$, for all $i \neq j$ satisfying $\alpha_i x_i = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$ and $y_j = \frac{1}{p_j}(m - \sum_{i \neq j} p_i y_i)$. For $i \neq j$ satisfying $\alpha_i x_i = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$, $y_i < x_i$ and $y_i = x_i$ if $\alpha_i x_i \neq \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$.

Thus, $\sum_{i \neq j} p_i y_i < \sum_{i \neq j} p_i x_i$ and hence $y_j > x_j$.

Clearly, $\min_{k \in \{1,\ldots,L\}} \alpha_k y_k = \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, $\max_{k \in \{1,\ldots,L\}} \alpha_k y_k = \alpha_j y_j > \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$.

Thus, for any solution $x \in B(p, m)$, to the above maximization problem, if $\max_{k \in \{1,\ldots,L\}} \alpha_k x_k > \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, then there exists a unique $j \in \{1,\ldots,L\}$ such that $\alpha_j x_j = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$. Q.E.D.

Lemma 2: Let $\beta \in [0, 1)$ and suppose x solves Problem H-β. If $j \notin \operatorname{argmax}_{k \in \{1,\ldots,L\}} \alpha_k x_k$, then $j \in \operatorname{argmin}_{k \in \{1,\ldots,L\}} \alpha_k x_k$.

Proof: By lemma 1, $\operatorname{argmax}_{k \in \{1,\ldots,L\}} \alpha_k x_k$ is a singleton. Without loss of generality suppose $\operatorname{argmax}_{k \in \{1,\ldots,L\}} \alpha_k x_k = \{1\}$.

Thus, $\alpha_1 x_1 = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$ and hence $\alpha_1 x_1 > \alpha_j x_j$ for all $j > 1$.

Suppose, there exists $j > 1$, such that $\alpha_1 x_1 > \alpha_j x_j > \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$. Without loss of generality suppose $j = 2$.

Let $y \in B(p,m)$ be such that $y_k = x_k$ for all $k > 2$, $y_2 = \frac{\min_{k \in \{1,\ldots,L\}} \alpha_k x_k}{\alpha_2}$ and $y_1 = x_1 + \frac{p_2(x_2 - y_2)}{p_1} > x_1$, since $x_2 > y_2$.

Thus, $(1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k y_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = (1-\beta)\alpha_1 y_1 + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = (1-\beta)\alpha_1 y_1 + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k > (1-\beta)\alpha_1 x_1 + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k = (1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, contradicting the optimality of x.

Thus, $\alpha_1 x_1 > \alpha_j x_j = \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$ for all $j > 1$. Q.E.D.

Lemma 3: Let $\beta \in [0, \frac{1}{2}]$ and suppose x solves Problem H-β.

Then, $\min_{k \in \{1,\ldots,L\}} \alpha_k x_k > 0$ implies $\frac{\alpha_j}{p_j} \leq \frac{\alpha_h}{p_h}$ for all $\alpha_h x_h = \min_{k \in \{1,\ldots,L\}} \alpha_k x_k > 0$ and $\alpha_j x_j = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k > 0$.

Proof: Suppose x∈B(p,m) suppose $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$. Thus, $\max_{k\in\{1,\ldots,L\}} \alpha_k x_k \geq \min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$.

Let $\alpha_h x_h = \min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$ and $\alpha_j x_j = \max_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$.

Let $\delta > 0$ be such that $x_h - \delta \geq 0$ and let y∈B(p,m) be such that $y_h = x_h - \delta$, $y_j = x_j + \frac{\delta p_h}{p_j}$, $y_k = x_k$ for $k \neq j,h$.

Then $\max_{k\in\{1,\ldots,L\}} \alpha_k y_k - \max_{k\in\{1,\ldots,L\}} \alpha_k x_k = \frac{\delta\alpha_j p_h}{p_j}$ and $\min_{k\in\{1,\ldots,L\}} \alpha_k y_k - \min_{k\in\{1,\ldots,L\}} \alpha_k x_k = \alpha_h\delta$.

Hence, if $\frac{\alpha_j}{p_j} > \frac{\alpha_h}{p_h}$ and $\beta\in[0, \frac{1}{2}]$, then $\{\beta[\min_{k\in\{1,\ldots,L\}} \alpha_k y_k] + (1-\beta)[\max_{k\in\{1,\ldots,L\}} \alpha_k y_k]\} - \{\beta[\min_{k\in\{1,\ldots,L\}} \alpha_k x_k] + (1-\beta)[\max_{k\in\{1,\ldots,L\}} \alpha_k x_k]\} = (1-\beta)\frac{\delta\alpha_j p_h}{p_j} - \beta\, \alpha_h\delta = \delta p_h[(1-\beta)\frac{\alpha_j}{p_j} - \beta\frac{\alpha_h}{p_h}] > 0$.

Thus, if x solves the above maximization problem in the statement of the lemma, then $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$ implies $\frac{\alpha_j}{p_j} \leq \frac{\alpha_h}{p_h}$ for all $\alpha_h x_h = \min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$ and $\alpha_j x_j = \max_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$. Q.E.D.

Lemma 4: Let $\beta\in[0, \frac{1}{2})$. If x solves Problem H-β, then it must be the case that $\max_{k\in\{1,\ldots,L\}} \alpha_k x_k > \min_{k\in\{1,\ldots,L\}} \alpha_k x_k$.

Proof: Suppose, $\max_{k\in\{1,\ldots,L\}} \alpha_k x_k = \min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$.

By lemma 3 and symmetry, it must be that $\frac{\alpha_j}{p_j} = \frac{\alpha_h}{p_h}$ for all h, j∈{1, …, L}.

Thus, $x_j = \frac{m}{\alpha_j \sum_{k=1}^{L}\frac{p_k}{\alpha_k}}$, for j = 1,…, L.

Let y∈B(p, m) be such that $y_j = x_j$ for j∈{3,…,L}, $y_2 = 0$, and $y_1 = \frac{m}{\alpha_1 \sum_{k=1}^{L}\frac{p_k}{\alpha_k}} + \frac{m p_2}{p_1\alpha_2 \sum_{k=1}^{L}\frac{p_k}{\alpha_k}}$.

Thus, $\min_{k\in\{1,\ldots,L\}} \alpha_k y_k = 0$ and $\max_{k\in\{1,\ldots,L\}} \alpha_k y_k = \frac{m}{\sum_{k=1}^{L}\frac{p_k}{\alpha_k}} + \frac{\alpha_1 m p_2}{p_1\alpha_2 \sum_{k=1}^{L}\frac{p_k}{\alpha_k}} = \frac{2m}{\sum_{k=1}^{L}\frac{p_k}{\alpha_k}}$, since $\frac{\alpha_j}{p_j} = \frac{\alpha_h}{p_h}$ implies $\frac{\alpha_1 m p_2}{p_1\alpha_2 \sum_{k=1}^{L}\frac{p_k}{\alpha_k}} = \frac{m}{\sum_{k=1}^{L}\frac{p_k}{\alpha_k}}$.

Thus, for $\beta[(0, \frac{1}{2})$, $(1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k y_k + \beta\min_{k\in\{1,\ldots,L\}} \alpha_k y_k = (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k y_k = \frac{2(1-\beta)m}{\sum_{k=1}^{L}\frac{p_k}{\alpha_k}} > \frac{m}{\sum_{k=1}^{L}\frac{p_k}{\alpha_k}} = (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k\in\{1,\ldots,L\}} \alpha_k x_k$.

Hence, for $\beta\in[0, \frac{1}{2})$, if x∈B(p, m) solves the above maximization problem, then it must be the case that $\max_{k\in\{1,\ldots,L\}} \alpha_k x_k > \min_{k\in\{1,\ldots,L\}} \alpha_k x_k$. Q.E.D.

Lemma 5: Let $\beta\in[0, \frac{1}{2})$. If x solves Problem H-β, then it must be the case that $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k = 0$.

Proof: Towards a contradiction suppose, $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$. By lemma 1, $\operatorname{armax}_{k\in\{1,\ldots,L\}} \alpha_k x_k$ is a singleton. Without loss of generality suppose $\operatorname{armax}_{k\in\{1,\ldots,L\}} \alpha_k x_k = \{1\}$. By lemma 4, $\alpha_1 x_1 > \min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$. Thus, $x_j > 0$ for all j.

By lemma 3, $\frac{\alpha_1}{p_1} \leq \frac{\alpha_j}{p_j}$ for all j > 1.

Suppose, $\frac{\alpha_1}{p_1} < \frac{\alpha_j}{p_j}$ for some j > 1.

Without loss of generality suppose, $\frac{\alpha_1}{p_1} < \frac{\alpha_2}{p_2}$.

Let $y \in B(p,m)$ be such that $y_j = x_j$ for all $j > 2$, $y_1 = \frac{\alpha_2 x_2}{\alpha_1} = \frac{\min_{k \in \{1,\ldots,L\}} \alpha_k x_k}{\alpha_1}$, $y_2 = x_2 + \frac{p_1(x_1 - y_1)}{p_2}$.

$\alpha_2 y_2 = \alpha_2 x_2 + \alpha_2 \frac{p_1(x_1-y_1)}{p_2} = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 \frac{p_1(x_1-y_1)}{p_2} - \alpha_1 x_1 > \alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 \frac{p_1(x_1-y_1)}{p_1} - \alpha_1 x_1 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_1(x_1 - y_1) - \alpha_1 x_1 = \alpha_1 x_1$, since $\alpha_2 x_2 = \alpha_1 y_1$.

Thus, $\max_{k \in \{1,\ldots,L\}} \alpha_k y_k = \alpha_2 y_2 > \alpha_1 x_1 = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k$ and $\alpha_1 y_1 = \alpha_j y_j = \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$ for all $j > 2$.

Thus, $(1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k y_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k y_k > (1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, contradicting the optimality of x.

Thus, it must be the case that $\frac{\alpha_1}{p_1} = \frac{\alpha_j}{p_j}$ for all $j > 1$.

Now, $(1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k = (1-\beta)\, \alpha_1 x_1 + \beta \alpha_2 x_2$, since $\operatorname{argmin}_{k \in \{1,\ldots,L\}} \alpha_k x_k = \{2,\ldots, L\}$.

This, time let $y \in B(p,m)$ be such that $y_1 = \frac{m}{p_1}$ and $y_j = 0$ for all $j > 1$.

$(1-\beta)\alpha_1 y_1 = (1-\beta)\alpha_1 \frac{m}{p_1} = (1-\beta)\frac{\alpha_1}{p_1} m \geq (1-\beta)\frac{\alpha_1}{p_1}(p_1 x_1 + p_2 x_2) = (1-\beta)\frac{\alpha_1}{p_1} p_1 x_1 + (1-\beta)\frac{\alpha_2}{p_2} p_2 x_2$, since $\frac{\alpha_1}{p_1} = \frac{\alpha_2}{p_2}$.

However, $1-\beta > \beta$ and $\alpha_2 x_2 > 0$ implies $(1-\beta)\frac{\alpha_1}{p_1} p_1 x_1 + (1-\beta)\frac{\alpha_2}{p_2} p_2 x_2 = (1-\beta)\alpha_1 x_1 + (1-\beta)\alpha_2 x_2 > (1-\beta)\alpha_1 x_1 + \beta \alpha_2 x_2$.

Thus, $(1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k y_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k y_k > (1-\beta)\max_{k \in \{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, contradicting the optimality of x.

Thus, it must be the case that $\min_{k \in \{1,\ldots,L\}} \alpha_k x_k = 0$. Q.E.D.

Lemma 6: Let $\beta \in [0, \frac{1}{2})$. If x solves Problem H-$\beta$, then it must be the case that $x_j = \frac{m}{p_j}$ for some $j \in \operatorname{argmax}_{k \in \{1,\ldots,L\}} \frac{\alpha_k}{p_k}$ and $x_k = 0$ for all $k \neq j$.

Proof: From lemma 5, it follows that if x solves Problem H-β, then it solves the problem.

Maximize $\max_{k \in \{1,\ldots,L\}} \alpha_k x_k$ subject to $x \in B(p,m)$.

The lemma follows immediately from this observation. Q.E.D.

Lemma 7: If $L > 2$ and portfolio x is such that $x_j = \frac{m}{p_j}$ for some $j \in \operatorname{argmax}_{k \in \{1,\ldots,L\}} \frac{\alpha_k}{p_k}$ and $x_k = 0$ for all $k \neq j$, then x solves Problem H-$\frac{1}{2}$.

Proof: Without loss of generality suppose $\{1\} = \operatorname{argmax}_{k \in \{1,\ldots,L\}} \frac{\alpha_k}{p_k}$.

By, lemma 2, $j \in \operatorname{argmin}_{k \in \{1,\ldots,L\}} \alpha_k x_k$ for all $j > 1$.

Let $y_1 = \frac{m}{p_1}$ and $y_j = 0$ for all $j > 1$.

Clearly $[\min_{k \in \{1,\ldots,L\}} \alpha_k y_k] + [\max_{k \in \{1,\ldots,L\}} \alpha_k y_k] = \frac{1}{2}\alpha_1 \frac{m}{p_1}$.

For $\delta > 0$, with $\delta \leq \frac{m}{p_1}$, let z be the L-vector defined as follows: $z_1 = \frac{m}{p_1} - \delta$, $z_j \geq 0$ for $j > 1$, such that $p_1\delta = \sum_{j=2}^{L} p_j z_j$. Suppose that for all $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$, it is the case that $z_j < \frac{m}{p_j}$.

Thus $\alpha_j z_j < m(\underset{k \in \{1,\ldots,L\}}{\max} \frac{\alpha_k}{p_k})$ for all $j = 1,\ldots,L$.

Case 1: $\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k = \alpha_1 z_1$.

Without loss of generality suppose, $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k = \alpha_2 z_2$.

Thus, $\alpha_2 z_2 + \alpha_1 z_1 = \alpha_2 z_2 + \alpha_1 \frac{m}{p_1} - \alpha_1\delta = \alpha_1 \frac{m}{p_1} - [\alpha_1\delta - \alpha_2 z_2]$.

Now, $\alpha_1\delta - \alpha_2 z_2 > \alpha_1\delta - \alpha_2(\frac{p_1\delta}{p_2})$, since when $L > 2$, then either $z_2 = 0 < \frac{p_1\delta}{p_2}$ or $z_2 > 0$ with $z_2 < \frac{p_1\delta}{p_2}$.

Thus, $\alpha_1\delta - \alpha_2 z_2 > \alpha_1\delta - \alpha_2(\frac{p_1\delta}{p_2}) = \alpha_1\delta - \alpha_2(\frac{p_1\delta}{p_2}) = \delta p_1[\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}] \geq 0$ since $1 \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$.

Hence, $\alpha_1\delta - \alpha_2 z_2 > 0$, so that $\alpha_2 z_2 + \alpha_1 z_1 = \alpha_1 \frac{m}{p_1} - [\alpha_1\delta - \alpha_2 z_2] < \alpha_1 \frac{m}{p_1}$.

Case 2: $\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k \neq \alpha_1 z_1$.

Thus $\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k = \alpha_i z_i$ for some $i \neq 1$ and $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k = \alpha_j z_j$, for some $j \neq i$.

Thus, $\alpha_i z_i > \alpha_1 z_1 = \underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k = \alpha_j z_j$, for some $j \neq i$.

If $\alpha_1 z_1 > \underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k$, then by considering the portfolio w with $w_1 = \frac{\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k}{\alpha_1}$, $w_i = z_i + \frac{p_1(z_1 - w_1)}{p_i}$, $w_k = z_k$ for all $k \in \{1,\ldots,L\} \setminus \{1,i\}$, we get $w \in B(p,m)$, $\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k w_k = \alpha_i w_i > \underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k$ and $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k w_k = \underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k$.

Thus, $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k w_k + \underset{k \in \{1,\ldots,L\}}{\max} \alpha_k w_k > \underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k + \underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k$.

Hence, if we can show that $[\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k y_k] + [\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k y_k] > [\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k] + [\underset{k \in \{1,\ldots,L\}}{max} \alpha_k z_k]$, where $\underset{k \in \{1,\ldots,L\}}{\max} \alpha_k z_k = \alpha_i z_i$ for some $i \neq 1$ and $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k = \alpha_1 z_1$, then we are done.

Without loss of generality suppose $i = 2$.

Since $L > 2$ and $\underset{k \in \{1,\ldots,L\}}{\min} \alpha_k z_k < \alpha_2 z_2$, it must be the case that $\alpha_1 z_1 \leq \alpha_3 z_3 \leq \alpha_2 z_2$.

If $z_1 = 0$, then

Thus, $\alpha_2 z_2 + \alpha_1 z_1 = \alpha_2 z_2 < m(\underset{k \in \{1,\ldots,L\}}{\max} \frac{\alpha_k}{p_k}) = \alpha_1 \frac{m}{p_1}$.

If $z_1 > 0$, then $z_3 > 0$ so that $z_2 < \frac{p_1\delta}{p_2}$.

Thus, $\alpha_2 z_2 + \alpha_1 z_1 < \alpha_2 \frac{p_1\delta}{p_2} + \alpha_1(\frac{m}{p_1} - \delta) = \alpha_1 \frac{m}{p_1} - \delta p_1(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}) \leq \alpha_1 \frac{m}{p_1}$, since $\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2} \geq 0$.

Hence, $\alpha_2 z_2 + \alpha_1 z_1 < \alpha_1 \frac{m}{p_1}$.

Thus, y solves Problem H-$\frac{1}{2}$. Q.E.D.

4 | Main Results

We provide below the main result with its proof.

Proposition 1: Let $\beta \in [0, \frac{1}{2})$. Then, x solves Problem H-β if and only if there exists $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$ such that $x_j = \frac{m}{p_j}$ and $x_k = 0$ for all $k \neq j$. Further if $L > 2$, then the equivalence holds for $\beta = \frac{1}{2}$ as well.

Proof: Let $\beta \in [0, \frac{1}{2})$. Suppose x solves Problem H-β. By lemma 6, it must be the case that $x_j = \frac{m}{p_j}$ for some $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$ and $x_k = 0$ for all $k \neq j$.

Let y be a portfolio, such that $y_i = \frac{m}{p_i}$ for some $i \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$ and $y_k = 0$ for all $k \neq i$.

Thus, $y \in B(p,m)$, $\max_{k \in \{1,\ldots,L\}} \alpha_k y_k = \max_{k \in \{1,\ldots,L\}} \alpha_k x_k = m \max_{k \in \{1,\ldots,L\}} \frac{\alpha_k}{p_k}$, $\min_{k \in \{1,\ldots,L\}} \alpha_k y_k = \min_{k \in \{1,\ldots,L\}} \alpha_k x_k = 0$.

Thus, $(1-\beta) \max_{k \in \{1,\ldots,L\}} \alpha_k y_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = (1-\beta) \max_{k \in \{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k \in \{1,\ldots,L\}} \alpha_k x_k$, and hence y solves Problem H-β.

Thus, x solves Problem H-β if and only if there exists $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$ such that $x_j = \frac{m}{p_j}$ and $x_k = 0$ for all $k \neq j$.

Now suppose $L > 2$. By lemma 7, if x is a portfolio such that $x_j = \frac{m}{p_j}$ for some $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$, and $x_k = 0$ for all $k \neq j$, then x solves H-$\frac{1}{2}$.

Now, suppose y solves H-$\frac{1}{2}$.

By lemmas 1 and 2, either $\max_{k \in \{1,\ldots,L\}} \alpha_k y_k = \min_{k \in \{1,\ldots,L\}} \alpha_k y_k$ or there exists a unique j such that

$\alpha_j y_j = \max_{k \in \{1,\ldots,L\}} \alpha_k y_k > \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = \alpha_i y_i$, for $i \neq j$.

If $\max_{k \in \{1,\ldots,L\}} \alpha_k y_k = \min_{k \in \{1,\ldots,L\}} \alpha_k y_k$, then $y_j = \frac{m}{\alpha_j \sum_{k=1}^{L} \frac{p_k}{\alpha_k}}$, for $j = 1,\ldots, L$.

Clearly $\frac{m}{\sum_{k=1}^{L} \frac{p_k}{\alpha_k}} < \frac{\alpha_j m}{p_j}$, for all $j = 1,\ldots, L$.

Further, $\min_{k \in \{1,\ldots,L\}} \frac{p_k}{\alpha_k} \leq \frac{p_j}{\alpha_j}$ for all $j = 1,\ldots, L$.

Thus, $L \min_{k \in \{1,\ldots,L\}} \frac{p_k}{\alpha_k} \leq \sum_{k=1}^{L} \frac{p_k}{\alpha_k}$.

Hence, for $L > 2$, $\min_{k \in \{1,\ldots,L\}} \frac{p_k}{\alpha_k} < \frac{1}{2} \sum_{k=1}^{L} \frac{p_k}{\alpha_k}$.

Thus, for $L > 2$, $\frac{m}{\sum_{k=1}^{L} \frac{p_k}{\alpha_k}} < \frac{1}{2} \frac{\alpha_j m}{p_j}$ for all $j \in \underset{k \in \{1,\ldots,L\}}{\text{argmax}} \frac{\alpha_k}{p_k}$.

This contradicts the optimality of y for H-$\frac{1}{2}$.

Thus, it must be that exists a unique j such that

$\alpha_j y_j = \max_{k \in \{1,\ldots,L\}} \alpha_k y_k > \min_{k \in \{1,\ldots,L\}} \alpha_k y_k = \alpha_i y_i$, for $i \neq j$.

Without loss of generality suppose, $j = 1$.

Thus, $\alpha_1 y_1 + \alpha_2 y_2 = m(\max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k})$ with $y \in B(p,m)$ and $\alpha_1 y_1 > \alpha_2 y_2$, which is not possible unless $\alpha_2 y_2 = 0$ and $\alpha_1 y_1 = m(\max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k})$.

But this implies $y_1 = \frac{m(\max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k})}{\alpha_1}$, $y_i = 0$ for all $i > 1$.

Since $p_1 y_1 \le m$, we get $m \ge p_1 y_1 = p_1(\frac{m(\max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k})}{\alpha_1})$, i.e., $\frac{\alpha_1}{p_1} \ge \max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k}$.

Thus, $\frac{\alpha_1}{p_1} = \max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k}$.

Further, $p_1 y_1 = p_1 \frac{m(\max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k})}{\alpha_1} = m$, since $\frac{\alpha_1}{p_1} = \max_{k\in\{1,\ldots,L\}} \frac{\alpha_k}{p_k}$.

This proves the proposition. Q.E.D.

Note: Tempting though it may be to conjecture otherwise, there does not seem to be a result analogous to Proposition 1, for continuous (fractional) knapsack problems (as in [13]), whereas for a linear utility profile $\alpha \in \mathbb{R}^L_{++}$ and price-wealth pair (p, m) the constraints are of the form $x \in B(p,m)$, $0 \le x_j \le 1$ for all $j \in \{1, \ldots, L\}$. The following example illustrates the problem.

Example 1: Let $L = 3$, $\alpha_1 > \alpha_2 > \alpha_3$, $p_j = 1$ for $j = 1, 2, 3$, $m = 2\frac{1}{2}$ and $\beta = \frac{1}{4}$.

The unique optimal solution to

Maximize $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, subject to $x_1 + x_2 + x_3 = 2\frac{1}{2}$, $0 \le x_j \le 1$ for $j = 1, 2, 3$ is

$x_1 = 1$, $x_2 = 1$, $x_3 = \frac{1}{2}$. Here $x_2 > x_3$.

However, the unique optimal solution to

Maximize $[\frac{1}{4}\min\{\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3\} + \frac{3}{4}\max\{\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3\}]$ subject to $x_1 + x_2 + x_3 = 2\frac{1}{2}$, $0 \le x_j \le 1$ for $j = 1, 2, 3$ is $x_1 = 1$, $x_2 = \frac{3}{2}(\frac{\alpha_3}{\alpha_2+\alpha_3})$, $x_3 = \frac{3}{2}(\frac{\alpha_2}{\alpha_2+\alpha_3})$ and now $x_2 < x_3$.

So, the optimal solutions for the two problems are different.

5 | Hurwicz Criterion and Max-Min

The above leaves open the question: what happens when a portfolio x that solves H-β is such that $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$? From Proposition 1 we know that this is possible provided $\beta \ge \frac{1}{2}$ and if $L > 2$, then only if $\beta > \frac{1}{2}$ and it is certainly the case when $\beta = 1$. The following proposition provides an answer to the question.

Proposition 2: For $\beta \in [\frac{1}{2}, 1)$ suppose x solves Problem H-β and satisfies $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$.

If for some $j \in \operatorname{argmax}_{k\in\{1,\ldots,L\}} \alpha_k x_k$, it is the case that $(1-\beta)\alpha_j \frac{m}{p_j} \ne (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta \min_{k\in\{1,\ldots,L\}} \alpha_k x_k$, then $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k = \max_{k\in\{1,\ldots,L\}} \alpha_k x_k = \alpha_j x_j$, with $x_j = \frac{m}{\alpha_j \sum_{k=1}^{L} \frac{p_k}{\alpha_k}}$ for all $j \in \{1,\ldots,L\}$.

Proof: For $\beta \in [\frac{1}{2}, 1)$ suppose x solves Problem H-β and satisfies $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$. We know that at any solution of H-β, the budget constraint must be binding. Thus, $\sum_{j=1}^{L} p_j x_j = m$. Further, $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k > 0$ implies $x_k > 0$ for $k \in \{1,\ldots, L\}$. Thus $p_k x_k > 0$ for $k \in \{1,\ldots, L\}$.

By lemma 1 we know that $\underset{k\in\{1,\ldots,L\}}{\operatorname{argmax}} \alpha_k x_k$ is a singleton. Without loss of generality suppose $\underset{k\in\{1,\ldots,L\}}{\operatorname{argmax}} \alpha_k x_k = \{1\}$. By lemma 2, it follows that $\underset{k\in\{1,\ldots,L\}}{\operatorname{argmin}} \alpha_k x_k = \{2,\ldots, L\}$.

Suppose $(1-\beta)\alpha_j\frac{m}{p_j} \neq (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k$ and towards a contradiction suppose $\max_{k\in\{1,\ldots,L\}} \alpha_k x_k > \min_{k\in\{1,\ldots,L\}} \alpha_k x_k$.

If $(1-\beta)\alpha_j\frac{m}{p_j} > (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k$, then x could not be a solution for H-β, since the portfolio whose j^{th} coordinate is $\frac{m}{p_j}$ and all other coordinates 0, satisfies the budget constraint and gives a higher value for the objective function of H-β than x does. Hence it must be that $(1-\beta)\alpha_j\frac{m}{p_j} < (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k$.

For $\varepsilon > 0$ satisfying $\alpha_1(x_1 - \varepsilon(\frac{m-p_1x_1}{p_1})) > (1+\varepsilon)\alpha_j x_j$ for all $j > 1$, consider the portfolio y, whose j^{th} coordinate for $j > 1$ is $(1+\varepsilon)x_j$ and the first coordinate is $x_1 - \varepsilon(\frac{m-p_1x_1}{p_1})$. Clearly $0 < x_1 - \varepsilon(\frac{m-p_1x_1}{p_1}) < x_1$ and $\alpha_j x_j < (1+\varepsilon)\alpha_j x_j$ for all $j > 1$.

$(\beta[\min_{k\in\{1,\ldots,L\}} \alpha_k y_k] + (1-\beta)[\max_{k\in\{1,\ldots,L\}} \alpha_k y_k]) - (\beta[\min_{k\in\{1,\ldots,L\}} \alpha_k x_k] + (1-\beta)[\max_{k\in\{1,\ldots,L\}} \alpha_k x_k]) = \varepsilon\beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k + \varepsilon(1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k - \varepsilon(1-\beta)\alpha_1\frac{m}{p_1} = \varepsilon[\beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k + (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k - (1-\beta)\alpha_1\frac{m}{p_1}] > 0$.

Since $y\in B(p,m)$, the optimality of x for H-β is contradicted.

Hence, $\min_{k\in\{1,\ldots,L\}} \alpha_k x_k = \max_{k\in\{1,\ldots,L\}} \alpha_k x_k = \alpha_j x_j$, with $x_j = \frac{m}{\alpha_j \sum_{k=1}^{L}\frac{p_k}{\alpha_k}}$ for all $j\in\{1,\ldots,L\}$. Q.E.D.

The following example illustrates that the requirement $(1-\beta)\alpha_j\frac{m}{p_j} \neq (1-\beta)\max_{k\in\{1,\ldots,L\}} \alpha_k x_k + \beta\min_{k\in\{1,\ldots,L\}} \alpha_k x_k$ in the statement of Proposition 2 is non-trivial.

Example 2: Let $L = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, $m = 1$, $p_1 = p_2 = 1$. Suppose $\beta = \frac{2}{3}$. For $(x_1, x_2)\in B(p,m)$ satisfying $x_1 + x_2 = 1$ and $x_1 \geq x_2$, we have $(1-\frac{2}{3})2x_1 \geq \frac{2}{3}x_2$ and $(1-\frac{2}{3})2x_1 + \frac{2}{3}x_2 = \frac{2}{3}$ and all of them solve the problem:

Maximize $[(1-\beta)\max_{k\in\{1,2\}} \alpha_k x_k + \beta\min_{k\in\{1,2\}} \alpha_k x_k]$

Subject to

$x_1 + x_2 \leq 1$,

$x_1 \geq 0, x_2 \geq 0$.

For the portfolio $(\frac{3}{4}, \frac{1}{4})$, $(1-\beta)\max_{k\in\{1,2\}} \alpha_k x_k = \frac{3}{2} > \frac{1}{6} = \beta\min_{k\in\{1,2\}} \alpha_k x_k > 0$, and yet $(\frac{3}{4}, \frac{1}{4})$ solves H-$\frac{2}{3}$.

6 | Conclusion

We demonstrate that, under "the equal ignorance principle," budget-constrained maximization based on the Hurwicz criterion for "non-probabilistic" uncertainty—where the degree of pessimism is less than half—reduces to expected utility maximization for a trader exhibiting state-dependent risk-neutrality. In this case, all wealth is invested in positive returns being available in a single state of nature, and nothing at all in other states.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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