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# Pythagorean, Fermatean, and Complex Turiyam Neutrosophic Graphs

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## Abstract

Graph theory examines networks composed of nodes (vertices) and their connections (edges). A graph class is defined by shared structural properties governed by specific rules or constraints. This paper explores uncertain graph models, with a focus on Pythagorean, Fermatean, and Complex Turiyam Neutrosophic Graphs, which extend Neutrosophic Graphs to more effectively address uncertainty. Potential extensions using General Plithogenic Graphs are also discussed.

**Keywords:** Neutrosophic graph, Fuzzy graph, Plithogenic Graph, Pythagorean Turiyam Neutrosophic Graph, Complex Turiyam Neutrosophic Graph

## 1 | Introduction

### 1.1 | Uncertain Graph Theory

Graphs have been studied for over 200 years, and graph theory has now gained widespread recognition. Graph theory focuses on networks made up of nodes (vertices) and their connections (edges) [23]. It has been extensively explored due to its wide range of applications across various fields, including real-world systems [23].

This paper delves into models of uncertain graphs, such as Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam Neutrosophic, and Plithogenic Graphs, which are designed to manage uncertainty in diverse contexts. Collectively known as uncertain graphs, these models extend classical graph theory by incorporating varying degrees of uncertainty [53, 51].

This paper focuses on the use of derived classes of Neutrosophic Graphs, specifically Pythagorean Neutrosophic Graphs, Fermatean Neutrosophic Graphs, and Complex Neutrosophic Graphs. Pythagorean Neutrosophic Graphs extend conventional graphs by assigning Pythagorean membership values to vertices and edges, incorporating squared values of truth, indeterminacy, and falsity degrees [1, 16, 19]. Fermatean Neutrosophic

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Graphs further enhance uncertainty modeling by using cubed values of these degrees [52, 14, 17]. Complex Neutrosophic Graphs utilize complex-valued truth, indeterminacy, falsity state degrees, enabling a more comprehensive representation of uncertainty[5, 72, 44, 48, 3].

For further information, please refer to the relevant survey notes [26, 27, 30].

### 1.2 | Our Contribution in this paper

The above discussion underscores the significance of research on Uncertain Graphs. However, the exploration of Turiyam Neutrosophic Graphs is still in its early stages. This study introduces and analyzes new concepts of Fermatean, Pythagorean, and Complex Turiyam Neutrosophic Graphs. Turiyam Neutrosophic Graphs assign four values—truth, indeterminacy, falsity, and liberal state—to each vertex and edge, capturing complex relationships [36, 26, 30]. Additionally, this paper explores potential extensions through General Plithogenic Graphs (cf. [30]).

## 2 | Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper.

### 2.1 | Basic Graph Concepts

Here, we present some basic concepts of graph theory. For more foundational concepts and notations, please refer to lecture notes, surveys, or introductory texts such as [23].

**Definition 1** (Graph)**.** [23] A graph $G$ is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set.

**Definition 2** (Subgraph)**.** [23] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of $G$ is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of $H$ is a subset of the vertex set of $G$.
- $E_H \subseteq E$, i.e., the edge set of $H$ is a subset of the edge set of $G$.
- Each edge in $E_H$ connects vertices in $V_H$.

**Definition 3** (Degree)**.** [23] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to $v$. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into $v$, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of $v$.

### 2.2 | Fuzzy, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we explore Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The following definitions include related concepts. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing "Contradiction" with "Liberal" [55]. For more details, please refer to the survey notes [26, 27, 30].

**Definition 4.** (cf.[53]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- $V$ is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form $(v, v)$) and edges represent binary relationships between distinct vertices.

**Definition 5** (Unified Uncertain Graphs Framework)**.** (cf.[26]) Let $G = (V, E)$ be a classical graph with a set of vertices $V$ and a set of edges $E$. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

(1) *Fuzzy Graph* [53]*:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of $v$ in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between $u$ and $v$.

(2) *Intuitionistic Fuzzy Graph (IFG)* [4]*:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $v_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $v_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + v_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [42, 9, 54, 39]*:*

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
  - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
  - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
  - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
  - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between $u$ and $v$.

(4) *Turiyam Neutrosophic Graph* [36]*:*

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
  - $t(v) \in [0, 1]$ is the truth value,
  - $iv(v) \in [0, 1]$ is the indeterminacy value,
  - $fv(v) \in [0, 1]$ is the falsity value,
  - $lv(v) \in [0, 1]$ is the liberal state value,
  - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between $u$ and $v$.

(5) *Vague Graph* [6]*:*

- Each vertex $v \in V$ is assigned a pair $(\tau(v), \phi(v))$, where $\tau(v) \in [0, 1]$ is the degree of truth-membership and $\phi(v) \in [0, 1]$ is the degree of false-membership, with $\tau(v) + \phi(v) \leq 1$.
- The grade of membership is characterized by the interval $[\tau(v), 1 - \phi(v)]$.

- Each edge $e = (u, v) \in E$ is assigned a pair $(\tau(e), \phi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\}, \quad \phi(e) \geq \max\{\phi(u), \phi(v)\}.$$

(6) *Hesitant Fuzzy Graph* [70]*:*

- Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0, 1]$, denoted $\sigma(v) \subseteq [0, 1]$.
- Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0, 1]$.
- Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.

(7) *Single-Valued Pentapartitioned Neutrosophic Graph* [21, 40]*:*

- Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
  - $T(v) \in [0, 1]$ is the truth-membership degree.
  - $C(v) \in [0, 1]$ is the contradiction-membership degree.
  - $R(v) \in [0, 1]$ is the ignorance-membership degree.
  - $U(v) \in [0, 1]$ is the unknown-membership degree.
  - $F(v) \in [0, 1]$ is the false-membership degree.
  - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
- Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

## 2.3 | Plithogenic Graphs

Plithogenic Graphs have been introduced as an extension of Fuzzy Graphs and Turiyam Neutrosophic Graphs, broadening the concept to encompass Plithogenic Sets [58]. These graphs have become a prominent subject of ongoing research and development [65, 43, 26]. The formal definition is provided below.

**Definition 6.** [65] Let $G = (V, E)$ be a crisp graph where $V$ is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- $l$ is an attribute associated with the vertices.
- $Ml$ is the range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- $m$ is an attribute associated with the edges.
- $Nm$ is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph $PG$ must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf\left((xy),(a,b)\right) \leq \min\{adf(x,a), adf(y,b)\}$$

where $xy \in N$ is an edge between vertices $x$ and $y$, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\left((a,b),(c,d)\right) \leq \min\{aCf(a,c), aCf(b,d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

**Example 7.** (cf.[30]) The following examples are provided.

- When $s = t = 1$, $PG$ is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, $PG$ is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, $PG$ is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, $PG$ is called a *Plithogenic quadripartitioned Neutrosophic Graph*.
- When $s = 5, t = 1$, $PG$ is called a *Plithogenic pentapartitioned Neutrosophic Graph*.
- When $s = 6, t = 1$, $PG$ is called a *Plithogenic hexapartitioned Neutrosophic Graph*.
- When $s = 7, t = 1$, $PG$ is called a *Plithogenic heptapartitioned Neutrosophic Graph*.
- When $s = 8, t = 1$, $PG$ is called a *Plithogenic octapartitioned Neutrosophic Graph*.
- When $s = 9, t = 1$, $PG$ is called a *Plithogenic nonapartitioned Neutrosophic Graph*.

## 2.4 | Pythagorean Graph

A *Pythagorean Fuzzy Graph (PFG)* is an extension of the traditional graph concept that incorporates the degrees of membership and non-membership based on Pythagorean fuzzy sets [8, 7, 66].

**Definition 8.** [8, 7, 66] A *Pythagorean Fuzzy Graph* is defined as a pair $G^{**} = (P, Q)$, where:

- $V$ is a non-empty set of vertices, and $E \subseteq V \times V$ is a set of edges.
- $P = (\nu_P, \xi_P)$ is a *Pythagorean fuzzy set* on $V$, such that for each vertex $u \in V$:

$$0 \leq \nu_P(u)^2 + \xi_P(u)^2 \leq 1,$$

where $\nu_P(u)$ and $\xi_P(u)$ denote the membership and non-membership degrees of the vertex $u$, respectively.

- $Q = (\nu_Q, \xi_Q)$ is a *Pythagorean fuzzy set* on $E$, such that for each edge $(u, v) \in E$:
$$0 \leq \nu_Q(u, v)^2 + \xi_Q(u, v)^2 \leq 1,$$
where $\nu_Q(u, v)$ and $\xi_Q(u, v)$ denote the membership and non-membership degrees of the edge $(u, v)$, respectively.

Additionally, the following conditions hold:

(1) For any vertices $u, v \in V$, if $(u, v) \in E$:
$$\nu_Q(u, v) \leq \min(\nu_P(u), \nu_P(v)),$$
$$\xi_Q(u, v) \geq \max(\xi_P(u), \xi_P(v)).$$

(2) If $\nu_Q(u, v) = 0$ and $\xi_Q(u, v) = 0$, then no edge exists between $u$ and $v$.

(3) An edge exists between $u$ and $v$ if any of the following holds:
- $\nu_Q(u, v) = 0$ and $\xi_Q(u, v) > 0$,
- $\nu_Q(u, v) > 0$ and $\xi_Q(u, v) = 0$,
- $\nu_Q(u, v) > 0$ and $\xi_Q(u, v) > 0$.

**Example 9** (Pythagorean Fuzzy Graph: Livestock Transportation Network)**.** Consider a transportation network for livestock where the vertices represent key locations (e.g., a farm, a market, and a processing plant) and the edges represent the transportation routes between them.

Let the set of vertices be:
$$V = \{\text{Farm A}, \text{ Market B}, \text{ Processing Plant C}\}.$$
Assign the following Pythagorean fuzzy membership and non-membership degrees to the vertices:
$$\begin{aligned}\nu_P(\text{Farm A}) &= 0.8, \quad \xi_P(\text{Farm A}) = 0.3,\\ \nu_P(\text{Market B}) &= 0.7, \quad \xi_P(\text{Market B}) = 0.5,\\ \nu_P(\text{Processing Plant C}) &= 0.9, \quad \xi_P(\text{Processing Plant C}) = 0.2.\end{aligned}$$
These values satisfy
$$0 \leq \nu_P(u)^2 + \xi_P(u)^2 \leq 1, \quad \forall u \in V.$$

Now, let the set of edges be
$$E = \{(\text{Farm A}, \text{Market B}), \ (\text{Market B}, \text{Processing Plant C}), \ (\text{Farm A}, \text{Processing Plant C})\}.$$
For each edge $(u, v) \in E$, assign the following Pythagorean fuzzy values:
$$\begin{aligned}\nu_Q(\text{Farm A}, \text{Market B}) &= 0.7, \quad \xi_Q(\text{Farm A}, \text{Market B}) = 0.5,\\ \nu_Q(\text{Market B}, \text{Processing Plant C}) &= 0.6, \quad \xi_Q(\text{Market B}, \text{Processing Plant C}) = 0.5,\\ \nu_Q(\text{Farm A}, \text{Processing Plant C}) &= 0.6, \quad \xi_Q(\text{Farm A}, \text{Processing Plant C}) = 0.3.\end{aligned}$$
These assignments are chosen so that for each edge $(u, v)$:
$$\nu_Q(u, v) \leq \min(\nu_P(u), \nu_P(v)) \quad \text{and} \quad \xi_Q(u, v) \geq \max(\xi_P(u), \xi_P(v)).$$
Thus, the structure $G^{**} = (P, Q)$ forms a valid Pythagorean Fuzzy Graph that models the efficiency and reliability of transportation routes in a livestock network.

A *Pythagorean Neutrosophic Graph (PNG)* is an extension of classical graph theory that combines the principles of Pythagorean fuzzy sets and neutrosophic sets, enabling more comprehensive modeling of uncertainty, indeterminacy, and truth [19, 38, 2].

**Definition 10.** (cf.[16]) A *Pythagorean Neutrosophic Graph* is defined as a pair $G = (V, E)$, where:

- $V$ is a set of vertices, $V = \{v_1, v_2, \dots, v_n\}$.
- Each vertex $v_i \in V$ is associated with three functions:

  - *Membership function* $\mu_1(v_i)$,
  - *Indeterminacy function* $\beta_1(v_i)$,
  - *Non-membership function* $\sigma_1(v_i)$,

  such that:

$$0 \leq \mu_1(v_i)^2 + \beta_1(v_i)^2 + \sigma_1(v_i)^2 \leq 2, \quad \forall v_i \in V.$$

- $E \subseteq V \times V$ is the set of edges, where each edge $(v_i, v_j) \in E$ is associated with three functions:
  - *Edge membership function* $\mu_2(v_i, v_j)$,
  - *Edge indeterminacy function* $\beta_2(v_i, v_j)$,
  - *Edge non-membership function* $\sigma_2(v_i, v_j)$,

  such that:

$$0 \leq \mu_2(v_i, v_j)^2 + \beta_2(v_i, v_j)^2 + \sigma_2(v_i, v_j)^2 \leq 2, \quad \forall (v_i, v_j) \in E.$$

Additionally, the following conditions hold:

(1) For any vertices $v_i, v_j \in V$, if $(v_i, v_j) \in E$:

$$\mu_2(v_i, v_j) \leq \min(\mu_1(v_i), \mu_1(v_j)),$$
$$\beta_2(v_i, v_j) \leq \min(\beta_1(v_i), \beta_1(v_j)),$$
$$\sigma_2(v_i, v_j) \geq \max(\sigma_1(v_i), \sigma_1(v_j)).$$

(2) If $\mu_2(v_i, v_j) = 0$, $\beta_2(v_i, v_j) = 0$, and $\sigma_2(v_i, v_j) = 0$, then no edge exists between $v_i$ and $v_j$.

**Example 11** (Pythagorean Neutrosophic Graph: Livestock Disease Transmission Network)**.** Consider a network that models the potential transmission of disease among livestock farms (cf.[20]). In this network, each vertex represents a farm and is characterized by three functions: membership (indicating the likelihood of infection), indeterminacy (reflecting uncertainty), and non-membership (indicating the likelihood of not being infected).

Let the set of vertices be:

$$V = \{\text{Farm X}, \ \text{Farm Y}, \ \text{Farm Z}\}.$$

Assign the following values to each vertex:

$$\begin{aligned} \mu_1(\text{Farm X}) &= 0.7, \quad \beta_1(\text{Farm X}) = 0.5, \quad \sigma_1(\text{Farm X}) = 0.6, \\ \mu_1(\text{Farm Y}) &= 0.6, \quad \beta_1(\text{Farm Y}) = 0.4, \quad \sigma_1(\text{Farm Y}) = 0.7, \\ \mu_1(\text{Farm Z}) &= 0.8, \quad \beta_1(\text{Farm Z}) = 0.3, \quad \sigma_1(\text{Farm Z}) = 0.4. \end{aligned}$$

These values satisfy:

$$0 \leq \mu_1(v)^2 + \beta_1(v)^2 + \sigma_1(v)^2 \leq 2, \quad \forall v \in V.$$

Let the edge set be:

$$E = \{(\text{Farm X}, \text{Farm Y}), \ (\text{Farm X}, \text{Farm Z}), \ (\text{Farm Y}, \text{Farm Z})\}.$$

For each edge $(v_i, v_j) \in E$, assign:

$$\begin{aligned} \mu_2(\text{Farm X}, \text{Farm Y}) &= 0.6, \quad \beta_2(\text{Farm X}, \text{Farm Y}) = 0.4, \quad \sigma_2(\text{Farm X}, \text{Farm Y}) = 0.7, \\ \mu_2(\text{Farm X}, \text{Farm Z}) &= 0.7, \quad \beta_2(\text{Farm X}, \text{Farm Z}) = 0.3, \quad \sigma_2(\text{Farm X}, \text{Farm Z}) = 0.6, \\ \mu_2(\text{Farm Y}, \text{Farm Z}) &= 0.5, \quad \beta_2(\text{Farm Y}, \text{Farm Z}) = 0.3, \quad \sigma_2(\text{Farm Y}, \text{Farm Z}) = 0.7. \end{aligned}$$

These edge values are chosen to satisfy:

$$\mu_2(v_i, v_j) \leq \min(\mu_1(v_i), \mu_1(v_j)), \quad \beta_2(v_i, v_j) \leq \min(\beta_1(v_i), \beta_1(v_j)),$$

and

$$\sigma_2(v_i, v_j) \geq \max(\sigma_1(v_i), \sigma_1(v_j)).$$

Thus, the graph $G = (V, E)$ forms a valid Pythagorean Neutrosophic Graph, providing a comprehensive model of uncertainty, indeterminacy, and infection risk in the context of disease transmission among livestock farms.

## 2.5 | Fermatean Neutrosophic Graph

A *Fermatean Neutrosophic Graph (FNG)* is an extension of the Fermatean neutrosophic set, incorporating the Fermatean degrees of truth, indeterminacy, and falsity into a graphical framework [52, 14, 46].

**Definition 12.** [15] Let $X$ be a universe of discourse. A *Fermatean Neutrosophic Set* $S$ on $X$ is defined as:

$$S = \left\{ (x, T_S(x), I_S(x), F_S(x)) \,\middle|\, x \in X \right\},$$

where:

- $T_S(x)$ is the truth-membership degree,
- $I_S(x)$ is the indeterminacy-membership degree,
- $F_S(x)$ is the falsity-membership degree,

and these degrees satisfy:

$$[T_S(x)]^3 + [I_S(x)]^3 + [F_S(x)]^3 \leq 2, \quad \forall x \in X.$$

**Definition 13.** A *Fermatean Neutrosophic Graph* is defined as a pair $G = (P, Q)$, where:

- $P$ is a Fermatean Neutrosophic Set defined on the set of vertices $V$,
- $Q$ is a Fermatean Neutrosophic Set defined on the set of edges $E \subseteq V \times V$.

For each vertex $u \in V$, we have:

$$[T_P(u)]^3 + [I_P(u)]^3 + [F_P(u)]^3 \leq 2, \quad \forall u \in V.$$

For each edge $(u, v) \in E$, we have:

$$[T_Q(u, v)]^3 + [I_Q(u, v)]^3 + [F_Q(u, v)]^3 \leq 2, \quad \forall (u, v) \in E.$$

The relationship between vertex and edge degrees is given by:

$$T_Q(u, v) \leq \min\{T_P(u), T_P(v)\},$$
$$I_Q(u, v) \leq \min\{I_P(u), I_P(v)\},$$
$$F_Q(u, v) \geq \max\{F_P(u), F_P(v)\}.$$

If $T_Q(u, v) = I_Q(u, v) = F_Q(u, v) = 0$, then $(u, v) \notin E$.

## 2.6 | Complex Neutrosophic Graph

A *Complex Neutrosophic Graph (CNG)* incorporates complex-valued degrees for truth, indeterminacy, and falsity, providing a richer representation of uncertainty in a graphical context [71, 11, 18].

**Definition 14.** [71] Let $X$ be a universe of discourse. A *Complex Neutrosophic Set* $S$ on $X$ is defined as:

$$S = \left\{ (x, T_S(x), I_S(x), F_S(x)) \,\middle|\, x \in X \right\},$$

where:

- $T_S(x) = r_S(x)\, e^{i\omega_S(x)}$, representing the complex truth-membership,
- $I_S(x) = t_S(x)\, e^{i\theta_S(x)}$, representing the complex indeterminacy-membership,
- $F_S(x) = k_S(x)\, e^{i\rho_S(x)}$, representing the complex falsity-membership.

These degrees satisfy:

$$[r_S(x)]^2 + [t_S(x)]^2 + [k_S(x)]^2 \leq 3, \quad \forall x \in X,$$

where $r_S(x), t_S(x), k_S(x) \in [0, 1]$ and $\omega_S(x), \theta_S(x), \rho_S(x) \in [0, 2\pi]$.

**Definition 15.** A *Complex Neutrosophic Graph* is defined as a pair $G = (P, Q)$, where:

- $P$ is a Complex Neutrosophic Set defined on the set of vertices $V$,
- $Q$ is a Complex Neutrosophic Set defined on the set of edges $E \subseteq V \times V$.

For each vertex $u \in V$, we have:

$$[r_P(u)]^2 + [t_P(u)]^2 + [k_P(u)]^2 \leq 3, \quad \forall u \in V.$$

For each edge $(u, v) \in E$, we have:

$$[r_Q(u, v)]^2 + [t_Q(u, v)]^2 + [k_Q(u, v)]^2 \leq 3, \quad \forall (u, v) \in E.$$

The relationship between vertex and edge degrees is given by:

$$\begin{aligned} r_Q(u, v) &\leq \min\{r_P(u), r_P(v)\}, \\ t_Q(u, v) &\leq \min\{t_P(u), t_P(v)\}, \\ k_Q(u, v) &\geq \max\{k_P(u), k_P(v)\}. \end{aligned}$$

If $r_Q(u, v) = t_Q(u, v) = k_Q(u, v) = 0$, then $(u, v) \notin E$.

## 3 | Result in this paper

In this section, we present the results of this paper.

### 3.1 | Pythagorean Turiyam Neutrosophic Graph

A *Pythagorean Turiyam Neutrosophic Graph* is an extension of the Turiyam Neutrosophic graph concept, incorporating Pythagorean fuzzy sets to handle uncertainty with four components: truth, indeterminacy, falsity, and liberal state degrees.

**Definition 16.** A *Pythagorean Turiyam Neutrosophic Graph* is a graph $G = (V, E)$ where:

- $V$ is a non-empty set of vertices,
- $E \subseteq V \times V$ is a set of edges.

Each vertex $v \in V$ is assigned four membership degrees:

- $t(v) \in [0, 1]$ is the truth-membership degree,
- $iv(v) \in [0, 1]$ is the indeterminacy-membership degree,
- $fv(v) \in [0, 1]$ is the falsity-membership degree,
- $lv(v) \in [0, 1]$ is the liberal state membership degree.

These degrees satisfy:

$$[t(v)]^2 + [iv(v)]^2 + [fv(v)]^2 + [lv(v)]^2 \leq 3, \quad \forall v \in V.$$

Each edge $e = (u, v) \in E$ is assigned four membership degrees:

- $t(u, v) \in [0, 1]$ is the truth-membership degree of the edge,
- $iv(u, v) \in [0, 1]$ is the indeterminacy-membership degree of the edge,
- $fv(u, v) \in [0, 1]$ is the falsity-membership degree of the edge,
- $lv(u, v) \in [0, 1]$ is the liberal state membership degree of the edge.

These degrees satisfy:

$$[t(u, v)]^2 + [iv(u, v)]^2 + [fv(u, v)]^2 + [lv(u, v)]^2 \leq 3, \quad \forall (u, v) \in E.$$

Additionally, the following conditions hold:

(1) For any edge $(u, v) \in E$:

$$t(u,v) \leq \min\{t(u), t(v)\},$$
$$iv(u,v) \leq \min\{iv(u), iv(v)\},$$
$$fv(u,v) \geq \max\{fv(u), fv(v)\},$$
$$lv(u,v) \leq \min\{lv(u), lv(v)\}.$$

(2) If $t(u,v) = iv(u,v) = fv(u,v) = lv(u,v) = 0$, then $(u,v) \notin E$.

**Theorem 17.** *A Pythagorean Turiyam Neutrosophic Graph is Turiyam Neutrosophic Graph.*

*Proof*: Obviously holds. □

**Theorem 18.** *Any Pythagorean Turiyam Neutrosophic Graph can be transformed into a Pythagorean Neutrosophic Graph by setting the liberal state degree to zero and appropriately scaling the degrees.*

*Proof*: Let $G = (V, E)$ be a Pythagorean Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$\mu(v) = \frac{t(v)}{\sqrt{\frac{3}{2}}}, \quad \beta(v) = \frac{iv(v)}{\sqrt{\frac{3}{2}}}, \quad \sigma(v) = \frac{fv(v)}{\sqrt{\frac{3}{2}}}.$$

Then,

$$[\mu(v)]^2 + [\beta(v)]^2 + [\sigma(v)]^2 = \frac{[t(v)]^2 + [iv(v)]^2 + [fv(v)]^2}{\frac{3}{2}} \leq 2.$$

Similarly, for each edge $(u, v) \in E$, define:

$$\mu(u,v) = \frac{t(u,v)}{\sqrt{\frac{3}{2}}}, \quad \beta(u,v) = \frac{iv(u,v)}{\sqrt{\frac{3}{2}}}, \quad \sigma(u,v) = \frac{fv(u,v)}{\sqrt{\frac{3}{2}}}.$$

Thus, $G$ becomes a Pythagorean Neutrosophic Graph. □

**Theorem 19.** *Any Pythagorean Turiyam Neutrosophic Graph can be transformed into a Pythagorean Fuzzy Graph by setting $iv(v) = fv(v) = lv(v) = 0$ for all $v \in V$ and scaling the truth degrees.*

*Proof*: Set $iv(v) = fv(v) = lv(v) = 0$. Then,

$$[t(v)]^2 \leq 3 \implies t(v) \leq \sqrt{3}.$$

Define $\nu(v) = \frac{t(v)}{\sqrt{3}}$, so $[\nu(v)]^2 \leq 1$. Similarly for edges, resulting in a Pythagorean Fuzzy Graph. □

**Theorem 20.** *Any subgraph of a Pythagorean Turiyam Neutrosophic Graph is also a Pythagorean Turiyam Neutrosophic Graph.*

*Proof*: Let $H = (V', E')$ be a subgraph of $G = (V, E)$, where $V' \subseteq V$ and $E' \subseteq E$. The membership degrees in $H$ are inherited from $G$ and satisfy the required conditions. Therefore, $H$ is a Pythagorean Turiyam Neutrosophic Graph. □

**Theorem 21.** *An edge $(u, v)$ exists in a Pythagorean Turiyam Neutrosophic Graph if and only if at least one of the degrees $t(u,v), iv(u,v), fv(u,v), lv(u,v)$ is greater than zero.*

*Proof*: By the definition, if all degrees are zero, $(u, v) \notin E$. Conversely, if any degree is positive, $(u, v) \in E$. □

**Theorem 22.** *In an undirected Pythagorean Turiyam Neutrosophic Graph, the edge degrees are symmetric:*

$$t(u,v) = t(v,u), \quad iv(u,v) = iv(v,u), \quad fv(u,v) = fv(v,u), \quad lv(u,v) = lv(v,u).$$

*Proof*: Since the graph is undirected, edges are unordered pairs, so the degrees are equal in both directions. □

**Theorem 23.** *For any vertex $v$:*

$$0 \leq t(v), iv(v), fv(v), lv(v) \leq 1, \quad [t(v)]^2 + [iv(v)]^2 + [fv(v)]^2 + [lv(v)]^2 \leq 3.$$

*Proof*: Directly follows from the definition of Pythagorean Turiyam Neutrosophic Graph. □

**Theorem 24.** *Under appropriate conditions, the union of two Pythagorean Turiyam Neutrosophic Graphs is a Pythagorean Turiyam Neutrosophic Graph.*

*Proof*: Define the union graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Assign degrees using the minimum of degrees from $G_1$ and $G_2$ to satisfy the Pythagorean condition. □

**Theorem 25.** *The Cartesian product of two Pythagorean Turiyam Neutrosophic Graphs is a Pythagorean Turiyam Neutrosophic Graph.*

*Proof*: Define the vertex set as $V = V_1 \times V_2$. For each vertex $(u, v)$, define degrees:

$$t(u, v) = t_1(u) \cdot t_2(v), \quad iv(u, v) = iv_1(u) \cdot iv_2(v), \quad fv(u, v) = fv_1(u) \cdot fv_2(v), \quad lv(u, v) = lv_1(u) \cdot lv_2(v).$$

These degrees satisfy the Pythagorean condition due to the properties of products of numbers in $[0, 1]$. □

**Theorem 26.** *In a Pythagorean Turiyam Neutrosophic Graph, the sum of the truth degrees of all vertices is greater than or equal to the sum of the truth degrees of all edges.*

*Proof*: Since $t(u, v) \leq \min\{t(u), t(v)\} \leq t(u)$ and $t(u, v) \leq t(v)$, summing over all edges and vertices gives the inequality. □

**Theorem 27.** *For any edge $(u, v)$:*

$$fv(u, v) \geq \max\{fv(u), fv(v)\}.$$

*Proof*: Given by the edge membership constraint in the definition. □

**Theorem 28.** *For any edge $(u, v)$:*

$$t(u, v) \leq \min\{t(u), t(v)\}.$$

*Proof*: Given by the edge membership constraint in the definition. □

## 3.2 | Fermatean Turiyam Neutrosophic Graph

A *Fermatean Turiyam Neutrosophic Graph* is an extension of the Fermatean Neutrosophic Graph that incorporates an additional component called the *liberal state degree*, allowing for a more nuanced representation of uncertainty with four components: truth, indeterminacy, falsity, and liberal state.

**Definition 29.** Let $X$ be a universe of discourse. A *Fermatean Turiyam Neutrosophic Set $A$* on $X$ is defined as:

$$A = \left\{ (x, T_A(x), I_A(x), F_A(x), L_A(x)) \;\middle|\; x \in X \right\},$$

where:

- $T_A(x) \in [0, 1]$ is the truth-membership degree,
- $I_A(x) \in [0, 1]$ is the indeterminacy-membership degree,
- $F_A(x) \in [0, 1]$ is the falsity-membership degree,
- $L_A(x) \in [0, 1]$ is the liberal state membership degree.

These degrees satisfy:

$$[T_A(x)]^3 + [I_A(x)]^3 + [F_A(x)]^3 + [L_A(x)]^3 \leq 3, \quad \forall x \in X.$$

**Definition 30.** A *Fermatean Turiyam Neutrosophic Graph* is a graph $G = (V, E)$ where:

- $V$ is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.

Each vertex $v \in V$ is assigned four membership degrees:

- $T(v) \in [0, 1]$ is the truth-membership degree,
- $I(v) \in [0, 1]$ is the indeterminacy-membership degree,
- $F(v) \in [0, 1]$ is the falsity-membership degree,
- $L(v) \in [0, 1]$ is the liberal state membership degree.

These degrees satisfy:

$$[T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3, \quad \forall v \in V.$$

Each edge $e = (u, v) \in E$ is assigned four membership degrees:

- $T(u, v) \in [0, 1]$ is the truth-membership degree of the edge,
- $I(u, v) \in [0, 1]$ is the indeterminacy-membership degree of the edge,
- $F(u, v) \in [0, 1]$ is the falsity-membership degree of the edge,
- $L(u, v) \in [0, 1]$ is the liberal state membership degree of the edge.

These degrees satisfy:

$$[T(u, v)]^3 + [I(u, v)]^3 + [F(u, v)]^3 + [L(u, v)]^3 \leq 3, \quad \forall (u, v) \in E.$$

Additionally, the following conditions hold for any edge $(u, v) \in E$:

$$\begin{aligned} T(u, v) &\leq \min\{T(u), T(v)\}, \\ I(u, v) &\leq \min\{I(u), I(v)\}, \\ F(u, v) &\geq \max\{F(u), F(v)\}, \\ L(u, v) &\leq \min\{L(u), L(v)\}. \end{aligned}$$

If $T(u, v) = I(u, v) = F(u, v) = L(u, v) = 0$, then $(u, v) \notin E$.

**Theorem 31.** *Any Fermatean Turiyam Neutrosophic Graph can be transformed into a Fermatean Neutrosophic Graph by setting the liberal state degree to zero and appropriately scaling the degrees.*

*Proof*: Let $G = (V, E)$ be a Fermatean Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$T'(v) = \frac{T(v)}{\sqrt[3]{\frac{3}{2}}}, \quad I'(v) = \frac{I(v)}{\sqrt[3]{\frac{3}{2}}}, \quad F'(v) = \frac{F(v)}{\sqrt[3]{\frac{3}{2}}}.$$

Then,

$$[T'(v)]^3 + [I'(v)]^3 + [F'(v)]^3 = \frac{[T(v)]^3 + [I(v)]^3 + [F(v)]^3}{\frac{3}{2}} \leq 2.$$

Similarly, for each edge $(u, v) \in E$, define:

$$T'(u, v) = \frac{T(u, v)}{\sqrt[3]{\frac{3}{2}}}, \quad I'(u, v) = \frac{I(u, v)}{\sqrt[3]{\frac{3}{2}}}, \quad F'(u, v) = \frac{F(u, v)}{\sqrt[3]{\frac{3}{2}}}.$$

Thus, $G' = (V, E)$ with these new degrees is a Fermatean Neutrosophic Graph. □

**Theorem 32.** *Any subgraph of a Fermatean Turiyam Neutrosophic Graph is also a Fermatean Turiyam Neutrosophic Graph.*

*Proof*: Let $G = (V, E)$ be a Fermatean Turiyam Neutrosophic Graph, and let $H = (V', E')$ be a subgraph of $G$, where $V' \subseteq V$ and $E' \subseteq E$.

For each vertex $v \in V'$, the membership degrees $T(v), I(v), F(v), L(v)$ are the same as in $G$. Since $G$ satisfies the conditions of a Fermatean Turiyam Neutrosophic Graph, the degrees satisfy:

$$[T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3, \quad \forall v \in V'.$$

Similarly, for each edge $(u, v) \in E'$, the degrees satisfy:

$$[T(u,v)]^3 + [I(u,v)]^3 + [F(u,v)]^3 + [L(u,v)]^3 \leq 3, \quad \forall (u,v) \in E'.$$

The relationships between the degrees of the vertices and edges also hold in $H$ since they are inherited from $G$.

Therefore, $H$ is also a Fermatean Turiyam Neutrosophic Graph. □

**Theorem 33.** *In an undirected Fermatean Turiyam Neutrosophic Graph, the degrees of the edges are symmetric:*

$$T(u,v) = T(v,u), \quad I(u,v) = I(v,u), \quad F(u,v) = F(v,u), \quad L(u,v) = L(v,u).$$

*Proof*: In an undirected graph, the edge $(u, v)$ is identical to the edge $(v, u)$. Therefore, the degrees associated with the edge must be the same regardless of the order of the vertices.

Thus, the degrees satisfy:

$$T(u,v) = T(v,u), \quad I(u,v) = I(v,u), \quad F(u,v) = F(v,u), \quad L(u,v) = L(v,u).$$

□

**Theorem 34.** *For any vertex $v$:*

$$0 \leq T(v), I(v), F(v), L(v) \leq 1, \quad [T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3.$$

*Proof*: By definition, $T(v), I(v), F(v), L(v) \in [0, 1]$ for all $v \in V$, and

$$[T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3.$$

This follows directly from the definition of a Fermatean Turiyam Neutrosophic Graph. □

**Theorem 35.** *An edge $(u, v)$ exists in a Fermatean Turiyam Neutrosophic Graph if and only if at least one of the degrees $T(u, v), I(u, v), F(u, v), L(u, v)$ is greater than zero.*

*Proof*: By definition, if $T(u, v) = I(u, v) = F(u, v) = L(u, v) = 0$, then $(u, v) \notin E$.

Conversely, if any of $T(u, v), I(u, v), F(u, v), L(u, v)$ is greater than zero, then $(u, v) \in E$.

Therefore, an edge exists between $u$ and $v$ if and only if at least one of the degrees is greater than zero. □

**Theorem 36.** *In a Fermatean Turiyam Neutrosophic Graph, the sum of the truth degrees of all vertices is greater than or equal to the sum of the truth degrees of all edges:*

$$\sum_{v \in V} T(v) \geq \sum_{(u,v) \in E} T(u, v).$$

*Proof*: Since for each edge $(u, v)$, $T(u, v) \leq \min\{T(u), T(v)\} \leq T(u)$ and $T(u, v) \leq T(v)$, the truth degree of the edge is less than or equal to the truth degree of each of its endpoints.

Therefore, summing over all edges:

$$\sum_{(u,v)\in E} T(u, v) \leq \sum_{(u,v)\in E} \min\{T(u), T(v)\} \leq \sum_{v\in V} T(v).$$

Thus:

$$\sum_{(u,v)\in E} T(u, v) \leq \sum_{v\in V} T(v).$$

□

**Theorem 37.** *Any Fermatean Turiyam Neutrosophic Graph can be transformed into a Fermatean Neutrosophic Graph by setting the liberal state degree to zero and appropriately scaling the degrees.*

*Proof*: Let $G = (V, E)$ be a Fermatean Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$T'(v) = \frac{T(v)}{\sqrt[3]{\frac{3}{2}}}, \quad I'(v) = \frac{I(v)}{\sqrt[3]{\frac{3}{2}}}, \quad F'(v) = \frac{F(v)}{\sqrt[3]{\frac{3}{2}}}.$$

Then:

$$[T'(v)]^3 + [I'(v)]^3 + [F'(v)]^3 = \frac{[T(v)]^3 + [I(v)]^3 + [F(v)]^3}{\frac{3}{2}} \leq 2.$$

Similarly, for each edge $(u, v) \in E$, define:

$$T'(u, v) = \frac{T(u, v)}{\sqrt[3]{\frac{3}{2}}}, \quad I'(u, v) = \frac{I(u, v)}{\sqrt[3]{\frac{3}{2}}}, \quad F'(u, v) = \frac{F(u, v)}{\sqrt[3]{\frac{3}{2}}}.$$

Set $L'(v) = L'(u, v) = 0$.

Thus, $G' = (V, E)$ with these new degrees is a Fermatean Neutrosophic Graph. □

**Theorem 38.** *Any Fermatean Turiyam Neutrosophic Graph can be transformed into a Pythagorean Turiyam Neutrosophic Graph by appropriately scaling the degrees.*

*Proof*: Let $G = (V, E)$ be a Fermatean Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$t(v) = \sqrt[3]{T(v)}, \quad iv(v) = \sqrt[3]{I(v)}, \quad fv(v) = \sqrt[3]{F(v)}, \quad lv(v) = \sqrt[3]{L(v)}.$$

Then:

$$[t(v)]^2 + [iv(v)]^2 + [fv(v)]^2 + [lv(v)]^2 = [T(v)]^{\frac{4}{3}} + [I(v)]^{\frac{4}{3}} + [F(v)]^{\frac{4}{3}} + [L(v)]^{\frac{4}{3}} \leq 3^{\frac{4}{3}}.$$

Since $3^{\frac{4}{3}} \approx 4.3267$, which is greater than 3, we normalize the degrees by dividing each by $3^{1/3}$:

$$t'(v) = \frac{t(v)}{3^{1/3}}, \quad iv'(v) = \frac{iv(v)}{3^{1/3}}, \quad fv'(v) = \frac{fv(v)}{3^{1/3}}, \quad lv'(v) = \frac{lv(v)}{3^{1/3}}.$$

Then:

$$[t'(v)]^2 + [iv'(v)]^2 + [fv'(v)]^2 + [lv'(v)]^2 \leq 3.$$

Similarly, for edges. Therefore, $G$ can be transformed into a Pythagorean Turiyam Neutrosophic Graph. □

**Theorem 39.** *Any Fermatean Turiyam Neutrosophic Graph can be transformed into a Turiyam Neutrosophic Graph by setting the degrees appropriately.*

*Proof*: In a Turiyam Neutrosophic Graph, the degrees satisfy:

$$T'(v) + I'(v) + F'(v) + L'(v) \leq 4.$$

Since in a Fermatean Turiyam Neutrosophic Graph:

$$[T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3,$$

and $T(v), I(v), F(v), L(v) \in [0, 1]$, the maximum value each can attain is 1. Therefore, the sum:

$$T'(v) + I'(v) + F'(v) + L'(v) \leq 4.$$

Thus, $G$ satisfies the conditions of a Turiyam Neutrosophic Graph. □

**Theorem 40.** *Any Fermatean Turiyam Neutrosophic Graph can be transformed into a Neutrosophic Graph by combining the indeterminacy and liberal state degrees.*

*Proof*: Let $G = (V, E)$ be a Fermatean Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$T'(v) = T(v), \quad I'(v) = I(v) + L(v), \quad F'(v) = F(v).$$

Since $T(v), I(v), F(v), L(v) \in [0, 1]$, their sum $T'(v) + I'(v) + F'(v) \leq 3$.

In a Neutrosophic Graph, the degrees satisfy:

$$0 \leq T'(v), I'(v), F'(v) \leq 1, \quad T'(v) + I'(v) + F'(v) \leq 3.$$

Therefore, $G$ can be transformed into a Neutrosophic Graph. □

## 3.3 | Complex Turiyam Neutrosophic Graph

A *Complex Turiyam Neutrosophic Graph* integrates complex-valued membership degrees for truth, indeterminacy, falsity, and liberal state, providing a richer framework to model uncertainty in graphs.

**Definition 41.** Let $X$ be a universe of discourse. A *Complex Turiyam Neutrosophic Set* $A$ on $X$ is defined as:

$$A = \left\{ (x, T_A(x), I_A(x), F_A(x), L_A(x)) \,\middle|\, x \in X \right\},$$

where:

- $T_A(x) = r_T(x)\, e^{i\omega_T(x)}$ is the complex truth-membership,
- $I_A(x) = r_I(x)\, e^{i\omega_I(x)}$ is the complex indeterminacy-membership,
- $F_A(x) = r_F(x)\, e^{i\omega_F(x)}$ is the complex falsity-membership,
- $L_A(x) = r_L(x)\, e^{i\omega_L(x)}$ is the complex liberal state membership.

Here, $r_T(x), r_I(x), r_F(x), r_L(x) \in [0,1]$ and $\omega_T(x), \omega_I(x), \omega_F(x), \omega_L(x) \in [0, 2\pi)$.

These degrees satisfy:

$$[r_T(x)]^2 + [r_I(x)]^2 + [r_F(x)]^2 + [r_L(x)]^2 \leq 4, \quad \forall x \in X.$$

**Definition 42.** A *Complex Turiyam Neutrosophic Graph* is defined as a pair $G = (V, E)$ where:

- $V$ is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.

Each vertex $v \in V$ is associated with four complex membership degrees:

- $T(v) = r_T(v)\, e^{i\omega_T(v)}$,
- $I(v) = r_I(v)\, e^{i\omega_I(v)}$,
- $F(v) = r_F(v)\, e^{i\omega_F(v)}$,
- $L(v) = r_L(v)\, e^{i\omega_L(v)}$.

These degrees satisfy:

$$[r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2 + [r_L(v)]^2 \leq 4, \quad \forall v \in V.$$

Each edge $e = (u, v) \in E$ is associated with four complex membership degrees:

- $T(u,v) = r_T(u,v)\, e^{i\omega_T(u,v)}$,
- $I(u,v) = r_I(u,v)\, e^{i\omega_I(u,v)}$,
- $F(u,v) = r_F(u,v)\, e^{i\omega_F(u,v)}$,
- $L(u,v) = r_L(u,v)\, e^{i\omega_L(u,v)}$.

These degrees satisfy:

$$[r_T(u,v)]^2 + [r_I(u,v)]^2 + [r_F(u,v)]^2 + [r_L(u,v)]^2 \leq 4, \quad \forall (u,v) \in E.$$

The relationship between vertex and edge degrees is given by:

$$\begin{aligned} r_T(u,v) &\leq \min\{r_T(u), r_T(v)\}, \\ r_I(u,v) &\leq \min\{r_I(u), r_I(v)\}, \\ r_F(u,v) &\geq \max\{r_F(u), r_F(v)\}, \\ r_L(u,v) &\leq \min\{r_L(u), r_L(v)\}. \end{aligned}$$

If $r_T(u,v) = r_I(u,v) = r_F(u,v) = r_L(u,v) = 0$, then $(u,v) \notin E$.

**Theorem 43.** *Any Complex Turiyam Neutrosophic Graph can be transformed into a Complex Neutrosophic Graph by setting the liberal state degree to zero and appropriately adjusting the modulus components.*

*Proof*: Let $G = (V, E)$ be a Complex Turiyam Neutrosophic Graph. For each vertex $v \in V$, define:

$$r'_T(v) = \frac{r_T(v)}{\sqrt{\frac{4}{3}}}, \quad r'_I(v) = \frac{r_I(v)}{\sqrt{\frac{4}{3}}}, \quad r'_F(v) = \frac{r_F(v)}{\sqrt{\frac{4}{3}}}.$$

Then,

$$[r'_T(v)]^2 + [r'_I(v)]^2 + [r'_F(v)]^2 = \frac{[r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2}{\frac{4}{3}} \leq 3.$$

Similarly, for each edge $(u,v) \in E$, define:

$$r'_T(u,v) = \frac{r_T(u,v)}{\sqrt{\frac{4}{3}}}, \quad r'_I(u,v) = \frac{r_I(u,v)}{\sqrt{\frac{4}{3}}}, \quad r'_F(u,v) = \frac{r_F(u,v)}{\sqrt{\frac{4}{3}}}.$$

Thus, $G' = (V, E)$ with these adjusted degrees is a Complex Neutrosophic Graph. □

**Theorem 44.** *Any subgraph of a Complex Turiyam Neutrosophic Graph is also a Complex Turiyam Neutrosophic Graph.*

*Proof*: Let $G = (V, E)$ be a Complex Turiyam Neutrosophic Graph, and let $H = (V', E')$ be a subgraph of $G$, where $V' \subseteq V$ and $E' \subseteq E$.

For each vertex $v \in V'$, the membership degrees $T(v), I(v), F(v), L(v)$ are the same as in $G$. Since $G$ satisfies the conditions of a Complex Turiyam Neutrosophic Graph, the degrees satisfy:

$$[r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2 + [r_L(v)]^2 \leq 4, \quad \forall v \in V'.$$

Similarly, for each edge $(u, v) \in E'$, the degrees satisfy:

$$[r_T(u,v)]^2 + [r_I(u,v)]^2 + [r_F(u,v)]^2 + [r_L(u,v)]^2 \leq 4, \quad \forall (u,v) \in E'.$$

The relationships between the degrees of the vertices and edges also hold in $H$ since they are inherited from $G$.

Therefore, $H$ is also a Complex Turiyam Neutrosophic Graph. □

**Theorem 45.** *For any vertex $v$:*

$$0 \leq r_T(v), r_I(v), r_F(v), r_L(v) \leq 1, \quad [r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2 + [r_L(v)]^2 \leq 4.$$

*Proof*: By definition, $r_T(v), r_I(v), r_F(v), r_L(v) \in [0, 1]$ for all $v \in V$, and

$$[r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2 + [r_L(v)]^2 \leq 4.$$

This follows directly from the definition of a Complex Turiyam Neutrosophic Graph. □

**Theorem 46.** *An edge $(u, v)$ exists in a Complex Turiyam Neutrosophic Graph if and only if at least one of the modulus components of the degrees $r_T(u, v), r_I(u, v), r_F(u, v), r_L(u, v)$ is greater than zero.*

*Proof*: By definition, if $r_T(u, v) = r_I(u, v) = r_F(u, v) = r_L(u, v) = 0$, then $(u, v) \notin E$.

Conversely, if any of $r_T(u, v), r_I(u, v), r_F(u, v), r_L(u, v)$ is greater than zero, then $(u, v) \in E$.

Therefore, an edge exists between $u$ and $v$ if and only if at least one of the modulus components is greater than zero. □

**Theorem 47.** *For any edge $(u, v)$:*

$$r_F(u, v) \geq \max\{r_F(u), r_F(v)\}.$$

*Proof*: By the edge conditions in the definition of a Complex Turiyam Neutrosophic Graph, we have:

$$r_F(u, v) \geq \max\{r_F(u), r_F(v)\}.$$

This follows directly from the definition. □

## 3.4 | General Plithogenic Graph

The General Plithogenic Graph is a generalization of the Plithogenic Graph (cf.[30, 49, 24]).

**Definition 48** (General Plithogenic Graph)**.** [30] Let $G = (V, E)$ be a classical graph, where $V$ is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

(1) *General Plithogenic Vertex Set* $PM$:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- $l$: Attribute associated with the vertices.
- $Ml$: Range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$: Degree of Contradiction Function (DCF) for vertices.

(2) *General Plithogenic Edge Set* $PN$:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- $m$: Attribute associated with the edges.
- $Nm$: Range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph $G^{GP}$ only needs to satisfy the following *Reflexivity and Symmetry* properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml \\ bCf(a,a) &= 0, & \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm \end{aligned}$$

**Theorem 49.** *Any General Plithogenic Graph $G^{GP}$ can be transformed into a Pythagorean Turiyam Neutrosophic Graph $G^{PT}$.*

*Proof*: (1) For each vertex $v \in M$, with $adf(v,a) = (d_1, d_2, d_3, d_4) \in [0,1]^4$, set:

$$t(v) = d_1, \quad iv(v) = d_2, \quad fv(v) = d_3, \quad lv(v) = d_4.$$

Ensure:

$$[t(v)]^2 + [iv(v)]^2 + [fv(v)]^2 + [lv(v)]^2 \leq 3.$$

(2) For each edge $e = (u,v) \in N$, with $bdf(e,b) = (d'_1, d'_2, d'_3, d'_4) \in [0,1]^4$, set:

$$t(u,v) = d'_1, \quad iv(u,v) = d'_2, \quad fv(u,v) = d'_3, \quad lv(u,v) = d'_4.$$

Ensure:

$$[t(u,v)]^2 + [iv(u,v)]^2 + [fv(u,v)]^2 + [lv(u,v)]^2 \leq 3.$$

(3) Adjust edge degrees to satisfy:

$$t(u,v) \leq \min\{t(u), t(v)\},$$
$$iv(u,v) \leq \min\{iv(u), iv(v)\},$$
$$fv(u,v) \geq \max\{fv(u), fv(v)\},$$
$$lv(u,v) \leq \min\{lv(u), lv(v)\}.$$

Thus, $G^{GP}$ can be transformed into a Pythagorean Turiyam Neutrosophic Graph $G^{PT}$. □

**Theorem 50.** *Any General Plithogenic Graph $G^{GP}$ can be transformed into a Fermatean Turiyam Neutrosophic Graph $G^{FT}$.*

*Proof*: Proceeding similarly to Theorem 1, but ensuring that the degrees satisfy:

$$[T(v)]^3 + [I(v)]^3 + [F(v)]^3 + [L(v)]^3 \leq 3.$$

□

**Theorem 51.** *Any General Plithogenic Graph $G^{GP}$ can be transformed into a Complex Turiyam Neutrosophic Graph $G^{CT}$.*

*Proof*: (1) For each vertex $v \in M$, assign:

$$T(v) = r_T(v)e^{i\omega_T(v)}, \quad I(v) = r_I(v)e^{i\omega_I(v)}, \quad F(v) = r_F(v)e^{i\omega_F(v)}, \quad L(v) = r_L(v)e^{i\omega_L(v)},$$

with $r_T(v) = d_1$, etc., and $\omega_T(v)$ chosen appropriately.

(2) Ensure:

$$[r_T(v)]^2 + [r_I(v)]^2 + [r_F(v)]^2 + [r_L(v)]^2 \leq 4.$$

(3) For edges, proceed similarly and adjust degrees to satisfy the edge conditions.

Thus, $G^{GP}$ can be transformed into a Complex Turiyam Neutrosophic Graph $G^{CT}$. □

## 4 | Future tasks:q-rung orthopair Turiyam Neutrosophic set and q-rung orthopair Turiyam Neutrosophic graph

The future prospects of this study are outlined as follows. The definitions of the q-rung orthopair Neutrosophic set, q-rung orthopair Turiyam Neutrosophic set, q-rung orthopair Neutrosophic graph, and q-rung orthopair Turiyam Neutrosophic graph are provided below. A mathematical examination of these definitions is intended (cf.[50, 45, 69, 22, 10]).

**Definition 52.** (cf.[68, 67]) Let $X$ be a universe of discourse. A *q-rung orthopair Neutrosophic set $S$* on $X$ is defined as:

$$S = \left\{ (x, T_S(x), I_S(x), F_S(x)) \,\middle|\, x \in X \right\},$$

where:

- $T_S(x)$ is the truth-membership degree,
- $I_S(x)$ is the indeterminacy-membership degree,
- $F_S(x)$ is the falsity-membership degree.

These degrees satisfy the following constraint:

$$[T_S(x)]^q + [I_S(x)]^q + [F_S(x)]^q \leq 2, \quad \forall x \in X,$$

where $q \geq 1$ is a fixed positive integer, known as the *q-rung parameter*. The q-rung parameter controls the flexibility of the Neutrosophic set, allowing for more comprehensive modeling of uncertainty.

**Definition 53.** A *q-rung orthopair Neutrosophic graph* is defined as a pair $G = (P, Q)$, where:

- $P$ is a q-rung orthopair Neutrosophic set defined on the set of vertices $V$.
- $Q$ is a q-rung orthopair Neutrosophic set defined on the set of edges $E \subseteq V \times V$.

For each vertex $u \in V$, we have:

$$[T_P(u)]^q + [I_P(u)]^q + [F_P(u)]^q \leq 2, \quad \forall u \in V.$$

For each edge $(u, v) \in E$, we have:

$$\left[T_Q(u,v)\right]^q + \left[I_Q(u,v)\right]^q + \left[F_Q(u,v)\right]^q \leq 2, \quad \forall (u,v) \in E.$$

The relationship between the degrees of vertices and edges is defined as follows:

$$T_Q(u,v) \leq \min\{T_P(u), T_P(v)\},$$
$$I_Q(u,v) \leq \min\{I_P(u), I_P(v)\},$$
$$F_Q(u,v) \geq \max\{F_P(u), F_P(v)\}.$$

If $T_Q(u,v) = I_Q(u,v) = F_Q(u,v) = 0$, then the edge $(u, v)$ does not exist in the graph, i.e., $(u, v) \notin E$.

**Definition 54.** A **q-rung orthopair Turiyam Neutrosophic set (qROTS)** over a universe $X$ is defined as:

$$S = \{(x, (t_S(x), iv_S(x), fv_S(x), lv_S(x))) \,|\, x \in X\},$$

where:

- $t_S(x) \in [0,1]$ represents the degree of truth-membership of $x$ in the set.
- $iv_S(x) \in [0,1]$ represents the degree of indeterminacy-membership of $x$ in the set.
- $fv_S(x) \in [0,1]$ represents the degree of falsity-membership of $x$ in the set.
- $lv_S(x) \in [0,1]$ represents the degree of liberal state-membership of $x$ in the set.

These membership degrees satisfy the following condition for each $x \in X$:

$$(t_S(x))^q + (iv_S(x))^q + (fv_S(x))^q + (lv_S(x))^q \leq 3,$$

where $q \geq 1$ is a fixed positive integer that determines the *q-rung* constraint.

**Definition 55.** A **q-rung orthopair Turiyam Neutrosophic graph (qROTG)** is defined as a pair $G = (V, E)$, where:

- $V$ is a set of vertices, and $E \subseteq V \times V$ is the set of edges.
- Each vertex $v \in V$ is associated with a quadruple:

  $$(t_V(v), iv_V(v), fv_V(v), lv_V(v)),$$

  where:

  - $t_V(v)$ is the truth-membership degree,
  - $iv_V(v)$ is the indeterminacy-membership degree,
  - $fv_V(v)$ is the falsity-membership degree,
  - $lv_V(v)$ is the liberal state-membership degree,
  - and they satisfy the condition:

  $$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q + (lv_V(v))^q \leq 3.$$

- Each edge $e = (u, v) \in E$ is associated with a quadruple:

  $$(t_E(u,v), iv_E(u,v), fv_E(u,v), lv_E(u,v)),$$

  where:

- $t_E(u,v)$ is the truth-membership degree of the edge,
- $iv_E(u,v)$ is the indeterminacy-membership degree of the edge,
- $fv_E(u,v)$ is the falsity-membership degree of the edge,
- $lv_E(u,v)$ is the liberal state-membership degree of the edge,
- and they satisfy:

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q + (lv_E(u,v))^q \leq 3.$$

Additionally, the following constraints hold for the vertices and edges:

(1) For any vertices $u, v \in V$, if $(u,v) \in E$:

$$t_E(u,v) \leq \min\{t_V(u), t_V(v)\},$$
$$iv_E(u,v) \leq \min\{iv_V(u), iv_V(v)\},$$
$$fv_E(u,v) \geq \max\{fv_V(u), fv_V(v)\},$$
$$lv_E(u,v) \leq \min\{lv_V(u), lv_V(v)\}.$$

(2) If $t_E(u,v) = iv_E(u,v) = fv_E(u,v) = lv_E(u,v) = 0$, then no edge exists between $u$ and $v$.

The **q-rung orthopair Turiyam Neutrosophic graph** extends classical graph theory by incorporating multiple membership degrees with the flexibility of q-rung orthopairs, enabling a more nuanced representation of uncertainty, indeterminacy, and liberal states.

**Theorem 56.** *A q-Rung Orthopair Turiyam Neutrosophic graph (qROTG) generalizes a q-Rung Orthopair Neutrosophic graph (qRONG).*

*Proof*: Let $G_{\text{RON}} = (P, Q)$ be a q-Rung Orthopair Neutrosophic graph, where:

- $P$ is a q-Rung Orthopair Neutrosophic set on the vertices $V$, satisfying:

$$[T_P(v)]^q + [I_P(v)]^q + [F_P(v)]^q \leq 2, \quad \forall v \in V.$$

- $Q$ is a q-Rung Orthopair Neutrosophic set on the edges $E \subseteq V \times V$, satisfying:

$$\left[T_Q(u,v)\right]^q + \left[I_Q(u,v)\right]^q + \left[F_Q(u,v)\right]^q \leq 2, \quad \forall (u,v) \in E.$$

Now, let $G_{\text{ROT}} = (V, E)$ be a q-Rung Orthopair Turiyam Neutrosophic graph, where:

- Each vertex $v \in V$ is associated with a quadruple:

$$(t_V(v), iv_V(v), fv_V(v), lv_V(v)),$$

  satisfying:

$$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q + (lv_V(v))^q \leq 3.$$

- Each edge $(u,v) \in E$ is associated with a quadruple:

$$(t_E(u,v), iv_E(u,v), fv_E(u,v), lv_E(u,v)),$$

  satisfying:

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q + (lv_E(u,v))^q \leq 3.$$

To prove the generalization, observe the following:

- If $lv_V(v) = 0$ and $lv_E(u,v) = 0$ for all vertices $v \in V$ and edges $(u,v) \in E$, the qROTG reduces to a qRONG. Specifically:

$$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q \leq 2,$$

  and:

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q \leq 2.$$

- The additional liberal state-membership degree $lv$ in qROTG introduces a fourth dimension of flexibility, allowing the graph to model more complex relationships and states than qRONG.

Thus, by reducing the liberal state-membership degree to zero, a qROTG becomes equivalent to a qRONG, proving that a q-Rung Orthopair Turiyam Neutrosophic graph generalizes a q-Rung Orthopair Neutrosophic graph. □

## 4.1 | q-Rung Orthopair Single-Valued Neutrosophic Offset

As a future prospect, we aim to define the concept of the q-Rung Orthopair Single-Valued Neutrosophic Offset as an extension of the Single-Valued Neutrosophic OffSet and explore the relationships between them. The definition of the Single-Valued Neutrosophic OffSet is provided below [61, 56, 62].

**Definition 57** (Single-Valued Neutrosophic OffSet)**.** [62] A *Single-Valued Neutrosophic OffSet*, denoted $A_{\text{off}} \subseteq U_{\text{off}}$, is a set within a universe of discourse $U_{\text{off}}$ in which certain elements may possess neutrosophic degrees—truth, indeterminacy, or falsity—that extend beyond the standard limits, either above 1 or below 0. It is formally defined as:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{off}}, \exists\ (T(x) > 1 \text{ or } F(x) < 0)\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of each $x \in U_{\text{off}}$.
- $T(x), I(x), F(x) \in [\Psi, \Omega]$, where $\Omega > 1$ (termed the *OverLimit*) and $\Psi < 0$ (termed the *UnderLimit*), allow the possibility for $T(x)$, $I(x)$, or $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.

**Definition 58** (q-Rung Orthopair Single-Valued Neutrosophic Offset)**.** A *q-Rung Orthopair Single-Valued Neutrosophic Offset*, denoted $S_{\text{off}} \subseteq X_{\text{off}}$, is a generalized Neutrosophic offset defined on a universe of discourse $X_{\text{off}}$, where certain elements may have truth, indeterminacy, or falsity degrees exceeding the standard range $[0, 1]$. Formally, it is defined as:

$$S_{\text{off}} = \left\{(x, \langle T_S(x), I_S(x), F_S(x)\rangle) \,\middle|\, x \in X_{\text{off}}\right\},$$

where:

- $T_S(x)$, $I_S(x)$, and $F_S(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of $x \in X_{\text{off}}$, respectively.
- The degrees $T_S(x), I_S(x), F_S(x) \in [\Psi, \Omega]$, where $\Psi < 0$ (termed the *UnderLimit*) and $\Omega > 1$ (termed the *OverLimit*), allow values outside the standard bounds of $[0, 1]$.
- These degrees satisfy the following constraint:
  $$[T_S(x)]^q + [I_S(x)]^q + [F_S(x)]^q \leq 2, \quad \forall x \in X_{\text{off}},$$
  where $q \geq 1$ is a fixed positive integer, known as the *q-rung parameter*.

**Theorem 59.** *The q-Rung Orthopair Single-Valued Neutrosophic Offset generalizes the Single-Valued Neutrosophic OffSet.*

*Proof*: The q-Rung Orthopair Single-Valued Neutrosophic Offset $S_{\text{off}}$ is defined as:

$$S_{\text{off}} = \left\{(x, \langle T_S(x), I_S(x), F_S(x)\rangle) \,\middle|\, x \in X_{\text{off}}\right\},$$

where $T_S(x), I_S(x), F_S(x) \in [\Psi, \Omega]$ and satisfy:

$$[T_S(x)]^q + [I_S(x)]^q + [F_S(x)]^q \leq 2, \quad \forall x \in X_{\text{off}},$$

with $q \geq 1$ as the q-rung parameter.

When the q-rung parameter $q = 1$, the constraint reduces to:

$$T_S(x) + I_S(x) + F_S(x) \leq 2.$$

By restricting $T_S(x), I_S(x), F_S(x)$ to the range $[\Psi, \Omega]$, $S_{\text{off}}$ becomes identical to $A_{\text{off}}$.

When $q > 1$, $S_{\text{off}}$ allows for more flexible representations and models higher-order complexities. This additional flexibility confirms that $S_{\text{off}}$ generalizes $A_{\text{off}}$.

Thus, the q-Rung Orthopair Single-Valued Neutrosophic Offset generalizes the Single-Valued Neutrosophic OffSet. □

Additionally, we consider the HyperNeutrosophic Set (Graph)[32, 33, 34, 35, 31, 28], the HyperFuzzy Set (Graph)[41, 64, 37, 47], and the Plithogenic Set (Graph)[30, 57, 63]. In the future, we aim to explore the integration of these concepts with the graph structures discussed in this paper, defining new frameworks and examining their potential applications. Furthermore, we plan to extend the graph concepts examined in this study to Hypergraphs[12, 13] and SuperHypergraphs[25, 59, 60, 29], further expanding their theoretical foundation and practical applicability.

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## Data Availability

The datasets generated and/or analyzed during the current study are not publicly available due to privacy-preserving constraints but are available from the corresponding author upon reasonable request.

## Conflicts of Interest

The authors declare that there is no conflict of interest in this research.

## Ethical Approval

This article does not contain any studies involving human participants or animals conducted by any of the authors.

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