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An Approach to Multi-Attribute Decision-Making Based on Single-Valued Neutrosophic Hesitant Fuzzy Aczel-Alsina Aggregation Operator

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Abstract: A single-valued Neutrosophic hesitant fuzzy set (SVNHFS) is a combination of a single-valued neutrosophic set (SVNS) and hesitant fuzzy set (HFS) that has been developed to address insufficient, unreliable, and vague environments in which each element has several possible options determined by the truthiness ($t_{и}$), indeterminacy ($i_{и}$) and falsity ($f_{и}$) value. By considering this, in this paper, we have proposed the Aczel-Alsina aggregation operator (AAAO) for SVNHFS, which is more flexible t-norm (Ŧ) and t-conorm (∇) than the other Ŧ, and ∇ due to the flexible nature of parameters to solve Multi-Attribute decision making (MADM) problems. Further, the score function (ş), accuracy function (ɑ), and certainty function (c) of SVNHFS have been defined. In this paper, we proposed the single-valued neutrosophic hesitant fuzzy Aczel-Alsina Weighted Averaging Operator (SVNHFAAWA), single-valued neutrosophic hesitant fuzzy Aczel-Alsina Weighted ordered Averaging Operator (SVNHFAAWOA), and single-valued neutrosophic hesitant fuzzy Aczel-Alsina hybrid averaging operator (SVNHFAAHA). To testify to the reliability and stability of the newly created aggregation operator (AO), an application of MADM has been discussed.

Keywords: Hesitant Fuzzy Sets; Single-valued Neutrosophic Hesitant Fuzzy Sets; Aczel-Alsina t-norm and t-conorm; Multi-attribute Decision-making.

1. Introduction

In the current era, MADM is a significant and prominent issue. These decisions are made based on multiple factors by assigning the weights to various factors, and all of these weight values are obtained from expert groups. In many cases, decision-makers assigned a rating value to each alternative within [0,1] based on the precise information. However, in real life, we face multiple challenges where the information is unclear due to the complexity and vagueness of the surrounding environment. So, to cope with these situations involving ambiguity and uncertainty, many theories have been defined such as the Fuzzy set (FS) theory [1], Intuitionistic fuzzy set (IFS) theory [2], Interval-Valued Intuitionistic Fuzzy set (IVIFS) [3] that deals with $t_{и}$ and $f_{и}$. Later on, many researchers proposed applications including robot selection [4], selection of effective diagnosis methods of tuberculosis [5], identification of eco-friendly transportation [6], and solar power-wind power plant site selection [7]. Still, these values were not useful for depicting the hesitancy and indeterminate values. To overcome such situations, the Neutrosophic set (NS) theory was proposed [8] which is capable of dealing with hesitancy and indeterminate values along with $t_{и}$ and $f_{и}$ values over the $(0^{-}, 1^{+})$, but as this interval is non-standard. Hence, it is challenging to implement real-world problems. Therefore, to reduce this difficulty, several categories of NS were proposed which involve

SVNS [9], Complex Neutrosophic set (CNS) [10], and Interval-Valued Neutrosophic set (IVNS) [11]. Various researchers utilized these categories for the evaluation of decision-making problems such as Jafar et al. [12] proposed the MADM problem under the NS environment, Jia [13] proposed the cross entropy measures of IVNS, Alqahtani et al. [14] discussed the application of hospital infrastructure design within the context of CNS, Li, and Zhang [15] suggested the SVNS and its application in computing, Abdullah [16] proposed the model for breast cancer classification under NS framework.

For being straightforward and useful for evaluating the practical application, aggregation operator (AO) was defined as generally the mathematical operations that transform a collection of numeric data into a single number. So, it attains the attention of the researcher and many researchers evaluated image processing, pattern recognition, machine learning, and the other various MADM problems using AOs. Imran et al. [17] proposed the AOs of NS based on the Prioritized Muirhead mean (PMM), Zhang and Ye [18] suggested the AOs of SVNS based on the Dombi power (DP) AOs, Latif et al. [19] defined the AAAO information under the SVNS environment, Rong et al. [20] generalize the SVNS power AOs based on Archimedean Ŧ and ∇. Schweizer-Sklar power aggregation operator (SSPAO) and its application in decision making was proposed by Liu et al. [21], Ashraf et al. [22] discussed Logarithmic hybrid AOs and its application. Due to the circumstance, operators assumed that the aggregate parameters are independent of one another and have no inter-relationship between them. However, in the real world, there is always an appropriate relation between these parameters. So, to understand the relations between these parametric values, the AOs like Muirhead mean (MM) and Bonferroni mean (BM) were defined as powerful tools for aggregating data and assisting us in decision-making. Akram et al. [23] proposed the MADM method based on the MM, Fahmi et al. [24] proposed the AOs of bi-polar NS based on the MM, and Liu [25] defined the 2-tuple linguistic NS for decision-making. Yager [26] introduced the concept of BM and discussed its application in decision-making, Surya et al. [27] suggested BM for q-ROFS, while geometric BM was introduced by Li et al. [28]. However, to deal with ambiguous and uncertain data, Aczel-Alsina (AA) is an important computational operator that favorably deals with such kind of information.

So, by concluding the above debate, we say that making decisions in reality is a more complicated and big task. Therefore, it is necessary to represent the ambiguous data more effectively to express the relationship between the parametric values to select the best alternative. Moreover, it is also a highly complicated task to execute any kind of AO based on the prior notions. The AAAO, in which the averaging and geometric aggregation operators are specific cases, is an efficient tool and provides the appropriate and feasible solution by aggregating the finite parametric values into a singleton set. Furthermore, SVNHFS [39] is more efficient and reliable than the IFS as it deals with the $t_{и}(\upsilon), i_{и}(\upsilon), f_{и}(\upsilon)$ values and has been utilized in many real-life scenarios. So, particularly, SVNHFS is suitable for analyzing ambiguous data by using the AOs, and the evaluation of the data which assists us in making decisions. This article aims to put forward the information of the proposed theory which is stated below;

- To present the AO of SVNHFS based on the AAAO.
- To explore the fundamental aspects and prominent properties.
- To demonstrate the new decision-making tool for determining the best alternative.
- To illustrate that, SVNHS is more reliable than IFS for dealing with ambiguous environments.
- To show the applicability of the proposed method, a comparison between the proposed and prior methods has been discussed.

Following is the format of the paper: In part 2, we discussed the definition of prior concepts. In 3, we proposed the theory of AA operations based on the SVNHFS environment including SVNHFAAWA, SVNHFAAOWA, and SVNHFAAHA, and also evaluated their properties. In 4, we provide the algorithm for solving the proposed approach. In section 5, we provide a MADM approach for suggested operators under the SVNHFS environment and illustrate this information with the help of some suitable examples. Finally, a comparison between the proposed approach and prior methodologies has been examined, followed by a discussion of the conclusion shown in part 6.

2. Preliminaries

Some basic concepts of SVNFS, Ŧ, $\mathcal{V}$ and AAAOs have been discussed in this section.

Definition 1. [19] Consider $\mathcal{X}$ to be the collection of a non-empty set and $\mathcal{v}$ represents the element in $\mathcal{X}$. An SVNS и on $\mathcal{X}$ is defined as;

$$и : \{\langle \mathcal{v}, (t_и(\mathcal{v}), i_и(\mathcal{v}), f_и(\mathcal{v})) | \mathcal{v} \in \mathcal{X} \rangle\} \tag{1}$$

s.t $0^- \leq t_{и(R)}(\mathcal{v}) + i_{и(R)}(\mathcal{v}) + f_{и(R)}(\mathcal{v}) \leq 3^+$; $0^- \leq t_{и(I)}(\mathcal{v}) + i_{и(I)}(\mathcal{v}) + f_{и(I)}(\mathcal{v}) \leq 3^+$

Example 1. Consider $\mathcal{X}$ consists of two elements i.e. $\{\mathcal{v}_1, \mathcal{v}_2\}$, s.t $t_и(\mathcal{v}_1) = 0.2$; $i_и(\mathcal{v}_1) = 0.1$; $f_и(\mathcal{v}_1) = 0.3$ and $t_и(\mathcal{v}_2) = 0.4$; $i_и(\mathcal{v}_2) = 0.3$; $f_и(\mathcal{v}_2) = 0.1$, then $и = \{\langle (\mathcal{v}_1, (0.2,0.1,0.3)), (\mathcal{v}_2, (0.4,0.3,0.1)) | \mathcal{v} \in \mathcal{X} \rangle\}$ is an SVNS.

Definition 2. [29] Consider $\mathcal{X}$ to be the collection of a non-empty set then a HFS (ɦ) on $\mathcal{X}$ is defined in terms of function f such that when applied, $\mathcal{X} \subset [0, 1]$.

$$ɦ: \{\langle \mathcal{v}, f_ɦ(\mathcal{v}) | \mathcal{v} \in \mathcal{X} \rangle\} \tag{2}$$

Where $f_ɦ(\mathcal{v})$ is the collection of different values of $[0, 1]$.

Example 2. Consider $\mathcal{X}$ consists of a single element i.e. $\{\mathcal{v}_1\}$, such that $f_ɦ(\mathcal{v}_1) = \{0.4,0.5,0.6\}$ then, $ɦ = \{\langle (\mathcal{v}_1, (0.4,0.5,0.6)) | \mathcal{v} \in \mathcal{X} \rangle\}$ is the hesitant fuzzy element (HFE).

Definition 3. [30] Suppose $\mathcal{X}$ be the non-empty set then a SVNHFS и on $\mathcal{X}$ is defined in terms of functions f and f' such that $\mathcal{X} \subset [0, 1]$.

$$и : \{\langle \mathcal{v}, (f_и(\mathcal{v}), f'_и(\mathcal{v}), f''_и(\mathcal{v})) | \mathcal{v} \in \mathcal{X} \rangle\} \tag{3}$$

Where $f_и(\mathcal{v})$, $f'_и(\mathcal{v})$ and $f''_и(\mathcal{v})$ are the finite subsets of [0,1] which represent the $t_и(\mathcal{v}), i_и(\mathcal{v}), f_и(\mathcal{v})$, s.t

$$\left(\vee(f_и(\mathcal{v})) + \wedge(f'_и(\mathcal{v}))\right) \leq 1; \ \left(\wedge(f_и(\mathcal{v})) + \vee(f'_и(\mathcal{v}))\right) \leq 1$$

Example 3. Consider $\mathcal{X}$ consist of a single element i.e. $\{\mathcal{v}_1\}$, such that $f_и(\mathcal{v}) = \{0.2,0.1,0.5\}$; $f'_и(\mathcal{v}) = \{0.3,0.6,0.7\}$; $f''_и(\mathcal{v}) = \{0.3,0.6,0.7\}$ then,

$$и = \{\langle \mathcal{v}, ((0.2,0.1,0.5), (0.3,0.6,0.7)) | \mathcal{v} \in \mathcal{X} \rangle\}$$

s.t

$$\vee\left(f_и(\mathcal{v})\right) = \vee(0.2,0.1,0.5) = 0.5$$
$$\vee\left(f'_и(\mathcal{v})\right) = \vee(0.3,0.6,0.7) = 0.7$$
$$\wedge\left(f_и(\mathcal{v})\right) = \wedge(0.2,0.1,0.5) = 0.1$$
$$\wedge\left(f'_и(\mathcal{v})\right) = \wedge(0.3,0.6,0.7) = 0.3$$

$$\left(\vee\left(f_и(\mathcal{v})\right) + \wedge\left(f'_и(\mathcal{v})\right)\right) = 0.5 + 0.3 = 0.8 \leq 1 \text{ and } \left(\wedge\left(f_и(\mathcal{v})\right) + \vee\left(f'_и(\mathcal{v})\right)\right) = 0.1 + 0.7 \leq 1$$

Is an SVNHFS и and $(f_и(\mathcal{v}), f'_и(\mathcal{v}))$ is a single-valued neutrosophic hesitant fuzzy element (SVNHFE).

Definition 4. [21] Consider $и: \{\langle \mathcal{v}, (f_и(\mathcal{v}), f'_и(\mathcal{v}), f''_и(\mathcal{v})) | \mathcal{v} \in \mathcal{X} \rangle\}$ be the SVNHFS, then the score value ȿ(и), accuracy value ɑ(и), and certainty value ϲ(и) is defined as;

$$ȿ(и) = \frac{1}{3}\left(\frac{1}{l}\sum_{j=1}^{l}\mu_j + \frac{1}{m}\sum_{j=1}^{m}(1-\eta_j) + \frac{1}{n}\sum_{j=1}^{n}(1-\nu_j)\right) \tag{4}$$

$$ɑ(и) = \frac{1}{l}\sum_{j=1}^{l}\mu_j - \frac{1}{n}\sum_{j=1}^{n}(1-\nu_j) \tag{5}$$

$$ϲ(и) = \frac{1}{l}\sum_{j=1}^{l}\mu_j \tag{6}$$

Where $f_и(\mathcal{v}) = \{\mu_1, \mu_2, \mu_3, \dots, \mu_l\}$; $\mu_1, \mu_2, \mu_3, \dots, \mu_l$ are the elements of $f_и(\mathcal{v})$; $f'_и(\mathcal{v}) = \{\eta_1, \eta_2, \eta_3, \dots, \eta_m\}$; $\eta_1, \eta_2, \eta_3, \dots, \eta_m$ are the elements of $f'_и(\mathcal{v})$, and $f''_и(\mathcal{v}) = \{\nu_1, \nu_2, \nu_3, \dots, \nu_n\}$; $\nu_1, \nu_2, \nu_3, \dots, \nu_n$ are the elements of $f''_и(\mathcal{v})$, and l, m, n shows the number of HFE in an SVNHFS.

Definition 5. [4] A mapping Ŧ: $[0,1] \times [0,1] \to [0,1]$ is said to be Ŧ if it holds the following properties.

i). Ŧ$(\Upsilon, 1) = \Upsilon$

ii). Commutativity: $\text{Ŧ}(\Upsilon, \Omega) = \text{Ŧ}(\Omega, \Upsilon)$
iii). Associativity : $\text{Ŧ}(\Upsilon, \varDelta(\Omega, \delta)) = \text{Ŧ}(\varDelta(\Upsilon, \Omega), \delta))$
iv). Monotonicity : if $\Omega \leq \delta$ then $\text{Ŧ}(\Upsilon, \Omega) \leq \text{Ŧ}(\Omega, \delta)$

Example 4. The example of Ŧ includes;

i). Drastic t-norm: $\text{Ŧ}_D(\Upsilon, \Omega) = \begin{cases} \Upsilon & ; \Omega = 1 \\ \Omega & ; \Upsilon = 1 \\ 0 & ; otherwise \end{cases}$

ii). Product t-norm: $\text{Ŧ}_P(\Upsilon, \Omega) = \Upsilon\Omega$

iii). Minimum t-norm: $\text{Ŧ}_M(\Upsilon, \Omega) = \wedge \Upsilon\Omega$

Definition 6. [4] A mapping $\nabla: [0,1] \times [0,1] \to [0,1]$ is said to be ∇ if it holds the following properties.

i). $\nabla(\Upsilon, 0) = \Upsilon$
ii). Commutativity: $\nabla(\Upsilon, \Omega) = \nabla(\Omega, \Upsilon)$
iii). Associativity : $\nabla(\Upsilon, \nabla(\Omega, \delta)) = \nabla(\nabla(\Upsilon, \Omega), \delta))$
iv). Monotonicity : if $\Omega \leq \delta$ then $\nabla(\Upsilon, \Omega) \leq \nabla(\Omega, \delta)$

Example 5. The example of ∇ includes;

i). Probabilistic t-conorm $\nabla_P(\Upsilon, \Omega) = \Upsilon + \Omega - \Upsilon\Omega$

ii). Drastic t-conorm $\nabla_D(\Upsilon, \Omega) = \begin{cases} \Upsilon & ; \Omega = 0 \\ \Omega & ; \Upsilon = 0 \\ 1 & ; otherwise \end{cases}$

iii). Lukasiewicz t-conorm $\nabla_L(\Upsilon, \Omega) = \wedge(\Upsilon + \Omega, 1)$

iv). Maximum t-conorm $\nabla_M(\Upsilon, \Omega) = \vee(\Upsilon, \Omega)$

Definition 7. [4] Aczel-Alsina t-norm $\left(\text{Ŧ}_{\text{P}}^{\text{ʒ}}\right)_{\text{ʒ}\in[0,\infty]}$ is determined by;

$$\text{Ŧ}_{\text{P}}^{\text{ʒ}}(\Upsilon, \Omega) = \begin{cases} \text{Ŧ}_D(\Upsilon, \Omega) & ; \text{ʒ} = 0 \\ \wedge(\Upsilon, \Omega) & ; \text{ʒ} = \infty \\ \varepsilon^{-((-\ln(\Upsilon))^{\text{ʒ}} + (-\ln(\Omega))^{\text{ʒ}})^{\frac{1}{\text{ʒ}}}} & ; otherwise \end{cases}$$

and Aczel-Alsina t-conorm $\left(\nabla_{\text{P}}^{\text{ʒ}}\right)_{\text{ʒ}\in[0,\infty]}$ is determined by;

$$\nabla_{\text{P}}^{\text{ʒ}}(\Upsilon, \Omega) = \begin{cases} \nabla_D(\Upsilon, \Omega) & ; \text{ʒ} = 0 \\ \vee(\Upsilon, \Omega) & ; \text{ʒ} = \infty \\ 1 - \varepsilon^{-((-\ln(1-\Upsilon))^{\text{ʒ}} + (-\ln(1-\Omega))^{\text{ʒ}})^{\frac{1}{\text{ʒ}}}} & ; otherwise \end{cases}$$

For limited cases, $\text{Ŧ}_{\text{P}}^{0} = \text{Ŧ}_D, \text{Ŧ}_{\text{P}}^{1} = \text{Ŧ}_P, \text{Ŧ}_{\text{P}}^{2} = \text{Ŧ}_M; \nabla_{\text{P}}^{0} = \nabla_D, \nabla_{\text{P}}^{1} = \nabla_P, \nabla_{\text{P}}^{2} = \nabla_M$. For each $\text{ʒ} \in [0, \infty]$, $\text{Ŧ}_{\text{P}}^{\text{ʒ}}$ and $\nabla_{\text{P}}^{\text{ʒ}}$ are the dual of each other, and $\text{Ŧ}_{\text{P}}^{\text{ʒ}}$ and $\nabla_{\text{P}}^{\text{ʒ}}$ are strictly increasing and strictly decreasing functions.

3. Aczel-Alsina Operations of SVNHFS

In this section, we will show the relationship between AA Operations with SVNHFS.

Definition 8. Consider $\text{и} = (\text{f}_{\text{и}}, \text{f}'_{\text{и}}, \text{f}''_{\text{и}}), \text{и}_1 = (\text{f}_{\text{и}1}, \text{f}'_{\text{и}1}, \text{f}''_{\text{и}1}), \text{и}_2 = (\text{f}_{\text{и}2}, \text{f}'_{\text{и}2}, \text{f}''_{\text{и}2})$ be the SVNHFSs, $\text{ʒ} \geq 1$ *and* $\varphi > 0$ then $\text{Ŧ}_{\text{P}}^{\text{ʒ}}$ and $\nabla_{\text{P}}^{\text{ʒ}}$ operations is defined as;

i). $\text{и} \oplus \text{и}_1 = \cup_{\text{f}_{\text{и}}, \text{f}'_{\text{и}}, \text{f}''_{\text{и}} \in \text{и}; \text{f}_{\text{и}1}, \text{f}'_{\text{и}1}, \text{f}''_{\text{и}1} \in \text{и}_1} \left(\begin{array}{c} 1 - \varepsilon^{-((-\ln(1-\text{f}_{\text{и}}))^{\text{ʒ}} + (-\ln(1-\text{f}_{\text{и}1}))^{\text{ʒ}})^{\frac{1}{\text{ʒ}}}}, \varepsilon^{-((-\ln(\text{f}'_{\text{и}}))^{\text{ʒ}} + (-\ln(\text{f}'_{\text{и}1}))^{\text{ʒ}})^{\frac{1}{\text{ʒ}}}}, \\ \varepsilon^{-((-\ln(\text{f}''_{\text{и}}))^{\text{ʒ}} + (-\ln(\text{f}''_{\text{и}1}))^{\text{ʒ}})^{\frac{1}{\text{ʒ}}}} \end{array} \right)$

ii). $$и \otimes и_1 = \cup_{f_и, f'_и, f''_и \in и; f_{и1}, f'_{и1}, f''_{и1} \in и_1} \begin{pmatrix} \varepsilon^{-((-\ln(f_и))^3 + (-\ln(f_{и1}))^3)^{\frac{1}{3}}}, 1 - \varepsilon^{-((-\ln(1-f'_и))^3 + (-\ln(1-f'_{и1}))^3)^{\frac{1}{3}}}, \\ 1 - \varepsilon^{-((-\ln(1-f''_и))^3 + (-\ln(1-f''_{и1}))^3)^{\frac{1}{3}}} \end{pmatrix}$$

iii). $$\varphi и = \cup_{f_и, f'_и, f''_и \in и} \left(1 - \varepsilon^{-(\varphi(-\ln(1-f_и))^3)^{\frac{1}{3}}}, \varepsilon^{-(\varphi(-\ln(f'_и))^3)^{\frac{1}{3}}}, \varepsilon^{-(\varphi(-\ln(f''_и))^3)^{\frac{1}{3}}}\right)$$

iv). $$и^{\varphi} = \cup_{f_и, f'_и, f''_и \in и} \left(\varepsilon^{-(\varphi(-\ln(f_и))^3)^{\frac{1}{3}}}, 1 - \varepsilon^{-(\varphi(-\ln(1-f'_и))^3)^{\frac{1}{3}}}, 1 - \varepsilon^{-(\varphi(-\ln(1-f''_и))^3)^{\frac{1}{3}}}\right)$$

Theorem 1. Let $и = (f_и, f'_и, f''_и), и_1 = (f_{и1}, f'_{и1}, f''_{и1}), и_2 = (f_{и2}, f'_{и2}, f''_{и2})$ be the SVNHFSs, then

i). $и \oplus и_1 = и_1 \oplus и$;
ii). $и \otimes и_1 = и_1 \otimes и$;
iii). $\varphi(и \oplus и_1) = \varphi и_1 \oplus \varphi и, \varphi > 0$;
iv). $(\varphi_1 + \varphi_2)и = \varphi_1 и + \varphi_2 и, \varphi_1, \varphi_2 > 0$;
v). $(и \otimes и_1)^{\varphi} = (и)^{\varphi} \otimes (и_1)^{\varphi}, \varphi > 0$;
vi). $и^{\varphi 1} \otimes и^{\varphi 2} = и^{(\varphi 1 + \varphi 2)}, \varphi_1, \varphi_2 > 0$;

3.1 *Single-Valued Neutrosophic Hesitant Fuzzy Aczel-Alsina Weighted Averaging Operator*

Definition 9. Assume $и_j (1 \leq j \leq k)$ be the collection of SVNHFEs, then SVNHFAAWA is:

$$SVNHFAAWA_{ш}(и_1, и_2, \ldots, и_k) = \oplus_{j=1}^{k} ш_j и_j = ш_1 и_1 \oplus ш_2 и_2 \oplus \ldots \oplus ш_j и_j \tag{7}$$

Where $ш = (ш_1, ш_2, \ldots, ш_k)^T$ is the weight vector of $и_j (1 \leq j \leq k)$, $и_j > 0, \sum_{j=1}^{k} ш_j = 1$

Theorem 2. Consider $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \leq j \leq k)$ be the collection of SVNHFEs, then the aggregated value by SVNHFAAWA operator is again a SVNHFEs;

$$SVNHFAAWA_{ш}(и_1, и_2, \ldots, и_k) = \oplus_{j=1}^{k} ш_j и_j$$

$$= \cup_{f_{и1}, f'_{и1}, f''_{и1} \in и_1, \ldots, f_{иj}, f'_{иj}, f''_{иj} \in и_j} \begin{pmatrix} 1 - \varepsilon^{-(\sum_{j=1}^{k} ш_j (-ln(1-f_{иj}))^3)^{\frac{1}{3}}}, \\ \varepsilon^{-(\sum_{j=1}^{k} ш_j (-ln(f'_{иj}))^3)^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{k} ш_j (-ln(f''_{иj}))^3)^{\frac{1}{3}}} \end{pmatrix} \tag{8}$$

Proof: To demonstrate Theorem 2, we have implemented the mathematical induction.

When $k = 2$, SVNHFAAWA operator is;

$$ш_1 и_1 = \left(1 - \varepsilon^{-(ш_1(-\ln(1-f_{и1}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_1(-\ln(f'_{и1}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_1(-\ln(f''_{и1}))^3)^{\frac{1}{3}}}\right),$$

$$ш_2 и_2 = \left(1 - \varepsilon^{-(ш_2(-\ln(1-f_{и2}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_2(-\ln(f'_{и2}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_2(-\ln(f''_{и2}))^3)^{\frac{1}{3}}}\right)$$

By definition 7,

$$SVNHFAAWA_{ш}(и_1, и_2) = ш_1 и_1 \oplus ш_2 и_2$$

$$= \left(1 - \varepsilon^{-(ш_1(-\ln(1-f_{и1}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_1(-\ln(f'_{и1}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_1(-\ln(f''_{и1}))^3)^{\frac{1}{3}}}\right)$$

$$\oplus \left(1 - \varepsilon^{-(ш_2(-\ln(1-f_{и2}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_2(-\ln(f'_{и2}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_2(-\ln(f''_{и2}))^3)^{\frac{1}{3}}}\right)$$

$$= \begin{pmatrix} 1 - \varepsilon^{-(ш_1(-\ln(1-f_{и1}))^3 + ш_2(-\ln(1-f_{и2}))^3)^{\frac{1}{3}}}, \varepsilon^{-(ш_1(-\ln(f'_{и1}))^3 + ш_2(-\ln(f'_{и2}))^3)^{\frac{1}{3}}}, \\ \varepsilon^{-(ш_1(-\ln(f''_{и1}))^3 + (ш_2(-\ln(f''_{и2}))^3)^{\frac{1}{3}}} \end{pmatrix}$$

$$\cup_{f_{и1}, f'_{и1} \in и_1, \ldots, f_{иj}, f'_{иj} \in и_j} \left(1 - \varepsilon^{-(\sum_{j=1}^{2} ш_j(-\ln(1-f_{иj}))^3)^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{2} ш_j(-\ln(f'_{иj}))^3)^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{2} ш_j(-\ln(f''_{иj}))^3)^{\frac{1}{3}}}\right)$$

For $k = m$

$$SVNHFAAWA_{ш}(и_1, и_2, \ldots, и_m) = \oplus_{j=1}^{m} ш_j и_j$$

$$= \cup_{f_{и1}, f'_{и1} \in и_1, \ldots, f_{иj}, f'_{иj} \in и_j} \begin{pmatrix} 1 - \varepsilon^{-(\sum_{j=1}^{m} ш_j(-\ln(1-f_{иj}))^3)^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{m} ш_j(-\ln(f'_{иj}))^3)^{\frac{1}{3}}}, \\ \varepsilon^{-(\sum_{j=1}^{m} ш_j(-\ln(f''_{иj}))^3)^{\frac{1}{3}}} \end{pmatrix}$$

Now, for $k = m + 1$

$$\mathrm{SVNHFAAWA}_{\mathrm{w}}(и_1, и_2, \dots, и_{m+1})$$
$$= \mathrm{SVNHFAAWA}_{\mathrm{w}}(и_1, и_2, \dots, и_m) \oplus \mathrm{w}_{m+1} и_{m+1} = \oplus_{j=1}^{m} \mathrm{w}_j и_j \oplus \mathrm{w}_{m+1} и_{m+1}$$
$$= \cup_{f_{и1}, f'_{и1} \in и_1, \dots, f_{иj}, f'_{иj} \in и_j} \begin{pmatrix} 1 - \varepsilon^{-(\sum_{j=1}^{m} \mathrm{w}_j (-\ln(1-f_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{m} \mathrm{w}_j (-\ln(f'_{иj}))^{3})^{\frac{1}{3}}}, \\ \varepsilon^{-(\sum_{j=1}^{m} \mathrm{w}_j (-\ln(f''_{иj}))^{3})^{\frac{1}{3}}} \end{pmatrix}$$
$$\oplus \cup_{f_{m+1}, f'_{m+1} \in и_{m+1}} \left(1 - \varepsilon^{-(\mathrm{w}_{m+1}(-\ln(1-f_{m+1}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\mathrm{w}_{m+1}(-\ln(f'_{m+1}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\mathrm{w}_{m+1}(-\ln(f''_{m+1}))^{3})^{\frac{1}{3}}}\right)$$
$$= \cup_{f_{и1}, f'_{и1} \in и_1, \dots, f_{m+1}, f'_{m+1} \in и_{m+1}} \left(1 - \varepsilon^{-(\sum_{j=1}^{m+1} \mathrm{w}_j (-\ln(1-f_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{m+1} \mathrm{w}_j (-\ln(f'_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{m+1} \mathrm{w}_j (-\ln(f''_{иj}))^{3})^{\frac{1}{3}}}\right)$$

Thus, it holds for any k.

Theorem 3. Consider $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \le j \le k)$ be the collection of SVNHFEs, then $\mathrm{SVNHFAAWA}_{\mathrm{w}}(и_1, и_2, \dots, и_k) = и$. i.e. (Idempotency)

Proof: As $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \le j \le k)$ be the collection of SVNHFEs, then the aggregated value by Eq. (8);

$$\mathrm{SVNHFAAWA}_{\mathrm{w}}(и_1, и_2, \dots, и_k) = \oplus_{j=1}^{k} \mathrm{w}_j и_j$$
$$= \cup_{f_{и1}, f'_{и1}, f''_{и1} \in и_1, \dots, f_{иj}, f'_{иj}, f''_{иj} \in и_j} \begin{pmatrix} 1 - \varepsilon^{-(\sum_{j=1}^{k} \mathrm{w}_j (-\ln(1-f_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-(\sum_{j=1}^{k} \mathrm{w}_j (-\ln(f'_{иj}))^{3})^{\frac{1}{3}}}, \\ \varepsilon^{-(\sum_{j=1}^{k} \mathrm{w}_j (-\ln(f''_{иj}))^{3})^{\frac{1}{3}}} \end{pmatrix}$$
$$= \left(1 - \varepsilon^{-((-\ln(1-f_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-((-\ln(f'_{иj}))^{3})^{\frac{1}{3}}}, \varepsilon^{-((-\ln(f''_{иj}))^{3})^{\frac{1}{3}}}\right)$$
$$= \left(1 - \varepsilon^{\ln(1-f_{иj})}, \varepsilon^{\ln(f'_{иj})}, \varepsilon^{\ln(f''_{иj})}\right) = (f_{и}, f'_{и}, f''_{и}) = и$$

Theorem 4. Consider $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \le j \le k)$ be the collection of SVNHFEs. Suppose $и_j^{-} = \wedge(и_1, и_2, \dots, и_k) = (f_{иj}^{-}, f'_{иj}{}^{-}, f''_{иj}{}^{-})$ and $и_j^{+} = \vee(и_1, и_2, \dots, и_k) = (f_{иj}^{+}, f'_{иj}{}^{+}, f''_{иj}{}^{+})$ s.t $f_{иj}^{-} = min(f_{иj}), f'_{иj}{}^{-} = max(f'_{иj}), f''_{иj}{}^{-} = \max(f''_{иj})$ and $f_{иj}^{+} = max(f_{иj}), f'_{иj}{}^{+} = min(f'_{иj}), f''_{иj}{}^{+} = \min(f''_{иj})$ then,

$$и_j^{-} \le SVNHFAAWA_{\mathrm{w}}(и_1, и_2, \dots, и_k) \le и_j^{+} \tag{9}$$

Which is the boundedness property.

Theorem 5. If $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \le j \le k)$ be the collection of SVNHFEs s.t $и_j \le и_j'$. Then,

$$SVNHFAAWA_{\mathrm{w}}(и_1, и_2, \dots, и_j) \le SVNHFAAWA_{\mathrm{w}}(и_1', и_2', \dots, и_j') \tag{10}$$

Which is a monotonicity property.

Additionally, we present the Single-valued Neutrosophic Hesitant fuzzy Aczel-Alsina ordered weighted averaging operator (SVNHFAAOWA).

Definition 10. Assume $и_j (1 \le j \le k)$ is the collection of SVNHFEs, then SVNHFAAOWA is a mapping:

$$\mathrm{SVNHFAAOWA}: и_k \to и$$

s.t. $\wp = (\wp_1, \wp_2, \dots, \wp_k)^T$ is the weight vector of $и_j (1 \le j \le k)$, $и_j > 0, \sum_{j=1}^{k} \wp_j = 1$,

$$SVNHFAAOWA_{\wp}(и_1, и_2, \dots, и_k) = \oplus_{j=1}^{k} \wp_j и_{ш(j)} = \wp_1 и_{ш(1)} \oplus \wp_2 и_{ш(2)} \oplus \dots \oplus \wp_j и_{ш(j)} \tag{11}$$

Where $(ш(1), ш(2), \dots, ш(j))$ are the permutations of $(1 \le j \le k)$; $и_{ш(j-1)} > и_{ш(j)}; \forall j = 1, 2, \dots, k$

Theorem 6. Consider $и_j = (f_{иj}, f'_{иj}, f''_{иj}), (1 \le j \le k)$ be the collection of SVNHFEs, then the aggregated value by SVNHFAAOWA operator is again a SVNHFEs;

$$SVNHFAAOWA_{\wp}(и_1, и_2, \dots, и_k) = \oplus_{j=1}^{k} \wp_j и_{ш(j)}$$

$$=\cup_{\mathrm{f}_{иɯ(1)},\mathrm{f}'_{иɯ(1)},\mathrm{f}''_{иɯ(1)}\in и_1,\dots,\mathrm{f}_{иɯ(j)},\mathrm{f}'_{иɯ(j)},\mathrm{f}''_{иɯ(j)}\in и_{ɯ(j)}}\left(\begin{array}{c}1-\varepsilon^{-(\sum_{j=1}^{k}\wp_j(-ln(1-\mathrm{f}_{иɯ(j)}))^{3})^{\frac{1}{3}}},\varepsilon^{-(\sum_{j=1}^{k}\wp_j(-ln(\mathrm{f}'_{иɯ(j)}))^{3})^{\frac{1}{3}}},\\ \varepsilon^{-(\sum_{j=1}^{k}\wp_j(-ln(\mathrm{f}''_{иɯ(j)}))^{3})^{\frac{1}{3}}}\end{array}\right)$$

(12)

Where $(ɯ(1),ɯ(2),\dots,ɯ(j))$ are the permutations of $(1\le j\le k)$; $и_{ɯ(j-1)}>и_{ɯ(j)};\forall j=1,2,\dots,k$

Theorem 7. (Idempotency) Consider $и_j=(\mathrm{f}_{иj},\mathrm{f}'_{иj},\mathrm{f}''_{иj}),(1\le j\le k)$ be the set of SVNHFNs, then

$$SVNHFAAOWA_{\wp}(и_1,и_2,\dots,и_k)=и \quad (12)$$

Theorem 8. (Boundedness) Consider $и_j=(\mathrm{f}_{иj},\mathrm{f}'_{иj},\mathrm{f}''_{иj}),(1\le j\le k)$ be the set of SVNHFNs. Suppose $и_j^-=\wedge(и_1,и_2,\dots,и_k)=(\mathrm{f}_{иj}^-,\mathrm{f}'^-_{иj},\mathrm{f}''^-_{иj})$ and $и_j^+=\vee(и_1,и_2,\dots,и_k)=(\mathrm{f}_{иj}^+,\mathrm{f}'^+_{иj},\mathrm{f}''^+_{иj})$ then,

$$и_j^-\le SVNHFAAOWA_{\wp}(и_1,и_2,\dots,и_k)\le и_j^+ \quad (13)$$

Theorem 9. If $и_j=(\mathrm{f}_{иj},\mathrm{f}'_{иj},\mathrm{f}''_{иj}),(1\le j\le k)$ be the set of SVNHFEs s.t $и_j\le и_j'$. Then,

$$SVNHFAAOWA_{\wp}(и_1,и_2,\dots,и_j)\le SVNHFAAOWA_{\wp}(и_1',и_2',\dots,и_j') \quad (14)$$

Which is a monotonicity property.

Theorem 10. Consider $и_j=(\mathrm{f}_{иj},\mathrm{f}'_{иj},\mathrm{f}''_{иj}),(1\le j\le k)$ and $и_j^*=(\mathrm{f}_{иj}^*,\mathrm{f}'^*_{иj},\mathrm{f}''^*_{иj});(1\le j\le k)$ be the two SVNHFAAOWA then;

$$SVNHFAAOWA_{\wp}(и_1,и_2,\dots,и_j)=SVNHFAAOWA_{\wp}(и_1^*,и_2^*,\dots,и_j^*) \quad (15)$$

Where $и_j^*(1\le j\le k)$ is any permutations of $и_j(1\le j\le k)$

Since by definition 9, we've seen that it weighs only the SVNHFEs which is a simple type of neutrosophic hesitant fuzzy numbers (NHFNs), and in definition 10, SVNHFAAOWA weights the ordered arrangement of NHFNs but in real life, we have to cope with both aspects at once. To overcome such aspects, we proposed the Single-valued Neutrosophic Hesitant fuzzy Aczel-Alsina hybrid averaging operator (SVNHFAAHA).

Definition 11. Assume $и_j(1\le j\le k)$ is the collection of SVNHFEs, then, SVNHFAAHA: $и_k\to и$ is;

$$SVNHFAAHA_{ɯ,\wp}(и_1,и_2,\dots,и_k)=\oplus_{j=1}^{k}ɯ_jи_{ɯ(j)}=ɯ_1и_{ɯ(1)}\oplus ɯ_2и_{ɯ(2)}\oplus\dots\oplus ɯ_jи_{ɯ(j)} \quad (16)$$

Such that $ɯ=(ɯ_1,ɯ_2,\dots,ɯ_k)^T$ is the weight vector of $и_j(1\le j\le k)$, $и_j>0,\sum_{j=1}^{k}ɯ_j=1$, $and\ \wp=(\wp_1,\wp_2,\dots,\wp_k)^T$ is the weight vector of $и_j(1\le j\le k)$, $и_j>0,\sum_{j=1}^{k}\wp_j=1$

Where $(ɯ(1),ɯ(2),\dots,ɯ(j))$ are the permutations of $(1\le j\le k)$; $и_{ɯ(j-1)}>и_{ɯ(j)};\forall j=1,2,\dots,k$

Theorem 11. Consider $и_j=(\mathrm{f}_{иj},\mathrm{f}'_{иj},\mathrm{f}''_{иj}),(1\le j\le k)$ be the set of SVNHFNs, then aggregated value by SVNHFAAHA operator is again a SVNHFN;

$$SVNHFAAHA_{ɯ}(и_1,и_2,\dots,и_k)=\oplus_{j=1}^{k}ɯ_jи_{ɯ(j)}$$

$$=\cup_{\mathrm{f}_{иɯ(1)},\mathrm{f}'_{иɯ(1)},\mathrm{f}''_{иɯ(1)}\in и_1,\dots,\mathrm{f}_{иɯ(j)},\mathrm{f}'_{иɯ(j)},\mathrm{f}''_{иɯ(j)}\in и_{ɯ(j)}}\left(\begin{array}{c}1-\varepsilon^{-(\sum_{j=1}^{k}ɯ_j(-ln(1-\mathrm{f}_{иɯ(j)}))^{3})^{\frac{1}{3}}},\\ \varepsilon^{-(\sum_{j=1}^{k}ɯ_j(-ln(\mathrm{f}'_{иɯ(j)}))^{3})^{\frac{1}{3}}},\varepsilon^{-(\sum_{j=1}^{k}ɯ_j(-ln(\mathrm{f}''_{иɯ(j)}))^{3})^{\frac{1}{3}}}\end{array}\right) \quad (17)$$

Theorem 12. The SVNHFAAWA and SVNHFAAOWA operators are the particular cases of the SVNHFAAHA operator.

Proof:

Consider $ɯ=\left(\frac{1}{k},\frac{1}{k},\dots,\frac{1}{k}\right)^T$ then;

$$SVNHFAAHA_{ɯ,\wp}(и_1,и_2,\dots,и_k)=\oplus_{j=1}^{k}ɯ_jи_{ɯ(j)}$$
$$=ɯ_1и_{ɯ(1)}\oplus ɯ_2и_{ɯ(2)}\oplus\dots\oplus ɯ_kи_{ɯ(k)} \quad (18)$$
$$=\frac{1}{k}\left(и_{ɯ(1)}\oplus и_{ɯ(2)}\oplus\dots\oplus и_{ɯ(k)}\right)$$

$$= \text{ш}_1 и_1 \oplus \text{ш}_2 и_2 \oplus \dots \oplus \text{ш}_j и_j = SVNHFAAWA_{\text{ш}}(и_1, и_2, \dots, и_k)$$

Consider $\wp = \left(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}\right)^T$ then;

$$SVNHFAAHA_{\text{ш},\wp}(и_1, и_2, \dots, и_k) = \oplus_{j=1}^{k} \text{ш}_j и_{ın(j)} = \wp_1 и_{ın(1)} \oplus \wp_2 и_{ın(2)} \oplus \dots \oplus \wp_k и_{ın(k)}$$

$$= \frac{1}{k}\left(и_{ın(1)} \oplus и_{ın(2)} \oplus \dots \oplus и_{ın(k)}\right)$$

$$= \wp_1 и_1 \oplus \wp_2 и_2 \oplus \dots \oplus \wp_j и_j = SVNHFAAOWA_{\wp}(и_1, и_2, \dots, и_k) \tag{19}$$

4. Evaluation of a Problem by MADM Approach

In this part, we discuss the MADM technique to solve the decision-making problem within the SVNHFS framework by putting suggested operators into execution. A case study from the actual world has also been discussed to illustrate it.

4.1 *Algorithm for Solving the Proposed Approach*

To evaluate the MADM technique, consider a collection of alternatives that are evaluated by considering the attributive values such that the values assigned to each alternative are in the form of SVNHFEs and the weights assigned to them are in real integers. In this scenario, let $e = \{e_1, e_2, e_3, \dots, e_j\}$ represent the collection of attributive values, $\text{ш} = \{\text{ш}_1, \text{ш}_2, \text{ш}_3, \dots, \text{ш}_j\}$ represents the weight value assigned to these attributes, such that $\text{ш}_k > 0, (k = 1,2,3, \dots, j)$; $\sum_{k=1}^{j} \text{ш}_k = 1$ and the collection of alternatives is represented by $\text{q}_k (k = 1,2,3, \dots, j)$. The value assigned by the decision-maker to each alternative $\text{q}_k (k = 1,2,3, \dots, j)$ corresponds to attributive values in the form of SVNHFE, and these assigned values are displayed in the form of matrix $\text{Ʀ} = (\eta_{ij})_{m \times n.}$

$$\text{Ʀ} = \begin{matrix} \\ \text{q}_1 \\ \text{q}_2 \\ \vdots \\ \text{q}_m \end{matrix} \begin{bmatrix} e_1 & e_2 & \dots & e_n \\ \eta_{11}^{(m)} & \eta_{12}^{(m)} & \dots & \eta_{1n}^{(m)} \\ \eta_{21}^{(m)} & \eta_{22}^{(m)} & \dots & \eta_{2n}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{m1}^{(m)} & \eta_{m2}^{(m)} & \dots & \eta_{mn}^{(m)} \end{bmatrix}$$

To evaluate the MADM problem, the following steps are followed:

i). In decision matrix (DM) $\text{Ʀ} = (\eta_{ij})_{m \times n}$, if there are two main types of criteria—benefit and cost types—using the following equations, all cost type parameters must be normalized into benefit type:

$$\eta_{ij} = \begin{cases} \eta_{ij}; & \text{cost type criteria} \\ (\eta_{ij})^c; & \text{benefit type criteria} \end{cases}$$

ii). Aggregate the DM by using the SVNHFAAWA AO;

$$SVNHFAAWA_{\text{ш}}(и_1, и_2, \dots, и_k) = \oplus_{j=1}^{k} \text{ш}_j и_j$$

$$= \cup_{\text{f}_{и1}, \text{f}'_{и1}, \text{f}''_{и1} \in и_1, \dots, \text{f}_{иj}, \text{f}'_{иj}, \text{f}''_{иj} \in и_j} \begin{pmatrix} 1 - \varepsilon^{-(\sum_{j=1}^{k} \text{ш}_j (-ln(1-\text{f}_{иj}))^{\mathfrak{Z}})^{\frac{1}{\mathfrak{Z}}}}, \\ \varepsilon^{-(\sum_{j=1}^{k} \text{ш}_j (-ln(\text{f}'_{иj}))^{\mathfrak{Z}})^{\frac{1}{\mathfrak{Z}}}}, \\ \varepsilon^{-(\sum_{j=1}^{k} \text{ш}_j (-ln(\text{f}''_{иj}))^{\mathfrak{Z}})^{\frac{1}{\mathfrak{Z}}}} \end{pmatrix} \tag{20}$$

iii). Evaluate the score value by using Eq.(4)

$$\text{ʂ}(и) = \frac{1}{3}\left(\frac{1}{l}\sum_{j=1}^{l} \mu_j + \frac{1}{m}\sum_{j=1}^{m}(1 - \eta_j) + \frac{1}{n}\sum_{j=1}^{n}(1 - \nu_j)\right) \tag{4}$$

If there is a variation in the score values we just go to step 4 but if there is no variation, then evaluate the $\text{ɑ}(и)$ by using Eq.(5), if the values of $\text{ɑ}(и)$are variate then go to step 4 otherwise evaluate the $\text{c}(и)$ by using Eq. (6).

iv). Rank $\text{q}_k (k = 1,2,3, \dots, j)$ and choose the most desired ones depending on the $\text{ʂ}(и)$.

5. Illustrative Example

Now, we will discuss the illustrative example of the above-mentioned approach.

5.1 *Case Study*

In the past few decades, smog has become the most alarming problem in Pakistan. Smog, the combination of smoke and fog, is produced when atmospheric ozone, water vapors, and air pollutants interact with each other. This smog is becoming significantly more severe due to poor air quality and a lot of polluting substances produced by vehicles and industries present in the air. It is extremely damaging to everything in nature, including individuals, creatures, and plants alike, and may lead to several harmful problems, some of which are life-threatening, such as lung cancer. The factors that contribute to the rise in the level of air pollutants in Pakistan include the use of heavy cars, industrial growth, urbanization, humid conditions, and lack of planting, etc.

In this scenario, we have discussed the best approach to minimize smog from the environment. Consider $q_i = \{q_1, q_2, q_3, q_4\}$ be the collection of the alternatives where

q_1: Hydrogen fuel additive; which improves the combustion cycles present in automobiles and reduces the level of air pollution.

q_2: Photocatalytic materials that can lower environmental pollution production.

q_3: Autonomous vehicles that can improve the efficiency of fuel and lower the emission levels of pollutants.

q_4: Air purification building towers which used to purify the air by absorbing the pollutants from the surroundings.

All these contribution of the technology is evaluated by the following parameters $e_i = \{e_1, e_2, e_3, e_4, e_5\}$ i.e. economic factor, risk factor, quality and reliability factor, environmental factor, and global impact on climate factor respectively.

The values assigned to each alternative correspond to each parametric value by the DM is recorded in the form of SVNHFE, displayed in Table 1.

Table 1. SVNHF DM.

	e_1	e_2	e_3	e_4	e_5
q_1	$\begin{pmatrix}\{0.30,0.50\},\\ \{0.40,1\},\\ \{0.1,0.3\}\end{pmatrix}$	$\begin{pmatrix}\{0.30,0.0001\},\\ \{0.2,0.1\},\\ \{0.40,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.50,0.70\},\\ \{0.40,0.40\},\\ \{0.50,0.50\}\end{pmatrix}$	$\begin{pmatrix}\{0.40,0.50\},\\ \{0.10,0.30\},\\ \{0.8,0.1\}\end{pmatrix}$	$\begin{pmatrix}\{0.20,0.0001\},\\ \{0.3,0.50\},\\ \{0.10,0.20\}\end{pmatrix}$
q_2	$\begin{pmatrix}\{0.40,0.60\},\\ \{0.20,0.20\},\\ \{0.10,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.10,0.20\},\\ \{0.10,0.40\},\\ \{0.20,0.30\}\end{pmatrix}$	$\begin{pmatrix}\{0.40,0.50\},\\ \{0.60,0.1\},\\ \{0.1,0.2\}\end{pmatrix}$	$\begin{pmatrix}\{0.50,0.3\},\\ \{0.20,0.15\},\\ \{0.1,0.3\}\end{pmatrix}$	$\begin{pmatrix}\{0.60,0.40\},\\ \{0.40,0.20\},\\ \{0.20,0.50\}\end{pmatrix}$
q_3	$\begin{pmatrix}\{0.30,0.60\},\\ \{0.10,0.15\},\\ \{0.30,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.10,0.50\},\\ \{0.30,0.20\},\\ \{0.30,0.20\}\end{pmatrix}$	$\begin{pmatrix}\{0.30,0.20\},\\ \{0.60,0.35\},\\ \{0.40,0.30\}\end{pmatrix}$	$\begin{pmatrix}\{0.20,0.40\},\\ \{0.50,0.20\},\\ \{0.3,0.10\}\end{pmatrix}$	$\begin{pmatrix}\{0.10,0.20\},\\ \{0.30,0.40\},\\ \{0.50,0.20\}\end{pmatrix}$
q_4	$\begin{pmatrix}\{0.50,0.20\},\\ \{0.20,0.15\},\\ \{0.20,0.50\}\end{pmatrix}$	$\begin{pmatrix}\{0.60,0.40\},\\ \{0.40,0.25\},\\ \{0.70,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.50,0.5\},\\ \{0.30,0.1\},\\ \{0.20,0.50\}\end{pmatrix}$	$\begin{pmatrix}\{0.60,0.70\},\\ \{0.10,0.30\},\\ \{0.30,0.50\}\end{pmatrix}$	$\begin{pmatrix}\{0.70,0.20\},\\ \{0.40,0.40\},\\ \{0.40,0.30\}\end{pmatrix}$

So, the steps of the suggested technique that were put into practice are below:

i). Since criteria values are of the same type, the DM does not need to be normalized.

ii). Now, for the given attributes assume that $\mathfrak{z} = 1$, $\omega = \{0.2, 0.4, 0.05, 0.25, 0.1\}$be the weight values assigned by the decision-makers. Aggregate the decision matrix by using the SVNHFAAWA aggregate operator by using Eq. (8), we get Table 2 as follows,

Table 2. Aggregated DM by using SVNHFAAWA.

Alternative	Aggregated decision matrix by using SVNHFAAWA
q_1	$(\{0.3287, 0.3107\}, \{0.2082, 0.1656\}, \{0.5101, 0.2519\})$
q_2	$(\{0.3525, 0.3607\}, \{0.1716, 0.2372\}, \{0.1414, 0.3277\})$
q_3	$(\{0.1793, 0.4629\}, \{0.2832, 0.2081\}, \{0.3202, 0.1971\})$
q_4	$(\{0.5890, 0.4549\}, \{0.2427, 0.2365\}, \{0.4088, 0.4345\})$

iii). Evaluating the $\varsigma(\text{и})$ by using Eq.(4), we get Table 3.

Table 3. $\varsigma(\text{и})$ of alternatives by using data from Table 2.

Alternative	Score Value by using SVNHFAAWA
q_1	0.5839
q_2	0.6392
q_3	0.6055
q_4	0.6202

iv). Rank the alternative according to decreasing $\varsigma(\text{и})$ as shown in Figure 1.

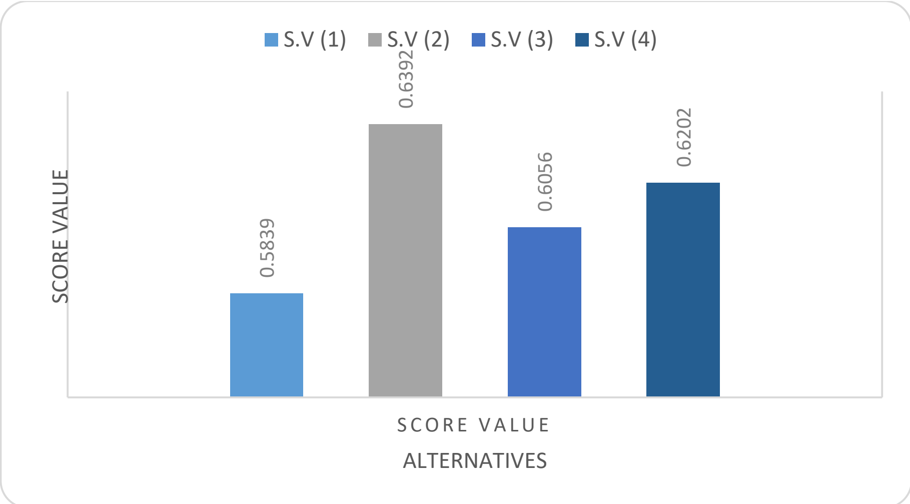

Figure 1. Ranking of Alternative by $\varsigma(\text{и})$.

So, to minimize the smog and air pollutants from the environment, a MADM problem is evaluated by AO under the SVNHF environment. Each decision-maker assigns values to each alternative, which are then stored in the form of a matrix and then evaluated by the proposed operator and gives the ranking of alternatives by using the score function. According to the proposed AO, q_2 is the best alternative which plays an important role in reducing the level of smog and making the air pollutant free. The graphical representation of the ranking of alternatives is displayed in Figure 1.

5.2 Further Discussion

The objective of the suggested AO is to show how each criterion value relates to the other and provide an option for dealing with the decision maker's extremely high or extremely low values, which may have an impact on the outcomes. To prove the flexibility and adaptability of the defined operators, we provide the scoring outcomes of the alternative by allocating various numbers to ʒ.

Table 4. Ranking values of alternative by assigning different values to $\mathfrak{z}$ under the proposed AO.

	Score Value				**Ranking**
	q_1	q_2	q_3	q_4	
$\mathfrak{z} = \mathbf{2}$	0.5028	0.4058	0.4379	0.4331	$q_1 > q_3 > q_4 > q_2$
$\mathfrak{z} = \mathbf{3}$	0.5521	0.4273	0.4797	0.4834	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{5}$	0.6015	0.4499	0.5226	0.5312	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{6}$	0.6165	0.4569	0.5356	0.544	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{10}$	0.6518	0.4745	0.5653	0.5709	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{20}$	0.6835	0.4937	0.5905	0.5933	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{50}$	0.7034	0.5075	0.6061	0.6074	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{100}$	0.7100	0.5121	0.6114	0.6120	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{150}$	0.7123	0.5136	0.6131	0.6136	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{350}$	0.7148	0.5154	0.6151	0.6153	$q_1 > q_4 > q_3 > q_2$
$\mathfrak{z} = \mathbf{500}$	0.7153	0.4825	0.6156	0.6157	$q_1 > q_4 > q_3 > q_2$

So, by putting the different values of $\mathfrak{z}$, we get the various alternatives which is illustrated in Table 4.

5.3 Comparison Analysis

In this section, we have compared our study with a variety of other methods and approaches to DM, to determine the reliability of the analyzed approach, shown in Table 5. For this purpose, we examined a few basic operators including SVNHF weighted averaging (SVNHFWA) and SVNHF weighted geometric (SVNHFWG) which are defined by Riaz et al. [31] and SVNHF Dombi weighted averaging (SVNHFDWA) and SVNHFD Dombi weighted geometric (SVNHFDWG) [32]. So, by considering the above data displayed in Table 1, we presented the comparison analysis in Table 5, which is also shown in Figure 2.

Moreover, this study was extended for the analysis of electric vehicles [33], capacitor data analysis [34], evaluation of medial best bearing ring [35], wind turbine development evaluation [36], and then by using the entropy measures [37], distance measures [38] and by TOPSIS approach [39]. It further can be extended to the set structures like [40-42].

Table 5. Comparison analysis.

Operator		**Score Values**	**Ranking**
SVNHFAAWA	Current work	0.5839,0.6392,0.6056,0.6202	$q_2 > q_4 > q_3 > q_1$
SVNHFWA	Riaz et al. [31]	0.6160,0.6392,0.6055,0.6231	$q_2 > q_4 > q_1 > q_3$
SVNHFWG	Riaz et al. [31]	0.5118,0.5978,0.5774,0.5798	$q_2 > q_4 > q_3 > q_1$
SVNHFDWA	Saha et al. [32]	0.6558,0.6571,0.6215,0.6482	$q_2 > q_1 > q_4 > q_3$
SVNHFDWG	Saha et al. [32]	0.4893,0.5775,0.5645,0.5618	$q_2 > q_3 > q_4 > q_1$

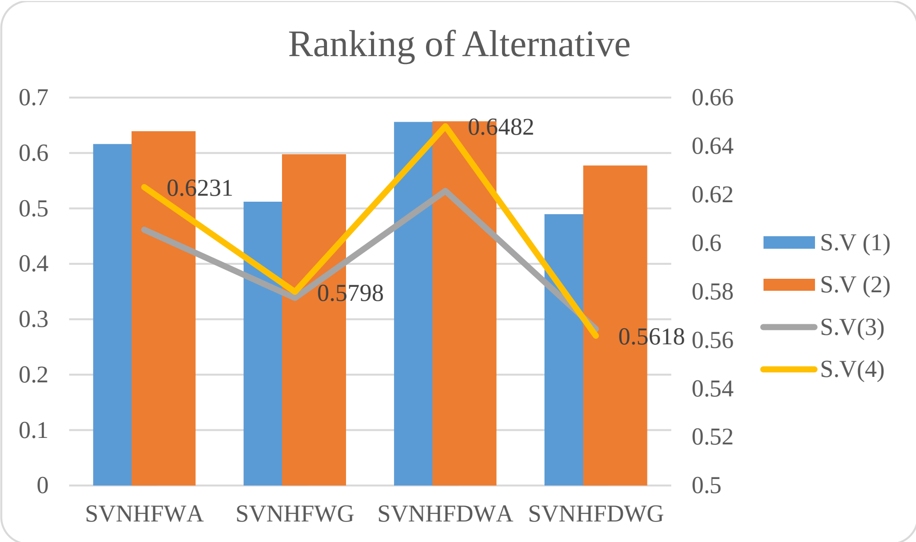

Figure 2. Graphical visualization of comparative analysis.

6. Conclusion

Since AOs are generally mathematical operations that aim to transform a set of numeric data into one numeric value AA plays an important role as a computational operator. So, given the information presented, we analyzed the following concepts:

i). We have presented the theory of SVNHFAAWA, SVNHFAAOWA, and SVNHFAAHA in the presence of AAAO based on the SVNHFS environment.

ii). We evaluate some of its properties which include Idempotency, monotonicity, and boundedness.

iii). We have provided a MADM approach for suggested techniques and put an example that will assist readers in understanding.

iv). Finally, we've demonstrated the applicability of the suggested method by showing the comparison between the proposed and prior methods.

The AAAO plays an important role as a computational operator which is favorably dealing with ambiguous information under the SVNHF environment. However, in various cases, it is not valid if the data was presented by the analyst in the form of complex SVNHF (CSVNHF) information as it cannot be processed by the proposed theory. Therefore, the theory of the CSVNHF set must be proposed in the future.

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

The authors declare no competing interests in the research.

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